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On the thermoelasticity with two porosities: asymptotic behavior

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Abstract

In this paper we consider the one-dimensional version of the thermoelasticity with two porous structures and porous dissipation on one or on both of them. We first give an existence and uniqueness result by means of semigroup theory. Exponential decay of the solutions is obtained when porous dissipation is assumed for each porous structure. Later we consider dissipation only on one of the porous structure and we prove that, under appropriate conditions on the coefficients, there exists undamped solutions. Therefore, asymptotic stability cannot be expected in general. However, we are able to give suitable sufficient conditions for the constitutive coefficients to guarantee the exponential decay of the solutions.

keywords: Existence and uniqueness of solutions, thermoelasticity with two porosities, decay estimates, semigroup argument, exponential decay

1 Introduction

Elastic material with double porosity are the aim of study of many works. These kind od materials are present in different real situations that can be found in geophysics, in civil engineering, and in biomechanics through applications to bones [1–5]. The first contributions in this line are due to Barenblatt *et al.* [1, 6], but, since then, other papers and books have been dedicated to describe the theoretical progress in this theory [2–4, 7–14].

In this context, it is assumed the existence of two porous structures: one is the macro-porosity, connected with the pores of the material, and the other is the micro-porosity, linked with the fissures of the skeleton. As Straughan pointed out [4], "a good example of this may be seen in the pictures in [15] where they show a pile of rocks, but the rocks themselves are full of

fissures (or cracks), and the macro porosity degrades over a period of ten years leaving a pile of finer material characteristic of the micro porous structure". In the literature it is usual to find relations of this theory with the Darcy law. The classical presentation of this theory involves the displacement, the pressure associated with the pores and the pressure associated with the fissures [4, 11, 14].

Nunziato and Cowin [16] proposed a theory to describe the behavior of porous solids materials. For these materials, there is a skeleton (or material matrix) that is elastic and where the interstices are void of material. These kinds of materials have been widely studied in the literature and different applications have been found, for instance in geological materials such as rocks and soils or in manufactured materials as ceramics and pressed powders. Nowadays, the theory of Nunziato and Cowin is commonly accepted as one of the non-classical elasticity theories. There is a huge amount of contributions referring this theory and, hence, it is not possible to cite them all. Nevertheless, some relevant contributions are cited in the paper. See, for example, references [17–27].

Following the ideas developed by Iesan and Quintanilla [28], which are based on the theory of Nunziato and Cowin, in this work we intend to study the problem determined when there is a double porosity structure in the material. It is worth mentioning that in our approach, there is not a fluid saturating the solid, instead the material void corresponds to the pore. We suppose the existence of two porous structures: one associated to the material pores and the other to the microporosity. The material skeleton supports both structures and the interactions between them are given by the constitutive equations. Both structures have influence one over the other, over the elastic deformations of the material matrix and over the heat conduction through the material. Obviously, the reciprocal is also true. That means that the heat conduction affects both structures as well. In other words: the porous structures and the heat conduction are coupled together. This theory is currently being studied [29–31].

The theory we are considering coincides with the classical thermoelasticity theory if the porous structures are not taken into account. In fact, this theory is one of the easier extensions of the classical thermoelasticty theory.

It is convenient to point out two facts. The first one is that this theory is new for materials with a double porous structure. The second one is that our approach is eminently theoretical. We think that the mathematical and physical analysis of the considered equations is important to decide the suitable application of the theory to real world situations. Indeed, some contributions have been done in this line of research, but we believe that further studies have to be carried out.

Finally, it is worth noting the mathematical similarities between the equations for porous-elastic materials and those for microstrech materials. Therefore, the equations that we propose in this paper can be also viewed as the equations describing a mixture of microstrech materials when their macroscopic structures coincide.

The only dissipative mechanism proposed in [28] was the thermal effect; however, no dissipation mechanisms were considered on the porosities. Here we consider dissipation mechanisms on the porosity. In fact, two cases will be analysed: first, when we have one dissipative mechanism on every porous structure and, secondly, when we only consider the dissipation on one porous structure. It is worth noting that we can see our theory as a particular sub-case of the theory proposed in [32]. Therefore, the interested reader can find the equations there. We study the one-dimensional problem and we want to clarify the asymptotic time behavior of the

solutions. In this sense, our work wants to be a continuation of the study developed by several authors concerning the porous elastic materials, as in the contributions [18–27].

In Section 2 we present the basic equations and assumptions of the theory. Existence and uniqueness of solutions are obtained in Section 3 by means of the semigroup theory. Exponential decay of solutions is obtained under the assumption of thermal dissipation and porous dissipation in each porous structure in Section 4. Section 5 is devoted to study the problem when the dissipation mechanisms are the thermal effect and the porous dissipation in only one structure. We will see that for suitable conditions on the parameters, we have undamped solutions, but we also give sufficient conditions to guarantee the exponential decay.

2 Basic Equations

In this section we recall the basic equations and settings for the theory of thermoelasticity with two porosities when we consider porous dissipation. We center our attention to the one-dimensional theory.

The basic equations for this theory can be obtained by means of the developments proposed in [28, 32]. A variational formulation of the same problem could be given with the methods proposed, e.g., in [33, 34]. In this sense, we may recall the basic evolution equations

$$\rho\ddot{u} = \Upsilon_x$$
$$K_1\ddot{\varphi} = \sigma_x + \delta \tag{2.1}$$
$$K_2\ddot{\psi} = \tau_x + \vartheta$$

together with the heat equation

$$\rho\dot{\eta} = q_x \tag{2.2}$$

and the constitutive equations

$$\Upsilon = \mu u_x + b\varphi + d\psi - \beta\theta$$
$$\sigma = \alpha\varphi_x + b_1\psi_x$$
$$\tau = b_1\varphi_x + \gamma\psi_x$$
$$\delta = -bu_x - \alpha_1\varphi - \alpha_3\psi + \gamma_1\theta - \varepsilon_1\dot{\varphi} - \varepsilon_2\dot{\psi} \tag{2.3}$$
$$\vartheta = -du_x - \alpha_3\varphi - \alpha_2\psi + \gamma_2\theta - \varepsilon_3\dot{\varphi} - \varepsilon_4\dot{\psi}$$
$$\rho\eta = \beta u_x + \gamma_1\varphi + \gamma_2\psi + c\theta$$
$$q = \kappa\theta_x$$

Here u represents the displacement, φ and ψ are the volume fractions for each porous structure, θ is the difference of temperature with respect the reference temperature, ρ is the mass density, K_1 and K_2 are the coefficients of inertia for each porous structure, Υ is the stress, σ and τ the equilibrated stress of each porous structure, δ and ϑ the equilibrated body force of each porous structure, η is the entropy, q the heat flux and $\mu, b, d, \beta, \alpha, b_1, \gamma, \alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \kappa$ and c are constitutive constants of the material.

It is prescribed by the basic principles that the mass density, the coefficients of inertia and the thermal capacity are positive. That is

$$\rho > 0, \ K_1 > 0, \ K_2 > 0, \ c > 0. \tag{2.4}$$

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The internal mechanical energy of the system is given by

$$W = \mu|u_x|^2 + 2bu_x\varphi + 2du_x\psi + \alpha|\varphi_x|^2 + \gamma|\psi_x|^2 + 2b_1\varphi_x\psi_x + \alpha_1|\varphi|^2 + \alpha_2|\psi|^2 + 2\alpha_3\varphi\psi. \quad (2.5)$$

To guarantee that the internal mechanical energy is positive, the following matrix should be positive definite

$$\begin{pmatrix} \mu & b & d & 0 & 0 \\ b & \alpha_1 & \alpha_3 & 0 & 0 \\ d & \alpha_3 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & \alpha & b_1 \\ 0 & 0 & 0 & b_1 & \gamma \end{pmatrix}. \quad (2.6)$$

That is, we need to assume

$$\mu > 0, \ \alpha > 0, \ \mu\alpha_1 > b^2, \ \mu\alpha_1\alpha_2 + 2bd\alpha_3 - d^2\alpha_1 - b^2\alpha_2 - \alpha_3^2\mu > 0, \ \alpha\gamma > b_1^2. \quad (2.7)$$

We also deduce that

$$\mu\alpha_2 > d^2, \ \alpha_1\alpha_2 > \alpha_3^2. \quad (2.8)$$

The mechanical dissipation of the system is given by

$$D_1 = \varepsilon_1|\dot\varphi|^2 + \varepsilon_4|\dot\psi|^2 + 2(\varepsilon_2 + \varepsilon_3)\dot\varphi\dot\psi. \quad (2.9)$$

In our mathematical study it also plays a relevant role the expression

$$D_2 = \kappa|\theta_x|^2 \quad (2.10)$$

and, in general, $D^* = D_1 + D_2$. If we want that D^* to be positive, we need to impose that

$$\kappa > 0, \ \varepsilon_1\varepsilon_4 > \frac{1}{4}(\varepsilon_2 + \varepsilon_3)^2, \ \varepsilon_1 > 0. \quad (2.11)$$

A limit case is obtained when $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0$, but $\kappa > 0$ and $\varepsilon_1 > 0$.

Following a variational approach, see e.g. [35], it would be possible to include the term D_2 in (2.5) and obtain the restriction $\kappa > 0$ in the list of restrictions in (2.7).

If we substitute the constitutive equations into the evolution equations and the heat equation, we obtain the system of field equations

$$\left. \begin{array}{l} \rho\ddot{u} = \mu u_{xx} + b\varphi_x + d\psi_x - \beta\theta_x \\ K_1\ddot\varphi = \alpha\varphi_{xx} + b_1\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi + \gamma_1\theta - \varepsilon_1\dot\varphi - \varepsilon_2\dot\psi \\ K_2\ddot\psi = b_1\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi + \gamma_2\theta - \varepsilon_3\dot\varphi - \varepsilon_4\dot\psi \\ c\dot\theta = \kappa\theta_{xx} - \beta\dot{u}_x - \gamma_1\dot\varphi - \gamma_2\dot\psi \end{array} \right\} \quad (2.12)$$

We study the solutions of the system (2.12) in $\mathcal{B} \times T$, where $\mathcal{B} = [0, \pi]$ and $T = [0, \infty)$. To have a well posed problem, we need to impose boundary and initial conditions. We assume homogeneous boundary conditions of Dirichlet type for the displacement and Neumann type in the other variables. That is,

$$u(x, t) = \varphi_x(x, t) = \psi_x(x, t) = \theta_x(x, t) = 0, \ \ x = 0, \pi \ \text{ and } \ t \in T. \quad (2.13)$$

We consider the initial conditions

$$u(x,0) = u_0(x),\, \dot{u}(x,0) = v_0(x),\; \varphi(x,0) = \varphi_0(x),\; \dot{\varphi}(x,0) = \zeta_0(x),$$
$$\psi(x,0) = \psi_0(x),\; \dot{\psi}(x,0) = \xi_0(x),\; \theta(x,0) = \theta_0(x),\;\; x \in \mathcal{B}. \tag{2.14}$$

It is known that for the problem determined by (2.12)–(2.14) we can always take solutions where φ, ψ and θ are constants. Thus, if we want to avoid this behavior, we need to impose that

$$\int_0^\pi \varphi^0(x)\, dx = \int_0^\pi \zeta^0(x)\, dx = \int_0^\pi \psi^0(x)\, dx = \int_0^\pi \xi^0(x)\, dx = \int_0^\pi \theta^0(x)\, dx = 0. \tag{2.15}$$

These conditions are assumed to guarantee that the solutions decay.

We now transform our initial boundary value problem into an ordinary differential equation on a suitable Hilbert space. We denote by $\mathcal{H}$ the Hilbert space

$$\Big\{ (u,v,\varphi,\zeta,\psi,\xi,\theta) \in H^1 \times L^2 \times H^1 \times L^2 \times H^1 \times L^2 \times L^2,$$
$$\int_0^\pi \varphi dx = \int_0^\pi \zeta dx = \int_0^\pi \psi dx = \int_0^\pi \xi dx = \int_0^\pi \theta dx = 0 \Big\} \tag{2.16}$$

with the inner product defined as follows: if $U^* = (u^*, v^*, \varphi^*, \zeta^*, \psi^*, \xi^*, \theta^*)$, then

$$\langle U, U^* \rangle_{\mathcal{H}} := \frac{1}{2} \int_0^\pi \Big[\mu u_x \overline{u_x^*} + \rho v \overline{v^*} + b u_x \overline{\varphi^*} + b \overline{u_x^*} \varphi + d u_x \overline{\psi^*} + d \overline{u_x^*} \psi + \alpha \varphi_x \overline{\varphi_x^*} + \gamma \psi_x \overline{\psi_x^*}$$
$$+ b_1 \varphi_x \overline{\psi_x^*} + b_1 \overline{\varphi_x^*} \psi_x + K_1 \zeta \overline{\zeta^*} + K_2 \xi \overline{\xi^*} + \alpha_1 \varphi \overline{\varphi^*} + \alpha_2 \psi \overline{\psi^*} + \alpha_3 \varphi \overline{\psi^*} + \alpha_3 \overline{\varphi^*} \psi + c \theta \overline{\theta^*} \Big] dx. \tag{2.17}$$

As usual, the superposed bar denotes the conjugate complex number.

We define the matrix operator

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu D^2}{\rho} & 0 & \frac{bD}{\rho} & 0 & \frac{dD}{\rho} & 0 & \frac{-\beta D}{\rho} \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ \frac{-bD}{K_1} & 0 & \frac{\alpha D^2 - \alpha_1}{K_1} & \frac{-\varepsilon_1}{K_1} & \frac{b_1 D^2 - \alpha_3}{K_1} & \frac{-\varepsilon_2}{K_1} & \frac{\gamma_1}{K_1} \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ \frac{-dD}{K_2} & 0 & \frac{b_1 D^2 - \alpha_3}{K_2} & \frac{-\varepsilon_3}{K_2} & \frac{\gamma D^2 - \alpha_2}{K_2} & \frac{-\varepsilon_4}{K_2} & \frac{\gamma_2}{K_2} \\ 0 & \frac{-\beta D}{c} & 0 & \frac{-\gamma_1}{c} & 0 & \frac{-\gamma_2}{c} & \frac{\kappa D^2}{c} \end{pmatrix} \tag{2.18}$$

where I is the identity operator and D denotes the derivative with respect to x. If $U = (u,v,\varphi,\zeta,\psi,\xi,\theta)$, then our initial boundary value problem can be written as

$$\frac{dU}{dt} = \mathcal{A}U,\; U_0 = (u_0, v_0, \varphi_0, \zeta_0, \psi_0, \xi_0, \theta_0). \tag{2.19}$$

The domain $\mathcal{D}$ of $\mathcal{A}$ is the set of $U \in \mathcal{H}$ such that $\mathcal{A}U \in \mathcal{H}$. It is clear that $\mathcal{D}$ is a dense subspace of $\mathcal{H}$.

3 Existence and uniqueness of solutions

In this section we present an existence and uniqueness result for the solutions of the problem determined by (2.12)–(2.15).

Lemma 3.1 *The operator $\mathcal{A}$ is dissipative. That is, for every $U \in \mathcal{D}$,*

$$Re\langle \mathcal{A}U, U \rangle \leq 0.$$

Proof. Taking into account the evolution equations, the divergence theorem and the boundary conditions, we have that

$$Re\langle \mathcal{A}U, U \rangle = -\frac{1}{2} \int_0^\pi D^* \, dx. \tag{3.1}$$

In view of assumptions (2.11), the lemma is proved. $\square$

Lemma 3.2 *The operator $\mathcal{A}$ satisfies that*

$$Range\,(\mathcal{A}) = \mathcal{H}.$$

Proof. We have to prove that, for every $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7) \in \mathcal{H}$, we can find $U \in \mathcal{H}$ such that $\mathcal{A}\mathcal{U} = \mathcal{F}$. That is, the following system

$$\left.\begin{aligned}
v &= f_1 \\
\frac{1}{\rho}\left[\mu D^2 u + bD\varphi + dD\psi - \beta D\theta\right] &= f_2 \\
\zeta &= f_3 \\
\frac{1}{K_1}\left[\alpha D^2\varphi + b_1 D^2\psi - bDu - \alpha_1\varphi - \alpha_3\psi + \gamma_1\theta - \varepsilon_1\zeta - \varepsilon_2\xi\right] &= f_4 \\
\xi &= f_5 \\
\frac{1}{K_2}\left[b_1 D^2\varphi + \gamma D^2\psi - dDu - \alpha_3\varphi - \alpha_2\psi + \gamma_2\theta - \varepsilon_3\zeta - \varepsilon_4\xi\right] &= f_6 \\
\frac{1}{c}\left[\kappa D^2\theta - \beta Dv - \gamma_1\zeta - \gamma_2\xi\right] &= f_7
\end{aligned}\right\} \tag{3.2}$$

has a solution. We will solve the system using Fourier series and so, we consider

$$f_i = \sum f_n^i \sin nx \ \text{ for } \ i = 1, 2 \ \text{ and } \ f_i = \sum f_n^i \cos nx \ \text{ for } \ i = 3, ..., 7$$

with

$$\sum n^2 \left(f_n^i\right)^2 < \infty, \ \text{ for } i = 1, 3, 5 \ \text{ and } \ \sum \left(f_n^i\right)^2 < \infty, \ \text{ for } i = 2, 4, 6, 7. \tag{3.3}$$

We want to find

$$u = \sum u_n \sin nx, \ v = \sum v_n \sin nx, \ \varphi = \sum \varphi_n \cos nx, \ \zeta = \sum \zeta_n \cos nx,$$

$$\psi = \sum \psi_n \cos nx, \ \xi = \sum \xi_n \cos nx, \ \theta = \sum \theta_n \cos nx.$$

From the first, third and fifth equations of system (3.2) it follows that $v_n = f_n^1$, $\zeta_n = f_n^3$ and $\xi_n = f_n^5$. From the seventh equation of (3.2), it is clear that

$$\theta_n = -\frac{1}{\kappa}\left(\frac{cf_n^7}{n^2} + \frac{\beta f_n^1}{n} + \frac{\gamma_1 f_n^3}{n^2} + \frac{\gamma_2 f_n^5}{n^2}\right). \tag{3.4}$$

In view of this equality and the conditions on f_n^i we see that $\sum \theta_n^2 < \infty$.

Now we consider the second, fourth and sixth equations of system (3.2) and we get

$$u_n = \frac{\theta_n d\beta(b_1^2 - \alpha\gamma)n^5 + dp_4(n)}{n^2 dq_5(n)}, \tag{3.5}$$

$$\varphi_n = \frac{n^3 L + p_2(n)}{q_5(n)} \tag{3.6}$$

and

$$\psi_n = \frac{n^3 M + r_2(n)}{q_5(n)}. \tag{3.7}$$

Here $p_4(n)$ is a polynomial in n of degree 4, $p_2(n)$ and $r_2(n)$ are polynomials in n of degree 2 and $q_5(n)$ is the polynomial

$$q_5(n) = n^5\mu\left(b_1^2 - \alpha\gamma\right) + n^3\left(-\alpha\alpha_2\mu - \alpha_1\gamma\mu + b^2\gamma - 2bb_1 d + 2\alpha_3 b_1\mu + \alpha d^2\right)$$
$$+ n\left(-\alpha_1\alpha_2\mu + \alpha_3^2\mu + \alpha_2 b^2 - 2\alpha_3 bd + \alpha_1 d^2\right). \tag{3.8}$$

Notice that $q_5(n)$ is a polynomial of degree 5 because $\mu\left(b_1^2 - \alpha\gamma\right) \neq 0$. The coefficients L and M are

$$L = b\beta\gamma\theta_n + \beta(-b_1 d\theta_n - b_1\mu(\varepsilon_3 f_3 + \varepsilon_4 f_5 + f_6 K_2 - \gamma_2\theta_n) - \gamma\mu(-\varepsilon_1 f_3 - \varepsilon_2 f_5 + \gamma_1\theta_n)$$
$$+ \gamma f_4 K_1\mu, \tag{3.9}$$

$$M = -b\beta b_1\theta_n - b_1\mu(\varepsilon_1 f_3 + f_4 K_1 + \varepsilon_2 f_5 - \gamma_1\theta_n) + \alpha\beta d\theta_n + \alpha\mu(\varepsilon_3 f_3\varepsilon_4 f_5 + f_6 K_2 - \gamma_2\theta_n). \tag{3.10}$$

We claim that $q_5(n)$ has no real roots. We write the polynomial as $q_5(n) = -n(An^4 + Bn^2 + C)$, where

$$A = \mu\left(\alpha\gamma - b_1^2\right), \tag{3.11}$$
$$B = \alpha\alpha_2\mu + \alpha_1\gamma\mu + 2bb_1 d - 2\alpha_3 b_1\mu - \alpha d^2 - \gamma b^2 \tag{3.12}$$

and

$$C = \alpha_1\alpha_2\mu - \alpha_2 b^2 - \alpha_3^2\mu + 2\alpha_3 bd - \alpha_1 d^2. \tag{3.13}$$

To prove the claim, it is sufficient to see that $A > 0$, $B > 0$ and $C > 0$. Directly from conditions (2.7) we obtain that $A > 0$ and $C > 0$. Again, using (2.7) we have

$$B = \alpha\left(\mu\alpha_2 - d^2\right) + \gamma\left(\mu\alpha_1 - b^2\right) + 2b_1(db - \mu\alpha_3)$$
$$\geq \alpha(\mu\alpha_2 - d^2) + \gamma(\mu\alpha_1 - b^2) - 2\sqrt{\alpha\gamma}\sqrt{(\mu\alpha_3 - bd)^2}. \tag{3.14}$$

On the other hand, multiplying the penultimate inequality of (2.7) by μ, we get

$$\mu^2\alpha_3^2 + b^2 d^2 - 2\mu\alpha_3 bd < \mu^2\alpha_1\alpha_2 + b^2 d^2 - \mu\left(\alpha_1 d^2 + \alpha_2 b^2\right). \tag{3.15}$$

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This is equivalent to

$$(\mu\alpha_3 - bd)^2 < \left(\mu\alpha_2 - d^2\right)\left(\mu\alpha_1 - b^2\right). \tag{3.16}$$

Since $\mu\alpha_1 > b^2$ and $\mu\alpha_2 > d^2$ (from (2.7) and (2.8) respectively), using (3.16) in (3.14) it follows that

$$\begin{aligned}
B &> \alpha\left(\mu\alpha_2 - d^2\right) + \gamma\left(\mu\alpha_1 - b^2\right) - 2\sqrt{\alpha\gamma}\sqrt{\mu\alpha_2 - d^2}\sqrt{\mu\alpha_1 - b^2} \\
&= \left(\sqrt{\alpha}\sqrt{\mu\alpha_2 - d^2} - \sqrt{\gamma}\sqrt{\mu\alpha_1 - b^2}\right)^2 \geq 0.
\end{aligned} \tag{3.17}$$

Then, $B > 0$ and the claim is proved. So, the denominators of u_n, φ_n and ψ_n do not vanish for any value of n.

Moreover, taking into account (3.3) and that $\sum\theta_n^2 < \infty$, it is not difficult to see that $\sum n^2 u_n^2 < \infty$, $\sum n^2\varphi_n^2 < \infty$ and $\sum n^2\psi_n^2 < \infty$.

It is worth noting that if $d = 0$, then the solution u_n given in (3.5) becomes

$$u_n = \frac{\beta n\theta_n - f_2\rho}{\mu n^2} - \frac{n^4 N + p_3(n)}{\mu n^2 q_5^*(n)}, \tag{3.18}$$

where $p_3(n)$ is a polynomial in n of degree 3, $q_5^*(n) = q_5(n)|_{d=0}$ and

$$N = b(b\beta\gamma\theta_n - b_1\mu(\varepsilon_3 f_3 + \varepsilon_4 f_5 + f_6 K_2 - \gamma_2\theta_n) + \gamma\mu(\varepsilon_1 f_3 + f_4 K_1 + \varepsilon_2 f_5 - \gamma_1\theta_n)). \tag{3.19}$$

As above, in this case we also obtain $\sum n^2(u_n)^2 < \infty$. $\square$

As a consequence of Lemmas 3.1 and 3.2 and the Lumer-Phillips corollary to the Hille-Yosida Theorem [36] we obtain the well-posedness of the problem.

Theorem 3.1 *The operator $\mathcal{A}$ generates a contraction semigroup $\{e^{t\mathcal{A}}\}_{t\geq 0}$ and, for $U_0 \in \mathcal{D}$, there exists a unique solution $U \in \mathcal{C}^1([0,\infty),\mathcal{H}) \cap \mathcal{C}^0([0,\infty),\mathcal{D})$ to the problem (2.19).*

It is worth noting that we only need that the dissipation D^* is not positive. Therefore, the theorem also applies when $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0$ and $\varepsilon_1 > 0$.

4 Exponential decay

In this section we prove the exponential decay of the solutions to the problem proposed previously. To prove such result we use the following characterization, going back to Gearhart, Huang and Prüß, see [36]. It is important to recall that in this section we assume that $\beta \neq 0$.

Theorem 4.1 *[36] Let $\{e^{t\mathcal{A}}\}_{t\geq 0}$ be a C_0-semigroup of contractions generated by the operator $\mathcal{A}$ in the Hilbert space $\mathcal{H}$. Then the semigroup is exponentially stable if and only if $i\mathbb{R} \subseteq \varrho(\mathcal{A})$ (resolvent set) and*

$$\overline{\lim}_{|\lambda|\to\infty}||(i\lambda I - \mathcal{A})^{-1}|| < \infty, \quad \lambda \in \mathbb{R}. \tag{4.1}$$

Theorem 4.2 *$\mathcal{A}$ generates a semigroup which is exponentially stable.*

Proof. Since $0 \in \varrho(\mathcal{A})$, following the arguments by Liu and Zheng (see [36], p.25), we assume that the imaginary axis is not contained in the resolvent set. Then there exists a real number $\varpi \neq 0$ with $||\mathcal{A}^{-1}||^{-1} \leq |\varpi| < \infty$ such that the set $\{i\lambda, |\lambda| < |\varpi|\}$ is in the resolvent of $\mathcal{A}$ and $\sup\{||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||, |\lambda| < |\varpi|\} = \infty$. Therefore, there exists a sequence of real numbers λ_n with $\lambda_n \to \varpi$, $|\lambda_n| < |\varpi|$ and a sequence of unit vectors $U_n = (u_n, v_n, \varphi_n, \zeta_n, \psi_n, \xi_n, \theta_n)$ in the domain of the operator $\mathcal{A}$ such that

$$||(i\lambda_n\mathcal{I} - \mathcal{A})U_n|| \to 0. \tag{4.2}$$

This implies

$$i\lambda_n u_n - v_n \to 0 \text{ in } H^1, \tag{4.3}$$

$$i\lambda_n v_n - \frac{1}{\rho}\left(\mu D^2 u_n + bD\varphi_n + dD\psi_n - \beta D\theta_n\right) \to 0 \text{ in } L^2, \tag{4.4}$$

$$i\lambda_n \varphi_n - \zeta_n \to 0 \text{ in } H^1, \tag{4.5}$$

$$i\lambda_n \zeta_n - \frac{1}{K_1}\left(-bDu_n + \alpha D^2\varphi_n - \alpha_1\varphi_n - \varepsilon_1\zeta_n + b_1 D^2\psi_n - \alpha_3\psi_n + \varepsilon_2\xi_n + \gamma_1\theta_n\right) \to 0 \text{ in } L^2, \tag{4.6}$$

$$i\lambda_n \psi_n - \xi_n \to 0 \text{ in } H^1, \tag{4.7}$$

$$i\lambda_n \xi_n - \frac{1}{K_2}\left(-dDu_n + b_1 D^2\varphi_n - \alpha_3\varphi_n - \varepsilon_3\zeta_n + \gamma D^2\psi_n - \alpha_2\psi_n - \varepsilon_4\xi_n + \gamma_2\theta_n\right) \to 0 \text{ in } L^2, \tag{4.8}$$

$$i\lambda_n \theta_n - \frac{1}{c}\left(-\beta Dv_n - \gamma_1\zeta_n - \gamma_2\xi_n + \kappa D^2\theta_n\right) \to 0 \text{ in } L^2. \tag{4.9}$$

Considering the inner product of $(i\lambda_n\mathcal{I} - \mathcal{A})\omega_n$ times ω_n in $\mathcal{H}$ and then, taking its real part, it yields $D\theta_n \to 0$, $\zeta_n \to 0$ and $\xi_n \to 0$ in L^2. In particular, $\theta_n \to 0$. Multiplying (4.6) by $\frac{K_1}{\lambda_n}\zeta_n$ and taking into account that Du_n, φ_n, ψ_n, ξ_n, and θ_n are bounded, and that $\zeta_n \to 0$, we obtain

$$\alpha\langle D^2\varphi_n, \frac{\zeta_n}{\lambda_n}\rangle + b_1\langle D^2\psi_n, \frac{\zeta_n}{\lambda_n}\rangle \to 0. \tag{4.10}$$

From (4.5), $\frac{\zeta_n}{\lambda_n} \sim i\varphi_n$. Then, (4.10) becomes

$$\alpha||D\varphi_n||^2 + b_1\langle D\psi_n, D\varphi_n\rangle \to 0. \tag{4.11}$$

Analogously, if we multiply (4.8) by $\frac{K_2}{\lambda_n}\zeta_n$, we get

$$b_1||D\varphi_n||^2 + \gamma\langle D\psi_n, D\varphi_n\rangle \to 0. \tag{4.12}$$

Considering (4.11) times γ minus (4.12) times b_1, it follows that

$$(\alpha\gamma - b_1^2)||D\varphi_n||^2 \to 0. \tag{4.13}$$

Since $\alpha\gamma - b_1^2 \neq 0$ (see 2.7), we have that $D\varphi_n \to 0$ in L^2.

In an analogous way, we multiply (4.6) by $\frac{K_1}{\lambda_n}\xi_n$ and (4.8) by $\frac{K_2}{\lambda_n}\xi_n$. Keeping in mind that $\xi \to 0$ and $\frac{\xi_n}{\lambda_n} \sim i\psi_n$, from (4.7) we obtain

$$\alpha\langle D\varphi, D\psi\rangle + b_1||D\psi||^2 \to 0 \tag{4.14}$$

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and

$$b_1\langle D\varphi, D\psi\rangle + \gamma||D\psi||^2 \to 0, \tag{4.15}$$

respectively. Thus, we find that

$$(\alpha\gamma - b_1^2)||D\psi_n||^2 \to 0 \tag{4.16}$$

and $D\psi_n \to 0$ in L^2.

Now we consider (4.9) times $\frac{c}{\lambda_n} Du$. Since $\theta_n \to 0$ and Du is bounded, we obtain

$$\beta\langle \frac{Dv_n}{\lambda_n}, Du_n\rangle - \kappa\langle D^2\theta_n, \frac{Du_n}{\lambda_n}\rangle \to 0. \tag{4.17}$$

On the other hand, (4.3) yields $\frac{v_n}{\lambda_n} \sim iu_n$. Using this equivalence in (4.17) we get

$$\beta\langle iDu_n, Du_n\rangle + \kappa\langle D\theta_n, \frac{D^2u_n}{\lambda_n}\rangle \to 0. \tag{4.18}$$

We claim that $\langle D\theta_n, \frac{D^2u_n}{\lambda_n}\rangle \to 0$. Since $D\theta_n \to 0$, it is enough to see that $\frac{D^2u_n}{\lambda_n}$ is bounded. In fact, from (4.4), we have that

$$i\rho v_n - \mu\frac{D^2u_n}{\lambda_n} \to 0. \tag{4.19}$$

As v_n is bounded, then $\frac{D^2u_n}{\lambda_n}$ is bounded as well. Therefore, (4.18) gives $||Du_n||^2 \to 0$.

Finally, multiplying (4.4) by $\rho\frac{v_n}{\lambda_n}$ we get

$$i\rho||v_n||^2 + \mu\langle Du_n, \frac{Dv_n}{\lambda_n}\rangle \to 0. \tag{4.20}$$

Using again the equivalence $\frac{v_n}{\lambda_n} \sim iu_n$ and the fact that $||Du_n||^2 \to 0$, we obtain $||v_n||^2 \to 0$. These behaviours contradict the hypothesis that the sequence U_n has unit norm.

Now, (4.1) is proved by a similar argument. If it is not true, then there exist a sequence λ_n with $|\lambda_n| \to \infty$ and a sequence of unit norm vectors $U_n = (u_n, v_n, \varphi_n, \zeta_n, \psi_n, \xi_n, \theta_n)$, in the domain of the operator $\mathcal{A}$, such that (4.2) holds. We can now follow the arguments used previously when the sequence λ_n is bounded. □

5   Dissipation only on one porous structure

In the previous section we have considered thermal dissipation and porous dissipation in each porous structure. We have seen the exponential decay of the solutions in that case. A natural question to ask is if only one porous dissipation with the thermal effect is sufficient to guarantee the exponential decay. This section is devoted to analyze this case. So, from now on, we assume $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0$ and $\varepsilon_1 = \varepsilon$.

The problem we want to study is determined by the system of equations

$$\left.\begin{array}{l} \rho\ddot{u} = \mu u_{xx} + b\varphi_x + d\psi_x - \beta\theta_x \\ K_1\ddot{\varphi} = \alpha\varphi_{xx} + b_1\psi_{xx} - bu_x - \alpha_1\varphi - \alpha_3\psi + \gamma_1\theta - \varepsilon\dot{\varphi} \\ K_2\ddot{\psi} = b_1\varphi_{xx} + \gamma\psi_{xx} - du_x - \alpha_3\varphi - \alpha_2\psi + \gamma_2\theta \\ c\dot{\theta} = \kappa\theta_{xx} - \beta\dot{u}_x - \gamma_1\dot{\varphi} - \gamma_2\dot{\psi} \end{array}\right\} \tag{5.1}$$

with the boundary and initial conditions (2.13)–(2.14).

As we pointed out previously, the existence and uniqueness Theorem 3.1 also applies to this case.

5.1 Undamped solutions

Notice that the problem determined by (5.1), (2.13)–(2.14) has undamped solutions. This fact implies that we can not expect the exponential decay in the general case.

We can find solutions of the form

$$(u, \varphi, \psi, \theta) = \left(A^* \sin nx \left\{ \begin{array}{c} \cos mt \\ \sin mt \end{array} \right\}, 0, B^* \cos nx \left\{ \begin{array}{c} \cos mt \\ \sin mt \end{array} \right\}, 0 \right), \tag{5.2}$$

for $n = 1, 2, 3...$ and a certain $m \in \mathbb{R}$, $m \neq 0$, which must be determined.

If solutions of this form exist, the following system must be satisfied:

$$\left. \begin{array}{r} -\rho A^* m^2 = -\mu A^* n^2 - dB^* n \\ 0 = -b_1 B^* n^2 - bn A^* - \alpha_3 B^* \\ -K_2 B^* m^2 = -\gamma B^* n^2 - dn A^* - \alpha_2 B^* \\ 0 = -\beta nm A^* - \gamma_2 m B^* \end{array} \right\} \tag{5.3}$$

First we consider the first and third equations of system (5.3). We write them as follows:

$$\left. \begin{array}{r} (\mu n^2 - \rho m^2) A^* + dn B^* = 0 \\ dn A^* + (\gamma n^2 + \alpha_2 - K_2 m^2) B^* = 0 \end{array} \right\} \tag{5.4}$$

We want to obtain $A^*, B^* \neq 0$ and $m \in \mathbb{R}$, $m \neq 0$. So, we need that the matrix of coefficients degenerates. That is,

$$\begin{vmatrix} \mu n^2 - \rho m^2 & dn \\ dn & \gamma n^2 + \alpha_2 - K_2 m^2 \end{vmatrix} = 0, \tag{5.5}$$

or

$$(\mu n^2 - \rho m^2)(\gamma n^2 + \alpha_2 - K_2 m^2) - d^2 n^2 = 0, \tag{5.6}$$

which is a biquadratic equation for m:

$$\rho K_2 m^4 - \left[\rho(\gamma n^2 + \alpha_2) + K_2 \mu n^2 \right] m^2 + \mu n^2(\gamma n^2 + \alpha_2) - d^2 n^2 = 0. \tag{5.7}$$

We can write

$$\rho K_2 m^4 - P m^2 + Q = 0, \tag{5.8}$$

where $P = \rho(\gamma n^2 + \alpha_2) + K_2 \mu n^2$ and $Q = \mu n^2(\gamma n^2 + \alpha_2) - d^2 n^2$. To prove the existence of undamped solutions we need to see the existence of a real number m satisfying this equation. Taking into account that the discriminant $P^2 - 4\rho K_2 Q > 0$, we obtain real solutions for m:

$$m = \pm \sqrt{\frac{P \pm \sqrt{P^2 - 4\rho K_2 Q}}{2\rho K_2}}. \tag{5.9}$$

Therefore, for each $n = 1, 2, 3...$ we have reals constants m and $A^*, B^* \neq 0$. Furthermore, taking some $\beta \neq 0$, there is $\gamma_2 = -\beta n A^* / B^*$ satisfying the fourth equation of system (5.3). Finally, it is clear that there are real numbers b_1, b and α_3 satisfying the second equation of system (5.3). So we can conclude the existence of cycle limits for our problem.

5.2   Sufficient conditions for exponential stability

In this subsection we prove the exponential decay of solutions to the problem determined by the system (5.1) together with the boundary and initial conditions (2.13)–(2.14) whenever at least one of the following assumptions (on the constitutive coefficients) is satisfied:

$$\text{H1:} \qquad \alpha_2 + \frac{b\gamma\gamma_2 - \beta\gamma\alpha_3 - b_1 d\gamma_2}{b_1\beta} < 0,$$

$$\text{H2:} \qquad \frac{\alpha_3\beta - \gamma_2 b}{b_1\beta} > -1.$$

Theorem 5.1 *Let us suppose that $\beta \neq 0$ and $b_1 \neq 0$. If assumptions H1 or H2 are satisfied, then the operator $\mathcal{A}$ generates a semigroup which is exponentially stable.*

Proof. We can follow the same argument as in Theorem 4.2. In our case we have:

$$i\lambda_n u_n - v_n \to 0 \text{ in } H_0^1, \tag{5.10}$$

$$i\rho\lambda_n v_n - \mu D^2 u_n - bD\varphi_n - dD\psi_n + \beta D\theta_n \to 0 \text{ in } L^2, \tag{5.11}$$

$$i\lambda_n \varphi_n - \zeta_n \to 0 \text{ in } H_0^1, \tag{5.12}$$

$$iK_1\lambda_n\zeta_n + bDu_n - \alpha D^2\varphi_n + \alpha_1\varphi_n + \varepsilon\zeta_n - b_1 D^2\psi_n + \alpha_3\psi_n - \gamma_1\theta_n \to 0 \text{ in } L^2, \tag{5.13}$$

$$i\lambda_n\psi_n - \xi_n \to 0 \text{ in } H_0^1, \tag{5.14}$$

$$iK_2\lambda_n\xi_n + dDu_n - b_1 D^2\varphi_n + \alpha_3\varphi_n - \gamma D^2\psi_n + \alpha_2\psi_n - \gamma_2\theta_n \to 0 \text{ in } L^2, \tag{5.15}$$

$$ic\lambda_n\theta_n + \beta Dv_n + \gamma_1\zeta_n + \gamma_2\xi_n - \kappa D^2\theta_n \to 0 \text{ in } L^2. \tag{5.16}$$

From the equation of the dissipation, we get

$$\zeta_n \to 0, \quad D\theta_n \to 0. \tag{5.17}$$

Using the same arguments as in the proof of Theorem 4.2, we get $D\varphi_n \to 0$ and so, $\varphi_n \to 0$.

The next step is to prove that $||D\psi_n|| \to 0$ and $||\psi_n|| \to 0$. This is the key point of the proof.

Here we consider (5.13) times $\frac{\xi_n}{\lambda_n}$ and we take into account that $\zeta_n \to 0$ and that ξ_n, Du_n, φ_n and θ_n are bounded. Therefore, we get

$$b\langle Du_n, \frac{\xi_n}{\lambda_n}\rangle - \alpha\langle D^2\varphi_n, \frac{\xi_n}{\lambda_n}\rangle - b_1\langle D^2\psi^2, \frac{\xi_n}{\lambda_n}\rangle + \alpha_3\langle \psi_n, \frac{\xi_n}{\lambda_n}\rangle \to 0. \tag{5.18}$$

From (5.14) we find that

$$i\psi_n \sim \frac{\xi_n}{\lambda_n}. \tag{5.19}$$

Using this equivalence in (5.18) and since $D\varphi_n \to 0$ and $D\psi_n$ is bounded, it follows that

$$b\langle Du_n, \psi_n\rangle + b_1||D\psi_n||^2 + \alpha_3||\psi_n||^2 \to 0. \tag{5.20}$$

We multiply (5.16) by $\frac{\psi_n}{\lambda_n}$ and following similar arguments to those used above, we get

$$\beta\langle \frac{Dv_n}{\lambda_n}, \psi_n\rangle + \gamma_2\langle \xi_n, \frac{\psi_n}{\lambda_n}\rangle \to 0. \tag{5.21}$$

From (5.10) it is clear that

$$iDu_n \sim \frac{Dv_n}{\lambda_n}. \tag{5.22}$$

Using this equivalence together with (5.19), expression (5.21) becomes

$$\beta\langle Du_n, \psi_n\rangle + \gamma_2||\psi_n||^2 \to 0. \tag{5.23}$$

Multiplying (5.15) by ψ_n and taking into account (5.14), we get

$$-\lambda_n^2 K_2||\psi_n||^2 + \alpha_2||\psi_n||^2 + \gamma||D\psi_n||^2 + d\langle Du_n, \psi_n\rangle \to 0. \tag{5.24}$$

We consider the system composed by (5.20), (5.23) and (5.24) and its associated determinant:

$$\Lambda = \begin{vmatrix} b_1 & b & \alpha_3 \\ 0 & \beta & \gamma_2 \\ \gamma & d & \alpha_2 - \lambda^2 K_2 \end{vmatrix} \tag{5.25}$$

In view of the assumption H1, we have that $\Lambda \neq 0$. Therefore $||D\psi_n|| \to 0$ and $||\psi_n|| \to 0$.

In case that assumption H2 is imposed, we can use the following argument. From (5.23) we get the equivalence

$$\langle Du_n, \psi_n\rangle \sim -\frac{\gamma_2}{\beta}||\psi_n||^2. \tag{5.26}$$

Since $b_1 \neq 0$, using the above equivalence in (5.20), it follows

$$||D\psi_n||^2 + \frac{1}{b_1}\left(\alpha_3 - \frac{\gamma_2 b}{\beta}\right)||\psi_n||^2 \to 0. \tag{5.27}$$

From the Poincaré inequality we obtain that

$$\int_0^\pi \left(1 + \frac{\alpha_3\beta - \gamma_2 b}{b_1\beta}\right)\psi_n^2 dx \to 0, \tag{5.28}$$

whenever

$$\frac{\alpha_3\beta - \gamma_2 b}{b_1\beta} > -1. \tag{5.29}$$

In this case, $||\psi_n||^2 \to 0$ and from (5.27), we have $||D\psi_n||^2 \to 0$.

Multiplying (5.15) by $\frac{\xi_n}{\lambda_n}$ and using the usual arguments, we get $||\xi_n||^2 \to 0$. Now, the proof is ended thanks, again, to the arguments of Theorem 4.2. $\square$

Acknowledgments

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