Differential operators and the Witten genus for projective spaces and Milnor manifolds

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Introduction

A *genus* (in the sense of Hirzebruch [4]) is a multiplicative invariant of cobordism classes of manifolds. Classical examples include the numerical invariants given by the signature and the $\hat{A}$ - and Todd genera. More recently genera were introduced which take as values modular forms on the upper half-plane, $\mathfrak{h} = \{ \tau \mid \text{Im}(\tau) > 0 \}$. The main examples are the elliptic genus ϕ_{ell} and the Witten genus ϕ_W ; we refer the reader to the texts [7] or [9] for details.

In this paper we concentrate on the Witten genus and the values it takes on certain manifolds much studied in the theory of genera: the complex and quaternionic projective spaces $\mathbb{CP}^{2k}$, $\mathbb{HP}^k$, and the Milnor manifolds $H_{i,j}$. The corresponding values for the elliptic genus are considered, for example, in [5].

Our first results are the following unexpected relations:

THEOREM A.

- (i) *the values of the Witten genus for the quaternionic projective manifolds $\mathbb{HP}^k$ and the Milnor manifolds $H_{3,2k-2}$ of the same dimension are*

related by

$$\binom{2k}{3} \phi_W(\mathbb{HP}^k) = 4(k-1)^3 \phi_W(H_{3,2k-2}), \quad (k \geq 1);$$

(ii) the values $\psi_k = \phi_W(H_{3,2k-2})$ satisfy the quadratic recurrence relation

$$\psi_k = \frac{4(k-2)\psi_1\psi_{k-1}}{2k-3} + 3 \sum_{\substack{i,j \geq 2 \\ i+j=k}} \left(\frac{5k-16}{k-3} - \frac{k(2k-1)}{ij} \right) \frac{\psi_i\psi_j}{(2i-1)(2j-1)}$$

for $k \geq 4$.

Considering derivatives with respect to the modular parameter τ we obtain also:

THEOREM B. *The Witten genus of the Milnor manifolds $H_{2,*}$ and the complex projective manifolds $\mathbb{CP}^{2*}$ are related by*

$$(2k+1) \phi_W(H_{2,2k-1}) = -\frac{1}{2\pi i} \frac{d}{d\tau} \phi_W(\mathbb{CP}^{2k}), \quad (k \geq 1).$$

In a subsequent article we will give a more geometric version of this result for a generalisation of ϕ_W to spin^c manifolds.

Taking suitable Fourier expansions in the variable $q = e^{2\pi i \tau}$, the elliptic and Witten genera can be regarded as taking values in the ring of power series $\mathbb{Q}[[q]]$, with constant term the classical $\hat{A}$ -genus. Theorem B, for example, says

$$(2k+1) \phi_W(H_{2,2k-1}) = -q \frac{d}{dq} \phi_W(\mathbb{CP}^{2k}),$$

hence $\hat{A}(H_{2,2k-1}) = 0$. Our third result gives more general vanishing and integrality results, for the higher order coefficients in the Witten genus.

THEOREM C.

(i) *the q -expansion of $\phi_W(\mathbb{HP}^k)$ is an element of $(k-1)^2 q^{k-1} \mathbb{Z}[[q]]$, for all $k \geq 1$.*

(ii) *the q -expansion of $\phi_W(H_{2t+1,2s})$ is an element of $q^s \mathbb{Q}[[q]]$, for all $s, t \geq 0$.*

The structure of the paper is as follows. In section 1, after recalling some of the classical theory of oriented genera ϕ , we introduce a family of even power series, where $r \geq 0$,

$$\omega_r(y) = y^{-2} + \sum_{k=1}^{\infty} a_k (1-k)^r y^{2k-2},$$

which are related to $g'(y)^{-2}$, where g is the logarithm of ϕ , and from whose coefficients the values $\phi(H_{i,j})$ may be quite easily obtained; in particular $\phi(H_{3,2k-2})$ is a certain rational multiple of a_k . The results of this section hold for *any* oriented genus ϕ ; in sections 2, 3 and 4 they are applied to the Witten genus ϕ_W , to prove Theorems A, B and C respectively.

In section 2 it is shown that the Weierstrass $\wp$ -function and the generating function for the values $\phi_W(\mathbb{H}P^k)$ are closely related to ω_1 and ω_2 respectively. The Weierstrass cubic equation is then used to derive a quadratic relation between the series ω_r . In section 3 the Weierstrass formalism is translated into that of Jacobi functions, and an alternative expression is found for the Witten genus in terms of the theta function ϑ_1 and the Dedekind η -function. The Jacobi function ϑ_1 satisfies a well-known partial differential equation from the theory of heat conduction, and a partial differential equation satisfied by the Witten genus is thus obtained. The final section is dedicated to investigating the Fourier coefficients of the various modular functions considered earlier. In particular the Fourier expansions of the a_k are shown to lie in $q^{k-1}\mathbb{Z}[[q]]$ for $k \geq 2$.

1 Oriented genera and formal series

We will consider the following classical families of manifolds.

- (i) the complex projective spaces, $\mathbb{C}P^{2k}$. These are not spin manifolds.
- (ii) the quaternionic projective spaces, $\mathbb{H}P^k$, which are spin manifolds of real dimension $4k$, but admit no almost complex structure.
- (iii) the Milnor manifolds, $H_{i,j}$, given as submanifolds of $\mathbb{C}P^i \times \mathbb{C}P^j$ defined by the equation in homogeneous coordinates

$$x_0 y_0 + x_1 y_1 + \cdots + x_r y_r = 0, \quad r = \min(i, j).$$

Abusing notation we will usually write M for the cobordism class $[M]$ of a manifold in the rational oriented cobordism ring $\Omega_*^{\text{SO}} \otimes \mathbb{Q}$. The sequence $\{\mathbb{C}P^{2k}\}$ gives a multiplicative basis for this ring, as do appropriate sequences of Milnor manifolds.

Let Λ be a $\mathbb{Q}$ -algebra and $\phi : \Omega^{\text{SO}} \otimes \mathbb{Q} \rightarrow \Lambda$ an *oriented genus* with characteristic series

$$Q(x) = 1 + \sum_{i=1}^{\infty} c_i x^{2i} \quad (c_i \in \Lambda).$$

The values of the genus ϕ on the above manifolds are determined by (and determine) the characteristic series in the following way:

$$\begin{aligned}\phi(\mathbb{CP}^{2k}) &= \text{coefficient of } x^{2k} \text{ in } Q(x)^{2k+1}, \\ \phi(\mathbb{HP}^k) &= \text{coefficient of } x^{2k} \text{ in } \frac{Q(x)^{2k+2}}{Q(2x)}, \\ \phi(H_{i,j}) &= \text{coefficient of } u^i v^j \text{ in } \frac{Q(u)^{i+1} Q(v)^{j+1}}{Q(u+v)} \cdot (u+v).\end{aligned}$$

The calculation of these values is not usually carried out directly from the characteristic series $Q(x)$, however, but rather using several related formal power series.

The *logarithm* of the genus is the odd power series $g(y)$ defined by $f(g(y)) = y$, the formal inverse of the series $f(x) = x/Q(x)$. The derivative of g gives the generating function for the values of ϕ on the complex projective manifolds:

$$g'(y) = 1 + \sum_{k=1}^{\infty} \phi(\mathbb{CP}^{2k}) y^{2k}$$

Further, the formal power series $h(y)$ and $H(u, v)$ defined by the formulae

$$\begin{aligned}f(2g(y)) &= \frac{2y h(y)}{g'(y)} && (\text{duplication}) \\ f(g(u) + g(v)) &= \frac{H(u, v)}{g'(u) g'(v)} && (\text{addition})\end{aligned}$$

are the generating functions for the values of ϕ on $\mathbb{HP}^k$ and $H_{i,j}$:

$$h(y) = \sum_{k \geq 0} \phi(\mathbb{HP}^k) y^{2k} \qquad H(u, v) = \sum_{i, j \geq 0} \phi(H_{i,j}) u^i v^j.$$

For the purposes of this note we consider also the following sequence of functions:

$$\omega_0(y) = \frac{1}{y^2 g'(y)^2}, \qquad \omega_r(y) = \mathcal{D}(\omega_{r-1})(y) \quad (r \geq 1).$$

Here $\mathcal{D}$ is the formal differential operator $\omega(y) \mapsto -\frac{y}{2} \omega'(y)$. In particular, one has

$$\omega_0(f(x)) = \left(\frac{\partial}{\partial x} \log f(x) \right)^2, \qquad \omega_1(f(x)) = -\frac{\partial^2}{\partial x^2} \log f(x). \quad (1)$$

Since g is odd, ω_0 can be written as a formal series of the form

$$\omega_0(y) = y^{-2} + a_1 + a_2 y^2 + a_3 y^4 + a_4 y^6 + a_5 y^8 + \dots, \quad (2)$$

and then for $r \geq 1$ each of the functions $\omega_r(y)$ can be written

$$\omega_r(y) = y^{-2} + \sum_{k=2}^{\infty} (1-k)^r a_k y^{2k-2}.$$

The series ω_0 in fact determines the genus ϕ .

Lemma 1.1 *The following relations hold between the values $\phi(\mathbb{CP}^{2k})$ and the coefficients a_k :*

$$\phi(\mathbb{CP}^{2k}) = \sum_{j=1}^k \frac{j-2k}{2k} a_j \phi(\mathbb{CP}^{2(k-j)}) = \sum_{j=1}^k \frac{(2j)!}{(-4)^j j!} \sum_{\mathcal{P}_k^j} \frac{a_1^{k_1} \dots a_k^{k_k}}{k_1! \dots k_k!}.$$

Here

$$\mathcal{P}_k^j = \left\{ (k_1, \dots, k_k) : k_i \geq 0, \sum i k_i = k, \sum k_i = j \right\},$$

the set of partitions of k of length j .

Proof. Compare coefficients in the series $g' \omega_1 = (g' + y g'') \omega_0$ and $y g' = \omega_0^{-\frac{1}{2}}$.

For fixed $i \geq 0$, consider also the generating function for ϕ on the Milnor manifolds $H_{i,j}$,

$$H_i(y) = \sum_{j=0}^{\infty} \phi(H_{i,j}) y^j. \quad (3)$$

Thus $H(u, v) = \sum H_i(u) v^i$. These series may also be expressed in terms of the functions ω_r .

Proposition 1.2 *There are polynomials $F_i \in \Lambda[x_1, \dots, x_i]$, $i \geq 0$, of total degree $\lfloor \frac{i}{2} \rfloor$, such that*

$$F_i(\omega_0, \dots, \omega_{i-1}) = \begin{cases} H_i, & i \text{ odd}, \\ \sqrt{\omega_0} H_i, & i \text{ even}. \end{cases}$$

Proof. Consider first the formal power series f_j for $j \geq 0$ defined by

$$f_j(y) = g'(y) f^{(j)}(g(y)),$$

where $f^{(j)}(x) = \frac{\partial^j}{\partial x^j} f(x)$. Then $\sqrt{\omega_0} f_0 = f_1 = 1$ and the following relations hold:

$$\begin{aligned}\sqrt{\omega_0} f_j &= -2\omega_0 \mathcal{D}(f_{j-1}) + (\omega_0 - \omega_1) f_{j-1}, \\ f_{j+1} &= -2\mathcal{D}(\sqrt{\omega_0} f_j) + \sqrt{\omega_0} f_j.\end{aligned}$$

One can now see inductively that $\sqrt{\omega_0} f_{2k}$ and f_{2k+1} are homogeneous polynomials of degree k in the ω_i ; thus the proposition will follow if we show that each H_i can be written as a linear combination of the series $\{f_j\}_{i-j \text{ even}, j \leq i}$. From the addition formula we have

$$\begin{aligned}H_i(u) &= \text{coefficient of } v^i \text{ in } g'(u)g'(v) f(g(u) + g(v)) \\ &= \text{coefficient of } v^{i+1} \text{ in } (i+1)g'(u) f^{(-1)}(g(u) + g(v)),\end{aligned}$$

where we write $f^{(-1)}(x)$ for $\int f(x) dx$. Let $c_{i,j}$ be the coefficient of v^{i+1} in $g(v)^{j+1}$; then

$$\begin{aligned}H_i(u) &= \sum_{j=0}^{\infty} c_{i,j} \cdot \text{coefficient of } x^{j+1} \text{ in } (i+1)g'(u) f^{(-1)}(g(u) + x) \\ &= \sum_{j=0}^{\infty} c_{i,j} \cdot \frac{i+1}{(j+1)!} f_j(u).\end{aligned}$$

Clearly $c_{i,j} = 0$ if $i-j$ is odd or if $j > i$.

Corollary 1.3 *The values of ϕ on $H_{2,2k-1}$ and $H_{3,2k-2}$ are given by*

$$\begin{aligned}\phi(H_{2,1}) &= 0, \\ \phi(H_{3,0}) &= \phi(\mathbb{CP}^2) = -\frac{1}{2}a_1,\end{aligned}$$

and for $k \geq 2$ by

$$\begin{aligned}2\phi(H_{2,2k-1}) &= \sum_{j=2}^k j a_j \phi(\mathbb{CP}^{2(k-j)}), \\ 6\phi(H_{3,2k-2}) &= k(2k-1)a_k.\end{aligned}$$

Proof. From proposition 1.2 the series $\sqrt{\omega_0} H_2$ and H_3 are linear polynomials in the ω_i ; explicitly we have $\sqrt{\omega_0} f_2 = \omega_0 - \omega_1$, $f_3 = -2(\omega_1 - \omega_2) + (\omega_0 - \omega_1)$ and

$$\begin{aligned}H_2 &= \left(\phi(\mathbb{CP}^2) + \frac{1}{2}(\omega_0 - \omega_1) \right) \omega_0^{-\frac{1}{2}}, \\ H_3 &= \frac{4}{3} \phi(\mathbb{CP}^2) + \frac{1}{6}(\omega_0 - 3\omega_1 + 2\omega_2).\end{aligned}$$

Comparing coefficients in the formal power series, the results follow.

2 The Witten genus and Weierstrass functions

The Witten genus ϕ_W takes values in the graded ring $\mathcal{M}_*$ of almost modular forms on the upper half-plane $\mathfrak{h} = \{\tau \in \mathbb{C} : \text{Im } \tau > 0\}$, and has characteristic series

$$Q_W(x) = \exp \left(\sum_{n=1}^{\infty} \frac{2}{(2n)!} G_{2n}(\tau) x^{2n} \right) = \frac{x e^{G_2(\tau) x^2}}{\sigma(x, \tau)} \quad (4)$$

see [10, p.221], [8, 5.1]. Here $G_{2n} \in \mathcal{M}_{2n}$ are given by (normalised) Eisenstein series,

$$G_{2n}(\tau) = \frac{(2n-1)!}{2(2\pi i)^{2n}} \sum_{0 \neq \gamma \in \mathbb{Z} + \mathbb{Z}\tau} \frac{1}{\gamma^{2n}},$$

and the elliptic function $\sigma(x, \tau)$ is related to the Weierstrass ζ - and $\wp$ -functions by:

$$\wp(x) = -\frac{\partial}{\partial x} \zeta(x), \quad \zeta(x) = \frac{\partial}{\partial x} \log \sigma(x).$$

For the corresponding series

$$f(x) = x/Q_W(x) = \sigma(x)/e^{G_2 x^2}$$

we therefore have

$$\wp(x) = -2G_2 - \frac{\partial^2}{\partial x^2} \log f(x). \quad (5)$$

Recall also the following relations from elliptic function theory [1, 13.13 (4,17)]:

$$\wp'(x)^2 = 4\wp(x)^3 - 20G_4\wp(x) - \frac{7}{3}G_6, \quad (6)$$

$$\wp(x) - \wp(z) = -\frac{\sigma(x+z)\sigma(x-z)}{\sigma(x)^2\sigma(z)^2}. \quad (7)$$

Proposition 2.1 *The functions ω_1 and ω_2 for the Witten genus satisfy*

$$\begin{aligned} \omega_1(y) &= 2G_2 + \wp(g(y)), \\ \omega_2(y) &= h(y)/y^2. \end{aligned}$$

Hence $\phi_W(\mathbb{HP}^k) = (k-1)^2 a_k$ for all $k \geq 0$.

Proof. The first equation follows from (1) and (5). Taking the limit as $z \rightarrow x$ of (7) gives

$$\wp'(x) = -\frac{\sigma(2x)}{\sigma(x)^4} = -\frac{f(2x)}{f(x)^4},$$

and combining this with the duplication formula we obtain

$$\wp'(g(y)) = -\frac{f(2g(y))}{y^4} = -\frac{2h(y)}{y^3 g'(y)}.$$

The remainder of the proposition now follows.

By proposition 1.2 the series H_j may also be expressed in terms of h or $\wp$; in particular corollary 1.3 gives Theorem A(i):

Corollary 2.2 *For all $k \geq 1$, the value of the Witten genus for the quaternionic projective space $\mathbb{HP}^k$ is a rational multiple of that for the Milnor manifold $H_{3,2k-2}$,*

$$k(2k-1)\phi_W(\mathbb{HP}^k) = 6(k-1)^2\phi_W(H_{3,2k-2}).$$

Now we can rewrite the Weierstrass cubic equation (6) as

$$4\omega_0\omega_2^2 = 4(\omega_1 - 2G_2)^3 - 20G_4(\omega_1 - 2G_2) - \frac{7}{3}G_6$$

Applying $\mathcal{D}$, removing the common factor of $4\omega_2$, and applying $\mathcal{D}$ once more, we have

$$2\omega_0\omega_4 + 3\omega_1\omega_3 + \omega_2^2 = 6(\omega_1 - 2G_2)\omega_2. \quad (8)$$

Now comparing coefficients of the power series on both sides one obtains:

Theorem 2.3 *For the Witten genus, the coefficients a_k satisfy the following quadratic recurrence relation for $k \geq 4$:*

$$a_k = -\frac{2(k-1)(k-2)}{k(2k-1)}a_1a_{k-1} + \frac{1}{2}\sum_{j=2}^{k-2}\left(\frac{j(k-j)(5k-16)}{k(2k-1)(k-3)} - 1\right)a_ja_{k-j}.$$

Theorem A(ii) now follows, since

$$\begin{aligned} \phi_W(H_{3,0}) &= -a_1/2 = 3G_2, \\ \phi_W(H_{3,2k-2}) &= k(2k-1)a_k/6, \end{aligned}$$

for $k \geq 2$.

3 The Witten genus and Jacobi functions

To prove Theorem B we introduce an alternative expression for the Witten genus in terms of Jacobi theta functions. We recall first [1, 13.20 (1)] that the Jacobi theta function $\vartheta_1(v) = \vartheta_1(v, \tau)$ is related to the σ - and ζ -functions considered above by

$$\sigma(x) = C \cdot \frac{\vartheta_1(x/2\pi i)}{\vartheta_1'(0)} \exp(\zeta(\pi i) x^2/2\pi i). \quad (9)$$

Here C is a complex constant and the null value $\vartheta_1'(0)$ is, up to a constant factor, the cube of the Dedekind η -function (compare [loc. cit. (9)], [6, p.141]). Furthermore ϑ_1 and η satisfy:

$$\begin{aligned} 4\pi i \frac{\partial}{\partial \tau} \vartheta_1(v, \tau) &= \frac{\partial^2}{\partial v^2} \vartheta_1(v, \tau) & (\text{heat equation}) \\ \frac{1}{2\pi i} \frac{d}{d\tau} \log \eta(\tau) &= -G_2(\tau). \end{aligned}$$

Proposition 3.1 *The power series f for the Witten genus can be expressed in terms of the Jacobi theta function ϑ_1 and the Dedekind eta function,*

$$f(x, \tau) = c \vartheta_1\left(\frac{x}{2\pi i}, \tau\right) \eta(\tau)^{-3},$$

for some $c \in \mathbb{C}$, and satisfies the partial differential equation

$$\frac{1}{2\pi i} \frac{\partial f}{\partial \tau} = 3 G_2 f + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}. \quad (10)$$

Proof. From (9) we have

$$f(x) = \frac{\sigma(x)}{e^{G_2 x^2}} = c \cdot \frac{\vartheta_1(x/2\pi i)}{\eta^3} \cdot \frac{\exp(\zeta(\pi i) x^2/2\pi i)}{\exp(G_2 x^2)}.$$

To see that $G_2 = \zeta(\pi i)/2\pi i$ we use the identities

$$\zeta(x) = \frac{\partial}{\partial x} \log \sigma(x) = \frac{f'(x)}{f(x)} + 2 G_2 x$$

together with the fact that $f'(\pi i) = 0$, which we will show from the Fourier expansions in section 4. Thus we have the first half of the proposition, and now the second follows from the differential equations satisfied by ϑ_1 and η . We can now prove

Proposition 3.2 *The series H_2 and g for the Witten genus satisfy*

$$\frac{1}{2\pi i} \frac{\partial}{\partial \tau} g(y, \tau) + H_2(y, \tau) = 0.$$

Proof. Using the notation of the proof of proposition 1.2 we know that

$$H_2(y) = c_{2,0} \cdot \frac{3}{1!} g'(y) y + c_{2,2} \cdot \frac{3}{3!} g'(y) f^{(2)}(g(y)),$$

where $c_{2,2} = 1$ and $c_{2,0} = \frac{1}{3} \phi_W(\mathbb{CP}^2) = G_2$. Substituting $y = f(x, \tau)$ and $g'(y)^{-1} = \partial f / \partial x$ gives

$$\frac{\partial f}{\partial x} \cdot H_2(f(x, \tau)) = 3 G_2 f + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}.$$

Now the identity $\partial f / \partial x \cdot \partial g / \partial \tau + \partial f / \partial \tau = 0$ and equation (10) give the result.

Since the generating functions for $\phi_W(H_{2,2k-1})$ and $\phi_W(\mathbb{CP}^{2k})$ are $H_2(y)$ and $\partial g / \partial y$ respectively, Theorem B follows from proposition 3.2.

4 Some remarks on the Fourier series

The characteristic series of the Witten genus has the following alternative form [10, p.221],

$$Q_W(x) = \frac{x}{e^{x/2} - e^{-x/2}} \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^n e^x)(1 - q^n e^{-x})},$$

which corresponds to (4) on taking the Fourier development in the variable $q = e^{2\pi i \tau}$.

Lemma 4.1 *The formal power series $f(x) = x/Q_W(x)$ may be written as*

$$f(x) = \sum_{k=0}^{\infty} b_{(k)} (e^{x/2} - e^{-x/2})^{2k+1}$$

where $b_{(k)} \in \mathbb{Z}[[q]]$ and has leading term $(-1)^k q^{k(k+1)/2}$.

Proof. Since $(1 - q^n e^x)(1 - q^n e^{-x}) = (1 - q^n)^2 - q^n (e^{x/2} - e^{-x/2})^2$ we have

$$f(x) = (e^{x/2} - e^{-x/2}) \prod_{n=1}^{\infty} (1 - b_n (e^{x/2} - e^{-x/2})^2),$$

where $b_n = q^n(1 - q^n)^{-2} = \sum_{r=1}^{\infty} r q^{rn}$. Let $b_{(k)} = (-1)^k \sum_{n_1 < \dots < n_k} b_{n_1} \dots b_{n_k}$; the lemma follows.

In particular $f'(\pi i) = 0$, completing the proof of proposition 3.1.

We can now consider the Fourier series corresponding to the a_k , the coefficients of ω_0 .

Proposition 4.2 *For $k \geq 2$, the q -expansion of a_k is an element of $q^{k-1}\mathbb{Z}[[q]]$.*

Proof. We can rewrite (2) as

$$1 + a_1 f(x)^2 + a_2 f(x)^4 + a_3 f(x)^6 + a_4 f(x)^8 + \dots = f'(x)^2$$

and by lemma 4.1 we obtain

$$1 + \sum_{i=1}^{\infty} a_i z^i \left(\sum b_{(k)} z^k \right)^{2i} = (1 + z/4) \left(\sum (2k+1) b_{(k)} z^k \right)^2,$$

where $z = (e^{x/2} - e^{-x/2})^2$. The coefficient of z in this equation gives $\frac{1}{4} + 6b_{(1)} = a_1$, and eliminating a_1 leaves

$$1 + \sum_{i=2}^{\infty} a_i z^i \left(\sum b_{(k)} z^k \right)^{2i} = \sum_{p,q \geq 0} (p(q+1)(4+z) + (1 - 6b_{(1)}z)) b_{(p)} b_{(q)} z^{p+q}$$

Comparing coefficients of z^k in this equation we find inductively that each a_k , $k \geq 2$, is a $\mathbb{Z}$ -linear combination of monomials $b_{(k_1)} \dots b_{(k_r)}$ with $k_1 + \dots + k_r = k - 1$ or k . Since the $b_{(k_i)}$ have integer coefficients and are divisible by q for each $k_i \geq 1$, it follows that for $k \geq 2$ the q -expansion of a_k also has integer coefficients and is divisible by q^{k-1} .

Corollary 4.3 *For all $k \geq 1$, the q -expansion of $\phi_W(\mathbb{HP}^k)$ is an element of $(k-1)^2 q^{k-1} \mathbb{Z}[[q]]$.*

Proof. Since $\phi_W(\mathbb{HP}^k) = (k-1)^2 a_k$.

Corollary 4.4 *The leading term of the q -expansion of $\phi_W(H_{2t+1,2s})$ has degree $\geq s$.*

Proof. Since the coefficient of y^{2s} in each function ω_r is a multiple of a_{s+1} , we see by proposition 1.2 that the coefficient of y^{2s} in H_{2t+1} is a linear combination of terms $a_{s_1+1} \dots a_{s_r+1}$ with $s_1 + \dots + s_r = s$. Since this coefficient

is $\phi(H_{2t+1,2s})$ by definition, and the q -expansion of each a_{s_i+1} is divisible by q^{s_i} , the result follows.

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