# Counting configuration-free sets in groups

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**Abstract**

We provide asymptotic counting for the number of subsets of given size which are free of certain configurations in finite groups. Applications include sets without solutions to equations in non-abelian groups, and linear configurations in abelian groups defined from group homomorphisms. The results are obtained by combining the methodology of hypergraph containers joint with arithmetic removal lemmas. Random sparse versions and threshold probabilities for existence of configurations in sets of given density are presented as well.

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## 1. Introduction

The study of sparse (and probabilistic) analogues of results in extremal combinatorics have become a very active area of research in extremal and random combinatorics (see e.g. the survey by Conlon [7]). One starting point is *Szemerédi Theorem* [39] on the existence of arbitrarily long arithmetic progressions in sets of integers with positive upper density. This seminal result and the tools arising in its many proofs have been enormously influential in the development

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of modern discrete mathematics. Nowadays a large proportion of the research in additive combinatorics is inspired by these achievements.

Sparse analogues of Szemerédi Theorem started in Kohayakawa, Rödl and Łuczack [18] by studying the threshold probability for a random set of the integer interval $[1, n]$ whose subsets of given density contain asymptotically almost surely (a.a.s.) 3–term arithmetic progressions. The extension of the result to $k$–term arithmetic progressions was a breakthrough obtained independently, and by different methods, by Conlon and Gowers [8] and by Schacht [31]. There is still another more recent proof based on combinatorial arguments due to Saxton, and Thomason [30] and by Balogh, Morris, and Samotij [3]. The approach in the above two papers is based on a methodology building on the structure of independent sets in hypergraphs. *Hypergraphs containers* (as it is named in [30]) provides a general framework to attack a wide variety of problems which can be encoded by uniform hypergraphs. The philosophy behind this method is that, for a large class of uniform hypergraphs which satisfy mild conditions, one can find a small collection of sets of vertices (which are called *containers*) which contain all independent sets of the given hypergraph, thus providing sensible upper bounds on the number of independent sets.

In addition to important applications in combinatorics, the two works above mentioned also contain arithmetic applications, providing in particular a new proof of the sparse Szemerédi Theorem. One important ingredient of these proofs, explicitly exposed in [3], is the so-called *Varnavides Theorem* [41]. This is the robust counterpart of Szemerédi Theorem: once a set has positive density, it does not only have one but a positive proportion of the total number of $k$–term arithmetic progressions. This phenomenon is the number theoretical counterpart of the supersaturation phenomenon in the graph setting.

Nowadays there is a rich theory dealing with these type of results, which are rephrased under the name of *Arithmetic Removal Lemmas*. The idea behind them can be traced back to the proof of Roth's Theorem by Ruzsa and Szemerédi [28] and was first formulated by Green [17] for a linear equation in an abelian group by using methods of Fourier analysis. The picture was complemented independently by Shapira [34] and by Král', Serra and Vena [21] by proving a removal lemma for linear systems in the integers. These results have been extended in several directions, including arithmetic removal lemmas for a single equation in non-abelian groups, for linear systems over finite fields and for integer linear systems over finite abelian groups (see [20, 34, 21, 22]).

These extensions of Green's Arithmetic Removal Lemma provide proofs of the Szemerédi Theorem in general abelian groups (see also [38]), but cannot handle the robust versions of the multidimensional Szemerédi Theorem (see for instance [36] on Furstenberg and Katznelson work [15]) or, more generally, the appearance and enumeration of finite configurations in dense subsets in abelian groups (as seen in Tao [40, Theorem B.1]). As a consequence, the above mentioned arithmetic removal lemmas cannot be used to show the sparse counterparts of these results (see [3, 8, 31]).

The main contribution of this paper is to combine the method of hypergraph containers with a removal lemma for group homomorphisms due to Vena [42],

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which unifies and extends previous results concerning arithmetic removal lemmas. This combination provides a more general and flexible counting result, which allows for new applications which are developed in paper. We next summarize the contents of the paper and its contributions.

Section 2 contains a version of the hypergraph container method in the more general framework of configuration systems. A system of configurations of degree $k$ on a ground set $G$ is a pair $(S, G)$ where $S \subset G^k$ is a subset of $k$–tuples of $G$. Our goal is to have a tool to bound the number of $S$-free subsets of $k$-tuples from $G$. We introduce some appropriate parameters on configuration systems and reformulate in Theorem 2.4 the counting result of [3] in this more general language, which will be needed for the applications developed later on in the paper. We define a class of systems of configurations, which we call *normal*, which satisfy some natural properties shared by all the configurations we meet in the arithmetic applications.

Section 3 extends to systems of configurations the study of the threshold probability for the stability in random subsets. Given a system $(S, G)$ of configurations, a subset $X \subset G$ is $(\delta, S)$–stable if, for every subset $X' \subset X$ with density at least $\delta$ in $X$, the set of $k$–tuples $(X')^k$ intersects $S$ ($X'$ contains a configuration form $S$.) We give the threshold probabilities for this notion of stability in random subsets of $G$ in Theorem 3.2 and Theorem 3.5 for normal systems of configurations. The transition probability for the stability, or 1–statement, can be deduced in a simple way from existing results. The transition probability for nonstability, or 0–statement, follows by carefully applying standard techniques, but it is more technically involved. Since stability is not a monotone property, the two transition probabilities may not coincide, and we provide a necessary condition on configurations systems to ensure that there is a threshold transition.

Section 4 discusses systems of configurations defined by kernels of homomorphisms in abelian groups. In this case the ground set $G$ is an abelian group and $S$ is the kernel of a group homomorphism $M : G^k \to G^m$. This extends in a natural way the study of solutions of linear systems in abelian groups, which has been considered in Saxton and Thomasson [30], and contains in particular the case of $k$–arithmetic progressions. The setting of group homomorphisms allows to treat much more general configurations in abelian groups, including the ones addressed in Tao [40, Theorem B.1], in particular the multidimensional Szemerédi Theorem. This general approach is one of the main motivations of this paper. In this Section we characterize the normal configurations arising from group homomorphisms. By using the Removal Lemma for homomorphisms from [42], we obtain the corresponding counting result for configurations arising from homomorphisms, Corollary 4.4. Moreover, we obtain the threshold probabilities for $(\delta, S)$–stability for these systems, Theorem 4.7, which in particular include the random sparse version of the multidimensional Szemerédi Theorem when the configurations considered involve possibly independent scalings $\{a + \lambda_1 F_1 + \lambda_2 F_2 + \cdots + \lambda_q F_q\}$ (e.g. Proposition 4.1 or Theorem 5.1).

Section 5 provides some additional examples of configurations in abelian groups which could not be treated before. Theorem 5.1 addresses the case of

rectangles in abelian groups where the sides belong to prescribed subgroups. We discuss extensions of the result to $d$–dimensional cubes and its connections to the Zarankiewicz problem and the conjecture of Sidorenko.

Section 6 is devoted to linear configurations in integer intervals. As usual, the case of integer intervals is reduced to the case of cyclic groups. The main result in this Section is Proposition 6.1 which shows that normality of systems of configurations in integer intervals can be reduced to the case of group homomorphisms in finite abelian groups. Through this reduction, the results from Section 4 can be translated to the integers.

Section 7 deals with the more studied case where the system of configurations corresponds to solutions of linear systems, namely, solutions of the equation $A\mathbf{x} = 0$ where $A$ is a $(k \times m)$ matrix with integer coefficients and $\mathbf{x} \in G^k$ where $G$ is a finite abelian group. We translate the normality conditions for systems of configurations to parameters of the matrix $A$, which results in the parameter $m_A$ introduced by Rödl and Ruciński in [24, Definition 1.1] and used in Schacht [31], Friedgut, Rödl and Schacht [12], and Saxton and Thomason [30]. This leads to the counting of the number of solution–free subsets of size $t$ of $[1, n]$ to the system of equations $A\mathbf{x} = 0$, Theorem 7.1, which is one of the outcomes of the framework presented in this paper. The upper bound on the number of solution–free sets of given size given by Theorem 7.1 provides a meaningful counting for all linear systems for which there are examples of solution–free sets with sublinear cardinality, as exemplified by the Behrend construction for the case of 3-term arithmetic progressions. Shapira [33] shows that almost all linear systems fall into this category. The stability of random subsets with respect to the configurations arising from linear systems follow straightforwardly from the results in Section 3 and 4. The corresponding threshold probability has been previously been obtained by Schacht [31, Theorem 2.4] and Saxton and Thomason [29, Theorem 12.3], and extends the sparse Szemerédi–type theorem of Conlon and Gowers [8, Theorem 1.12]. The section concludes with a brief discussion in the case of linear systems over finite fields.

Section 8 considers configurations arising from solutions to single equations in nonabelian groups. The main result of the section, Theorem 8.1, provides counting of solution–free sets of given size and the random sparse version.

The paper concludes with a discussion of further research in Section 9.

## 2. Counting Configuration–free sets: Definitions and Main Result

We start recalling the main theorem from [3], Theorem 2.1 below. This is the statement which eventually leads to counting the number of independent sets in hypergraphs. We use it to count solution–free sets in configuration systems as stated in Theorem 2.4, which is the main goal in this Section and will be the main tool in the applications discussed in the coming sections.

Let $H = (V, E)$ be a $k$–uniform hypergraph with $v(H)$ vertices and $e(H)$ edges. A family $\mathcal{F}$ of subsets of $V(H)$ is said to be *increasing* if, given $A \in \mathcal{F}$ and $B \subset V(H)$ with $A \subset B$, then $B \in \mathcal{F}$. Given an increasing family $\mathcal{F}$ of

subsets of $V(H)$, the hypergraph $H$ is said to be $(\mathcal{F}, \varepsilon)$-*dense* if

$$e(H[A]) \geq \varepsilon\, e(H), \quad \text{for each } A \in \mathcal{F},$$

where $H[A]$ stands for the hypergraph induced by the vertices in $A$. The degree $d_H(T)$ of a set $T \subset V(H)$ is the number of edges of $H$ which contain $T$ and

$$\Delta_\ell(H) = \max\left\{ d_H(T) : T \subset \binom{V(H)}{\ell} \right\}.$$

The family of independent sets of $H$ is denoted by $\mathcal{I}(H)$.

**Theorem 2.1** (Balogh, Morris, Samotij, Theorem 2.2 in [3]). *For every $k \in \mathbb{N}$ and all positive $c$ and $\varepsilon$, there exists a positive constant $C = C(k, \varepsilon, c)$ such that the following holds. Let $H$ be a $k$-uniform hypergraph and let $\mathcal{F} \subset 2^{V(H)}$ be an increasing family of sets such that $|A| > \varepsilon v(H)$ for all $A \in \mathcal{F}$. Suppose that $H$ is $(\mathcal{F}, \varepsilon)$–dense and $p \in (0, 1)$ is such that, for every $\ell \in [1, k]$,*

$$\Delta_\ell(H) \leq c p^{\ell-1} \frac{e(H)}{v(H)}.$$

*Then there is a family $\mathcal{S} \in \binom{V(H)}{\leq C p v(H)}$ and functions $f : \mathcal{S} \to 2^{V(H)} \setminus \mathcal{F}$ and $g : \mathcal{I}(H) \to \mathcal{S}$ such that*

$$g(I) \subset I \quad \text{and} \quad I \setminus g(I) \subset f(g(I)).$$

Roughly speaking, Theorem 2.1 reads as follows: in a $k$-uniform hypergraph $H$ satisfying certain natural conditions, each independent set $I$ of $H$ contains an small subset $g(I)$ (its *fingerprint*) such that all sets labeled with the same fingerprint are essentially contained in a single (small) set $f(g(I))$. The notion of hypergraph containers was developed independently by Saxton and Thomason in [30].

We next introduce the main definitions used in this paper, as well as the version of Theorem 2.1 we will use in the applications. We start with the notion of system of configurations.

**Definition 2.2** (System of configurations). *Let $k$ be a positive integer, let $G$ be a finite set and let $S \subset G^k$. The pair $(S, G)$ is said to be a* system of configurations *of degree $k$.*

We say that $(g_1, \ldots, g_k) \in G^k$ is a *solution* of the system $(S, G)$ if $(g_1, \ldots, g_k) \in S$. We denote by $S^{(j)}$ the subset of solutions $(g_1, \ldots, g_k) \in S$ which have precisely $j$ different values. For a given set $U = \{u_1, \ldots, u_m\} \subset [1, k]$, let $\pi_U$ denote the projection

$$\pi_U : G^k \to G^m$$
$$(g_1, \ldots, g_k) \mapsto (g_{u_1}, \ldots, g_{u_m})$$

which keeps the coordinates indexed by the elements in $U$. For $i \in [1, k]$, let us define the $i-th$ $(S, G)-degree$ $of$ $freedom$ as

$$\alpha_i = \max_{\substack{U \subset [1,k] \\ |U|=i}} \max_{(g_1,\ldots,g_i) \in G^i} \left\{ \left| S \cap \pi_U^{-1}(g_1, \ldots, g_i) \right| \right\}.$$

The $i$–th degree of freedom is an upper bound for the number of solutions which share a given $i$–th tuple. This notion appears naturally in this context and can be found for instance in Rödl and Ruciński [24]. It plays the role of the edge density in the study of subgraphs in the random graph model of Erdős and Rényi [9].

Additionally, we define the $restricted$ $i-th$ $(S, G)-degree$ $of$ $freedom$ as the quantity

$$\alpha_i^k = \max_{\substack{U \subset [1,k] \\ |U|=i}} \max_{(g_1,\ldots,g_i) \in G^i} \left\{ \left| S^{(k)} \cap \pi_U^{-1}(g_1, \ldots, g_i) \right| \right\}.$$

The following definition is inspired by Varnavides Theorem [41], which gives a robust version of Roth's Theorem [25]. It describes the supersaturation phenomenon:

**Definition 2.3** (Varnavides property, V-property)**.** *The system of configurations* $(S, G)$ *of degree $k$ is said to fulfill the* Varnavides property*, or V-property, if for every $\varepsilon > 0$ there exist a $\gamma = \gamma(\varepsilon, k)$ such that, for any $X \subset G$ with $|X| \geq \varepsilon|G|$,*

$$|X^k \cap S| \geq \gamma|S|.$$

*A sequence of systems $\{(S_i, G_i)\}_{i \geq 1}$ of degree $k$ is said to satisfy the V-property if $\gamma$ is the same function for each member of the family and only depends on $\varepsilon$.*

The next result is an adaptation of [3, Lemma 4.2] in order to count the number of solution-free sets for systems of configurations.

**Theorem 2.4** (Counting independent sets for configuration systems)**.** *Let $k$ be a fixed positive integer and $\delta > 0$. Let $(S, G)$ be a system of configurations of degree $k$ satisfying the V-property with function $\gamma$. Write $n = |G|$. For each $i \in [1, k]$, let $\alpha_i^k$ be the restricted $i-th$ $(S, G)-degree$ of freedom.*

*Assume that each subset of $G$ with more than $\frac{\delta n}{2}$ elements contains a configuration in $S^{(k)}$. Then, for each $t$ such that*

$$t \geq C \frac{|G|}{\delta} \max_{\ell \in [2,k]} \left\{ \left( \frac{\alpha_\ell^k}{\alpha_1^k} \frac{1}{k} \binom{k}{\ell} \right)^{\frac{1}{\ell-1}} \right\} \text{ and } t \leq \frac{\delta n}{2} \tag{1}$$

*where*

$$C = C\left(k, \frac{\xi}{|S^{(k)}|}, \alpha_1^k \frac{(k-1)! \, |G|}{|S^{(k)}|}\right), \text{ with } \xi = \max\left\{(\gamma - 1)|S| + |S^{(k)}|, \frac{\delta n}{2}\right\},$$

*is the constant from Theorem 2.1, there are at most*

$$t \left[\frac{2e}{\delta^2}\right]^{\delta t} \binom{\delta n}{t}$$

*sets of size $t$ with no solution in $S^{(k)}$. If we assume that $\delta = \min\{\beta/2, 1/40\}$, then the bound can be rewritten as*

$$\binom{\beta n}{t}.$$

Theorem 2.4 deals with sets with no solutions having its entries pairwise distinct. Some solutions may still be contained in the set but then at least two of its entries coincide. In order to apply the result to count sets free of solutions with some identical entries, one can construct a different configuration system obtained by identifying these equal entries. Let us also recall that the constant $C$ in Theorem 2.4 depends on $\gamma$ through the $(\mathcal{F}, \varepsilon)$-dense condition in Theorem 2.1.

*Proof.* The proof follows the lines of the arguments of [3, Lemma 4.2]. We include the details for completeness. We consider the $k$-uniform hypergraph $H$ whose vertex set is $V(H) = G$. Observe that each solution $\mathbf{x} \in S^{(k)}$ (all the variables having different values) of the system $(S, G)$ defines a set of size $k$ in $G$ (namely, forgetting the order of the variables). We define the edge set as

$$E(H) = \Big\{\{x_1, \ldots, x_k\} : (x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \in S^{(k)} \text{ for a permutation } \sigma \text{ on } k \text{ elements}\Big\}$$

by using the identification between vertices of the hypergraph and elements of the group. Thus $v(H) = n$ and $e(H)$ satisfies

$$\frac{|S^{(k)}|}{k!} \leq e(H) \leq |S^{(k)}|. \tag{2}$$

Observe that every independent set of $H$ defines a solution–free set of the configuration system $(S, G)$ restricted to the set of solutions $S^{(k)}$.

Consider now the family of sets with more than $\delta n$ vertices: $\mathcal{F} = \{F \subseteq V(H) : |F| \geq \delta n\}$. We shall show that $H$ satisfies the conditions of Theorem 2.1 with respect to the family $\mathcal{F}$. The family $\mathcal{F}$ is clearly increasing. Since $(S, G)$ satisfies the V-property, given any set $F \in \mathcal{F}$, there are more than $\gamma |S|$ solutions involving elements in $F$. In particular, there are at least

$$\gamma |S| - (|S| - |S^{(k)}|) = (\gamma - 1)|S| + |S^{(k)}|$$

solutions whose entries are pairwise distinct and belong to $F$. On the other hand, since each set of size $\frac{\delta n}{2}$ contains a solution in $S^{(k)}$, then any set of size at least $\delta n$ contains at least $\frac{\delta n}{2}$ solutions in $S^{(k)}$. Hence, there are at least $\xi = \max\left\{(\gamma - 1)|S| + |S^{(k)}|, \frac{\delta n}{2}\right\}$ solutions in $S^{(k)} \cap F^k$.

Therefore there are at least $\xi/k!$ edges in each set $F$. The total number of edges is $e(H)$, which satisfies the relations in (2). Let $\varepsilon = \xi/e(H)$. Then $H$ is $(\mathcal{F}, \varepsilon)$-dense with an $\varepsilon$ such that

$$\frac{\xi}{|S^{(k)}|} \leq \varepsilon \leq \frac{\xi}{|S^{(k)}|} k!.$$

Let us now check the conditions concerning the degrees. For each $\ell \in [1, k]$ we have that

$$\Delta_\ell(H) \leq \alpha_\ell^k \binom{k}{\ell}$$

as this is the maximum number of solutions in $S^{(k)}$ containing a given subset of $\ell$ vertices. Choose $c$ with

$$c = k\alpha_1^k \frac{v(H)}{e(H)} \geq k\alpha_1^k \frac{|G|}{|S^{(k)}|}.$$

Then

$$\Delta_1(H) \leq c \frac{e(H)}{v(H)}.$$

The parameter $p$ in Theorem 2.1 is chosen as

$$p = \max_{\ell \in [2, k]} \left\{ \left( \frac{1}{c} \frac{v(H)}{e(H)} \alpha_\ell^k \binom{k}{\ell} \right)^{\frac{1}{\ell - 1}} \right\}.$$

Then we have

$$\Delta_\ell(H) \leq cp^{\ell - 1} \frac{e(H)}{v(H)}.$$

Therefore, we can apply Theorem 2.1, which combined with the arguments in the proof of [3, Lemma 4.2] gives the bound as stated on the number of independent sets of the graph, and hence the bound on the number of sets not intersecting $S^{(k)}$.

Let us remark that $C_{2.1}(k, \varepsilon, c)$ is increasing whith $c$, and increasing when $\varepsilon$ is decreasing. Indeed, it is shown in the proof of [3, Theorem 2.2] that

$$C_{2.1}(k, \varepsilon, c) = (k - 1) \left( \frac{1}{\delta} \log \left( \frac{1}{\varepsilon} \right) + 1 \right),$$

where $\delta = (ck2^{k+1})^{-k}$ (see the proof of [3, Proposition 3.1]). This justifies the definition of $C$ in the statement and finishes the proof. $\square$

To conclude this section, let us shortly discuss the nature of the system of configurations we consider in this paper. All of them share the following natural properties:

**Definition 2.5.** *A sequence $\{(S_i, G_i)\}_{i \geq 1}$ of configuration systems of degree $k$ is said to be* normal *with function $\gamma$ if*

*C1 The $G_i$'s are finite and growing in size with $i$.*

*C2 $\lim_{i \to \infty} \frac{|G_i|}{|S_i|} = 0$ and $\lim_{i \to \infty} \frac{|G_i|^k}{|S_i|} = \infty$.*

*C3 $\lim_{i \to \infty} \frac{|S_i^{(k)}|}{|S_i|} = 1$.*

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*C4 Each $(S_i, G_i)$ satisfies the V–property with a function $\gamma = \gamma_{2.3}(\delta)$ universal for all the systems in the sequence.*

The above conditions reflect the asymptotic nature of the results. Condition C2 states that the number of configurations asymptotically exceeds the trivial ones for invariant systems (namely, the ones with constant entries), and that these configurations impose non–trivial restrictions, hence its number is asymptotically smaller than the whole set of possible configurations. All the invariant systems of configurations arising from integer matrices with two more columns than rows satisfy Condition C2. Although not crucial for many applications, these conditions are specially needed when dealing with the random sparse analogues we treat in Section 3. Condition C3 ensures that most of the solutions have pairwise distinct entries. This is again a common feature in most applications and one can reduce to configurations systems satisfying it by identifying some entries. Additionally, in most of our applications, $G_i$ has a group structure and $S_i$ is induced by a group homomorphism. The latter constraint can be relaxed in the non–abelian setting.

## 3. Random sparse results

We address in this section the study of random sparse models. Our objective is to extend known sparse analogues of extremal results in additive combinatorics to the context of the normal systems of configurations introduced in Section 2. The main properties that we study are the following, which extend the notion of $\delta-$Roth property of Kohayakawa, Łuczak and Rödl [18] described in the Introduction.

**Definition 3.1.** *Let $(S, G)$ be a system of configurations of degree $k$. We say that a set $X \subseteq G$ is the $(\delta, S)-$stable if, for every subset $X' \subset X$ with $|X'| \geq \delta|X|$, we have $X'^k \cap S \neq \emptyset$. We also say that $X$ is $(\delta, S)_k-$stable if in addition $X'^k \cap S^{(k)} \neq \emptyset$.*

We study the previous properties in the binomial random model: fix a probability $p$ (that may depend on $|G|$), and consider the binomial random set $[G]_p$ built by choosing independently each element of $G$ with probability $p$. We observe that both the $(\delta, S)$-stable and the $(\delta, S)_k$-stable properties are *not* monotone increasing (as they are not closed by supersets). However, we will show that there is a threshold phenomenon for these properties given that the configuration system has some uniformity properties.

We prove the $0-$statement and the $1-$statement separately. The $0-$statement is proved in Theorem 3.7 by combining Lemma 3.5 and Lemma 3.4. We first consider the less technically involved 1–statement.

**Theorem 3.2** (1–statement). *Let $\delta > 0$ and let $\{(S_i, G_i)\}_{i \geq 1}$ be a normal sequence of system of configurations of degree $k$ with function $\gamma$. Write $n_i = |G_i|$ and assume that every set with more than $\delta n_i/2$ elements has a solution in $S_i^{(k)}$*

and that $(\gamma - 1)|S_i| + |S_i^{(k)}| > 0$ for all $i$. Let

$$p_{(S_i,G_i)} = \max_{\ell \in [2,k]} \left( \frac{\alpha_\ell^k(S_i, G_i)}{\alpha_1^k(S_i, G_i)} \right)^{\frac{1}{\ell-1}}.$$

Then there exists $C = C(\delta, \gamma, k) > 0$ such that

$$\lim_{i \to \infty} \mathbb{P}([G_i]_p \text{ is } (\delta, S_i) - \text{stable}) = 1, \text{ if } p \geq Cp_{(S_i,G_i)}. \tag{3}$$

*Proof.* The proof follows exactly the lines of Corollary 4.1 in [3], adapted here to the terminology of systems of configurations.

We write $(S, G)$ as a generic configuration system $(S_i, G_i)$ of degree $k$ and $|G| = n$. Assume that $p \geq Cp_{(S,G)}$ with $C = C_{2.4} \max_{\ell \in [2,k]} \left( \frac{1}{k} \binom{k}{\ell} \right)^{\frac{1}{\ell-1}}$. Write $t = \frac{\delta}{2}pn$ and $\beta = \delta/2 \cdot e^{-1/\delta-1}$. Let $X_t$ be the random variable counting the number of subsets of $[G]_p$ of size $t$ without configurations. Let $\mathcal{E}$ be the event that $[G]_p$ does not have the $(\delta, S)-$stability. Hence:

$$\mathbb{P}(\mathcal{E}) \leq \mathbb{P}\left( \mathcal{E} \cap \left[ |[G]_p| \geq \frac{1}{2}pn \right] \right) + \mathbb{P}\left( \mathcal{E} \cap \left[ |[G]_p| < \frac{1}{2}pn \right] \right) \tag{4}$$

$$< \mathbb{P}\left( \mathcal{E} \cap \left[ |[G]_p| \geq \frac{1}{2}pn \right] \right) + \mathbb{P}\left( |[G]_p| < \frac{1}{2}pn \right)$$

$$< \mathbb{P}(X_t > 0) + e^{-pn/8},$$

where we have used Chernoff's inequality to exponentially bound $\mathbb{P}(|[G]_p| < \frac{1}{2}pn)$. Let us finally bound the probability $\mathbb{P}(X_t > 0)$. We are under the assumptions of Theorem 2.4, hence, for $n$ large enough,

$$\mathbb{P}(X_m > 0) \leq \mathbb{E}[X_t] \leq \binom{\beta n}{t}p^t \leq \left( \frac{\beta epn}{t} \right)^t = \left( \frac{2\beta e}{\delta} \right)^t = e^{-t/\delta}.$$

Putting this bound together with the upper bound obtained in (4) gives that for $p \geq p_{(S,G)}$ we have that

$$\mathbb{P}([G]_p \text{ has the } (\delta, S)-\text{stable}) \geq 1 - e^{-t/\delta} - e^{-pn/8}$$

as we wanted to show. $\square$

We next deal with the 0–statement. In this case we are only interested in solutions with pairwise distinct entries, that is, we only consider $S^{(k)}$ as the solution set. The strategy to prove the 0−statement is based on an application of the Alteration Method (see e.g. [2, Chapter 3]) and follows the proof of the 0−statement of [31, Theorem 2.3]. In particular, we use the fact that the random variable $|[G]_p|$ is asymptotically concentrated around its expected value $p|G|$. Additionally, we need to assume some concentration around the expected values of the different projections of the solution set:

**Definition 3.3** (Concentration for configuration systems). *Let $\mathcal{G} = \{(S_i, G_i)\}_{i \geq 1}$ be a sequence of configuration systems with $n_i = |G_i|$, and let $0 \leq p \leq 1$ (which may depend on $n_i$). We say that $\mathcal{G}$ is* concentrated *for $p$ if, for every $\varepsilon, \varepsilon' > 0$, there exists an $n_0 = n_0(\varepsilon, \varepsilon', k)$ such that*

$$\mathbb{P}\left(\left|\left|\pi_U\left(S_i^{(k)}\right) \cap [G_i]_p^{|U|}\right| - p^{|U|}\left|\pi_U\left(S_i^{(k)}\right)\right|\right| \geq \varepsilon p^{|U|} |\pi_U\left(S_i^{(k)}\right)|\right) \leq \varepsilon',$$

*for each $n_i \geq n_0$ and each $U \subset [1, k]$ with $|U| \geq 2$.*

In other words, $|\pi_U\left(S_i^{(k)}\right) \cap [G_i]_p^{|U|}|$ is asymptotically concentrated around its expectation. Let us observe that a general sequence of systems of configurations may fail to satisfy such concentration. For instance, a system of configurations in which one variable only takes a single value in $G_i$. However, for sequences $\mathcal{G}$ which are concentrated in the sense of Definition 3.3, we can obtain a $0-$statement for a wide range of values for $p$. The proof is divided into two parts. As it will be shown, the V-property is not required neither in Theorem 3.4 nor in Theorem 3.5.

We start proving the 0-statement for small values of $p$. In this case only the second part of Condition C2 in the definition of a normal sequence is needed.

**Theorem 3.4** (0−statement for configurations, small $p$). *Let $0 < \delta < 1$ and let $\{(S_i, G_i)\}_{i \geq 1}$ be a sequence of systems of configurations of degree $k$. Let $\beta_i = \frac{|G_i|^k}{|S_i^{(k)}|}$ and assume that $\lim_{i \to \infty} \beta_i = \infty$. If*

$$p \leq f(\beta_i, |G_i|) \, |S_i^{(k)}|^{-1/k}$$

*for some $f$ with $f(\beta_i, |G_i|) \to \infty$ as $|G_i| \to \infty$, then*

$$\lim_{i \to \infty} \mathbb{P}([G_i]_p \text{ is } (\delta, S_i)_k-\text{stable}) = 0.$$

*Proof.* Let $Z = |[G_i]_p|$ and $Y = |[G_i]_p^k \cap S_i^{(k)}|$. We have $\mathbb{E}(Z) = p|G_i|$ and $\mathbb{E}(Y) = p^k |S_i^{(k)}|$. We distinguish two cases depending on the choice of $p$.

Let $b = \log(\log(\beta_i))$, choose $h = \sqrt{b}$. Assume first that $p \leq h|G_i|^{-1}$, so that $\mathbb{E}(Z) \leq h$, and write

$$\mathbb{P}(Y \geq 1) = \mathbb{P}([Y \geq 1] \wedge |[G_i]_p| > b) + \mathbb{P}([Y \geq 1] \wedge |[G_i]_p| \leq b).$$

By Markov inequality

$$\mathbb{P}([Y \geq 1] \wedge |[G_i]_p| > b) \leq \mathbb{P}\left(|[G_i]_p| > b\right) \leq \frac{\mathbb{E}(Z)}{b} \leq \frac{h}{b} \to 0 \; (i \to \infty).$$

To bound the second term observe that

$$\mathbb{P}([Y \geq 1] \wedge |[G_i]_p| \leq b) = \sum_{j=k}^{b} \mathbb{P}([Y \geq 1] \wedge |[G_i]_p| = j)$$

$$\leq \sum_{j=k}^{b} \sum_{X \in \binom{G_i}{j}} \sum_{x \in X^k} \underbrace{\mathbb{P}(x \in S_i^{(k)} \cap [G_i]_p^k | [G_i]_p = X)}_{1 \text{ if } x \in S_i^{(k)}; \ 0 \text{ if } x \notin S_i^{(k)}} \underbrace{\mathbb{P}([G_i]_p = X)}_{p^j(1-p)^{|G_i|-j}}$$

$$= \sum_{j=k}^{b} \sum_{x \in S_i^{(k)}} \left| \left\{ X \ni \{x\} \mid X \in \binom{G_i}{j} \right\} \right| p^j(1-p)^{|G_i|-j}$$

$$= \sum_{j=k}^{b} |S_i^{(k)}| \binom{|G_i| - k}{j - k} p^j(1-p)^{|G_i|-j}$$

$$\leq b! \sum_{j=k}^{b} \frac{|S_i^{(k)}|}{|G_i|^k} |G_i|^j p^j(1-p)^{|G_i|-j}$$

$$\overset{h>1}{\leq} (b+1)! \frac{|S_i^{(k)}|}{|G_i|^k} h^b$$

$$\leq \frac{1}{\beta_i} b^{3b/2} = \frac{[\log(\log(\beta_i))]^{\frac{3}{2}\log(\log(\beta_i))}}{\beta_i} \to 0 \ (i \to \infty)$$

Therefore, for this range of $p$ the random set $[G_i]_p$ has no solutions with high probability.

Assume now that $p \geq h|G_i|^{-1}$ and $p \leq f\ |S_i^{(k)}|^{-1/k}$. Let $d = 2h$. In this range we will show that $[G_i]_p$ contains few solutions and there is a large subset which contains none, so the random set is not $(\delta, S_i)_k$–stable. By Markov inequality,

$$\mathbb{P}(Y > a) \leq \frac{\mathbb{E}(Y)}{a} \leq \frac{f^k}{a}. \tag{5}$$

Since $Z = |[G_i]_p|$ is a binomial random variable $\mathrm{Var}(Z) = |G_i|p(1-p)$. Moreover, $\mathbb{E}(Z) \overset{|G_i| \to \infty}{\to} \infty$ as $h \overset{|G_i| \to \infty}{\to} \infty$. Then Chebyshev's inequality can be used to show

$$\mathbb{P}(|Z - \mathbb{E}(Z)| \geq \mathbb{E}(Z)/2) \leq \frac{4(1-p)}{|G_i|p} \leq \frac{4(1-p)}{h} \leq \frac{4}{h} \overset{|G_i| \to \infty}{\to} 0 \tag{6}$$

hence

$$\lim_{i \to \infty} \mathbb{P}\left(Z \geq \frac{\mathbb{E}(Z)}{2}\right) = 1 \Rightarrow \lim_{i \to \infty} \mathbb{P}\left(Z \geq \frac{h}{2}\right) = 1 \Rightarrow \lim_{i \to \infty} \mathbb{P}\left(|[G_i]_p| \geq d\right) = 1 \tag{7}$$

Now, writing $a = \frac{(1-\delta)d}{2}$ and

$$f = \sqrt[2k]{a} = \sqrt[2k]{(1-\delta)\sqrt{\log(\log(\beta_i))}},$$

we get that (5), the value we are interested in, tends to 0 as $i$ increases, $f$ goes to infinity with $i$, and we can delete all the solutions, with probability tending to 1, be removing at most $(1-\delta)d$ elements (as the number of solutions is, with asymptotically high probability, smaller than $(1-\delta)d/2$, so we can delete one element in $[G_i]_p$ per each solution.) By (7), the deletion of $(1-\delta)d$ elements leaves $[G_i]_p$, with asymptotically high probability, with more than $\delta|[G_i]_p|$ elements, hence showing that the set is, a.a.s., not $(\delta, S_i)_k$−stable. $\square$

The next theorem studies the regime when $p$ is large:

**Theorem 3.5** (0−statement for configurations, large $p$). *Let $0 < \delta < 1$ and let $\{(S_i, G_i)\}_{i \geq 1}$ be a family of systems of configurations of degree $k$. Write*

$$p_{(S_i, G_i)} = \min \left\{ \max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} \left( \frac{|G_i|}{|\pi_U(S_i^{(k)})|} \right)^{\frac{1}{|U|-1}}, 1 \right\}. \tag{8}$$

*If $p$ is such that*

$$\max \left\{ |G_i|^{-1}, |S_i^{(k)}|^{-1/k} \right\} = o(p) \ and \ p \leq cp_{(S_i, G_i)} \tag{9}$$

*for some constant $c := c(\delta, k, \{(S_i, G_i)\}_{i \geq 1}) > 0$ universal for the family, and $\{(S_i, G_i)\}_{i \geq 1}$ satisfies Definition 3.3 with $p$, then*

$$\lim_{i \to \infty} \mathbb{P}([G_i]_p \ is \ (\delta, S_i)_k\text{−stable}) = 0.$$

*Proof.* Let $(S, G)$ denote a generic system of configurations of degree $k$ in the family. Recall that $\mathbb{E}(|[G]_p|) = p|G|$. Also, for $U \subset [1, k]$, $|U| \geq 2$, we have that $\mathbb{E}\left( \left| [G]_p^{|U|} \cap \pi_U(S^{(k)}) \right| \right) = p^{|U|} |\pi_U(S^{(k)})|$. Consider now

$$p' := p'(U) = \left( \frac{|G|}{|\pi_U(S^{(k)})|} \right)^{\frac{1}{|U|-1}} \ and \ \ p'' = p' \frac{1-\delta}{4}.$$

Assume that $p' < 1$ for each choice of $U$. Observe that

$$p'|G| = p'^{|U|} |\pi_U(S^{(k)})|.$$

This equality tells us that the expected number of elements in the random set $[G]_{p'}$ equals the expected number of solutions. Observe also that

$$\left( \frac{1-\delta}{4} \right)^{|U|-1} p''|G| = (p'')^{|U|} |\pi_U(S^{(k)})|.$$

Let us analyze the random set $[G]_{p''}$. By Definition 3.3, with probability tending to 1 as $|G| \to \infty$, we have

$$|\pi_U(S^{(k)}) \cap [G]_{p''}^{|U|}| \leq \sqrt{2}(p'')^{|U|} |\pi_U(S^{(k)})|.$$

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Observe also that, by the first condition of (9) and Chebyshev's inequality, asymptotically almost surely

$$|[G]_{p''}| \geq \frac{1}{\sqrt{2}} p''|G|.$$

If there exists a set of relative density $\geq \delta$ in $[G]_{p''}$ with no solutions, then the set $[G]_{p''}$ will not satisfy the $(\delta, S)_k$-stable property. On one hand, there are, asymptotically almost surely, at most

$$\sqrt{2}(p'')^{|U|}|\pi_U(S^{(k)})|$$

configurations in $\pi_U(S^{(k)}) \cap [G]_{p''}^{|U|}$. Additionally,

$$\sqrt{2}(p'')^{|U|}|\pi_U(S^{(k)})| = \left(\frac{1}{4}\right)^{|U|-2} \frac{\sqrt{2}}{4}(1-\delta)^{|U|-1}p''|G| \leq \frac{1}{\sqrt{2}} \frac{1-\delta}{2}p''|G|.$$

Therefore, we can apply the Alteration Method in the following way: by avoiding a set of size $\frac{1}{\sqrt{2}} \frac{1-\delta}{2}p''|G|$ from $[G]_{p''}$, we can find a subset with no configurations in $\pi_U(S^{(k)}) \cap [G]_{p''}^{|U|}$. As the remaining part of $[G]_{p''}$ after removing at most $\frac{1}{\sqrt{2}} \frac{1-\delta}{2}p''|G|$ has relative size larger than $\delta$, we can find sets of relative density larger than $\delta$ (asymptotically almost surely). Therefore, the $(\delta, S_i)_k$-stability property is fulfilled with probability tending to 0.

Since this can be done for any choice of $U$, we can take the maximum of all these probabilities $p'$ to find the maximum probability for which the $(\delta, S)_k$-stability is fulfilled with probability asymptotically 0 for some $U$ (and, hence, for the system as a whole). Thus considering the probability

$$\max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} p'(U) = \max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} \left(\frac{|G|}{|\pi_U(S^{(k)})|}\right)^{\frac{1}{|U|-1}}$$

and $c = \frac{1-\delta}{4}$ the result holds.

Finally assume that $p_{(S,G)} = 1$ (that is, $p' \geq 1$). Then, by equation (8), $|\pi_V(S^{(k)})| \leq |G|$ for some $V \subset [1,k]$, $|V| \geq 2$. Let $p_{(S,G)} = 1$ and $c = (1-\delta)/4$ as before. We shall select a set of relative density $\geq \delta$ from $[G]_p$ that avoids $\pi_V(S^{(k)})$ completely.

Observe that each $x \in \pi_V(S^{(k)})$ considers $|V| \geq 2$ elements in $G$, each of its different components. Hence, there exists a set of size at most $|G|/2$ such that, if $[G]_p$ avoids it, then $[G]_p^{|V|} \cap \pi_V(S^{(k)}) = \emptyset$ (this set $Q$ is iteratively constructed by selecting the value of $G \setminus Q_{l-1}$ found in more elements of $\pi_V(S^{(k)}) \setminus Q_{l-1}^{|V|}$ regardless of the index of the coordinate, adding such element to $Q_{l-1}$ to form $Q_l$, and iterate until $(G \setminus Q)^{|V|} \cap \pi_V(S^{(k)}) = \emptyset$). By the choice of $c$, determining such set will be possible asymptotically almost surely. This completes the proof of the statement and the remaining range for the $0-$statement. $\square$

*3.1. Uniformity*

The order of magnitude on the probability in Theorem 3.2 and in Theorem 3.5 do no match for a general configuration system, as the Example 3.1 below shows. The following natural notion ensures an equality between the probabilities coming from Theorem 3.2 and Theorem 3.5, as Theorem 3.7 below shows. In particular, Example 3.1 does not satisfy the following definition.

**Definition 3.6** (($\rho, k$)–uniformity). *Given $\rho > 0$, the system $(S, G)$ is said to be $(\rho, k)$–uniform (or $\rho-$uniform if $k$ is clear from the context) if, for each $U \subset [1, k]$ with $|U| \geq 2$, and for each $(x_1, \ldots, x_{|U|}) \in \pi_U(S^{(k)})$, then*

$$\left| \pi_U^{-1}((x_1, \ldots, x_{|U|})) \cap S^{(k)} \right| \geq \rho \max_{(y_1, \ldots, y_{|U|}) \in \pi_U(S^{(k)})} \left\{ \left| \pi_U^{-1}((y_1, \ldots, y_{|U|})) \cap S^{(k)} \right| \right\}.$$

In other words, the number of solutions projected to an element is, up to a constant factor, the same as its maximum number.

**Theorem 3.7** (0−statement, uniform systems). *Let $\{(S_i, G_i)\}_{i \geq 1}$ be a sequence of systems of configurations. Write*

$$p_{(S_i, G_i)} = \max_{\ell \in [2, k]} \left( \frac{\alpha_\ell^k}{\alpha_1^k} \right)^{\frac{1}{\ell - 1}}.$$

*Assume that $\{(S_i, G_i)\}_{i \geq 1}$*

*(i) satisfies Definition 3.3 for $p$ with*

$$\max \left\{ |G_i|^{-1}, |S_i^{(k)}|^{-1/k} \right\} = o(p) \text{ and } p \leq c p_{(S_i, G_i)} \tag{10}$$

*(ii) is $(\rho, k)$–uniform, with the same $\rho > 0$ for each of the systems.*

*Then there exist constants $0 < c < C$, $c$ and $C$ depending on $\delta$ and on $k$, such that*

$$\lim_{i \to \infty} \mathbb{P}([G_i]_p \text{ is } (\delta, S_i)_k - \text{stable}) = 0, \text{ if } p \text{ satisfies } (10),$$

*Proof.* Pick a generic system $(S, G)$. Assume $\pi_{\{i\}}(S^{(k)}) = G$ for all $i \in [1, k]$. Then

$$\frac{|S^{(k)}|}{|G|} = \alpha_1^k. \tag{11}$$

In any system we have, by the definition of $\alpha_\ell^k$,

$$\alpha_\ell^k \geq \max_{\substack{U \subseteq [1,k] \\ |\bar{U}| = \ell}} \frac{|S^{(k)}|}{|\pi_U(S^{(k)})|}.$$

If the system is $\rho-$uniform then by Definition 3.6

$$\rho \alpha_\ell^k \leq \max_{\substack{U \subseteq [1,k] \\ |\bar{U}| = \ell}} \frac{|S^{(k)}|}{|\pi_U(S^{(k)})|}.$$

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Now we put everything together and substitute these expressions in Theorem 3.5. The result follows as the constant depend on $\rho$ and the precise value of the index $\ell$ that gives the maximum in $p_{(S,G)}$, hence the absolute $C$, can be obtained as a function of $k$ and $\rho$.

If $\pi_{\{i\}}(S^{(k)}) \subsetneq G$, then if $|\pi_{\{i\}}(S^{(k)})| \geq |G|/2$, the proof goes through by adjusting the $C$ by a constant in (11). Otherwise, the 0-statement is also fulfilled by the argument at the end of Theorem 3.5 and by observing that $p_{(S_i,G_i)} \leq 1$ by the definitions of $\alpha_\ell^k$. The other cases for $p$ are covered by Proposition 3.4. $\square$

As we shall see, the systems of configuration discussed in sections 4-8 satisfy Definition 3.3 and Definition 3.6. However, our results do not clarify if the system has a gap between the probabilities of the 0-statement and the 1-statement if the system is not $(\rho, k)$–uniform. For a further discussion on how some conditions might be relaxed, the reader is referred to the approach taken by Schacht [31] and the discussion in Balogh, Morris and Samotij [3, page 692].

**Example 3.1** (Non uniform system of configurations). The following is an example of a system for which the threshold probabilities from Theorem 3.2 and Theorem 3.5 do not match.

Let
$$A = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -7 \end{pmatrix}$$

and consider the following system of configurations $\overline{S} = S \cup S'$ with $S = A^{-1}(0) \subset \mathbb{Z}_q^8$, $q$ a large prime, and $S' \subset \mathbb{Z}_q^8$ having the following properties:

- $|S'| = q^c$, $c = 9/2$.

- $\pi_{\{4,5,6\}}(S') = \{(1,2,3)\}$.

- For $U \subseteq [1,2,3] \cup [7,8]$, $|\pi_U(S')| = \Theta(q^{c\frac{|U|}{5}})$ and $\pi_U^{-1}(x) = \Theta(\max_{g \in \pi_U(S')}\{|\pi_U^{-1}(g)|\})$

- all the elements in $S'$ have all the coordinates pairwise different.

Such a set $S'$ can be created by choosing elements in $\mathbb{Z}_q^5$ uniformly and random with probability $\frac{q^c}{q^5}$ and discarding the (few) unwanted elements: the resulting set well have the desired properties with high probability. The set $S'$ has the following shape: $S' = \{(*, *, *, 1, 2, 3, *, *) \; : \; * \in \mathbb{Z}_q\}$, has size $q^c$ and it is uniformly distributed throughout the coordinates $[1,2,3] \cup [7,8]$.

The value of $p_{S_i,G_i}$ for the 1–statement given in Theorem 3.2 is

$$\max_{\ell \in [2,8]} \left(\frac{\alpha_\ell^k}{\alpha_1^k}\right)^{\frac{1}{\ell-1}} = \theta\left(q^{-\frac{1}{4}}\right) = \left(\frac{\alpha_3^k}{\alpha_1^k}\right)^{1/2},$$

whereas the one for the 0–statement from Theorem 3.4 is

$$\max_{\substack{U \subseteq [1,k] \\ |\overline{U}| \geq 2}} \left(\frac{|\mathbb{Z}_p|}{|\pi_U(S^{(k)} \cup S')|}\right)^{\frac{1}{|U|-1}} = \theta\left(q^{-\frac{2}{3}}\right) = \left(\frac{q}{q^3}\right)^{\frac{1}{3}}$$

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| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\alpha_i^k$ | $\theta(q^6)$ | $\theta(q^{4.5})$ | $\theta(q^{4.5})$ | $\theta(q^{3.6})$ | $\theta(q^{2.7})$ | $\theta(q^{1.8})$ | $\theta(q^{0.9})$ | $\theta(1)$ |
| $\chi_i^k$ | | $\theta(q^2)$ | $\theta(q^{2.7})$ | $\theta(q^3)$ | $\theta(q^4)$ | $\theta(q^5)$ | $\theta(q^6)$ | $\theta(q^6)$ |

Table 1: The values of the parameters in Example 3.1

(the values of the parameters are recorded in Table 3.1.)

In this example there are two different features involved. One one hand, $S'$ is such that one projection onto the coordinate variables $\{4, 5, 6\}$ has only one possible solution. Additionally, this solution has many preimages. This forces the Theorem 3.2 to pick a larger probability than it should. On the other hand, there are projections $\pi_U$ that have more values than what they should. This forces the probability in Theorem 3.5 to be smaller and compensate for this.  □

## 4. Homomorphisms of finite abelian groups

In this section we study sets free of *configurations arising from homomorphisms* between finite abelian groups. Given $G$ a finite abelian group and $M$ a group homomorphism $M : G^k \to G^m$, then we say that the system of configurations $(S, G)$ *arise* from $M$ if $S = M^{-1}(0)$ (or, in general, $S = M^{-1}(g)$, with $g \in G^m$, but we shall restrict ourselves to $g = 0$.) The group homomorphism $M$ is said to be *invariant* if, for every $x \in G$, the vector $(x, \dots, x)$ belongs to $S = M^{-1}(0)$.

Throughout this section we restrict ourselves to system of configurations arising from invariant homomorphisms. They are a quite general class of systems of configurations that satisfy the V-property as Lemma 4.3 below shows.

As an illustration of the configurations that can be encoded by the group homomorphism setting, we next show the canonical form of invariant homomorphisms in the integers. This will be later exploited in Section 6 to deal with configurations in integer intervals. For a given homomorphism $M : [\mathbb{Z}^m]^{k_1} \to [\mathbb{Z}^m]^{k_2}$, we can consider it as a linear system defined by $mk_1 \times mk_2$ integer values (see [42]). Let $e_i$ denote the vector with 1 in the $i$-th coordinate and 0 in the rest. Since $(e_i, \dots, e_i) \in S \cap [1, i]^m$ for each $i \in [1, m]$, then we observe that, in each of the $mk_2$ equations, the variables corresponding with some $i$-th coordinate should sum to 0 for each coordinate and each equation.

**Proposition 4.1.** *The solutions of each invariant system* $M : [\mathbb{Z}^m]^k \to [\mathbb{Z}^m]^k$ *(namely, elements* $(x_1, \dots, x_k) \in S = M^{-1}(0)$*) can be codified by equation of the form*

$$
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0 \\ \vdots \\ x_0 \end{pmatrix} + \lambda_1 \begin{pmatrix} F_{1,1} \\ F_{1,2} \\ \vdots \\ F_{1,k} \end{pmatrix} + \cdots + \lambda_q \begin{pmatrix} F_{q,1} \\ F_{q,2} \\ \vdots \\ F_{q,k} \end{pmatrix}
$$

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*for fixed $q$, $F_{i,j} \in \mathbb{Z}^m$ (depending on $M$) and some $\lambda_i \in \mathbb{Z}$ and $x_0 \in \mathbb{Z}^m$.*

*Proof.* One direction is clear (the right to left), as those configurations are linear and invariant. On the other direction, we proceed as in the case of linear systems of equations to find a basis. Consider the solution set and relate two solutions if their difference is an element as $(x_0, \ldots, x_0)$. There is an extra solution (more than just the class of the zero). Take the representative with minimal $l_1$ norm and name it $F_1$. Now consider the modulus of the solution set by the possible sum of $(x_0, \ldots, x_0) + \lambda_1 F_1$ for each $\lambda_1 \in \mathbb{Z}$. Do the same as there are more than just one class (the class of zero). The process should end as the maximum number of degrees of freedom is $mk_1$ (both these numbers depends on $M$). $\square$

Let us show an illustrative example.

**Example 4.1.** Consider the invariant homomorphisms $A : [\mathbb{Z}^2]^3 \to [\mathbb{Z}^2]^2$ given by

$$
A = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.
$$

The homomorphism $A$ allows us to codify the 2-dimensional simplices in $\mathbb{Z}^2$: sets of 3 points of the type $((x, y), (x + a, y), (x, y + a))$ for $x, y, a \in \mathbb{Z}$. If we are interested in configurations with $a \geq 0$, we can consider a configuration symmetric with respect to the $(x, y)$ such as: $((x, y), (x + a, y), (x, y + a), (x - a, y), (x, y - a))$. These configuration systems cannot be codified using an integer matrix with three columns (one per each point).

We start proving the V-property for invariant homomorphism systems, Lemma 4.3, using the following arithmetic removal lemma from [42]:

**Theorem 4.2** (Removal lemma for homomorphisms, Theorem 2 in [42]). *Let $G$ be a finite abelian group and let $k$ be a positive integer. Let $M : G^k \to G^k$ be a group homomorphism. Let $g \in G^k$. Let $X_i \subset G$ for $i = [1, k]$, and $X = X_1 \times \cdots \times X_m$ and let $S = M^{-1}(g)$.*
*For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, k) > 0$ such that, if*

$$|S \cap X| < \delta |S|,$$

*then there are sets $X_i' \subset \left[ X_i \cap \pi_{\{i\}}(S) \right]$ with $|X_i'| < \varepsilon |\pi_{\{i\}}(S)|$ and*

$$S \cap (X \setminus X') = \emptyset, \quad \text{where } X \setminus X' = (X_1 \setminus X_1') \times \cdots \times (X_m \setminus X_m').$$

As a consequence of this result we have the following:

**Lemma 4.3** (V-property for invariant homomorphism systems). *Let $(S, G)$ be a system of configurations arising from an invariant homomorphism $M$ of degree $k$. Then, for every $\varepsilon > 0$ there exist $\gamma = \gamma(\varepsilon, k) > 0$ such that, for any set $A \subset G$ with $|A| > \varepsilon |G|$, then $|A^k \cap S| \geq \gamma |S|$.*

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*Proof.* We use Theorem 4.2 and proceed by contradiction. To destroy all the configurations we should remove, at least $\varepsilon/k$ elements in each of the copies of $A$ that form the cartesian product $S^k$. This is true because, for each $s \in A$, $(s, \ldots, s) \in S$. Therefore, there are more than $\delta_{4.2}(\varepsilon/k, k)|S|$ solutions with all its elements in $S^m$, hence proving the result with $\gamma(\varepsilon, k) = \delta_{4.2}(\varepsilon/k, k)$. $\qquad\square$

We can now apply Theorem 2.4 to configurations arising from invariant group homomorphisms.

**Corollary 4.4.** *Let $k$ a fixed integer and a $\delta > 0$. Then for any normal sequence $\{(S_i, G_i)\}_{i \geq 1}$ of configuration systems coming from invariant homomorphisms, there exists $n_0$ depending on $\delta$ such that if $|G_i| > n_0$ then for each $t$ such that*

$$t \geq C \frac{|G_i|}{\delta} \max_{\ell \in [2,k]} \left\{ \left( \frac{\alpha_\ell^k}{\alpha_1^k} \frac{1}{k} \binom{k}{\ell} \right)^{\frac{1}{\ell-1}} \right\} \text{ and } t \leq \frac{\delta |G_i|}{2}$$

*there are*

$$t \left[ \frac{2e}{\delta^2} \right]^{\delta t} \binom{\delta |G_i|}{t}$$

*sets of size $t$ with no solution in $S_i^{(k)}$.*

*Proof.* We use Theorem 2.4, so all the conditions should be verified. Observe that, by Lemma 4.3 the system satisfies the V-property with a uniform function depending only on $k$, $\gamma = \gamma_{2.3}(\delta, k)$. The condition $\xi = (\gamma - 1)|S| + |S^{(k)}| > 0$ and the fact that any set of size $\delta|G|/2$ has a solution with all the variables being different is satisfied for an $n_0$ sufficiently large depending on $k$ and $\delta$ (because of the $\gamma$), and on how fast C3 approaches 1. $C$ is a constant depending on $k$ and on $n_0$. Indeed, normality implies that $C$ depends on how fast the ratio between the different solutions found in any set of relative size $\delta$ and the whole solution set approaches 1. $\qquad\square$

We next show the concentration property for invariant homomorphism systems.

**Proposition 4.5** (Concentration for invariant homomorphism systems)**.** *Let $\{S_i, G_i\}_{i \geq 1}$ be an normal family of systems arising from invariant homomorphisms. Let $p = p(i) \in [0, 1]$ be such that*

(i) $\lim_{i \to \infty} p(i)|G_i| = \infty$.

(ii) $\lim_{i \to \infty} p(i)|\pi_U(S_i)|^{1/|U|} = \infty$ *for each $U \subset [1, k]$, $|U| \geq 2$.*

*Then the sequence satisfies Definition 3.3 for $p$.*

Let us remark that Condition (ii) implies $\lim_{i \to \infty} p(i)|S_i^{(k)}|^{1/k} = \infty$.

*Proof.* Our strategy is to use the Second Moment Method in the form of [2, Corollary 4.3.4]. In most of the proof, we use $G$ instead of $G_i$ to simplify the notation and the little-$o$ refer to the asymptotics for $i \to \infty$. Let $X$ denote the random variable that counts the number of solutions in $S^{(k)} \cap [G]_p^k$. We write $X$ as $X = X_1 + \cdots + X_r$, $r = |S^{(k)}|$, where $X_i$ is the indicator random variable that the $i$-th solution in $S^{(k)}$ belongs to $[G]_p^k$. Denote by $A_i$ the event associated to $X_i$. We also write $i \sim j$ if $A_i$ and $A_j$ are not independent. As it is proven in [2, Corollary 4.3.4], it is enough to show that $\Delta = \sum_{i \sim j} \mathbb{P}(A_i \cap A_j) = o(\mathbb{E}(X^2))$ (for a certain range of $p$) in order to conclude that $X$ is concentrated around its average value.

We compute $\Delta$ grouping the pairs of events $(A_i, A_j)$ according to the size of the non-empty intersection. Write $\Delta = \sum_{\ell=1}^k \Delta_\ell$, where $\Delta_i$ contains the summands $i \sim j$ in $\Delta$ such that $|A_i \cap A_j| = \ell$. Observe that this non-empty intersection can occur in several ways ($A_i$ is a set, while $X_j$ is an ordered tuple), hence a constant correcting factor (bounded above and below by functions of $k$) has to be considered when computing $\Delta$.

We start bounding $\Delta_1$. Observe that $\Delta_1 \leq kp^{2k-1}|S^{(k)}|\frac{|S^{(k)}|}{|G|}$ up to a constant depending on $k$. Indeed, in the bound for $\Delta_1$, the factor $k$ stands for the position of the common variable, $p^{2k-1}$ is the probability of having precisely these $p^{2k-1}$ different elements in $|A_i \cup A_j|$, the factor $|S^{(k)}|$ stands as we are checking for all the solutions $A_i$ (and then we are seeing how many solutions $A_j$ are there satisfying the first condition). Finally, the factor $\frac{|S^{(k)}|}{|G|}$ is an upper bound on the number of solutions $A_j$ that share 1 element with $A_i$ in the $s$-th position. Observe that the factor $\frac{|S^{(k)}|}{|G|}$ arises as the system is invariant, hence the maps $+g : S^{(k)} \to S^{(k)}$ such that $x \to x + (g, \ldots, g)$ for each $g \in G$ creates a partition in $S^{(k)}$ according to the value on any (fixed) coordinate of the solution set (even if $S^{(k)}$ is empty). As some of the $\leq k!\frac{|S^{(k)}|}{|G|}$ solutions $x_j$ that have $x_{j,\ell} \in A_j$ for the $\ell$ coordinate (fixed at the beginning) may also have $x_{j,\ell'} \in A_j$ for some $\ell' \neq \ell$, the bound presented is an upper bound (up to a constant depending on $k$).

We argue now on $\Delta_2$. The arguments for $\Delta_l$, $l \geq 3$ are similar. For $\Delta_2$, there are $\binom{k}{2}$ possible choices for the coordinates to be shared. Also, there are $\pi_U(S^{(k)})$, with $U = \{i,j\}$, $|U| = 2$ possible pairs of values $x_i = g_1$ and $x_j = g_2$. Observe that each of the $S^{(k)}$ solutions have, at most, $\frac{|S|}{|\pi_U(S)|}$ elements sharing the same pair of $(x_i = g_1, x_j = g_2)$. Indeed, the number of elements in $S$ that have $(x_i, x_j)$ as the $i$-th and the $j$-th variables respectively is, either 0, or the size of the homogeneous system with the addition of the equations $x_i = 0$ and $x_j = 0$ (each solution is the sum of one in the new homogeneous system plus a particular solution with the addition of $x_i = g_1$, $x_j = g_2$). Hence, $|S|/|\pi_{\{i,j\}}(S)|$ is the appropriate number when it is different from 0. Since $\frac{S}{\pi_U(S)} \leq \frac{S}{\pi_U(S^{(k)})}$

for each $U \in \binom{[k]}{2}$, the factor in $\Delta_2$ is, at most, a constant depending on $k$ times

$$p^{2k-2} \sum_{U \in \binom{[k]}{2}} \frac{|S|}{|\pi_U(S^{(k)})|} |S^{(k)}|.$$

By normality, $\frac{|S|}{|\pi_U(S^{(k)})|}$ is asymptotically equivalent to $\frac{|S^{(k)}|}{|\pi_U(S^{(k)})|}$. Summing all bounds for $\Delta_l$, $1 \leq l \leq k$ we can bound $\Delta$ by

$$\Delta \leq c(k) \sum_{s=1}^{k} p^{2k-s} \sum_{U \in \binom{[k]}{s}} \frac{|S|}{|\pi_U(S^{(k)})|} |S^{(k)}|,$$

for a certain constant $c(k)$ only depending on $k$. If

$$p|G|^{2k-s} \frac{|S|}{|\pi_U(S^{(k)})|} |S^{(k)}| = o(p|G|^{2k}|S^{(k)}|^2) = o(\mathbb{E}(X)^2), \tag{12}$$

then we can use [2, Corollary 4.3.4] to show the result. Expression (12) follows from $\lim_{i \to \infty} \frac{|S_i|}{|S_i^{(k)}|} = 1$, and that, by hypothesis, $o(p(i)) = \frac{1}{|G_i|}$ (so that $\mathbb{E}(X) \to \infty$), and $o(p(i)) = |\pi_U(S_i^{(k)}|^{-1/|U|}$ for each $U \subset [1, k]$. This completes the proof. $\square$

Let us observe that the Condition C4 from normality has not been needed for Proposition 4.5. Additionally, the invariant systems that satisfy C3 also satisfy C1.

**Proposition 4.6.** *Any sequence of invariant homomorphisms systems $\{(S_i, G_i)\}_{i \geq 1}$ of degree $k$ such that $\lim_{i \to \infty} \frac{|S_i^{(k)}|}{|S_i|} = 1$ (Condition C3) satisfies the uniformity condition from Definition 3.6.*

The proof of Proposition 4.6 is contained in the argument to prove Proposition 4.5 when we see the bound on the number of solutions that each equation has.

After all the preliminary results have been shown, we are ready to prove the main result of this section.

**Theorem 4.7** (Threshold function for homomorphism systems). *Let $1 > \delta > 0$ be a positive real number, and let $\{(S_i, G_i)\}_{i \geq 1}$ be an normal sequence of systems of degree $k$ arising from homomorphisms of finite abelian groups. Let*

$$p_{(S_i, G_i)} = \max_{\ell \in [2,k]} \left( \frac{\alpha_\ell^k(S_i, G_i)}{\alpha_1^k(S_i, G_i)} \right)^{\frac{1}{\ell-1}}. \tag{13}$$

*Then, there exist constants $c_1, c_2$, depending on $\delta$ and $k$ such that*

$$\lim_{i \to \infty} \mathbb{P}([G_i]_p \text{ is } (\delta, S_i)_k-\text{stable}) = \begin{cases} 1 & \text{if } p \geq c_1 p_{(S_i, G_i)}, \\ 0 & \text{if } p < c_2 p_{(S_i, G_i)}. \end{cases}$$

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Let us comment that the normal condition (more precisely, the first part of C2, together with C3) guaranties that the solution set in the invariant systems has asymptotically more solutions than the trivial ones. Additionally, Condition C3 allows the use of Theorem 3.2.

*Proof.* We put all the pieces gathered in Section 3 and from the beginning of the current section. We use $(S, G)$ to denote a generic $(S_i, G_i)$ in the proof. The little-$o$'s are referred to asymptotic behaviour when $i \to \infty$ (or $|G| \to \infty$).

First, Theorem 3.2 can be applied to find the constant $c_1$ and show the first part of the result as $\lim_{i \to \infty} |G_i| = \infty$ and the system satisfy the V-property thanks to Lemma 4.3. Indeed, Lemma 4.3 warranties that $\gamma$ only depends on $k$ and on $\delta$, but is independent on the particular homomorphisms or group, hence it is uniform for the family of homomorphisms. The first part of C2, together with C3 allows us to warranty that, for $|G|$ large enough depending on $\delta$ and $k$ (and on how fast the limit goes to $\infty$), any set of density $\delta|G|/2$ has solutions in $S^{(k)}$ and condition $\xi_{2.4}/|S^{(k)}| > a > 0$ is satisfied for some $a$ depending on $\delta$ and on the family of systems.

Let us now show the 0-statement. In this case, there is a discrepancy between the range of probabilities given by Proposition 4.5 and those used in the hypothesis of Theorem 3.7 (or Theorem 3.5) that we address in the following argument. The uniformity of the systems of configurations coming from homomorphisms, Definition 3.6, is satisfied by Proposition 4.6. Hence, up to a constant factor depending on $k$ and on the family of systems, we can use (8) as an alternative definition of $p_{(S,G)}$. Let $c_2 = c_{3.5}$ be the constant arising in the upper bound on the probability coming from Theorem 3.5, which contains the constant $c_2$ according to the chosen definition of $p_{(S,G)}$. Observe that due to the invariance of the system, the argument from the proof of Proposition 4.5 regarding $\Delta_1$, together with a combination of the first part of C2 and C3 implies that

$$\max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} \left( \frac{|G_i|}{|\pi_U(S^{(k)})|} \right)^{\frac{1}{|U|-1}} \leq 1,$$

hence

$$p \leq c_2 p_{(S,G)} = c_2 \min \left\{ \max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} \left( \frac{|G_i|}{|\pi_U(S^{(k)})|} \right)^{\frac{1}{|U|-1}}, 1 \right\} = c_2 \max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} \left( \frac{|G|}{|\pi_U(S^{(k)})|} \right)^{\frac{1}{|U|-1}}.$$

(14)

Since $|\pi_U(S^{(k)})| \leq |G|^{|U|}$ and $|\pi_{[1,k]}(S^{(k)})| = o(|G|^k)$, then

$$o \left( \max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} |\pi_U(S^{(k)})|^{-\frac{1}{|U|}} \right) = |G|^{-1} \text{ and } o \left( \max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} \left( \frac{|G|}{|\pi_U(S^{(k)})|} \right)^{\frac{1}{|U|-1}} \right) = |G|^{-1}.$$

(15)

Observe that for each $U$

$$\left(\frac{|G|}{|\pi_U(S^{(k)})|}\right)^{\frac{1}{|U|-1}} \geq |\pi_U(S^{(k)})|^{-\frac{1}{|U|}}. \tag{16}$$

Indeed, if

$$\left(\frac{|G|}{|\pi_U(S^{(k)})|}\right)^{\frac{1}{|U|-1}} < |\pi_U(S^{(k)})|^{-\frac{1}{|U|}} \tag{17}$$

for some $U$, then

$$|\pi_U(S^{(k)})| > |G|^{|U|}, \tag{18}$$

which contradicts $|\pi_U(S^{(k)})| \leq |G|^{|U|}$. Hence

$$\max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} \left(\frac{|G|}{|\pi_U(S^{(k)})|}\right)^{\frac{1}{|U|-1}} \geq \max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} |\pi_U(S^{(k)})|^{-\frac{1}{|U|}}. \tag{19}$$

Now observe that, if

$$\max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} \left(\frac{|G|}{|\pi_U(S^{(k)})|}\right)^{\frac{1}{|U|-1}} \approx \max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} |\pi_U(S^{(k)})|^{-\frac{1}{|U|}}, \tag{20}$$

then (16) implies that

$$\left(\frac{|G|}{|\pi_{U_0}(S^{(k)})|}\right)^{\frac{1}{|U_0|-1}} \approx |\pi_{U_0}(S^{(k)})|^{-\frac{1}{|U_0|}} \tag{21}$$

for the $U_0$ giving the maximum. Hence by the argument from (17) to (18) we obtain

$$\max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} |\pi_U(S^{(k)})|^{-\frac{1}{|U|}} = |\pi_{U_0}(S^{(k)})|^{-\frac{1}{|U_0|}} \approx |G|^{-1},$$

which contradicts the second part of (15). Therefore (19) and the impossibility of (20) implies

$$o\left(\max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} \left(\frac{|G|}{|\pi_U(S^{(k)})|}\right)^{\frac{1}{|U|-1}}\right) = \max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} |\pi_U(S^{(k)})|^{-\frac{1}{|U|}} \tag{22}$$

holds. Let us now compare $p$ with the right-hand side of Equation (22). Assume first that

$$o(p) = \max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} |\pi_U(S^{(k)})|^{-\frac{1}{|U|}}. \tag{23}$$

Relation (23) implies that

$$o(p) = |S^{(k)}|^{-\frac{1}{k}}. \tag{24}$$

Therefore, we can use Proposition 4.5 to show concentration around the mean in this interval. In particular, we can apply Theorem 3.5 to obtain the 0-statement in the region defined by (14) and (23) as there is an asymptotic gap between the two by (22), and additionally (24) holds.

Assume now that

$$\max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} |\pi_U(S^{(k)})|^{-\frac{1}{|U|}} = |S^{(k)}|^{1/k}.$$

Then Proposition 3.4 allows to complete the remaining range of probabilities left by Assumption (23).

In the cases where $\max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} |\pi_U(S^{(k)})|^{-\frac{1}{|U|}} > |S^{(k)}|^{1/k}$, consider $U_0$ to be the set for which

$$\max_{\substack{U \subseteq [1,k] \\ |U| \geq 2}} |\pi_U(S^{(k)})|^{-\frac{1}{|U|}} = |\pi_{U_0}(S^{(k)})|^{1/|U_0|},$$

and consider a new family of systems of configurations $(S, G) = (\pi_{U_0}(S^{(k)}), G)$. To conclude, observe that if $|\pi_{U_0}(S^{(k)})|^{1/|U_0|} = o(|G|)$, then we can apply the reasoning of Proposition 3.4 to this new system of configurations to show the 0-statement (possibly with an adjustment in the constant $c_2$). The case $|\pi_{U_0}(S^{(k)})|^{1/|U_0|} \approx |G|$ cannot occur by (15). $\qquad\square$

## 5. Rectangles in abelian groups and some variations

The following theorem illustrates an application of Theorem 2.4 which cannot be directly obtained form the previously existing tools.

**Theorem 5.1** (Rectangles in abelian groups). *Let $\{G_i\}_{i \geq 1}$ be a sequence of finite abelian groups, $H_i, K_i$ subgroups of $G_i$ and such that $|H_i|, |K_i|, |G_i| \to \infty$. For each $\delta > 0$ with $\delta < 1/40$ there exist $C = C(\delta)$ and $i_0 > 0$, depending on the family $\{G_i, H_i, K_i\}_{i \geq 1}$ and on $\delta$, for which the following holds. Let*

$$S_i = \{(x, x+a, x+b, x+a+b) : x \in G_i, a \in H_i, b \in K_i\}$$

*be the set of configurations. Assume that $\max\{|H_i|, |K_i|\} \leq (|S_i^{(4)}|/|G_i|)^{2/3}$. For each $i \geq i_0$ the number of sets free of configurations in $S_i^{(4)}$ and with cardinality $t$ such that*

$$t > \frac{C}{\delta} \left( \frac{|G_i|^4}{|S_i^{(4)}|} \right)^{1/3}$$

*is bounded from above by $\binom{2\delta|G_i|}{t}$.*

*Proof of Theorem 5.1.* First let us observe that the solution set is isomorphic to the group $G_i \times H_i \times K_i$, which is a subgroup of $G_i^3$ (in $G_i^4$). Indeed,

$$S_i = \{(x_1, x_2, x_3, x_4) : x_2 - x_1 \in H_i, \, x_3 - x_1 \in K_i, \, x_4 - x_1 = x_2 - x_1 + x_3 - x_1\},$$

which can be defined in terms of the kernel of an homomorphism between abelian groups. Assume $\max\{|K_i|, |H_i|\} = |K_i|$. All the hypothesis of Corollary 4.4 are satisfied, so we shall check that, under the hypothesis $\max\{|H_i|, |K_i|\} \leq (|S_i^{(4)}|/|G_i|)^{2/3}$, then

$$C' \frac{|G_i|}{\delta} \max_{\ell \in [2,4]} \left\{ \left( \frac{\alpha_\ell^4}{\alpha_1^4} \frac{1}{4} \binom{4}{\ell} \right)^{\frac{1}{\ell-1}} \right\} = \frac{C}{\delta} \left( \frac{|G_i|^4}{|S_i^{(4)}|} \right)^{1/3}.$$

To compute $\alpha_2^4$ we shall compute, for $U \subset [1,4]$ with $|U| = 2$, $\max_{(g_1,g_2) \in G_i} |S_i^{(4)}(A_i, G_i) \cap \pi_U^{-1}(g_1, g_2)|$. If $U$ is $\{1,2\}$, $\{1,3\}$, $\{2,4\}$ or $\{3,4\}$, then the size of the preimage is, approximately, $|K_i|$ or $|H_i|$ respectively. In the other cases the sizes depends, essentially, on $K_i \cap H_i$, hence they are smaller. Hence $\alpha_2^4 \approx \max\{|K_i|, |H_i|\} = |K_i|$. The second equality holds by assumption.

Let us compute $\alpha_3^4$: if all the elements are different, there is always just one choice as given 3 points of the above rectangle, the 4th point is always computed uniquely. Hence $\alpha_3^4 = 1$. Therefore, also $\alpha_4^4 = 1$. Consequently,

$$\max_{\ell \in [2,4]} \left\{ \left( \frac{\alpha_\ell^4}{\alpha_1^4} \right)^{\frac{1}{\ell-1}} \right\} \approx \max \left\{ \left( \frac{|G_i|}{|S_i^{(4)}|} \right)^{\frac{1}{3}}, \left( \frac{|G_i|}{|S_i^{(4)}|} \right)^{\frac{1}{2}}, \frac{|K_i||G_i|}{|S_i^{(4)}|} \right\}.$$

Since $S_i^{(4)} \geq |G_i|$ and $\max\{|H_i|, |K_i|\} \leq (|S_i^{(4)}|/|G_i|)^{2/3}$, then

$$\max_{\ell \in [2,4]} \left\{ \left( \frac{\alpha_\ell^4}{\alpha_1^4} \right)^{\frac{1}{\ell-1}} \right\} \approx \left( \frac{|G_i|}{|S_i^{(4)}|} \right)^{\frac{1}{3}}$$

and the result follows from Corollary 4.4. $\qquad\square$

Let us discuss briefly the case when $G_i = \mathbb{Z}_i^2$, $H_i = \mathbb{Z}_i \times \{0\}$, $K_i = \{0\} \times \mathbb{Z}_i$, or more generally when $G_i = H_i \times K_i$ (with $|H_i| = |K_i| = i$). Then the size $t$ in Theorem 5.1 satisfies that $t > C'' i^{4/3}$, for a certain constant $C''$. Observe that a set $X \subset G_i$ defines a bipartite graph in the following way: for each $(h, k) \in G_i = H_i \times K_i$ construct the bipartite graph with vertex set $V = H_i \cup K_i$ by adding an edge between $h$ and $k$. If $X$ is solution-free, then the corresponding bipartite graph does not contain cycles of length 4. In this situation, it is well known that the maximum number of edges in a bipartite graph (with bipartition of size $i \times i$) without cycles of length four is $i^{3/2} + o(i^{3/2})$ as a consequence of a classical result of Kővari, Sós and Turán [19]. This shows that for $C i^{4/3} < t < c i^{3/2}$, there is a wide range of values of $t$ to which Theorem 5.1 gives a meaningful bound.

Theorem 5.1 can be generalized to higher dimensional cubes or to other configurations in a two-dimensional cartesian product with more points. More precisely, given $\{G_i\}_{i\geq 1}$ a sequence of finite abelian groups and $K_i$, $H_i$ two subgroups (whose sizes grow when $i \to \infty$), we can consider configurations $S_i$ of the form

$$S_i = \{(x, x + a_1, \ldots, x + a_r, x + b_1, x + b_1 + a_1, \ldots, x + b_1 + a_r, \ldots, x + b_r + a_r) :$$
$$x \in G_i, a_j \in H_i, b_j \in K_i\}, \tag{25}$$

which generalizes the configuration studied in Theorem 5.1. The same techniques used to prove Theorem 5.1 show that for $0 < \delta < 1/40$ and assuming that

$$\max\{|H_i|, |K_i|\} \leq \left(\frac{|G_i|}{|S_i^{(k)}|}\right)^{\frac{r+1}{(r+1)^2-1}},$$

then there exists a constant $C$, that depends on $\delta$ and in the family $\{G_i, H_i, K_i\}_{i\geq 1}$, and an integer $i_0$ such that the following holds: for $t$ such that

$$t > \frac{C}{\delta}\left(\frac{|G_i|^{(r+1)^2}}{|S_i^{(k)}|}\right)^{\frac{1}{(r+1)^2-1}}, \tag{26}$$

then the number of solution-free sets of $G_i$ size $t$ (for $i > i_0$) is bounded from above by $\binom{2\delta|G_i|}{t}$. Observe also that this result can be even more extended when dealing with an asymmetric version, namely considering $r + 1$ elements in $H_i$ and $s + 1$ elements in $K_i$.

Similarly as for Theorem 5.1, when we particularize (25) to $G_i = H_i \times K_i$, $H_i \cap K_i = \{0\}$ and $|H_i| = |K_i| = i$, then each set $X \subset G_i$ without solutions in $S_i$ defines a bipartite graph with $|H_i|$ vertices on each stable set without $K_{r+1,r+1}$ as a subgraph. Even more, each point of the type $x + a_i + b_j$ corresponds to an edge, hence we can obtain results for any bipartite graph. However, in this case, the lower bounds such as (26) are more involved and depend heavily on the particular system of configurations considered.

This connects the configurations codified by (25) with the classical Zarankiewicz problem [19]. The problem is to study the function $\text{zex}(n, K_{t,s})$ counting the largest number of edges in a bipartite graph with $n$ vertices on each stable set which excludes $K_{t,s}$ ($s \geq t$) as a subgraph. Some partial results are known for the general Zarankiewicz problem: an upper bound $O(n^{2-1/t})$ was obtained by Kővári, Sós and Turán in [19]. It is conjectured that this upper bound gives the correct order of magnitude, but the problem of finding lower bounds (namely, explicit constructions) has been shown to be more difficult. Some instances are known to close the gap in the order of magnitude: Erdős, Rényi and Sós [10] found a lower bound for $\text{zex}(n, K_{2,t})$ which match the upper bound, Brown [6] and Füredi [13] proved the right order of magnitude for $\text{zex}(n, K_{3,3})$. Finally the case $\text{zex}(n, K_{s,t})$ with $s \geq (t-1)!+1$ was proved by Alon, Rónyai and Szabó [1].

Continuing in the bipartite graph case, Balogh and Samotij showed in [4] that the $\log_2$ of the total number of subgraphs on $n$ vertices without a $K_{s,t}$ is bounded above by $cn^{2-1/t}$ with an explicit constant that depends on $s$ and $t$. The upper bound from [4] is more accurate than what can be obtained with Theorem 5.1 (or its equivalent for $K_{r+1,r+1}$). For bipartite graphs on $n$ vertices, one can obtain the V-property using the known cases of Sidorenko's conjecture [35], which states that, in a graph with edge-density $d$, there are $d^{e(H)}n^{e(H)}$ graph homomorphisms from a bipartite graph with $e(H)$ edges. Sidorenko's conjecture is known to hold for complete bipartite graphs (see [37]), hence giving much better bounds than [42, Theorem 1].

Let us mention still another generalization of Theorem 5.1 by exploiting the homomorphism setting. Let $G$ be a finite abelian group, $H$ a subgroup of $G$ and $\phi : H \to G$ an injective group homomorphism with $a \neq \pm\phi(a)$ for each $a \in H$. We consider the configuration set of 'slanted squares' defined by $\{(x, x+a, x+\phi(a), x+a+\phi(a)) : x \in G, a \in H\}$, which includes the rhombuses $\{((x,y), (x+a_1, y+a_2), (x+a_2, y+a_1), (x+a_1+a_2, y+a_1+a_2)) : x, y, a_1, a_2 \in \mathbb{Z}_i\}$. Other geometric structures that we may consider are isosceles triangles where the uneven side is located along the $x$-axis $\{((x,y), (x+a_1, y+a_2), (x-a_1, y+a_2)) : x, y, a_1, a_2 \in \mathbb{Z}_i\}$, or all the possible right-angled triangles $\{((x,y), (x+a_1, y+a_2), (x+a_2, y-a_1)) : x, y, a_1, a_2 \in \mathbb{Z}_i\}$. All these configurations can be treated in a similar way as the case of rectangles of Theorem 5.1.

## 6. Configurations in $[1,n]^m$

In this section we consider linear configurations in $[1,n]^m$. These linear configurations arise from group homomorphisms $M : [\mathbb{Z}^m]^k \to [\mathbb{Z}^m]^k$ with the invariant property (namely, $M(x, \ldots, x) = 0$ for each $x \in \mathbb{Z}^m$.) Even though $[1,n]^m$ is not a group, we shall see that the proof of Corollary 4.4 can be adapted to obtain an analogous result in this case. Let $S_n = M^{-1}(0) \cap ([1,n]^m)^k$ denote the set of configurations and $S_n^{(k)}$ the sets of configurations where all the points are different.

We observe that, by the invariance of $M$, the set $S_n$ is invariant by translations as long as the boundary conditions are preserved. In other words, if $(x_1, \ldots, x_k) \in S_n$, $x_i \in [1,n]$, then $(x_1 + x, \ldots, x_k + x) \in S_n$ for every $x \in [-n,n]^m$ with sufficiently small coordinates so that $x_i + x \in [1,n]^m$ for each $i$. Similarly, $S_n$ is invariant by small dilations: $(\lambda x_1, \ldots, \lambda x_k) \in S_n$ for every $\lambda \in [1,n]$ sufficiently small so that $\lambda x_i \in [1,n]$ for each $x_i$. In particular the maximum in the definition of the parameter $\alpha_1$ is achieved at the point $(1, \ldots, 1) \in [1,n]^m$.

The next Proposition shows that study of configurations in $[0,1]^n$ can be reduced to the one for configurations arising from invariant homomorphisms of cyclic groups, which allows for application of the results stated in Section 4.

**Proposition 6.1** ($\alpha_i$ in the $m$-dimensional cube). *Given an invariant homomorphism $M : [\mathbb{Z}^m]^k \to [\mathbb{Z}^m]^k$, and the sequence of systems of configurations $\{(S_i, [1,i]^m)\}_{i \geq 1}$, with $S_i = M^{-1}(0) \cap ([1,i]^m)^k$, satisfying the V-property with*

$\gamma_{4.3}$ *depending on $M$ and uniform for the family, there exist a $\lambda > 0$ and a constant $c > 0$, both depending on $M$, such that*

$$\alpha_i(S_n, [1, n]^m) \leq \alpha_i(\overline{S}_{\lambda n}, \mathbb{Z}_{\lambda n}^m) \leq c\alpha_i(S_n, [1, n]^m), \qquad (27)$$

*where $\overline{S}_{\lambda n}$ is the kernel of the natural restriction of $M$, a matrix of integers, to $M_n : [\mathbb{Z}_{\lambda n}^m]^k \to [\mathbb{Z}_{\lambda n}^m]^k$.*

*Proof.* Since $M$ is a fixed matrix of integers, there exists a $\lambda \in \mathbb{Z}^+$ such that the following holds. Given $M_{\lambda i} : [\mathbb{Z}_{\lambda i}^m]^k \to [\mathbb{Z}_{\lambda i}^m]^k$, the natural restriction to $\mathbb{Z}_{\lambda i}^m$ of $M$, then $\mathbf{x} \in \overline{S}_{\lambda i} \cap ([1, i]^m)^k$ if and only if $\mathbf{x} \in S_i$ (this shows $\alpha_i^k(S_n, [1, n]^m) \leq \alpha_i^k(\overline{S}_{\lambda n}, \mathbb{Z}_{\lambda n}^m)$). That is, we obtain $\lambda = \lambda(M)$ (the $\lambda$ of the statement) as the minimal value for which, if all the variables have coordinate values in the first $i$ positive integers, then any equation can be read in the integer setting and not in the cyclic group setting. Now we apply Lemma 4.3 on $M_{\lambda i}$ for the set $B_i = [1, i]^m \subset \mathbb{Z}_{\lambda i}^m$ as $|B_i| = \frac{|\mathbb{Z}_{\lambda i}^m|}{\lambda^m}$ (thus $B_i$ represents a positive proportion of $\mathbb{Z}_{\lambda i}^m$). By the choice of $\lambda$, $\prod_{j=1}^k B_i^k \cap \overline{S}_{\lambda i} = S_i$. This shows the first part of the result.

To prove the second part we observe that, as in the proof of Proposition 4.5, when we predetermine some of the values for the variables for the system $M_n$, the number of solutions is either 0 (if the predetermined values renders the system incompatible), or the same number of solutions as in the homogeneous case. In particular, if we select some predetermined values that can be completed to a full solution, the number of solutions projected only depends on the indices of the variables selected. By the first part of the statement already proved, and an averaging argument, we conclude that, for each $U \subset [1, k]$ there exists a $c = c(M)$ and a solutions with all the variables in $[1, n]^m$ with the following property: the number of solutions projected to it when from $S_n$ is a constant $c$ away from the ones projected from $\overline{S}_{\lambda n}$. Therefore, for every $i$, $\alpha_i(\overline{S}_{\lambda n}, \mathbb{Z}_{\lambda n}^m) \leq c\alpha_i(S_n, [1, n]^m)$. $\qquad \square$

*Remark* 6.2. If instead of $\alpha_i$ we consider $\alpha_i^k$ the results will be similar as, if $x_i \neq x_j$ in the given subsystem and we impose the equation $x_i = x_j$, then the difference in terms of the sizes of the solution sets is at least $n$. Therefore we have that $\alpha_i^k \approx \alpha_i$ in this case.

By Observation 6.2, Proposition 6.1 and assuming the normality of the family of homomorphism systems induced by $M$, we obtain analogous results to Proposition 4.5, Theorem 4.7 and Corollary 4.4 in the case of configurations in integer intervals. Therefore, we obtain counting and random-sparse-analogue results for sets in $[1, n]^m$ free of $m$-dimensional simplices

$$\{((x_1, \ldots, x_m), (x_1 + a, \ldots, x_m), \ldots, (x_1, \ldots, x_m + a)) \mid x_i, x_i + a \in [1, n]\} \quad (28)$$

(multidimensional Szemerédi), or other homothetic-to-a-point linear structures.

In some cases, we want to consider configurations systems where some of the $a_i$ from Proposition 4.1 are non-negative (such as when we ask in (28) for $a \geq 0$). By considering symmetric configurations (configuration containing the vectors

with $+a_i$ and $-a_i$ for every $i$ in Proposition 4.1), we obtain the V-property for these restricted configurations. Furthermore, the $\alpha_i$ in the restricted case are, up to a multiplicative constant, equal to their unrestricted counterparts. Indeed, the total number of solutions is, up to the factor $(1/2)$ raised to the power given by the number $a_i$'s asked to have a specific sign. Similarly as in the proof of Proposition 6.1, an averaging argument shows that, up to a multiplicative constant depending on the number of coordinates and on the number of points in the configuration, the maximum number of solutions projected to a partial solution with $a_i$ restricted in sign is the same as in the case of the homomorphism that has been restricted.

## 7. Linear systems of equations on abelian groups

In this subsection we study in detail the important case of linear systems. Let $G$ be an abelian group. Following the language of Module Theory, for $g \in G$, $n \in \mathbb{N}$, we define $ng = g + \overset{(n)}{\dots} + g$ (the definition can be extended to negative $n$). Let $A$ be $k \times m$ matrix with integer entries, $(x_1, \dots, x_k) \in G^k$ and $\mathbf{x} = (x_1, \dots, x_k)^{\mathrm{T}}$. We consider the group homomorphism from $G^k$ to $G^m$ defined by the matrix multiplication $(x_1, \dots, x_k) \mapsto A\mathbf{x}$. This homomorphism defines a system of configurations whose solutions are the elements $\mathbf{x} \in G^k$ such that $A\mathbf{x} = 0$. As usual, we refer to these type of homomorphisms as linear system of equations. In all this section we assume that $A$ has maximum rank.

We start by discussing results in the integer scenario in 7.1, where we relate our framework with [24]. Later, we develop on the case when dealing with finite fields and finite abelian groups in 7.2. Let us mention that similar results appeared in the work of Saxton and Thomason in [30].

### 7.1. The integer case

Following [24], we assume that $A$ is *irredundant*, namely for each pair of indices $i \neq j$, there exist a solution $(x_1, \dots, x_k)$ with $x_i \neq x_j$. In particular, irredundancy implies that $S^{(k)} \subset A^{-1}(0)$ is non-empty. This naturally relates with the condition of *abundancy* introduced in [30]. Indeed, an abundant matrix is irredundant, but not every irredundant matrix is abundant. For instance $x_1 - 2x_2 = 0$ is irreducible but not abundant. See for example [26] for a wide variety of explicit instances studied in the literature fitting with this setting.

We are also interested when taking coordinates in the interval $[1, n]$ (or in some cases in $[-n/2, n/2]$) instead of $\mathbb{Z}$ to obtain quantitative results. As discussed in Section 6 we reduce the problem of integers to the case of cyclic groups by using Proposition 6.1.

We say that $A$ satisfies the *strong column condition* if the sum of the columns of $A$ is zero. This is equivalent to the fact that $A$ induces an invariant homomorphism. Frankl, Graham, Rödl [11] (see also [32, Theorem 6.1]) proved the V-property in this particular case (which the authors call *density regular* property). More precisely, for every $\delta > 0$ there exists $n_0 = n_0(\delta)$ and an $\epsilon = \epsilon(\delta, A) > 0$ with the following property: for every $n \geq n_0$ and for every set $X \subset [1, n]$ with

$|X| \geq \delta n$, the linear system $A\mathbf{x} = 0$ satisfying the strong columns condition has at least $\epsilon n^{k-\ell}$ solutions with $\mathbf{x} \in A^{-1}(0) \cap [1,n]^k = S^{(k)}$, where $\ell$ is the rank of the matrix $A$. Moreover, if the matrix $A$ does not satisfy the strong column condition then $A$ is not density regular. Indeed, one can obtain that solutions given by the V-property could be assume to have pairwise different components (see for instance [26]). Complementarily, using the fact that $A$ is irredundant we can deduce that $\xi_{2.4}/|S^{(k)}| > a > 0$, for some $a$ depending on $A$ and on $\delta$, is satisfied and Theorem 2.4 applies in this setting. Irredundancy implies that the system $A'$ formed by $A$ when any equation $x_i = x_j$ is added reduces the rank by 1 while keeping the same number of variables. Since the number of variables is bounded by $k$, we obtain that $|A'^{-1}(0) \cap [1,n]^k| = O(n^{k-\ell-1})$ hence $|S^{(k)}| \approx |S| \approx n^{k-\ell}$.

The main result of this section is Theorem 7.1, which extends [3, Theorem 1.1]. In order to state it, let us define the parameter $m_A$ defined by Rödl and Ruciński in [24, Definition 1.1]:

$$m_A = \max_{q \in [2,k]} \max_{\substack{B \subset [1,k] \\ |B| = q}} \frac{q-1}{q-1+h_B-\ell} \tag{29}$$

where $\ell$ is the rank of the matrix $A$, $h_B$ stands for the rank of the matrix $A^B$ (namely, the submatrix of $A$ where the columns indexed by $B \subset [1,k]$ have been deleted). As Theorem 7.1 shows, the right order of magnitude of the lower bound for $t$ in Theorem 2.4 is given by $n^{1-1/m_A}$.

**Theorem 7.1.** *Let $A$ be a $k \times m$ irredundant matrix, $k > m$, with integer entries and satisfying the strong columns condition. For every positive $\beta$ there exists constants $C = C(A, \beta)$ and $n_0 = n_0(A, \beta)$ such that if $n \geq n_0$ and $t \geq Cn^{1-1/m_A}$, then the number of solution–free subsets of size $t$ of $[1,n]$ to the system of equations $A\boldsymbol{x} = 0$ is at most*

$$\binom{\beta n}{t}.$$

In [24] the parameter $m_A$ was shown to be the right threshold for the probability that any partition of a random set contains a monochromatic solution $Ax = 0$, $A$ satisfying the strong columns condition. Furthermore, the parameter $m_A$ was already exploited in the work of Schacht e.g [31, Theorem 2.4], Friedgut, Rödl and Schacht [12, Theorem 1.1] where they extend the result of [24] to partition regular matrices, and Saxton and Thomason [30, Theorem 2.10].

*Proof.* In order to prove Theorem 7.1, we show that our specification of $t$ in terms of the different coefficients $\{\alpha_l^k\}_{1 \leq l \leq k}$ gives exactly $m_A$. In particular, we see that $n^{1-1/m_A}$ is a multiplicative constant away from the first part of (1) and then use Theorem 2.4. As in the defintion of $m_A$, let $\ell$ be the rank of the matrix $A$.

Fix $1 \leq l \leq k$. Observe first that, from linearity of the equation $A\mathbf{x} = 0$ and the fact that $A$ is irredundant, $\alpha_l$ and $\alpha_l^k$ have the same order of magnitude:

when fixing $l$ coordinates of $\mathbf{x}$ in positions indexed by $B \subset [1,k]$, $l = |B|$, the new system of equations $A^B \mathbf{y} = \mathbf{b}$ has either 0 solutions or a number that only depends on the rank of $A^B$. Assume that there exists a solution $\mathbf{x}_0 = (\mathbf{x}_{\overline{B}}, \mathbf{x}_B)$ with all coordinates being different such that $A^B \mathbf{x}_B = -A^{\overline{B}} \mathbf{x}_{\overline{B}} = \mathbf{b}$. Each solution $\mathbf{y}_B \in S(A^B, \mathbb{Z})$ of $A^B \mathbf{y}_B = 0$ for which $\mathbf{y}_B + \mathbf{x}_B$ has some repeated coordinates, say indexed by $i$ and $j$, must satisfy an additional equation of the type $y_{B,i} - y_{B,j} = x_{B,j} - x_{B,i} = d \neq 0$; for $A^B$ irredundant for the pair $(i,j)$, the previous additional constrain implies lowering the rank by 1 thus meaning asymptotically less solutions (lowered by a factor of $1/n$); for $A^B$ not being irredundant for the pair $(i,j)$, the addition of the equation $y_{B,i} - y_{B,j} = d$ makes the new system incompatible, so no solution added to $\mathbf{x}_B$ will equate the coordinates $i$ and $j$. Hence, most (asymptotically all) of the solutions that project to $\mathbf{x}_{\overline{B}}$ have all different coordinates. An analogous reasoning can be applied when there is no solution $\mathbf{x}_0 = (\mathbf{x}_{\overline{B}}, \mathbf{x}_B)$ with all coordinates being different such that $A^B \mathbf{x}_B = -A^{\overline{B}} \mathbf{x}_{\overline{B}} = \mathbf{b}$ to observe that the orders of magnitude of the number of solutions that are projected to $\mathbf{x}_{\overline{B}}$ are the same regardless of whether all the coordinates are asked to be different or no. Therefore, by taking the maximum over all $B$ of fixed size $l$ we conclude that $\alpha_l$ and $\alpha_l^k$ must have the same order of magnitude. Compare this argument with the one in the general setting of homomorphism developed in the proof of Proposition 4.5 (See the argument for $\Delta_2$ in the proof of Proposition 4.5).

We can study then the modified parameter

$$\max_{l \in [2,k]} \left\{ \left( \frac{\alpha_l}{\alpha_1} \right)^{\frac{1}{l-1}} \right\} = c \max_{l \in [2,k]} \left\{ \left( \frac{\alpha_l}{\alpha_1} \frac{1}{k} \binom{k}{l} \right)^{\frac{1}{l-1}} \right\} \tag{30}$$

for some $c$ depending on $k$. Observe that left hand side of (30) is the value appearing in Theorem 2.4 but where $\alpha_l^k$ have been replaced by $\alpha_l$. When the solutions of $A\mathbf{x} = 0$ are restricted with $\mathbf{x} \in [1,n]^k$ or ($\mathbf{x} \in [-n/2, n/2]^k = G^k$), we have that $|A^{-1}(0) \cap [1,n]^k| = c_A n^{k-\ell}$. Let $S_n = A^{-1}(0) \cap [1,n]^k$ and consider

$$\alpha_i = \max_{\substack{B \subset [1,k] \\ |B|=i}} \max_{(g_1,\ldots,g_i) \in G^i} \left\{ |S_n \cap \pi_B^{-1}(g_1,\ldots,g_i)| \right\}.$$

Observe that, fixing $B \subset [1,k]$ with $|B| = i$ then

$$\max_{(g_1,\ldots,g_i) \in G^i} \left\{ |S_n \cap \pi_B^{-1}(g_1,\ldots,g_i)| \right\} = c_{A,B,i}\, n^{k-i-h_B},$$

as $k-i-h_B$ are the degrees of freedom: the difference between the free variables, $k-i$, minus the rank of the matrix, or the number of relations/valid equations between the variables. Therefore,

$$\alpha_i = \max_{\substack{B \subset [1,k] \\ |B|=i}} \left\{ c_{A,B,i}\, n^{k-i-h_B} \right\} = c'_{A,i} \max_{\substack{B \subset [1,k] \\ |B|=i}} n^{k-i-h_B}.$$

It is not difficult to see that $\alpha_1 = c_A n^{k-1-\ell}$ as, if the matrix is irredundant, there is a variable for which, if we fix it, the rank of the new matrix (with one

less column), does not change. Indeed, if the matrix is irredundant, $m \geq k+1$ as there is at least one non-zero solution to $A\mathbf{x} = 0$. If the matrix is full rank there is one $k \times k$ full rank submatrix. The claim follows. We conclude substituting everything in (30) that

$$\max_{i \in [2,k]} \left\{ \left( \frac{c'_{A,i} \max_{\substack{B \subset [1,k] \\ |B|=i}} n^{k-i-h_B}}{c_A n^{k-1-\ell}} \right)^{\frac{1}{i-1}} \right\} = c'''_A \max_{i \in [2,k]} \left\{ \left( \max_{\substack{B \subset [1,k] \\ |B|=i}} n^{k-i-h_B-k+1+\ell} \right)^{\frac{1}{i-1}} \right\}$$

$$= \max_{i \in [2,k]} \max_{\substack{B \subset [1,k] \\ |B|=i}} n^{\frac{-i-h_B+1+\ell}{i-1}} = \max_{i \in [2,k]} \max_{\substack{B \subset [1,k] \\ |B|=i}} n^{\frac{-i-h_B+1+\ell}{i-1}}.$$

Hence the above quantity is maximal whenever $\frac{-i-h_B+1+\ell}{i-1}$ is maximal on the appropriate domain. But $\frac{-i-h_B+1+\ell}{i-1}$ is maximal if and only if

$$-\frac{1}{\frac{-i-h_B+1+\ell}{i-1}} = \frac{i-1}{i+h_B-1-\ell}$$

is maximal, which is precisely the quantity $m_A$. Hence, we have found the relation between $\{\alpha_l^k\}_{1 \leq l \leq k}$ (which have the same order of magnitude as $\{\alpha_l\}_{1 \leq l \leq k}$) and $m_A$.

As mentioned at the beginning of the section, the V-property is provided by [11], and the irredundancy implies that $\xi_{2.4}/|S^{(k)}| > a > 0$. Hence, we obtain Theorem 7.1 as a corollary of Theorem 2.4. □

Let us see a direct consequence of Theorem 7.1. Denote by $ex(A, n)$ the size of the largest subset $F \subset [1, n]$ which contains no non-trivial solution of the equation $A\mathbf{x} = 0$. By a trivial solution, following Ruzsa [27] and Shapira [33], we mean a solution which has constant value on variables whose coefficients add up to zero. Computing $ex(A, n)$ is not obvious and depends heavily on $A$. For linear equations ($m = 1$) the situation can be illustrated as follows. Let $L = (a_1, \ldots, a_k)$. Ruzsa names the linear equation $L \cdot \mathbf{x} = 0$ to be of *genus g* if $g$ is the size of the largest partition of the coefficient set such that the sum of coefficients in each part is zero. In [27] he proves that $ex(L, n) \ll n^{1/g}$.

Theorem 7.1 presents an upper bound for the number of solution-free subsets of size $t$ with $t \geq Cn^{1-1/m_A}$ (notice that $m_A > 0$, hence $1 - 1/m_A < 1$). For equations of genus $g > 0$, the size of the largest solution-free set is of the order $n^{1/g}$. Hence, we are interested in the cases when $1/g > 1 - 1/m_A$. It is known that the number of linear systems equations for which the extremal free sets can be (almost)-linear is not negligible. Furthermore, Shapira [33] shows that almost all linear systems of equations satisfying the strong columns condition have sharp sublinear (Behrend–type) examples which are solution–free. More precisely, let $\mathcal{A}(k, m, h)$ be the set of $k \times m$ matrices $A$ with integer coefficients such that

(i) all coefficients in $A$ are bounded in absolute value by $h$,

(ii) $A$ satisfies the strong columns condition, so the coefficients of every row in $A$ sum to zero.

(iii) $m \leq k - \lceil \sqrt{2k} \rceil + 1$.

**Theorem 7.2** (Shapira [33])**.** *Let $k \geq 6$ and $h$ be positive integers. There are $c(k,h)$ and $c(k)$ such that all but at most $c(k)/h$ of the matrices in $\mathcal{A}(k,m,h)$ satisfy a Behrend–type lower bound:*

$$ex(A,n) \geq ne^{-c(k,h)\sqrt{\log n}}.$$

Roughly speaking, Theorem 7.2 tells that the large majority of system of equations have a Behrend–type lower bound for $ex(A,n)$. Consequently, Theorem 7.1 gives (for almost all systems) a non-trivial tight bound for the number of solution-free sets in the regime

$$n^{1-1/m_A} \leq t \leq ne^{-c(A)\sqrt{\log n}}.$$

*Remark* 7.3. Saxton and Thomason prove similar results for linear systems of equations using the container methodology in [30], which slightly differs from [3]. See Subsection 7.2 for a detailed explanation of their results.

Let us finally discuss the random sparse counterpart. The arguments for invariant homomorphisms in Section 4 apply in this framework and Definitions 3.3 and 3.6 apply. Moreover, the previous arguments had shown that, for each $\delta > 0$ the threshold probability for the $(\delta, A^{-1}(0) \cap [1,n]^k)_k$-stability is

$$p_A = n^{-1/m_A}.$$

This result was obtained by Schacht [31, Theorem 2.4] and Saxton and Thomason [29, Theorem 12.3], and extends the sparse Szemerédi–type of Conlon and Gowers [8, Theorem 1.12].

### 7.2. Linear system of equations in finite abelian groups

In this subsection we discuss results for linear system of equations over finite fields and abelian groups. In both cases the V-property holds as a consequence of the V-property for group homomorphisms, and also from previous works of Král', Serra and Vena [21, 22].

In the finite field setting, the computations are essentially the same as in the integer case, giving rise to the very same constant $m_A$ as defined in Equation (29). Hence, this gives the analogue of Theorem 7.1 for equations over finite fields. For finite fields we could consider matrices with coefficients over the field and the results would also be analogous. Again, in this setting we have lower bounds for the size of sets $X \subset \mathbb{F}_p^n$ avoiding solutions to a given equation $A\mathbf{x} = 0$: when $n$ is fixed and $p$ tends to infinity, Behrend-type constructions from integers transfer easily to constructions in $\mathbb{F}_p^n$. However, the case with $p$ fixed and $n$ tending to infinite had been less studied and there are only few families studied in this context, see for instance [23].

Let us finally shortly describe this setting over general abelian groups, which slight differs from the previous cases. In this context, as pointed out in [29], the rank of a matrix over an abelian group is not well defined, and hence Equation (29) must be computed by other sources. See [29, Section 10] for the details of the right parameter $m_A$ in this context. In [29, Theorem 10.3], the authors obtain an upper bound on the number of solution-free sets in finite abelian groups for systems of configurations arising from integer matrices satisfying a condition on the *determinantal* of the matrix. This condition is not necessary due to the arithmetic removal lemma for group homomorhisms we are using as it extends [22, Theorem 1].

## 8. Configurations in non-abelian groups

In this subsection we discuss still another family of examples arising from equations on non-abelian groups. To simplify notation, we write $e$ for a generic identity element on a group $G$. The main theorem we can prove is the following:

**Theorem 8.1.** *Let $r_1, \ldots, r_k$ be fixed positive integers and $r = r_1 + \cdots + r_k$. Let $\{G_i\}_{i \geq 1}$ be a sequence of groups with unit element $e$. Assume that the exponent of $G_i$ is a divisor of $r$ and that for every $j$, $\gcd(r_j, |G_i|) = 1$. Then, for each $\delta > 0$ with $\delta \leq \min\{\beta/2, 1/40\}$ there exist positive constants, $i_0$, $c_1$ and $c_2$, $C$ such that the following hold: for each $i \geq i_0$ and $t$ in the margin*

$$\frac{C}{\delta} k^{-\frac{1}{k-1}} |G_i|^{\frac{1}{k-1}} \leq t \leq \frac{\delta}{2} |G_i|,$$

*there are at most*

$$\binom{\beta |G_i|}{t}$$

*sets in $G_i$ which are free of solutions to the equation*

$$x_1^{r_1} \cdots x_k^{r_k} = e. \tag{31}$$

*Furthermore, given $S_i$ the solution set induced by (31) and $p_{(S_i, G_i)} = |G_i|^{-\frac{k-2}{k-1}}$ then*

$$\lim_{i \to \infty} \mathbb{P}([G_i]_p \text{ is } (\delta, S_i)_k\text{-stable}) = \begin{cases} 1 & \text{if } p \geq c_1 p_{(S_i, G_i)}, \\ 0 & \text{if } p < c_2 p_{(S_i, G_i)}. \end{cases}$$

*Proof.* Define $S_i = \{(x_1, \ldots, x_k) \in G_i^k : x_1^{r_1} \ldots x_k^{r_k} = e\}$. Observe that such equation does *not* emerge from any group homomorphism, and that the group setting is general (the groups are not necessarily abelian). This system of configurations satisfies the conditions of our framework: as it was shown by Král', Serra and Vena in [20], for each $\delta > 0$, $(S_i, G_i)$ satisfies the V-property with a certain function $\gamma_{[20]}(\delta, k)$. As $|S_i| = |S_i^{(k)}|(1 + o(1))$, it is obvious that $(\gamma_{[20]}(\delta, k) - 1)|S_i| + |S_i^{(k)}| > a|S_i^{(k)}| > 0$ (for some $a$ depending on $\delta$ and on $k$), hence $\xi_{2.4}/|S_i^{(k)}| > a > 0$ is satisfied and we are under the assumptions of

Theorem 2.4 (note that it is also true that every subset of size greater than $\delta|G_i|/2$ contains a configuration in $S_i^{(k)}$).

We can easily compute the parameters $\alpha_i$. Note that as $\gcd(r_s, |G_i|) = 1$ then the function $f_r : G_i \rightarrow G_i$, with $f_{r_s}(g) = g^{r_s}$ is a bijection. Hence, when fixing a set of $s$ variables on the equation $x_1^{r_1} \cdots x_k^{r_k} = e$, then number of solutions is equal to $|G_i|^{k-i-1}$. Consequently $\alpha_i = |G_i|^{k-i-1}$ and $\alpha_i^k = |G_i|^{k-i-1}(1 + o(1))$. The last equality holds because the number of solutions with $i$ fixed components and repeated variables is $o(|G_i|^{k-i-1})$. Finally, for $i = k$ we have $\alpha_k = \alpha_k^k = 1$ (a single equation has always solutions with all different components). Indeed, these arguments hold due to the fact of dealing with a single equation. Hence, the margin for $t$ for which Theorem 2.4 applies is

$$\frac{C}{\delta} k^{-\frac{1}{k-1}} |G_i|^{\frac{1}{k-1}} \le t \le \frac{\delta}{2}|G_i|, \tag{32}$$

where $C$ is the constant stated in Theorem 2.4.

Finally, let us consider now the random counterpart. In the particular case of a single equation it is straightforward to show both the concentration and the uniformity properties, namely Definitions 3.3 and 3.6 (indeed, one can argue exactly in the same way as in the case of a single equation on the abelian setting, which is covered by the group homomorphism setting). This gives a threshold probability function equal to $p_{(S_i,G_i)} = |G_i|^{-\frac{k-2}{k-1}}$. $\quad\square$

*Remark* 8.2. As it is shown, in [20], the V-property is also satisfied for certain system of equations in non-abelian groups which are *graph representable* (see [20, Section 3]). So a similar analysis could be also done for the corresponding systems of configurations.

## 9. Further research

In this paper we have provided a wide variety of examples in which we can combine the hypergraph container technique jointly with supersaturation results arising from removal lemmas in different scenarios. Let us mention that families arising from non-linear configurations (in which a V-property also exists) can be studied as well. Some of these have been covered in [3, 8] such as the polynomial extension of Szemerédi Theorem due to Bergelson and Leibman [5] (see also [16]).

On the other side, there are configurations which still fall beyond the reach of our methods. Let us mention a couple of them. First, let us discuss an example very similar in shape to the one studied in Theorem 5.1. In [14] the authors study the maximum number of 1 in a $n \times n$ matrix with entries in $\{0, 1\}$ without certain configurations. By configuration we mean a given partial matrix with 1's and blanks as entries: there is a certain ordering in the position of the 1's. These configurations can not be in general described as given by group homomorphisms, due to the existence of an ordering. Hence, in this situation we do not have a V-property arising from our setting. Let us mention that the problem considered in [14] is a natural generalization of the Zarankievicz

problem considered in [19]: both problems coincide when the pattern is the all-ones grid, then any pattern in the matrix coincides with a copy of a complete bipartite graph in a given bipartite graph.

As a second example, in the context of groups, Solymosi in [36, Theorem 2.2] proved by means of the Triangle Removal Lemma of Ruzsa and Szemerédi [28] that for every $\delta > 0$ there is a threshold $n_0 \in \mathbb{N}$ such that if $G$ is a finite group of order $|G| \geq n_0$ then any set $B \subset G \times G$ with $|B| = \delta|G|^2$ contains three elements $(a, b)$, $(a, c)$, $(e, f)$ such that $ab = ec$ and $ac = ef$. However, the V-property for this configuration is not known.

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