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Modeling Group Assessments by means of Hesitant Fuzzy Linguistic Term Sets

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Abstract

Hesitant linguistic term sets have been introduced to capture the human way of reasoning using linguistic expressions involving different levels of precision. In this paper, a lattice structure is provided to the set of hesitant fuzzy linguistic term sets by means of the operations intersection and connected union. In addition, in a group decision making framework, hesitant fuzzy linguistic descriptions are defined to manage situations in which decision makers are assessing different alternatives by means of hesitant fuzzy linguistic term sets. Based on the introduced lattice structure, two distances between hesitant fuzzy linguistic descriptions are defined. These metric structures allow distances between decision makers to be computed. A centroid of the decision making group is proposed for each distance to model group representatives in the considered group decision making framework.

Keywords: Linguistic modeling, Group decision making, Uncertainty and fuzzy reasoning, Hesitant linguistic term sets.

Introduction

Different approaches have been developed in the fuzzy set literature involving linguistic modeling to handle the imprecision and uncertainty inherent in human preference reasoning [4, 9, 10, 12, 16]. In addition, several extensions of classic fuzzy sets theory have been established to include different levels of precision or multi-granularity in linguistic modeling [3, 7, 14]. Hesitant Fuzzy Linguistic Term Sets (HFLTSs) were introduced to capture the human way of reasoning involving different levels of precision. To this end, a set of linguistic expressions is defined based on the concept of hesitance [14].

L-fuzzy sets are considered as a generalization of the classic fuzzy sets with range values of membership functions in a lattice L [6]. Classic fuzzy sets can be considered as a special case of the L-fuzzy sets with L = [0, 1]. The relation between L-fuzzy sets

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and other extensions of fuzzy sets, such as intuitionistic fuzzy sets and interval-valued fuzzy sets, has been analyzed in several studies [3, 18].

In this paper, we define a lattice structure on the set of HFLTSs over a set of linguistic terms, $\mathcal{H}_{\mathcal{S}}$, based on the literature related to absolute order-of-magnitude spaces with different levels of precision or multi-granularity [5, 13, 17]. This allows us to consider hesitant fuzzy linguistic descriptions (HFLDs) as L-fuzzy sets based on this lattice. The set $\mathcal{F}_{\mathcal{H}}$ of all the $\mathcal{H}_{\mathcal{S}}$-fuzzy sets is also introduced.

In group assessment processes where decision makers (DMs) are assessing different alternatives by means of hesitant fuzzy linguistic term sets, the assessments provided by each DM are modeled as a HFLD. To study differences between HFLDs representing the assessments of each DM of a group, we present two distances in $\mathcal{H}_{\mathcal{S}}$ between HFLTSs, and their associated distances in $\mathcal{F}_{\mathcal{H}}$ between HFLDs.

Taking into consideration the different perspectives of the DMs in the decision-making group, we present a HFLD that characterizes the group via an aggregation of linguistic preferences. In addition, a centroid of the group is presented for each distance in $\mathcal{F}_{\mathcal{H}}$, as the HFLD that minimizes the addition of distances to the HFLDs of all the DMs in the group. Distances between HFLDs are used to measure differences between the DMs.

The rest of this paper is organized as follows: first, Section 1 presents the lattice of hesitant fuzzy linguistic term sets. The concept of hesitant fuzzy linguistic description is introduced in Section 2. In Section 3, two distances between HFLDs are defined by means of two distances between HFLTSs. A new approach for group preference modeling based on an aggregation of HFLDs and the distances between them is presented in Section 4. Finally, Section 5 contains the main conclusions and lines of future research.

1. The Lattice of Hesitant Fuzzy Linguistic Term Sets

In this section, we briefly review some basic concepts related to HFLTSs [1, 13, 14, 15]. This enables us to provide the set of HFLTSs with a lattice structure, to define a partial order and a compatibility relation in this set.

From here on, let S be a finite totally ordered set of linguistic terms, $\mathcal{S} = \{a_1, \ldots, a_n\}$, with $a_1 < \ldots < a_n$.

Definition 1 ([14]) A *hesitant fuzzy linguistic term set (HFLTS)* over $\mathcal{S}$ is a subset of consecutive linguistic terms of S, i.e. $\{x \in \mathcal{S} \mid a_i \leq x \leq a_j\}$, for some $i, j \in \{1, \ldots, n\}$ with $i \leq j$.

The HFLTS $\mathcal{S}$ is called the *full HFLTS*. and it is also denoted by the symbol ?. Moreover, the empty set $\{\} = \emptyset$ is also considered as a HFLTS and it is called the *empty HFLTS*.

From now on, the non-empty HFLTS $H = \{x \in \mathcal{S} \mid a_i \leq x \leq a_j\}$ is also denoted by $[a_i, a_j]$. If $i = j$, $[a_i, a_i]$ is the singleton $\{a_i\}$. The set of all HFLTSs over S is denoted by $\mathcal{H}_{\mathcal{S}}$:

$$\mathcal{H}_{\mathcal{S}} = \{[a_i, a_j] \mid i, j \in \{1, \ldots, n\}, i \leq j\} \cup \{\emptyset\}$$

A simple calculation proves that the cardinality of $\mathcal{H}_\mathcal{S}$ is $|\mathcal{H}_\mathcal{S}| = 1 + \frac{n(n+1)}{2}$.

The union and complement [14] are not closed operations on the set $\mathcal{H}_\mathcal{S}$. Indeed, the union of two non-empty HFLTSs $[a_i, a_j]$ and $[a_{i'}, a_{j'}]$ is a HFLTS if and only if $[a_i, a_j] \cap [a_{i'}, a_{j'}] \neq \emptyset$ or $i = j'+1$ or $i' = j+1$. On the other hand, the complement of a non-empty HFLTS $[a_i, a_j]$ is a HFLTS if and only if $i = 1$ or $j = n$. The intersection of HFLTSs is a closed binary operation on the set $\mathcal{H}_\mathcal{S}$.

The *connected union,* $\sqcup$, of HFLTSs [13] is a closed binary operation on the set $\mathcal{H}_\mathcal{S}$, which is defined as follows:

Definition 2 The *connected union of two HFLTSs* is the least element of $\mathcal{H}_\mathcal{S}$, based on the subset inclusion relation $\subseteq$, that contains both HFLTSs.

As proven in the general case of order-of-magnitude spaces over a well-ordered (finite or infinite) set [13], the binary operations intersection and connected union provide a lattice structure to the set $\mathcal{H}_\mathcal{S}$ of HFLTSs.

Proposition 1 $(\mathcal{H}_\mathcal{S}, \sqcup, \cap)$ is a lattice.

PROOF. The two operations $\sqcup$ and $\cap$ are clearly idempotent and commutative. The intersection is associative, and $(H_1 \sqcup H_2) \sqcup H_3 = H_1 \sqcup (H_2 \sqcup H_3)$ because $H_1 \sqcup (H_2 \sqcup H_3)$ is the least element that contains $H_1 \sqcup H_2$ and H_3. The two absorption laws can be checked in a similar straightforward manner.

The lattice $(\mathcal{H}_\mathcal{S}, \sqcup, \cap)$ is not distributive. A counterexample in which the property $H_1 \cap (H_2 \sqcup H_3) = (H_1 \cap H_2) \sqcup (H_1 \cap H_3)$ does not hold is given in the case where $\mathcal{S}$ has at least three linguistic terms, considering $a_1, a_2, a_3 \in \mathcal{S}$ such that $a_1 < a_2 < a_3$ and the following three HFLTSs: $H_2 = \{a_1\}, H_1 = \{a_2\}, H_3 = \{a_3\}$.

The partial order $\leq$ in the lattice is given by: $H_1 \leq H_2 \Leftrightarrow H_1 \sqcup H_2 = H_1 \Leftrightarrow H_1 \cap H_2 = H_2 \Leftrightarrow H_1 \supseteq H_2$. Therefore, this order is the inverse subset inclusion relation and we call it *to be less or equally precise than.*

Definition 3 For any non-empty HFLTS $[a_i, a_j]$ and $[a_{i'}, a_{j'}]$, we say that $[a_i, a_j]$ is *less or equally precise than* $[a_{i'}, a_{j'}]$ if and only if $[a_i, a_j] \supseteq [a_{i'}, a_{j'}]$, i.e., $i \leq i'$ and $j' \leq j$.

Then, the least element in the lattice $\mathcal{H}_\mathcal{S}$ is $0_{\mathcal{H}_\mathcal{S}} = ? = \mathcal{S}$ because it is the least precise HFLTS, and the greatest element is $1_{\mathcal{H}_\mathcal{S}} = \emptyset$ because $H \supseteq \emptyset$ for all $H \in \mathcal{H}_\mathcal{S}$.

In Figure 1 the diagram of the lattice $(\mathcal{H}_\mathcal{S}, \sqcup, \cap)$ is depicted.

The relation *to be compatible* between non-empty HFLTSs is defined, inspired by the concept of qualitative equality in absolute order-of-magnitude qualitative spaces [17] as follows:

Definition 4 For any non-empty HFLTS $[a_i, a_j]$ and $[a_{i'}, a_{j'}]$, we say that $[a_i, a_j]$ and $[a_{i'}, a_{j'}]$ are *compatible* if and only if $[a_i, a_j] \cap [a_{i'}, a_{j'}] \neq \emptyset$, i.e., $j \geq i'$ or $j' \geq i$.

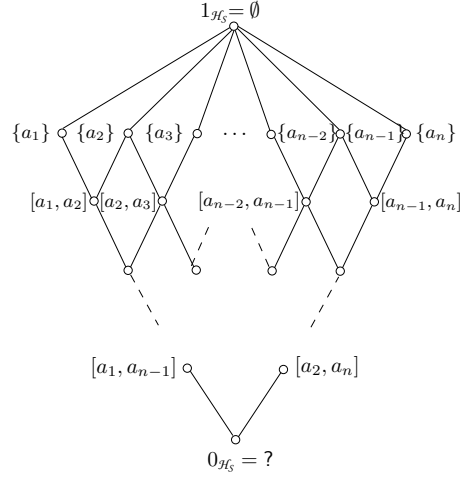Let us consider a simple example to illustrate the above definitions.

Figure 1: Diagram of the lattice $(\mathcal{H}_{\mathcal{S}}, \sqcup, \cap)$ [13]

Example 1 Given the set of linguistic terms: $\mathcal{S} = \{a_1, a_2, a_3, a_4\}$ with: $a_1 =$ *slightly good* , $a_2 =$ *moderately good* , $a_3 =$ *very good* , $a_4 =$ *extremely good* , and the following HFLTSs:

$\{a_3\}$ = *very good,*
$[a_1, a_3] =$ *not extremely good,*
$? = [a_1, a_4] =$ *unknown, and*
$\{a_1\} =$ *slightly good,*

identifying $a_i = \{a_i\}, \forall i = \{1, 2, 3, 4\}$.

The relation to be *less or equally precise than* among the first three HFLTSs gives: $? \supseteq [a_1, a_3] \supseteq \{a_3\}$. However, $\{a_1\}$ are $\{a_3\}$ are not comparable by this relation. In addition, $\{a_3\}$, $[a_1, a_3]$ and $?$ are *compatible* since their pairwise intersections are non-empty, while $\{a_1\}$ and $\{a_3\}$ are not compatible.

Two distances between HFLTSs will be introduced in Section 3 based on the properties of the lattice $(\mathcal{H}_{\mathcal{S}}, \sqcup, \cap)$.

2. Hesitant Fuzzy Linguistic Descriptions

The concept of an *L-fuzzy set on a non-empty set* Λ was introduced by Goguen in [6] as a function $f : \Lambda \to L$, where L is a lattice. This concept is applied to the case of the lattice $(\mathcal{H}_{\mathcal{S}}, \sqcup, \cap)$ of HFLTSs over a finite totally ordered set of linguistic terms $\mathcal{S}$ in the following definitions.

Definition 5 An $\mathcal{H}_{\mathcal{S}}$-*fuzzy set on* Λ is a function $F_H : \Lambda \to \mathcal{H}_{\mathcal{S}}$.

Note that any $f : \Lambda \to \{0, 1\}$ defines an ordinary set or crisp set on Λ, that is, a subset of Λ, whose characteristic function is f. If $f : \Lambda \to [0, 1]$, then f defines a

fuzzy set on Λ, where for each $\lambda \in \Lambda$, $f(\lambda)$ is the degree of membership of λ. We can therefore consider an $\mathcal{H}_\mathcal{S}$-fuzzy set as a function $F_H : \Lambda \to \mathcal{H}_\mathcal{S}$ that assigns to each element of Λ a HFLTS from $\mathcal{H}_\mathcal{S}$ instead of a degree of membership.

Definition 6 The *set $\mathcal{F}_\mathcal{H}$ of $\mathcal{H}_\mathcal{S}$-fuzzy sets on Λ* is:

$$\mathcal{F}_\mathcal{H} = (\mathcal{H}_\mathcal{S})^\Lambda = \{F_H \mid F_H : \Lambda \to \mathcal{H}_\mathcal{S}\}.$$

Definition 7 A *Hesitant fuzzy linguistic description (HFLD) of the set Λ by $\mathcal{H}_\mathcal{S}$* is an $\mathcal{H}_\mathcal{S}$-fuzzy set F_H on Λ such that for all $\lambda \in \Lambda$, $F_H(\lambda)$ is a non-empty HFLTS, i.e., $F_H(\lambda) \in \mathcal{H}_\mathcal{S} - \{\emptyset\}$.

From now on, the set Λ will represent a set of alternatives, and a HFLD will be used to model a DM's assessment of the alternatives in Λ. Note that missing values (such as DK/NA/REF) will be considered as ?.

Example 2 Following Example 1, and given the same set $\mathcal{S}$ of linguistic terms, let us consider $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, then

$$
\begin{array}{ccc}
F_H : & \Lambda & \longrightarrow & \mathcal{H}_\mathcal{S} \\
& \lambda_1 & \mapsto & \{a_3\} \\
& \lambda_2 & \mapsto & [a_1, a_3] \\
& \lambda_3 & \mapsto & ? \\
& \lambda_4 & \mapsto & \{a_1\}
\end{array}
$$

is a HFLD of the set Λ.

3. Distances between Hesitant Fuzzy Linguistic Descriptions

In order to define a first distance between HFLDs, that measures differences in the assessments of DMs, we previously consider the following distance between non-empty HFLTSs:

Definition 8 Given $H_1, H_2 \in \mathcal{H}_\mathcal{S} - \{\emptyset\}$, the *distance D_1 between H_1 and H_2* is defined as:
$$D_1(H_1, H_2) = card(H_1 \sqcup H_2) - card(H_1 \cap H_2)$$

As proven in the case of order-of-magnitude spaces over a finite well-ordered set in [13], D_1 fulfills the distance requirements. This distance between non-empty HFLTSs induces a distance between HFLDs as follows:

Definition 9 Let us consider F_H^1 and F_H^2 two HFLDs of a finite set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{H}_\mathcal{S}$, with $F_H^1(\lambda_i) = H_i^1$ and $F_H^2(\lambda_i) = H_i^2$, for all $i \in \{1, \ldots, r\}$. Then, the *distance $D_1^\mathcal{F}$ between these two HFLDs* is defined as:

$$D_1^\mathcal{F}(F_H^1, F_H^2) = \sum_{t=1}^{r} D_1(H_t^1, H_t^2) \tag{1}$$

Expression (1) provides a distance in the set $(\mathcal{H}_\mathcal{S} - \{\emptyset\})^\Lambda$, i.e., a distance between HFLDs. In fact, each HFLD F_H of the set Λ by $\mathcal{H}_\mathcal{S}$ can be identified with the r-dimensional vector $(H_1, \ldots, H_r) \in (\mathcal{H}_\mathcal{S} - \{\emptyset\})^r = (\mathcal{H}_\mathcal{S} - \{\emptyset\}) \times \cdots \times (\mathcal{H}_\mathcal{S} - \{\emptyset\})$ whose components are $H_i = F_H(\lambda_i)$, for all $i \in \{1, \ldots, r\}$. Therefore the set $(\mathcal{H}_\mathcal{S} - \{\emptyset\})^\Lambda$ can be identified with the Cartesian product $(\mathcal{H}_\mathcal{S} - \{\emptyset\})^r$, and the Cartesian product of metric spaces is a metric space using the product distance and the city-block norm, which in this case results in Formula (1).

Remark 1 The maximum value for D_1 between two HFLTSs from $\mathcal{H}_\mathcal{S} - \{\emptyset\}$, where $\mathcal{S} = \{a_1, \ldots, a_n\}$, is n. This case is given, for instance, when $H_1 = \{a_1\}$ and $H_2 = \{a_n\}$, among others. Consequently, the maximum value for $D_1^\mathcal{F}$ between two HFLDs of the set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ is $r \cdot n$.

Let us consider a simple example to illustrate the computation of this distance between HFLDs.

Example 3 Let us consider $\mathcal{S} = \{a_1, a_2, a_3, a_4\}$ as in Examples 1 and 2, and F_H^1 and F_H^2 two HFLDs of the set $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ by $\mathcal{H}_\mathcal{S}$ given in Table 1.

Table 1: HFLDs F_H^1 and F_H^2.

	F_H^1	F_H^2
λ_1	$\{a_3\}$	$\{a_1\}$
λ_2	$[a_1, a_3]$	$\{a_4\}$
λ_3	?	$[a_2, a_4]$
λ_4	$\{a_1\}$	$\{a_4\}$

Therefore:

$$D_1^\mathcal{F}(F_H^1, F_H^2) = \sum_{t=1}^4 (card(H_t^1 \sqcup H_t^2) - card(H_t^1 \cap H_t^2)) =$$

$$(3 - 0) + (4 - 0) + (4 - 3) + (4 - 0) = 12.$$

In this case, the distance between two HFLDs ranges from 0 to 16, which gives us a reference to frame the obtained result.

To capture differences among pairs of HFLTSs that are at the same distance D_1, we introduce the following measure of agreement that takes into consideration the gap between a pair of HFLTSs:

Definition 10 Given $H_1, H_2 \in \mathcal{H}_\mathcal{S} - \{\emptyset\}$, the *concordance between H_1 and H_2* is defined as:

$$\mathcal{C}(H_1, H_2) = \begin{cases} card(H_1 \cap H_2) & \text{if} & H_1 \cap H_2 \neq \emptyset \\ -card((H_1 \sqcup H_2) \cap \overline{H_1} \cap \overline{H_2}) & \text{if} & H_1 \cap H_2 = \emptyset \end{cases}$$

where $\overline{H} = \{x \in \mathcal{S} \mid x \notin H\}$ is the complement of H with respect to $\mathcal{S}$.

It is straightforward to see that if $H_1 = [a_i, a_j]$ and $H_2 = [a_{j+k}, a_l]$, with $k > 0$, then $\mathcal{C}(H_1, H_2) = -(k-1)$. Moreover, notice that the concordance between two HFLTSs is positive if and only if the two HFLTSs are compatible. In addition, the aim of the concordance is to consider how much in common two HFLTSs have or how big is the gap between them in case that they have nothing in common. According to the concordance, we present a new distance between non-empty HFLTSs as:

Definition 11 Given $H_1, H_2 \in \mathcal{H}_\mathcal{S} - \{\emptyset\}$, the *distance D_2 between H_1 and H_2 is defined as:*

$$D_2(H_1, H_2) = card(H_1 \sqcup H_2) - \mathcal{C}(H_1, H_2)$$

In order to prove that D_2 is a distance, we will see that it is equivalent to the geodesic distance in the graph $\mathcal{H}_\mathcal{S} - \{\emptyset\}$, based on measuring the length of the shortest path between two elements of the graph [8]. In $\mathcal{H}_\mathcal{S} - \{\emptyset\}$, the shortest path between two HFLTSs can always be reached passing through the connected union of both of them. In Figure 2, we can see, as an example, the shortest path between $\{a_1\}$ and $[a_2, a_3]$ working with $\mathcal{S} = \{a_1, a_2, a_3, a_4\}$. In this case, the length of the shortest path is 3.
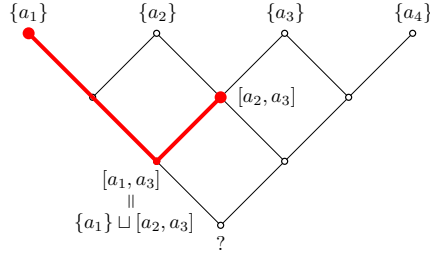

Figure 2: Shortest path between $\{a_1\}$ and $[a_2, a_3]$.

Lemma 1 D_2 can be equivalently expressed as:

$$D_2(H_1, H_2) = 2 \cdot card(H_1 \sqcup H_2) - card(H_1) - card(H_2)$$

PROOF. We see that $2 \cdot card(H_1 \sqcup H_2) - card(H_1) - card(H_2) = card(H_1 \sqcup H_2) - card(H_1 \cap H_2) + card((H_1 \sqcup H_2) \cap \overline{H_1} \cap \overline{H_2})$. Indeed, if $H_1 \cap H_2 \neq \emptyset$, both parts are equal to $card(H_1 \sqcup H_2) - card(H_1 \cap H_2)$, while if $H_1 \cap H_2 = \emptyset$, then both parts are $card(H_1 \sqcup H_2) + card((H_1 \sqcup H_2) \cap \overline{H_1} \cap \overline{H_2})$.

Proposition 2 D_2 is equivalent to the geodesic distance in the graph $\mathcal{H}_\mathcal{S} - \{\emptyset\}$.

PROOF. By Lemma 1, $D_2(H_1, H_2) = 2 \cdot card(H_1 \sqcup H_2) - card(H_1) - card(H_2) = (card(H_1 \sqcup H_2) - card(H_1)) + (card(H_1 \sqcup H_2) - card(H_2)) = \ell(H_1, H_1 \sqcup H_2) +$

$\ell(H_2, H_1 \sqcup H_2) = \ell(H_1, H_2)$, where $\ell(H, H')$ is the length of the shortest path from H to H'.

Once we have proved that D_2 is a distance between HFLTSs, we can use it to define an associated distance between HFLDs, analogously to what we did for D_1:

Definition 12 Let us consider F_H^1 and F_H^2 two HFLDs of a set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{H}_\mathcal{S}$, with $F_H^1(\lambda_i) = H_i^1$ and $F_H^2(\lambda_i) = H_i^2$, for all $i \in \{1, \ldots, r\}$. Then, the *distance $D_2^\mathcal{F}$ between these two HFLDs* is defined as:

$$D_2^\mathcal{F}(F_H^1, F_H^2) = \sum_{t=1}^r D_2(H_t^1, H_t^2)$$

Remark 2 The maximum value for D_2 between two HFLTSs from $\mathcal{H}_\mathcal{S} - \{\emptyset\}$, where $\mathcal{S} = \{a_1, \ldots, a_n\}$, is $2n - 2$. This case is given only when $H_1 = \{a_1\}$ and $H_2 = \{a_n\}$. Consequently, the maximum value for $D_2^\mathcal{F}$ between two HFLDs of the set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ is $r \cdot (2n - 2)$.

In order to illustrate this new distance, let us see the following example:

Example 4 Let F_H^1 and F_H^2 be the two HFLDs from Example 3 of the set Λ by $\mathcal{H}_\mathcal{S}$ given in Table 1. Therefore:

$$D_2^\mathcal{F}(F_H^1, F_H^2) = \sum_{t=1}^4 (card(H_t^1 \sqcup H_t^2) - \mathcal{C}(H_t^1, H_t^2)) =$$

$$(3 - (-1)) + (4 - 0) + (4 - 3) + (4 - (-2)) = 15.$$

In this case, the distance between two HFLDs ranges from 0 to 24, which gives us a reference to frame the obtained result.

The two distances that have been proposed can be compared as follows:

Proposition 3 Given two non-empty HFLTSs, H_1 and H_2, from $\mathcal{H}_\mathcal{S} - \{\emptyset\}$,

$$D_1(H_1, H_2) \le D_2(H_1, H_2).$$

PROOF. It is enough to rewrite $D_2(H_1, H_2)$ as:

$$D_2(H_1, H_2) = card(H_1 \sqcup H_2) - card(H_1 \cap H_2) + card((H_1 \sqcup H_2) \cap \overline{H_1} \cap \overline{H_2}) =$$

$$D_1(H_1, H_2) + card((H_1 \sqcup H_2) \cap \overline{H_1} \cap \overline{H_2}) \ge D_1(H_1, H_2).$$

Proposition 3 can be generalized to the distance between HFLDs as follows:

Proposition 4 Given two HFLDs, F_H^1 and F_H^2, of a set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$,

$$D_1^\mathcal{F}(F_H^1, F_H^2) \le D_2^\mathcal{F}(F_H^1, F_H^2).$$

PROOF. Taking into account Definitions 9 and 12, then, by Proposition 3, the proof becomes trivial.

To illustrate these two propositions, let us summarize the results from Examples 3 and 4 in the following table:

Table 2: Distances between HFLDs F_H^1 and F_H^2.

	F_H^1	F_H^2	$D_1(H_t^1, H_t^2)$	$D_2(H_t^1, H_t^2)$
λ_1	$\{a_3\}$	$\{a_1\}$	3	4
λ_2	$[a_1, a_3]$	$\{a_4\}$	4	4
λ_3	?	$[a_2, a_4]$	1	1
λ_4	$\{a_1\}$	$\{a_4\}$	4	6
			$D_1^{\mathcal{F}}(F_H^1, F_H^2) = 12$	$D_2^{\mathcal{F}}(F_H^1, F_H^2) = 15$

4. Modeling Group Assessments

In this section, we analyze from two different perspectives how to summarize the assessments given by a group of DMs that are assessing a set of alternatives by means of HFLTSs. To this end, the lattice structure of $\mathcal{H}_{\mathcal{S}}$-fuzzy sets and the distances defined in Section 3 are considered to aggregate the DMs' assessments of alternatives.

We consider two possible representatives to summarize the group's assessments: Firstly, the connected union in $\mathcal{H}_{\mathcal{S}}$-fuzzy sets and secondly, the HFLD of the set of alternatives Λ that minimizes the addition of distances to the assessments of all the DMs in the group, with respect to the two distances presented in Section 3, $D_1^{\mathcal{F}}$ and $D_2^{\mathcal{F}}$.

The connected union among $\mathcal{H}_{\mathcal{S}}$-fuzzy sets can be considered as a reasonable way to model the group assessment, because it provides a HFLD compatible with all the HFLDs in the group for all the alternatives. Notice that the intersection among $\mathcal{H}_{\mathcal{S}}$-fuzzy sets cannot be used to model the group assessments because some of its values may result in the null HFLTS. If so, the intersection would not be a HFLD.

Definition 13 Let Λ be a set of alternatives and G a group of k DMs. Let $F_H^1, \ldots, F_H^k$ be the HFLDs of Λ provided by the DMs in G. The *HFLD of Λ associated to the connected union of the assessments in group G* is defined as:

$$
\begin{array}{rcl}
F_H^G: & \Lambda & \longrightarrow & \mathcal{H}_{\mathcal{S}} - \{\emptyset\} \\
& \lambda & \mapsto & F_H^G(\lambda) = F_H^1(\lambda) \sqcup \ldots \sqcup F_H^k(\lambda)
\end{array}
$$

However, this way of representing the group's assessment tends very fast to ? in most of cases, because it is very sensitive to outliers. In addition, it does not consider the precision that DMs in the group use. For this reason, to solve these drawbacks, a representative of the group of DMs as a *centroid of the group* is defined by means of the concept of a distance as follows:

Definition 14 Let Λ be a set of r alternatives, G a group of k DMs and $F_H^1, \ldots, F_H^k$ the HFLDs of Λ provided by the DMs in G, then, for any distance $D^{\mathcal{F}}$ in $\mathcal{F}_{\mathcal{H}}$, a *centroid of the group with respect to $D^{\mathcal{F}}$* is:

$$F_H^C = arg \min_{F_H^X \in (\mathcal{H}_{\mathcal{S}} - \{\emptyset\})^r} \sum_{t=1}^k D^{\mathcal{F}}(F_H^X, F_H^t),$$

identifying each HFLD F_H with the vector $(H_1, \ldots, H_r) \in (\mathcal{H}_{\mathcal{S}} - \{\emptyset\})^r$, where $F_H(\lambda_i) = H_i$, for all $i = 1, \ldots, r$.

In the specific case of the two distances presented in Section 3, $D_1^{\mathcal{F}}$ and $D_2^{\mathcal{F}}$, the corresponding centroids will be denoted as $F_{H\,1}^C$ and $F_{H\,2}^C$ respectively.

Note that, for a given distance, more than one HFLD can produce the minimum value for the sum of distances in the above definition. Thus, a group of DMs can have more than one centroid with respect to the same distance. In addition, neither the HFLD of the connected union nor those of the centroids of the group with respect to any distance have necessarily to coincide with any HFLD provided by a DM in the group.

Example 5 Following Examples 1, 2, 3 and 4, where $\mathcal{S} = \{a_1, a_2, a_3, a_4\}$ and $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, let us consider a group G of five DMs. The HFLDs of the set Λ by $\mathcal{H}_{\mathcal{S}}$ corresponding to the DMs in G are given in Table 3 (columns from 2 to 6). Column 7 shows the HFLD associated to the connected union, F_H^G, columns 8 and 9 show the two centroids of the group, $F_{H\,1}^{C_1}$ and $F_{H\,1}^{C_2}$, according to $D_1^{\mathcal{F}}$, and in the last column we find the unique centroid of the group, $F_{H\,2}^C$, with respect to $D_2^{\mathcal{F}}$. An exhaustive search has been conducted to obtain the centroids of the group $F_{H\,1}^{C_1}$, $F_{H\,1}^{C_2}$ and $F_{H\,2}^C$.

Table 3: The HFLDs in G corresponding to Example 5.

	F_H^1	F_H^2	F_H^3	F_H^4	F_H^5	$\boldsymbol{F_H^G}$	$\boldsymbol{F_{H\,1}^{C_1}}$	$\boldsymbol{F_{H\,1}^{C_2}}$	$\boldsymbol{F_{H\,2}^{C_1}}$
λ_1	$\{a_3\}$	$\{a_1\}$	$[a_1, a_2]$	$[a_1, a_3]$	$\{a_2\}$	$[\boldsymbol{a_1, a_3}]$	$[\boldsymbol{a_1, a_2}]$	$[\boldsymbol{a_1, a_2}]$	$[\boldsymbol{a_1, a_2}]$
λ_2	$[a_1, a_3]$	$\{a_4\}$	$\{a_1\}$	$\{a_4\}$	$\{a_1\}$	?	$\{\boldsymbol{a_1}\}$	$\{\boldsymbol{a_1}\}$	$[\boldsymbol{a_1, a_3}]$
λ_3	?	$[a_2, a_4]$	$\{a_3\}$	$\{a_1\}$	$[a_3, a_4]$	?	$[\boldsymbol{a_3, a_4}]$	$[\boldsymbol{a_2, a_4}]$	$[\boldsymbol{a_2, a_4}]$
λ_4	$\{a_1\}$	$\{a_4\}$	$[a_3, a_4]$	$\{a_3\}$	$[a_3, a_4]$	?	$[\boldsymbol{a_3, a_4}]$	$[\boldsymbol{a_3, a_4}]$	$[\boldsymbol{a_3, a_4}]$

Note that, as it can be seen in Table 3, the considered group of DMs has two centroids according to $D_1^{\mathcal{F}}$ that just differ in their values corresponding to λ_3: $[a_3, a_4]$ and $[a_2, a_4]$. However, since $[a_2, a_4] \supseteq [a_3, a_4]$, one can choose $F_{H\,1}^{C_1}$ as the most precise centroid representing the group with respect to $D_1^{\mathcal{F}}$.

Figure 3 depicts, for each element in Λ, a graphical representation of the HFLTSs given by the DMs in G, together with the HFLTSs corresponding to the HFLD associated to the connected union, F_{HG}, and to the three centroids of the group, $F_{H\,1}^{C_1}$, $F_{H\,1}^{C_2}$ and $F_{H\,2}^{C}$.
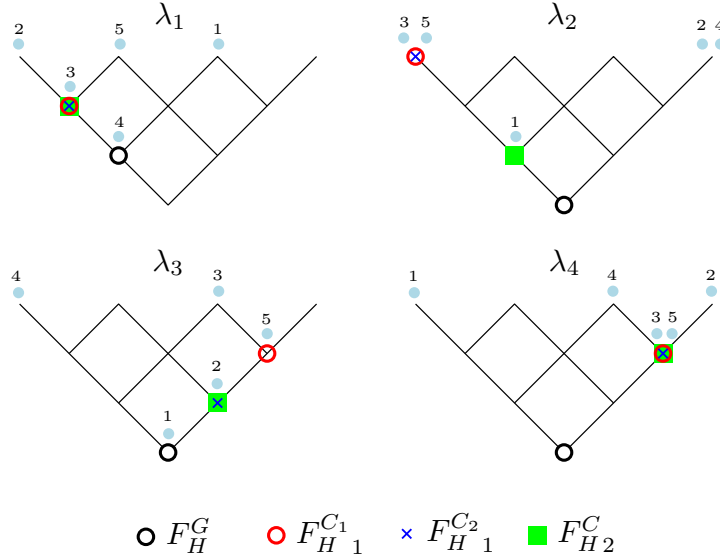

Figure 3: Graphical representation of Example 5.

Finally, Tables 4 and 5 present the matrices of distances, with respect to $D_1^{\mathcal{F}}$ and $D_2^{\mathcal{F}}$ respectively, computed for each pair of HFLDs in the group G expanded with F_H^G and the corresponding centroids for each distance: $F_{H\,1}^{C_1}$ and $F_{H\,1}^{C_2}$ in the first case, and $F_{H\,2}^{C}$ in the second case.

We can observe similar results by analyzing the values of the distances provided in Tables 4 and 5. Assessments corresponding to the descriptions given by DM 3 and DM 5 are the closest ones with respect to both distances. In the same way, the most distant pairs of assessments correspond to the pairs: DM 1 and DM 2, DM 1 and DM 3 and DM 1 and DM 4 with respect to $D_1^{\mathcal{F}}$, Whilst according to $D_2^{\mathcal{F}}$, the tie is broken and the most distant ones are DM 1 and DM 2. We can also observe that assessments provided by DM 3 and DM 5 are the closest ones to the centroids of the group in both cases. Finally, the assessment corresponding to the descriptions given by DM 1 is the closest one to the assessment associated to the connected union with respect to both distances. It is also one of the two most distant assessments from the centroids of the group, together with the assessment given by DM 4.

Table 4: Distances $D_1^{\mathcal{F}}$ between DMs, the connected union and the centroids $F_{H\ 1}^{C_1}$ and $F_{H\ 1}^{C_2}$.

$D_1^{\mathcal{F}}$	F_H^1	F_H^2	F_H^3	F_H^4	F_H^5	F_H^G	$F_{H\ 1}^{C_1}$	$F_{H\ 1}^{C_2}$
F_H^1	0	12	12	12	10	6	11	10
F_H^2		0	8	8	8	10	7	6
F_H^3			0	9	2	9	1	2
F_H^4				0	11	9	10	10
F_H^5					0	9	1	2
F_H^G						0	8	7
$F_{H\ 1}^{C_1}$							0	1
$F_{H\ 1}^{C_2}$								0

Table 5: Distances $D_2^{\mathcal{F}}$ between DMs, the connected union and the centroid $F_{H\ 2}^{C}$.

$D_2^{\mathcal{F}}$	F_H^1	F_H^2	F_H^3	F_H^4	F_H^5	F_H^G	$F_{H\ 2}^{C}$
F_H^1	0	15	12	13	11	6	9
F_H^2		0	10	8	10	9	6
F_H^3			0	12	2	9	4
F_H^4				0	14	9	10
F_H^5					0	9	4
F_H^G						0	5
$F_{H\ 2}^{C}$							0

Note that these distances matrices quantifying the similarity in between pairwise linguistic expressions could be used in other pattern recognition contexts, such as clustering, classification or ranking [2, 11]. In addition, the use of HFLTSs will allow the definition of fuzzy outputs able to capture the inherent complexity underlying in end-users' opinions

5. Conclusions

This paper proposes a theoretical framework to model group assessments on the basis of HFLTSs. To this aim, the concept of distance between DMs in group decision-making when DMs' assessments are expressed using HFLTSs is studied. This concept allows similarities and differences among DMs' opinions to be analyzed.

From a well-ordered set $\mathcal{S}$ of linguistic terms, the set of hesitant fuzzy linguistic term sets $\mathcal{H}_\mathcal{S}$ has been provided with two closed aggregation operations, connected union and intersection, which are suitable to be used on reasoning and comparisons. In addition, the two operations provide $\mathcal{H}_\mathcal{S}$ with a lattice structure. The hesitant fuzzy linguistic descriptions of a set Λ are defined as $\mathcal{H}_\mathcal{S}$-fuzzy sets.

Two distances between HFLDs have been proposed. The first distance, D_1, is built directly from connected union and intersection. The second distance, D_2, coincides with D_1 in the case that there is a non-empty intersection between the considered pair of HFLTSs and, intuitively, corresponds to adding the gap between them to D_1 if their intersection is empty.

Finally, the concept of centroid of a decision-making group is introduced by minimizing the addition of distances to the assessments of all the DMs in the group. The two proposed distances are used to do a further study of the corresponding centroids, which can be used as representatives of the opinions of the group of DMs. Moreover, the distances between each DM and the centroid can be considered as a measure of agreement within the group. Lastly, most dissident DMs in the group can be easily identified by means of distances to the centroid.

The proposed structure based on distances and centroids is not only limited to decision making scenarios. It provides a general model suitable for comparing opinions between end-users in general when expressed in terms of ordered linguistic terms.

Future research is oriented towards three main directions. First, the design of an algorithm for the computation of the proposed centroids of a decision-making group. Second, based on the proposed centroids, a study will be addressed to analyze risk measurement and validity assurance of the actions derived from a decision outcome. This analysis will be oriented towards the improvement of consensus reaching processes by focussing in the dissident DMs. Finally, a real case study will be conducted in the marketing research area to analyze customers preferences in a retailing context.

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