Whitehead problems for words in $\mathbb{Z}^m \times F_n$

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Abstract

We solve the Whitehead problem for automorphisms, monomorphisms and endomorphisms in $\mathbb{Z}^m \times F_n$ after giving an explicit description of each of these families of transformations.

We generically call *Whitehead problems* for a finitely presented group G the problems consisting in, given two objects (of the same certain suitable kind $\mathcal{O}$) in G , and a family $\mathcal{F}$ of transformations, decide whether there exists an element in $\mathcal{F}$ sending one object to the other. Specifically we will write $\text{WhP}(\mathcal{O}, \mathcal{F})$ to mean the Whitehead problem with objects in $\mathcal{O}$ and transformations in $\mathcal{F}$, i.e.

$$\text{WhP}(\mathcal{O}, \mathcal{F}) \equiv \text{¿} \exists \varphi \in \mathcal{F} \text{ such that } o_1 \xrightarrow{\varphi} o_2?_{(o_1, o_2 \text{ in } \mathcal{O})}.$$

It is customary to include as a part of the problem, the search of one of such transformations, in case that there exists. So will we.

The kind of “objects in G ” usually considered include elements (i.e. words in the generators), subgroups, and conjugacy classes, as well as tuples of these; while the typical families of transformations (which we will always think acting on the right) are those of automorphisms, monomorphisms, epimorphisms and endomorphisms of G ; we denote them respectively by $\text{Aut } G$, $\text{Mon } G$, $\text{Epi } G$ and $\text{End } G$.

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This whole family of problems arise from the seminal $\text{WhP}(F_n, \text{Aut } F_n)$, where F_n denotes the free group on n generators. It was proposed and solved by Whitehead in [6] using a (now classical) technique called peak-reduction.

In this note we will deal with Whitehead problems for words in finitely generated free-abelian times free groups (see [4] for full details). In sake of notational easiness we will hereafter usually abbreviate $G = \mathbb{Z}^m \times F_n$. Concretely we will solve $\text{WhP}(G, \text{Aut } G)$, $\text{WhP}(G, \text{Mon } G)$ and $\text{WhP}(G, \text{End } G)$. It is not surprising that the (already solved) corresponding problems for $\mathbb{Z}^m$ and F_n emerge when considering Whitehead problems for G .

For the free-abelian groups the problems considered become those of the existence of solutions (of certain type) for integer matrix equations of the form $\mathbf{a} \cdot \mathbf{X} = \mathbf{b}$. This can be easily decided using linear algebra.

Proposition 1. *Let $m \geq 1$, then*

- (i) $\text{WhP}(\mathbb{Z}^m, \text{Aut } \mathbb{Z}^m)$ is solvable.
- (ii) $\text{WhP}(\mathbb{Z}^m, \text{Mon } \mathbb{Z}^m)$ is solvable.
- (iii) $\text{WhP}(\mathbb{Z}^m, \text{End } \mathbb{Z}^m)$ is solvable. □

The same problems for the free group F_n are much more complicated. As mentioned above, the case of automorphisms was already solved by Whitehead back in the 30's of the last century. The case of endomorphisms can be solved by writing a system of equations over F_n (with unknowns being the images of a given free basis for F_n), and then solving it by the powerful Makanin's algorithm. Finally, the case of monomorphisms was recently solved by Ciobanu and Houcine.

Theorem 2. *Let $n \geq 2$, then*

- (i) [Whitehead, [6]] $\text{WhP}(F_n, \text{Aut } F_n)$ is solvable.
- (ii) [Ciobanu-Houcine, [1]] $\text{WhP}(F_n, \text{Mon } F_n)$ is solvable.
- (iii) [Makanin, [5]] $\text{WhP}(F_n, \text{End } F_n)$ is solvable. □

So, the auto, mono and endo Whitehead problems (for words) are solvable for both $\mathbb{Z}^m$ and F_n . For $G = \mathbb{Z}^m \times F_n$ though, these problems turn out to be more than the mere juxtaposition of the corresponding problems for its factors. That is because the endomorphisms of G are more than pairs of endomorphisms of $\mathbb{Z}^m$ and F_n as well. It is not difficult to obtain a complete description of them imposing the preservation of the (commutativity) relations defining G .

Proposition 3. *The endomorphisms of $G = \mathbb{Z}^m \times F_n$ are given by*

$$\Psi_{\phi, \mathbf{Q}, \mathbf{P}} : (\mathbf{a}, u) \mapsto (\mathbf{a}\mathbf{Q} + \mathbf{u}\mathbf{P}, u\phi_{\mathbf{a}})$$

where $\mathbf{u} = u^{\mathbf{ab}} \in \mathbb{Z}^n$, $\mathbf{Q}$ and $\mathbf{P}$ are integer matrices, and $\phi_{\mathbf{a}}: F_n \rightarrow F_n$ is either

- (i) an endomorphism $\phi: F_n \rightarrow F_n$ (independent from $\mathbf{a}$), or
- (ii) a map $u \mapsto w^{\alpha(\mathbf{a}, \mathbf{u})}$ where w is a non-proper power in $F_n \setminus \{1\}$ and $\alpha(\mathbf{a}, \mathbf{u}) = \mathbf{a}\mathbf{l}^\top + \mathbf{u}\mathbf{h}^\top \in \mathbb{Z}$ for certain $\mathbf{l} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ and $\mathbf{h} \in \mathbb{Z}^n$.

We will refer to them as type (I) and type (II) endomorphisms of G respectively. $\square$

Note that if $n = 0$ then type (I) and type (II) endomorphisms do coincide. Otherwise, it turns out that type (II) endomorphisms are a sort of degenerated case corresponding to a free contribution from the abelian part while all the injective and exhaustive endomorphisms of G are of type (I). Indeed, viewing $\mathbf{Q}$ as the endomorphism of $\mathbb{Z}^m$ given by right multiplying by $\mathbf{Q}$, we have the following quite natural characterization (note that the matrix $\mathbf{P}$ plays absolutely no role in this matter).

Proposition 4. *Let Ψ be an endomorphism of $G = \mathbb{Z}^m \times F_n$, with $n \geq 2$. Then,*

- (i) Ψ is a monomorphism if and only if it is of type (I) with ϕ a monomorphism of F_n and $\mathbf{Q}$ a monomorphism of $\mathbb{Z}^m$ (i.e. $\det \mathbf{Q} \neq 0$).
- (ii) Ψ is an epimorphism if and only if it is of type (I) with ϕ an epimorphism of F_n and $\mathbf{Q}$ an epimorphism of $\mathbb{Z}^m$ (i.e. $\det \mathbf{Q} = \pm 1$). $\square$

The hopfianity of $\mathbb{Z}^m$ and F_n together with this last proposition provide immediately the following results.

Corollary 5. $\mathbb{Z}^m \times F_n$ is hopfian and not cohopfian. $\square$

Corollary 6. *An endomorphism of $G = \mathbb{Z}^m \times F_n$ ($n \geq 2$) is an automorphism if and only if it is of type (I) with $\phi \in \text{Aut}(F_n)$ and $\mathbf{Q} \in \text{GL}_m(\mathbb{Z})$. $\square$*

Now we have the ingredients to prove the main result of this note.

Theorem 7. *Let $G = \mathbb{Z}^m \times F_n$ with $m \geq 1$ and $n \geq 2$, then*

- (i) $\text{WhP}(G, \text{Aut } G)$ is solvable.
- (ii) $\text{WhP}(G, \text{Mon } G)$ is solvable.
- (iii) $\text{WhP}(G, \text{End } G)$ is solvable.

Sketch of the proof. We are given two elements $(\mathbf{a}, u), (\mathbf{b}, v) \in G$, and have to decide whether there exists an automorphism (resp. monomorphism, endomorphism) of $\mathbb{Z}^m \times F_n$ sending one to the other; and in affirmative case, find one of them.

Using the previous descriptions for each type of transformations in $\mathbb{Z}^m \times F_n$ and separating the free-abelian and free parts, our problems reduce to deciding whether there exist integer matrices $\mathbf{P}, \mathbf{Q}$ and a transformation ϕ of F_n ($\mathbf{Q}$ and ϕ of certain kind depending on the case, see proposition 4) such that the two following independent conditions hold.

$$\left. \begin{aligned} u\phi &= v \\ \mathbf{a}\mathbf{Q} + \mathbf{u}\mathbf{P} &= \mathbf{b} \end{aligned} \right\} \begin{aligned} (1) \\ (2) \end{aligned}$$

Note that the subproblem associated to condition (1) becomes respectively the already solved $\text{WhP}(F_n, \text{Aut } F_n)$, $\text{WhP}(F_n, \text{Mon } F_n)$ and $\text{WhP}(F_n, \text{End } F_n)$ in the cases of autos, monos, and endos of type (I), and is straightforward to check for endos of type (II). Thus, if there is not any ϕ solving each of these problems (for F_n), then the corresponding problem (for G) has no solution either, and we are done.

Otherwise, the decision method provides such a ϕ and our target reduces to solving the subproblem associated to condition (2): given arbitrary elements $\mathbf{a} \in \mathbb{Z}^m$ and $\mathbf{u} \in \mathbb{Z}^n$, decide whether there exist integer matrices $\mathbf{P}$ and $\mathbf{Q}$ (satisfying $\det \mathbf{Q} \neq 0$ in the case of monos, and $\det \mathbf{Q} = \pm 1$ in the case of autos) such that $\mathbf{a}\mathbf{Q} + \mathbf{u}\mathbf{P} = \mathbf{b}$.

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{u} = \mathbf{0}$, these are well known results in linear algebra, otherwise write $\alpha = \gcd(\mathbf{a}) \neq 0$ and $\mu = \gcd(\mathbf{u}) \neq 0$. Then, the problems reduce to test whether the following linear system of equations

$$\left. \begin{aligned} \alpha x_1 &+ \mu y_1 &= b_1 \\ &\vdots &\vdots \\ \alpha x_m &+ \mu y_m &= b_m \end{aligned} \right\} \quad (3)$$

has integral solutions $x_1, \dots, x_m, y_1, \dots, y_m \in \mathbb{Z}$ (with no extra condition in the case of endos, satisfying $(x_1, \dots, x_m) \neq \mathbf{0}$ in the case of monos, and satisfying $\gcd(x_1, \dots, x_m) = 1$ in the case of autos).

So, for the case of endos the decision is a standard argument in linear algebra. In the case of monomorphisms, the condition $(x_1, \dots, x_m) \neq \mathbf{0}$ turns out to be superfluous and the same standard argument works, while the more involved case of autos become an exercise in elementary arithmetic and is decidable as well.

Finally, observe that in any of the affirmative cases, we can use the description in proposition 3 to reconstruct a transformation Ψ (of the corresponding type) such that $(\mathbf{a}, u)\Psi = (\mathbf{b}, v)$. $\square$

We note that, very recently, a new version of the classical peak-reduction theorem has been developed by M. Day [3] for an arbitrary partially commutative group (see also [2]). These techniques allow the author to solve the Whitehead problem for this kind of groups, in its variant relative to tuples of conjugacy classes and automorphisms. As far as we know, $\text{WhP}(G, \text{Mon } G)$ and $\text{WhP}(G, \text{End } G)$ remain unsolved for a general partially commutative group G . Our theorem 7 is a small contribution into this direction, solving these problems for free-abelian times free groups in a direct and self-contained form.

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