Denumerants of 3-numerical semigroups

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Abstract. Denumerants of numerical semigroups are known to be difficult to obtain, even with small embedding dimension of the semigroups. In this work we give some results on denumerants of 3-semigroups $S = \langle a, b, c \rangle$. The time efficiency of the resulting algorithms range from $O(1)$ to $O(c)$. Closed expressions are obtained under certain conditions.

Key words: Denumerant, L-shapes, numerical semigroup, factorization.

1 Introduction

Let $\mathbb{N}_0$ be the set of non negative integers. Given a set $A = \{a_1, \dots, a_n\} \subset \mathbb{N}_0$, $\gcd(a_1, \dots, a_n) = 1$, the n -numerical semigroup $S = S(A)$ generated by A is defined by $\langle a_1, \dots, a_n \rangle = \{x_1a_1 + \dots + x_na_n : x_1, \dots, x_n \in \mathbb{N}_0\}$. If A is a minimal set of generators then S has *embedding dimension* $e(S) = n$. An element $m \in S$ has a *factorization* $(t_1, \dots, t_n)$ in S if $m = t_1a_1 + \dots + t_na_n$. The set of factorizations of m in S is denoted by $\mathcal{F}(m, S) = \{(x_1, \dots, x_n) \in \mathbb{N}_0^n : x_1a_1 + \dots + x_na_n = m\}$. The *denumerant* of m in S is the cardinality $d(m, S) = |\mathcal{F}(m, S)|$.

In this work we give some results on denumerants of generic elements of embedding dimension three numerical semigroups $S = \langle a, b, c \rangle$. Algorithms of time-cost ranging from $O(1)$ to $O(c)$ (in the worst case) are also derived. We use *minimum distance diagrams* related to S as a main tool. In particular, we use the main results given in [1].

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2 Tools

From now on we denote the equivalence class of n modulo N as $[n]_N$. Given $m \in S \setminus \{0\}$, the *Apéry set* of m in S is $\text{Ap}(m, S) = \{s \in S : s - m \notin S\}$. This set encodes many properties of semigroup. It can be shown that $|\text{Ap}(m, S)| = m$ and $\text{Ap}(m, S) = \{l_0, \dots, l_{m-1}\}$ with $l_k \in [k]_m$ for $0 \leq k < m$.

Minimum Distance Diagrams, $\mathcal{H}$, of numerical semigroups are used to study many distance-related and distribution of the semigroup elements. The minimum distance diagrams related to $S = \langle a_1, \dots, a_n \rangle$ are sets of cardinality $|\mathcal{H}| = a_n$ of unitary cubes $\llbracket i_1, \dots, i_{n-1} \rrbracket = [i_1, i_1 + 1] \times \dots \times [i_{n-1}, i_{n-1} + 1]$ in the first orthant of $\mathbb{R}^{n-1}$ with coordinates $(i_1, \dots, i_{n-1}) \in \mathbb{N}_0^{n-1}$ representing factorizations of the elements of $\text{Ap}(a_n, S)$, where $[s, t] = \{r \in \mathbb{R} : s \leq r \leq t\}$. Each element $l_k \in \text{Ap}(a_n, S)$ is represented by exactly one cube $\llbracket i_1, \dots, i_{n-1} \rrbracket$ in $\mathcal{H}$ with $i_1 a_1 + \dots + i_{n-1} a_{n-1} = l_k$. Given any $m \in S$, with $m \in [l_k]_{a_n}$, we call the *basic coordinates* of m with respect to $\mathcal{H}$ to $(x_1, \dots, x_{n-1}) \in \mathbb{N}_0^{n-1}$ if $\llbracket x_1, \dots, x_{n-1} \rrbracket \in \mathcal{H}$ and $x_1 a_1 + \dots + x_{n-1} a_{n-1} = l_k$. We also call the *basic factorization* of m in S with respect to $\mathcal{H}$ to $(x_1, \dots, x_{n-1}, \frac{m-l_k}{a_n}) \in \mathcal{F}(m, S)$.

These diagrams were used in [1] to study some aspects on factorization and catenary degree of 3-numerical semigroups $S = \langle a, b, c \rangle$. In particular, we recall here some nomenclature and results. Diagrams related to S are L-shaped or rectangles (considered as degenerated L-shapes with $wy = 0$) and are denoted by $\mathcal{H} = L(l, h, w, y)$, where these entries are the lengths of the sides $0 \leq w < l$, $0 \leq y < h$, $\gcd(l, h, w, y) = 1$ and $lh - wy = c$. The diagram $\mathcal{H}$ tessellates $\mathbb{R}^2$ by translation through the vectors $\mathbf{u} = (l, -y)$ and $\mathbf{v} = (-w, h)$. See [2,3] for more details.

For the parameters $\delta = (la - yb)/c$ and $\theta = (hb - wa)/c$, we have $\delta \geq 0$, $\theta \geq 0$, $\delta + \theta > 0$, $\delta + y > 0$ and $\theta + w > 0$ whenever $\mathcal{H} = L(l, h, w, y)$ is a minimum distance diagram of $S = \langle a, b, c \rangle$.

Theorem 1 ([1, Th. 2]). *Given $\mathcal{H} = L(l, h, w, y)$ a minimum distance diagram of $S = \langle a, b, c \rangle$ and $m \in S$, let (x_0, y_0, z_0) be the basic factorization of m in S respect to $\mathcal{H}$, then*

$$d(m, S) = 1 + \left\lfloor \frac{z_0}{\delta + \theta} \right\rfloor + \sum_{k=0}^{\left\lfloor \frac{z_0}{\delta + \theta} \right\rfloor} (S_k + T_k),$$

with

$$S_k = \begin{cases} \left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor & \text{if } \delta = 0, \\ \left\lfloor \frac{z_0 - k(\delta + \theta)}{\delta} \right\rfloor & \text{if } y = 0, \\ \min \left\{ \left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor, \left\lfloor \frac{z_0 - k(\delta + \theta)}{\delta} \right\rfloor \right\} & \text{if } \delta y \neq 0, \end{cases}$$

and

$$T_k = \begin{cases} \left\lfloor \frac{x_0 + k(l-w)}{w} \right\rfloor & \text{if } \theta = 0, \\ \left\lfloor \frac{z_0 - k(\delta + \theta)}{\theta} \right\rfloor & \text{if } w = 0, \\ \min\left\{ \left\lfloor \frac{x_0 + k(l-w)}{w} \right\rfloor, \left\lfloor \frac{z_0 - k(\delta + \theta)}{\theta} \right\rfloor \right\} & \text{if } \theta w \neq 0. \end{cases}$$

This theorem does not give an efficient algorithm for computing denumerants. The reason is not the expression of minima appearing in S_k when $\delta y \neq 0$ or in T_k when $\theta w \neq 0$, these minima can be left out for $k \geq \left\lceil \frac{yz_0 - \delta y_0}{\delta(h-y) + y(\delta + \theta)} \right\rceil$ in S_k when $\delta y \neq 0$ and for $k \geq \left\lceil \frac{wz_0 - \theta x_0}{\theta(l-w) + w(\delta + \theta)} \right\rceil$ in T_k when $\theta w \neq 0$. The reason is the expression of the sum. Essentially we have to compute sums like

$$\sum_{k=0}^N \left\lfloor \frac{\alpha \pm k\beta}{q} \right\rfloor = \sum_{k=0}^N (\bar{\alpha} \pm k\bar{\beta}) + \sum_{k=0}^N \left\lfloor \frac{s \pm kt}{q} \right\rfloor,$$

for $\alpha, \beta, \bar{\alpha}, \bar{\beta}, q \in \mathbb{N}$, with $q \neq 0$, $\alpha = \bar{\alpha}q + s$, $0 \leq s < q$, and $\beta = \bar{\beta}q + t$, $0 \leq t < q$.

3 Trying $\sum_{k=0}^N \left\lfloor \frac{s \pm kt}{q} \right\rfloor$

The efficiency of computing denumerants using these sums is compromised because of the number of terms to be added N , $O(N) = O(z_0)$. The coordinate z_0 of a basic factorization can become very large. Thus, we need to save add up all those terms.

In this section we try the sum $\sum_{k=0}^N \left\lfloor \frac{s+kt}{q} \right\rfloor$ with $0 \leq s, t < q$. The sum with the minus sign can be solved using an analogous method by symmetry. The idea of the method we propose here is making this addition as a Lebesgue-like discrete integration. Let us consider the graph of the function $f(x) = \left\lfloor \frac{s+tx}{q} \right\rfloor$ and pay attention to the constant parts of the graph. Let us denote the interval $I_m = [x_m, x_{m+1})$ where the function f has constant value m , that is $x_m = \frac{mq-s}{t}$ (if $t = 0$ the entire sum equals zero). Then, some main properties hold:

- (1) There are $M + 1$ different intervals, with $M = \left\lfloor \frac{s+Nt}{q} \right\rfloor$. Each interval I_m has length $x_{m+1} - x_m = \frac{q}{t}$ except, possibly, I_0 and/or I_M .
- (2) $\left\lfloor \frac{q}{t} \right\rfloor \leq |I_m \cap \mathbb{N}| \leq \left\lceil \frac{q}{t} \right\rceil$ except, possibly, I_0 and/or I_M .
- (3) Those intervals I_m with $x_m \in \mathbb{N}_0$ are iS-type intervals. Intervals with $|I_m \cap \mathbb{N}| = \left\lceil \frac{q}{t} \right\rceil$ are mS-type intervals. Intervals iS are also mS except, possibly, I_M . There can be mS-intervals that are not iS-intervals.
- (4) The behaviour of intervals at the same relative position between two consecutive iS-intervals is the same. There are $t - 1$ intervals between two

consecutive iS-intervals. The behaviour of intervals I_m is periodic of period t . Depending on the values s, t and q , this period is not the minimum one.

Fig. 1. Graph of f with $s = 0$, $t = 3$, $q = 10$

Figure 1 shows the graph of f with parameters $s = 0$, $t = 3$ and $q = 10$. Note the 3-periodic behaviour of the intervals I_m , $m \in \{0, \dots, 9\}$. In this case, only the iS-intervals contain $\lceil \frac{q}{t} \rceil = 4$ integral values, except I_M (with $M = 9$) for $N = 30$ that is only one integral point $I_9 = \{9\}$.

Fig. 2. Graph of f with $s = 5$, $t = 3$, $q = 15$

We study two main cases, $\gcd(t, q) > 1$ and $\gcd(t, q) = 1$. Figure 1 is the simple behaviour of the second case. The first case can be studied for $t \mid q$ and $t \nmid q$. When $t \mid q$, all intervals have the same behavior except, possibly, the first and the last ones; this is the case of Figure 2. Note that all intervals are of type mS and there is no iS-intervals, essentially the existence of iS-intervals depends on the value s . When $t \nmid q$ and $\gcd(t, q) = g > 1$, depending on the value of s appear iS-intervals and the period is not t but t/g ; Figure 3 shows this case.

Depending on the behavior of intervals I_m the value of the sum can be arranged in an almost closed expression that sometimes results in a closed expression. The following results take into account all cases (that are more than the examples appearing in the three figures mentioned above).

From now on set $M = \left\lfloor \frac{s+Nt}{q} \right\rfloor$ and $x_M = \frac{Mq-s}{t}$.

Fig. 3. Graph of f with $s = 0$, $t = 6$, $q = 14$

Theorem 2 (Case $t \mid q$). Assume $q = \bar{q}t$. Then,

$$\sum_{k=0}^N \left\lfloor \frac{s+kt}{q} \right\rfloor = \bar{q} \frac{M(M-1)}{2} + M(N - \lfloor x_M \rfloor + 1).$$

Theorem 2 is a simple case for applying the Lebesgue-like discrete summation. Thi theorem give closed expressions for denumerants.

Now we give the cases $t \nmid q$ with $\gcd(t, q) = 1$ or $\gcd(t, q) = g > 1$. To describe it we must define another type of intervals. Define $\hat{s}$ and $\hat{q}$ by

$$q = \bar{q}t + \hat{q}, \quad s = \bar{s}t + \hat{s}, \quad 0 \leq \hat{s}, \hat{q} < t.$$

We say that I_m is an hS-interval if $(\hat{s} - m\hat{q})(\bmod t) < \hat{q}$. We denote the sets $I = \{i_1, \dots, i_\alpha\}$, $J = \{j_1, \dots, j_n\}$, $K = \{k_1, \dots, k_v\}$ of ordered indices of all hS-type intervals taken from some region to be defined. We also denote the sums $S_I = \sum_{l=1}^\alpha i_l$, $S_J = \sum_{l=1}^n j_l$, $S_K = \sum_{l=1}^v k_l$, $S = \bar{q} \frac{(M-1)M}{2} + M(N - \lfloor x_M \rfloor + 1)$ and $\mathbb{S} = \sum_{k=0}^N \left\lfloor \frac{s+kt}{q} \right\rfloor$.

Theorem 3 (Case $t \nmid q$ and $\gcd(t, q) = 1$). Set $m_0 \equiv q^{-1}s(\bmod t)$ with $m_0 \in \{0, \dots, t-1\}$ and $u = \lfloor \frac{M-m_0-1}{t} \rfloor$.

- (a) If $m_0 = 0$ and $u \leq 0$, or $m_0 = M$, take $K \subset [1, M-1]$. Then $\mathbb{S} = S + S_K$.
- (b) If $m_0 = 0$ and $u > 0$, take $J \subset [0, t-1]$ and $K \subset [ut, M-1]$. Then $\mathbb{S} = S + uS_J + nt \frac{(u-1)u}{2} + S_K$.
- (c) If $0 < m_0 < M$ and $m_0 + t \geq M$, take $I \subset [1, m_0-1]$ and $K \subset [m_0, M-1]$. Then $\mathbb{S} = S + S_I + S_K$.
- (d) If $0 < m_0 < M$ and $m_0 + t < M$, take the sets $I \subset [1, m_0-1]$ and $J \subset [m_0, m_0+t-1]$. Take also $K \subset [m_0+ut, M-1]$ whenever $u > 0$ and $K = \emptyset$ otherwise (thus $S_K = 0$). Then $\mathbb{S} = S + S_I + uS_J + nt \frac{(u-1)u}{2} + S_K$.

Theorem 4 (Case $t \nmid q$ and $\gcd(t, q) = g > 1$). Set $\tilde{t} = t/g$. Take $J \subset [0, \tilde{t} - 1]$. Set $m_0 = j_1$ and $u = \left\lfloor \frac{M-m_0-1}{\tilde{t}} \right\rfloor$.

(a) If $m_0 \geq M$, then $\mathbb{S} = S$.

(b) If $0 \leq m_0 < M$, take $K \subset [m_0 + u\tilde{t}, M - 1]$. Then $\mathbb{S} = S + uS_J + n\tilde{t}\frac{(u-1)u}{2} + S_K$.

Theorems 3 and 4 do not give closed expressions for denumerants because of the computation of the sets of indices. This task has order $O(t)$ in the worst case. However, the resulting algorithm is efficient. Similar results can be obtained to compute the sum $\sum_{k=0}^N \left\lfloor \frac{s-kt}{q} \right\rfloor$.

4 Application example

Each 3-semigroup has its own denumerant's information encoded in the related L-shapes. As an example, let us consider for instance $S = \langle 121, 1111, 2323 \rangle$ with related L-shape $L(101, 23, 0, 11)$. Thus, $\delta = 0$ and $\theta = 25553$.

This semigroup belongs to the general case $\delta = w = 0$, according with Theorem 1 (there are exactly five generic cases $\{\delta = 0, w = 0\}$, $\{\delta = 0, w \neq 0\}$, $\{\theta = 0, y = 0\}$, $\{\theta = 0, y \neq 0\}$ and $\{\delta \neq 0, \theta \neq 0\}$). Let us consider the case $\{\delta = 0, w = 0\}$ (then $y\theta \neq 0$). Assume $S = \langle a, b, c \rangle$ and $\mathcal{H} = L(l, h, 0, y)$ belongs to this case. Then, the denumerant is given by

$$d(m, S) = 1 + \left\lfloor \frac{z_0}{\theta} \right\rfloor + \sum_{k=0}^{\left\lfloor \frac{z_0}{\theta} \right\rfloor} \left(\left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor + \left\lfloor \frac{z_0 - k\theta}{\theta} \right\rfloor \right),$$

where (x_0, y_0, z_0) is the basic factorization of $m \in S$ with respect to $\mathcal{H}$. Set $y_0 = \overline{y_0}y + \hat{y}_0$ and $h - y = \overline{n}y + \hat{n}$ with $0 \leq \hat{y}_0, \hat{n} < y$. Then,

$$\sum_{k=0}^{\lambda_0} \left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor = (1 + \lambda_0)[\overline{y_0}] + \frac{1}{2}\lambda_0\hat{n} + \sum_{k=0}^{\lambda_0} \left\lfloor \frac{\hat{y}_0 + k\hat{n}}{y} \right\rfloor,$$

with $\lambda_0 = \left\lfloor \frac{z_0}{\theta} \right\rfloor$. Similarly,

$$\sum_{k=0}^{\lambda_0} \left\lfloor \frac{z_0 - k\theta}{\theta} \right\rfloor = \sum_{k=0}^{\lambda_0} (\lambda_0 - k) = \frac{1}{2}\lambda_0(1 + \lambda_0).$$

Therefore, the denumerant is given by

$$d(m, S) = (1 + \lambda_0)[1 + \overline{y_0}] + \frac{1}{2}\lambda_0(1 + \hat{n}) + \sum_{k=0}^{\lambda_0} \left\lfloor \frac{\hat{y}_0 + k\hat{n}}{y} \right\rfloor.$$

For $S = \langle 121, 1111, 2323 \rangle$, let us take the family of elements of $m_t \in S$ given by the basic factorization $(x_0, y_0, z_0) = (1, 1, t)$, i.e. $m_t = 1232 + 2323t$. Then,

in this case, the parameters are $\overline{y}_0 = 0$, $\hat{y}_0 = 1$, $\overline{n} = 1$, $\hat{n} = 1$, $y = 11$ and $\lambda_0 = \lfloor \frac{t}{11} \rfloor$. Thus, the denumerant is

$$d(m_t, S) = (1 + \lfloor \frac{t}{11} \rfloor)^2 + \sum_{k=0}^{\lfloor \frac{t}{11} \rfloor} \left\lfloor \frac{1+k}{11} \right\rfloor.$$

Applying the results given in Section 2, we can take large values for t . For instance, when $t = 10^{10^3}$, the denumerant $d(m_t, \langle 121, 1111, 2323 \rangle)$ is

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86401202103681442373725632370418508815236113947008384865900
18355482581937070575687706067297170372656872124127106355013
83705399205261030784757611250382814450458614863741996576336
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12806317148265826264562297662155082803832709199101981830544
850830347007054848234787152293363588331442400219780
    
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