ISSN 1560-3547, Regular and Chaotic Dynamics, 0000, Vol. 00, No. 0, pp. 1–7. © RCD Editorial Office, 2016.

Zeroth Poisson homology, foliated cohomology and perfect Poisson manifolds

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Received September 5, 2017; accepted Month XX, 20XX

Abstract—We prove that for compact regular Poisson manifolds, the zeroth homology group is isomorphic to the top foliated cohomology group and we give some applications. In particular, we show that for regular unimodular Poisson manifolds top Poisson and foliated cohomology groups are isomorphic. Inspired by the symplectic setting, we define what is a perfect Poisson manifold. We use these Poisson homology computations to provide families of perfect Poisson manifolds.

MSC2010 numbers: 53D17, 53C12

DOI: 10.0000/S1560354700000012

Keywords: Poisson homology, foliated cohomology

1. INTRODUCTION

The description of geometric and algebraic properties of infinite dimensional groups and algebras is an important mathematical problem. Some of these groups are linked to physical problems, giving an additional motivation to pursue their study. An important example is that of the group of volume preserving transformations of a manifold. By Arnold's work [1] their Riemannian geometry is deeply related to fluid mechanics. Another relevant example is the group of Hamiltonian diffeomorphisms of phase spaces. Since Calabi's proof on the perfectness of the Lie algebra of Hamiltonian vector fields on a symplectic manifold [8], much has been done on the study of the group of Hamiltonian transformations of a symplectic manifold [3, 21]. However, little is known on the structure of the group of Hamiltonian transformations of a Poisson manifold.

In this paper we address Calabi's question for a regular Poisson manifold, that is, the study of the subalgebra of commutators of its Poisson algebra. It is well-known that such subalgebra is captured by the zeroth Poisson homology of the Poisson manifold. Poisson homology and cohomology groups are the main algebraic invariants of a Poisson manifold. However, it is in general very difficult to discuss their structure, and even less to provide explicit computations ([26], [18],[14], [22], [16]). The characteristic foliation of a regular Poisson manifold is the (regular) foliation integrating the Hamiltonian directions. Thus, properties of the commutator subalgebra are necessarily tied to

[1] Eva Miranda is supported by the Catalan Institution for Research and Advanced Studies via an ICREA Academia 2016 Prize, by a *Chaire d'Excellence* de la Fondation Sciences Mathématiques de Paris, and partially supported by the Ministerio de Economía y Competitividad project with reference MTM2015-69135-P/FEDER and by the Generalitat de Catalunya project with reference 2014SGR634. This work is supported by a public grant overseen by the French National Research Agency (ANR) as part of the *"Investissements d'Avenir"* program (reference: ANR-10-LABX-0098). Part of this research has been supported by an Ajut mobilitat 2017-EPSEB. We thank the Centre de Recerca Matemàtica CRM-Barcelona for their hospitality in July 2017 when this project has been finalized.

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properties of the characteristic foliation. Our main result in this paper follows closely Calabi's original arguments to prove that the zeroth cohomology group of a compact regular Poisson manifold is entirely controlled by the behaviour of the characteristic foliation:

Theorem 1. *Let (M, π) be a compact regular Poisson manifold of rank $2n$, and let $\mathcal{F}_\pi$ denote its characteristic foliation. Then there is a canonical isomorphism of vector spaces:*

$$H_0^\pi(M) \longrightarrow H^{2n}(\mathcal{F}_\pi)$$

Our result provides a de Rham approach of the zeroth Poisson homology group which is more amenable to explicit computations. Also under the unimodularity assumption[1)] it identifies the top Poisson and foliated cohomology groups:

Corollary 1. *Let (M, π) be an orientable unimodular compact regular Poisson manifold of dimension m and rank $2n$. Then there is an isomorphism of cohomology groups:*

$$H_\pi^m(M) \to H^{2n}(\mathcal{F}_\pi)$$

It has been recently shown [6, 13] that unimodular Poisson manifolds appear as geometrical model for the critical set (collision) of several problems in celestial mechanics which were largely studied by Arnold [2] (see also [10]).

The structure of the paper is as follows: in Section 1 we prove Theorem 1. In Section 2 we discuss applications of Theorem 1 and exhibit various examples where the group $H_0^\pi(M)$ is computed. This provides families of examples of perfect Poisson manifolds.

2. COMMUTATORS AND FOLIATED COHOMOLOGY

Notation: In this paper $H_\pi^k(M)$ will denote the degree k Poisson cohomology group [18], $H_k^\pi(M)$ with denote the degree k homology group as defined by Brylinski [7], and $H^k(\mathcal{F}_\pi)$ will denote the degree k foliated cohomology group of $(M, \mathcal{F}_\pi)$. When we refer to top foliated cohomology groups we will mean $H^m(\mathcal{F}_\pi)$, where m is the rank of the (characteristic) foliation.

We begin with the proof of Theorem 1:

Proof 1. We regard the Poisson structure π as a closed, non-degenerate foliated 2-form $\omega_{\mathcal{F}_\pi} \in \Omega^2(\mathcal{F}_\pi)$, where closedness is with respect to the foliated de Rham differential $d_{\mathcal{F}_\pi}$.

Recall that by definition:

$$H_0^\pi(M) := \frac{C^\infty(M)}{\{C^\infty(M), C^\infty(M)\}}.$$

We define the map:

$$\begin{array}{rcl} \phi : C^\infty(M) & \longrightarrow & \Omega^{2n}(\mathcal{F}_\pi) \\ f & \longmapsto & f\dfrac{\omega_{\mathcal{F}_\pi}^n}{n!}. \end{array} \tag{2.1}$$

Hence to prove the theorem we must show the equality

$$\phi(\{C^\infty(M), C^\infty(M)\}) = d_{\mathcal{F}_\pi}(\Omega^{2n-1}(\mathcal{F}_\pi)).$$

To check the inclusion $\phi(\{C^\infty(M), C^\infty(M)\}) \subset d_{\mathcal{F}_\pi}(\Omega^{2n-1}(\mathcal{F}_\pi))$ we use Leibniz's rule (which holds in the foliated setting whenever they involve vector fields tangent to the foliation):

$$L_{X_f}(g\frac{\omega_{\mathcal{F}_\pi}^n}{n!}) = (L_{X_f}g)\frac{\omega_{\mathcal{F}_\pi}^n}{n!} + gL_{X_f}\frac{\omega_{\mathcal{F}_\pi}^n}{n!}.$$

[1)] Poisson manifolds admitting an invariant measure with respect to Hamiltonian vector fields are called unimodular (unimodularity can be captured using Poisson cohomology).

Since the leafwise Liouville volume form $\frac{\omega^n_{\mathcal{F}_\pi}}{n!}$ is Hamiltonian invariant the following equality holds

$$\{f,g\}\frac{\omega^n_{\mathcal{F}_\pi}}{n!} = L_{X_f}(g\frac{\omega^n_{\mathcal{F}_\pi}}{n!}) = d_{\mathcal{F}_\pi}(i_{X_f}g\frac{\omega^n_{\mathcal{F}_\pi}}{n!}),$$

as we wanted to prove.

To prove the other inclusion we consider $\gamma \in d_{\mathcal{F}_\pi}(\Omega^{2n-1}(\mathcal{F}_\pi))$. Because M is compact we can choose a finite open cover U_i, $1 \leq i \leq d$, such that each U_i is in the domain of Weinstein local coordinates (as the splitting theorem guarantees [29]) and let us fix a partition of unity $\beta_i, 1 \leq i \leq d$, subordinated to this open cover.

By setting $\gamma_i := \beta_i\gamma$, then it is clear that our problem reduces to showing that $d_{\mathcal{F}_\pi}\gamma_i$ can be written as a sum of commutators supported in U_i.

If we let $x_1, \ldots, x_{2n} \in C^\infty(U_i)$ be the (symplectic) Weinstein local coordinates, then in U_i we can write:

$$\gamma_i = \sum_{i=1}^{2n} h_j dx_1 \wedge \cdots \wedge d\hat{x}_i \wedge \cdots \wedge dx_{2n}, \quad h_j \in C^\infty(U_i).$$

Therefore:

$$d_{\mathcal{F}_\pi}\gamma_i = \sum_{j=1}^{2n}(-1)^{j-1}n!\frac{\partial h_j}{\partial x_j}\frac{\omega^n_{\mathcal{F}_\pi}}{n!}$$

We define $g_j := \beta x_j$, where β is supported in U_i and equals 1 in the support of γ_j. Then

$$\sum_{j=1}^{n}(-1)^{2j-1}n!\{h_{2j-1}, g_{2j}\} + \sum_{j=1}^{n}(-1)^{2j-1}n!\{h_{2j}, g_{2j-1}\} = \sum_{j=1}^{2n}(-1)^{j-1}n!\frac{\partial h_j}{\partial x_j},$$

and this finishes the proof of the theorem.

Theorem 1 has an interesting consequence: let $(M, \mathcal{F})$ be a foliated compact manifold and consider the set of Poisson structures whose characteristic foliation is $\mathcal{F}$. Observe that this subset of the leafwise closed and non-degenerate 2-forms is open in the compact open topology among the set of closed forms. The interesting observation is that any Poisson structure on this set has zeroth Poisson homology group canonically isomorphic to $H^{\mathrm{top}}(\mathcal{F})$ so the zeroth Poisson homology group only depends on the characteristic foliation.

Remark 1. Our strategy to prove Theorem 1 is analogous to that of Calabi [8]. Lichnerowicz [18] also used the same approach to prove that if $U \subset M$ is contractible and contained in the domain of Weinstein coordinates, then $C^\infty(U) = \{C^\infty(U), C^\infty(U)\}$. However, his proof is entirely local and does not make any reference to the foliated de Rham differential.

Our result can also be derived from a spectral sequence argument to compute Poisson homology [26]. We believe our proof is more transparent and direct.

Remark 2. This isomorphism generalizes Brylinski's isomorphism [7] at any degrees $H^\pi_m(M^{2n}) \cong H^{2n-m}(M^{2n})$ for compact symplectic manifolds.

Remark 3. A different connection between foliated cohomology and Poisson cohomology already appears in [27] where the modular class (see subsection 3.1) is related to the Reeb class of the symplectic foliation. This is made more precise by [4] where an injection $H^1(\mathcal{F}) \to H^1_\pi(M)$ is exhibited.

3. APPLICATIONS

In this section we describe several applications of the main theorem.

3.1. Unimodular Poisson manifolds

For a given volume form Ω on an oriented Poisson manifold M the associated modular vector field is defined [27] as the following derivation:

$$C^\infty(M) \to \mathbb{R} : f \mapsto \frac{\mathcal{L}_{X_f}\Omega}{\Omega}.$$

It is a Poisson vector field and it preserves the volume form. Given two different choices of volume form Ω, the resulting modular vector fields differ by a Hamiltonian vector field thus defining the same class in the first Poisson cohomology group. This class is known as the **modular class** of the Poisson manifold. A Poisson manifold is called **unimodular** if its modular class vanishes.

We recall from [28] the duality between Poisson cohomology and homology for this class of Poisson manifolds.

Theorem 2 (Evens-Lu-Weinstein [28]). *Let (M, π) be an m-dimensional orientable unimodular compact regular Poisson manifold then,*

$$H^k_\pi(M) \cong H^{m-k}_\pi(M).$$

As a consequence of Theorems 1 and 2 we obtain the following theorem which relates Poisson and foliated cohomology of top degree:

Corollary 2. *Let (M, π) be an orientable unimodular compact regular Poisson manifold of dimension m and rank $2n$. There is an isomorphism of cohomology groups:*

$$H^m_\pi(M) \to H^{2n}(\mathcal{F}_\pi). \tag{3.1}$$

As we remarked in the introduction, for a general Poisson manifold little can be said on its of Poisson (co)homology groups. However, the are some important types of Poisson structures for which much more can be said, and, in particular, isomorphism (3.1) holds true. As these types of Poisson manifolds are unimodular, isomorphism (3.1) can be obtained by simply applying Corollary 2. We illustrate this with two examples:

3.1.1. ***Example 1: Cosymplectic structures***

Cosymplectic manifolds have widely been studied related to different problems in Differential Geometry and Topology (see for instance de survey [9] and [5]). They are also ubiquitous in Poisson Geometry since their geometrical data determine a codimension-one symplectic foliation. Examples of cosymplectic manifolds can be found in [15] and [16] since the critical set of a b-symplectic manifold has naturally cosymplectic structures associated to them. In particular so are the collision set [13] of the geometrical models for the n-body problem and other problems in celestial mechanics [6] as studied by Arnold [2].

A **cosymplectic structure** on a manifold M^{2n+1} is given by a pair of closed forms $(\theta, \eta) \in \Omega^1(M) \times \Omega^2(M)$ for which $\theta \wedge \eta^n$ is a volume form on M. Cosymplectic manifolds are naturally endowed with a corank-one regular Poisson structure whose characteristic foliation integrated the kernel of α.

Theorem 3 (Osorno, [23]). *The Poisson cohomology of the Poisson structure π associated to a cosymplectic manifold can be given in terms of the foliated cohomology as follows:*

$$H^k_\pi(M) \simeq H^k(\mathcal{F}_\pi) \oplus H^{k-1}(\mathcal{F}_\pi).$$

For the top Poisson cohomology group the above theorem gives: $H^{2n+1}_\pi(M) \cong H^{2n}(\mathcal{F})$. Since for a cosymplectic manifold $\alpha \wedge \eta^n$ is a Hamiltonian invariant volume form, we can obtain this isomorphism for M compact by simply applying Corollary 2.

3.1.2. Example 2: Poisson manifolds of s-proper type

A Poisson manifold is said to be of compact type if the Lie algebroid $A = (T^*M, \sharp)$ associated to the Poisson structure π with anchor map $\sharp : T^*M \to TM, \sharp(\alpha) := \pi(\alpha, \cdot)$ integrates to a compact *symplectic Lie groupoid* [11].

Hodge-type isomorphisms hold for Poisson manifolds of compact type [11]:

$$H^k_\pi(M) \cong H^{top-k}_\Pi(M).$$

As Poisson manifolds of compact type are unimodular [11], one deduces:

Theorem 4 (Crainic-Fernandes-Martinez,[11]). *For orientable regular Poisson manifolds of compact type there exist isomorphisms:*

$$H^\pi_k(M) \cong H^k_\pi(M).$$

Theorem 4 in degree zero says that $H^\pi_0(M)$ is isomorphic to the Casimirs of (M, π). As the characteristic foliation of a regular Poisson manifold of compact type is compact and has finite holonomy [11], the Casimirs are isomorphic to the foliated cohomology group of top degree, this giving (3.1). Once more, unimodularity suffices to obtain Theorem 4.

3.2. Perfect Poisson manifolds

We would like to finish this note by going back to the discussion in the introduction. Calabi's original motivation was proving that the Lie algebra group of symplectomorphisms is perfect.

If we consider the Lie algebra of Hamiltonian vector fields we may want to see that it is perfect in a direct way. Because it is easier to study Poisson brackets of functions rather than Lie brackets of vector fieldsl we will compare the set of Hamiltonian vector fields with the set of functions in the symplectic setting.

For a symplectic manifold (M, ω) the Lie algebra of Hamiltonian vector fields is perfect. The mapping that sends a function to its Hamiltonian vector field, $f \mapsto X_f$, defines the short exact sequence of Lie algebras,

$$0 \to \mathbb{R} \to C^\infty(M) \to \mathrm{ham}(M, \omega) \to 0.$$

It follows that $\mathrm{ham}(M, \omega)$ is perfect if and only if

$$C^\infty(M) = \{C^\infty(M), C^\infty(M)\} + \mathbb{R}.$$

We may use the quotient $\frac{C^\infty(M)}{\{C^\infty(M), C^\infty(M)\}}$ as a first approach to this equality. This quotient coincides with the zeroth Poisson homology $H^\pi_0(M)$. From Brylinki [7] the zeroth Poisson homology of a symplectic manifold is isomorphic to the top de Rham cohomology class which, in turn, is of dimension 1 in the compact case. Thus a symplectic manifold satisfies the equality above and a compact symplectic manifold is perfect.

For a Poisson manifold (M, π) the analogous question is whether the Lie algebra of Hamiltonian vector fields associated to a given Poisson structure $\mathrm{ham}(M, \pi)$ is perfect. From the discussion above in the symplectic setting we adopt the following definition of **perfect Poisson manifold.**

Definition 1. *A Poisson manifold is perfect if and only if, the following equality holds:*

$$C^\infty(M) = \{C^\infty(M), C^\infty(M)\} + \mathbb{R}.$$

Thus perfectness of the Lie algebra of Hamiltonian vector fields is connected to the zeroth-Poisson homology group $H^\Pi_0(M)$ of a Poisson manifold.

In particular the discussion above proves.

Theorem 5. *If $H^\pi_0(M) = 0$ then (M, π) is perfect.*

This provides a wide class of examples. Let us construct one of them:

3.2.1. Example: A corank one Poisson manifold which is perfect and not unimodular

We consider the 3-dimensional compact Poisson manifold with symplectic foliation defined as follows:

- Following [17] we denote by $\mathbb{T}^3_A$ the mapping torus associated to the diffeomorphism on $\mathbb{T}^2$ given by a matrix in $SL(2,\mathbb{Z})$. In other words, on the product $\mathbb{T}^2 \times \mathbb{R}$ consider the action: $(m,t) \to (A(m), t+1)$ for $A \in SL(2,\mathbb{R})$ and we identify (m,t) with $(A(m), t+1)$. For this construction we assume A to be a hyperbolic matrix i.e., $|Tr(A)| > 2$.
- In $\mathbb{T}^3_A$ consider the 2-dimensional foliation as follows: For an hyperbolic element A there are two real eigenvalues with irrational slope. We denote by λ one of the eigenvalues and by α the irrational slope. The foliation by lines of irrational slope α on the torus $\mathbb{T}^2$ is invariant by the action described above and thus descends to the quotient $\mathbb{T}^3_A$. Denote by $\mathcal{F}$ the foliation generated in this way on $\mathbb{T}^3_A$.
- Observe that $\mathbb{T}^3_A$ is naturally endowed with a Poisson structure whose induced symplectic structure on each of the leaves is given by the area form.
- In [17] the foliated cohomology of $\mathbb{T}^3_A$ is computed. In particular:

  Theorem 6 (El Kacimi, [17]). $H^2(\mathcal{F}) = 0$.

As a consequence of Theorems 1 and 6 we obtain the following Poisson homology computation,

Corollary 3. *For the Poisson manifold $\mathbb{T}^3_A$ described above, $H^\pi_0(\mathbb{T}^3_A) = 0$ and, in particular, the Poisson manifold $\mathbb{T}^3_A$ is perfect.*

Remark 4. As observed already in [17] since the top foliated cohomology group of this foliation does not vanish then it does not admit any transverse density. In particular, the codimension-one foliation constructed above is not unimodular.

For unimodular Poisson manifolds the restrictions on the zeroth homology group for the Poisson manifold to be perfect are weaker.

Recall that an unimodular Poisson manifold admits a Hamiltonian invariant volume. In [27] it is shown that that for μ any Hamiltonian invariant volume form and $C^\infty_\mu(X)$ the hyperplane of $C^\infty(X)$ of zero mean functions with respect to μ, one has

$$\{C^\infty(X), C^\infty(X)\} \subset C^\infty_\mu(X) \tag{3.2}$$

In particular, for unimodular Poisson manifolds we obtain,

Theorem 7. *If a Poisson manifold (M,π) is unimodular and satisfies $\dim H^\pi_0(M) = 0,1$ then it is perfect.*

From this theorem we may recover that any compact symplectic manifold is perfect (as we already proved above) because it is unimodular and $\dim H^\pi_0(M) = 1$.

Example 1. In particular, as an application of theorem 7 a Poisson manifold (M_1,π_1) given by any cosymplectic manifold in dimension 3 with a compact symplectic leaf which is a 2-sphere S^2 (and thus all of them are, [16]) is perfect.

We may produce product-type examples (and counter-examples) from the ones above using a Künneth-type formula for foliated cohomology.

The following statement was proved for sheaf cohomology in the context of Geometric Quantization [20]. When applied to the limit case $\omega \to 0$ it yields the following Künneth formula for foliated cohomology of the product foliated manifolds $(M_1,\mathcal{F}_1) \times (M_1,\mathcal{F}_2)$.

Theorem 8 (Miranda-Presas [20]). *There is an isomorphism*

$$H^n(\mathcal{F}_1 \times \mathcal{F}_2) \cong \bigoplus_{p+q=n} H^p(\mathcal{F}_1) \otimes H^q(\mathcal{F}_2),$$

whenever the foliated cohomology associated to $\mathcal{F}_1$ has finite dimension, M_1 is compact and M_2 admits a good *covering.*

From this theorem we deduce:

Example 2. The product of two different compact cosymplectic manifolds (M_1, π_1) in example 1 is perfect.(This includes the product of cosymplectic manifolds with symplectic manifolds).

Example 3. The product of $(\mathbb{T}^3_A, \pi)$ constructed in section 3.2.1 with any compact symplectic manifold is perfect.

Example 4. The product of any compact cosymplectic manifold with the Poisson manifold $(\mathbb{T}^3_A, \pi)$ constructed in section 3.2.1 is perfect.

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