SPARSE SETS, LOWNESS AND HIGHNESS*

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Abstract. We develop the notions of “generalized lowness” for sets in PH (the union of the polynomial-time hierarchy) and of “generalized highness” for arbitrary sets. Also, we develop the notions of “extended lowness” and “extended highness” for arbitrary sets. These notions extend the decomposition of NP into low sets and high sets developed by Schöning [15] and studied by Ko and Schöning [9].

We show that either every sparse set in PH is generalized high or no sparse set in PH is generalized high. Further, either every sparse set is extended high or no sparse set is extended high. In both situations, the former case corresponds to the polynomial-time hierarchy having only finitely many levels while the latter case corresponds to the polynomial-time hierarchy extending infinitely many levels.

Key words. polynomial-time hierarchy, generalized lowness and highness, extended lowness and highness, sparse sets

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Introduction. Recent studies on the structure of intractable sets have shown that the notion of NP-completeness is not compatible with the notion of being sparse unless the polynomial-time hierarchy has only finitely many levels. Berman [4] showed that tally sets cannot be $\leq_m^P$ -complete for NP unless $P = NP$. Fortune [6] showed that co-sparse sets cannot be $\leq_m^P$ -complete for NP unless $P = NP$. Berman and Hartmanis [5] had conjectured that sparse sets cannot be $\leq_m^P$ -complete for NP unless $P = NP$ and Mahaney [13] showed that this conjecture was true. Other results due to Long [10] and to Karp and Lipton [7] showed that if there exist sparse sets with certain properties of being hard or complete with respect to certain reducibilities, then the polynomial-time hierarchy “collapses,” that is, $PH = \Sigma_n^P$ for some n .

In [15] Schöning developed a new approach to such questions. He considered a decomposition of the class NP which depends on the number of distinct levels in the polynomial-time hierarchy. Call a set A in NP “low” if for some n , $\Sigma_n^P(A) \subseteq \Sigma_n^P$, and call a set B in NP “high” if for some n , $\Sigma_{n+1}^P \subseteq \Sigma_n^P(B)$. Thus, if A is low, then with respect to the operator Σ_n^P , A does not encode the power of a quantifier, but if A is high, then A does encode the power of a quantifier. It is easy to see that if there is a set in NP that is both high and low, then the polynomial-time hierarchy collapses. On the other hand, if the polynomial-time hierarchy does not collapse, then the collection of low sets and the collection of high sets are disjoint, and, furthermore, there are sets in NP that are neither high nor low. The structure of the low and high hierarchies in NP reflects the structure of those sets in $NP - P$ that are not $\leq_m^P$ -complete for NP. Ko and Schöning [9] have shown that the sets in NP with polynomial-size circuits (hence, also sparse sets in NP) are low in NP. Thus, if a set in NP with polynomial-size circuits is high, then the polynomial-time hierarchy collapses.

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The arguments of Ko and Schöning regarding lowness depend heavily on the fact that the sets under consideration are in NP. Here we consider notions of “low” and “high” that do not depend on the set being in NP. Call a set A “generalized low” if there exist i and j with $i \leq j$ such that $\Sigma_i^P(A) \subseteq \Sigma_j^P$; clearly, a generalized low set is in some level of the polynomial-time hierarchy. Every set in $\text{PH} = \bigcup_{i \geq 0} \Sigma_i^P$ is generalized low since $A \in \Sigma_i^P$ implies $\Sigma_i^P(A) \subseteq \Sigma_{2i}^P$. Call a set A “generalized high” if there exist k and l with $k > l$ such that $\Sigma_k^P \subseteq \Sigma_l^P(A)$. Notice that if A is $\leq_m^P$ -complete for Σ_j^P , then $\Sigma_{j+1}^P = \Sigma_1^P(A)$ so that A is generalized high. It is clear that if the polynomial-time hierarchy collapses, then every set in PH is generalized high since $\text{PH} = \Sigma_k^P$ implies that if $A \in \text{PH}$, then $\Sigma_{k+i}^P = \Sigma_k^P(A)$ for all $i \geq 0$. On the other hand, if the polynomial-time hierarchy does not collapse, then the structure of the class of generalized low sets in PH reflects the structure of $\text{PH} - \{L\}$ for some i , L is $\leq_m^P$ -complete for Σ_i^P . Clearly, every set in PH is both generalized low and generalized high if the polynomial-time hierarchy collapses. It is shown here that if there is a set that is simultaneously generalized low and generalized high, then the polynomial-time hierarchy collapses. Thus, either every set in PH is both generalized low and generalized high or no set in PH is both generalized low and generalized high.

In the earlier studies [9], [15] subsets of NP with certain quantitative properties were considered (e.g., sparse sets, sets with polynomial-size circuits). Thus we are led to restrict attention to sets in PH with these same quantitative properties. We are interested in finding classes $\mathcal{C}_1$ of sets in PH such that if every set in $\mathcal{C}_1$ is generalized high, then the polynomial-time hierarchy collapses, and classes $\mathcal{C}_2$ of sets in PH such that if one set in $\mathcal{C}_2$ is generalized high, then the polynomial-time hierarchy collapses (so as noted above, every set $\mathcal{C}_2$ is generalized high). Here we focus on the class of sparse sets in PH. One of our main results is that either *every* sparse set in PH is generalized high or *no* sparse set in PH is generalized high. The former case corresponds to the notion of the polynomial-time hierarchy collapsing while the latter corresponds to the notion that there are infinitely many levels in the polynomial-time hierarchy. The same result holds for the class of sets with polynomial-size circuits.

The notion of “generalized low” lifts the idea of “low” sets in NP to sets with similar properties in PH. One can lift this notion to arbitrary sets, not just those in PH, if one has a way of comparing them with respect to NP. This is done in the following way.

Call a set A “extended low” if for some n , $\Sigma_n^P(A) \subseteq \Sigma_{n-1}^P(A \oplus \text{SAT})$. Call a set A “extended high” if for some n , $\Sigma_n^P(A \oplus \text{SAT}) \subseteq \Sigma_n^P(A)$. We show that every sparse set (whether or not it is in PH) is extended low. Further, if any sparse set is extended high, then the polynomial-time hierarchy collapses. Another of our main results is that either *every* sparse set is extended high or *no* sparse set is extended high. Again, the former case corresponds to the collapse of the polynomial-time hierarchy which the latter case corresponds to that hierarchy extended to infinitely many levels.

Our purpose here is to understand the role that structural notions such as sparsity, having polynomial-size circuits, etc., have in determining the underlying structure of complexity classes, with emphasis on the properties of feasibly computable sets. We believe that our results shed new light on the role of sparse sets in determining the structure of the polynomial-time hierarchy.

2. Preliminaries. It is assumed that the reader is familiar with the basic concepts from the theories of automata, computability, and formal languages. Some of the concepts that are most important for this paper are reviewed here, and notation is established.

For a string w , $|w|$ denotes the length of w . The empty string is denoted by ϵ , $|\epsilon| = 0$.

For a set S , $\|S\|$ denotes the cardinality of S .

It is assumed that all sets of strings are taken over some fixed alphabet Σ that includes $\{0, 1\}$. If $A \subseteq \Sigma^*$, then $\bar{A} = \Sigma^* - A$.

For sets $A, B \subseteq \Sigma^*$, the *join* of A and B is defined as $A \oplus B = \{0x \mid x \in A\} \cup \{1y \mid y \in B\}$.

For a set S and an integer $n \geq 0$, let $S_{\leq n}$ denote $\{x \in S \mid |x| \leq n\}$.

Let $<$ denote any standard polynomial time computable total order defined on Σ^* . We consider $c(\cdot)$ to be an encoding function. For a finite set $S \subseteq \Sigma^*$, let $c(S)$ denote some standard encoding of S into a single string in Σ^* . We use the notation $\langle \cdot, \cdot \rangle$ to denote a polynomial-time pairing function on Σ^* and generalize it to n -tuples in the usual way. Observe that it is possible given $\langle x, c(S) \rangle$ to determine in polynomial time whether $x \in S$.

We assume standard definitions of oracle machines and relativized complexity classes specified by deterministic or nondeterministic oracle machines that are time-bounded or space-bounded. See [8], [11], [15], or [18] for details. In particular, Long [11] has studied the notion of "strong nondeterministic" reducibility. Define $A \leq_T^{\text{SN}} B$ if $A \in \text{NP}(B)$ and $\bar{A} \in \text{NP}(B)$. This is equivalent to $\text{NP}(A) \subseteq \text{NP}(B)$.

For an integer $k > 1$, a polynomial p , and a set L , we may define a set A as follows: $x \in A$ if and only if $(\exists y_1)_p \cdots (Q_k y_k)_p (\langle x, y_1, \dots, y_k \rangle \in L)$. The quantifiers are intended to alternate between existential and universal so that if k is even, then Q_k is universal, and if k is odd, then Q_k is existential. For each i , the domain of y is bounded: $|y_i| \leq p(|x|)$. Similarly, we may define a set B as follows: $x \in B$ if and only if $(\forall y_1)_p \cdots (Q_k y_k)_p (\langle x, y_1, \dots, y_k \rangle \in L)$. Now Q_k is universal if k is odd and existential if k is even.

Now we review some well-known properties of the polynomial-time hierarchy. We refer the reader to the papers of Stockmeyer [17] and Wrathall [18] where the properties of this hierarchy were first described.

DEFINITION 2.1. (a) Let A be a set. Define $\Sigma_0^P(A) = \Pi_0^P(A) = \Delta_0^P(A) = P(A)$, and for each integer $i \geq 0$, define $\Delta_{i+1}^P(A) = P(\Sigma_i^P(A))$, $\Sigma_{i+1}^P(A) = \text{NP}(\Pi_i^P(A))$, and $\Pi_{i+1}^P(A) = \text{co} - \Sigma_{i+1}^P(A)$. The structure $\{(\Delta_i^P(A), \Sigma_i^P(A), \Pi_i^P(A))\}$ is the *polynomial-time hierarchy relative to A*. Define $\text{PH}(A) = \bigcup_{i \geq 0} \Sigma_i^P(A)$.

(b) For each integer $i \geq 0$, define $\Delta_i^P = \Delta_i^P(\phi)$, $\Sigma_i^P = \Sigma_i^P(\phi)$, and $\Pi_i^P = \Pi_i^P(\phi)$. The structure $\{(\Delta_i^P, \Sigma_i^P, \Pi_i^P)\}_{i \geq 0}$ is the *polynomial-time hierarchy*. Define $\text{PH} = \bigcup_{i \geq 0} \Sigma_i^P$.

Recall that for each $k \geq 1$, a set A is in Σ_k^P if and only if there is a set $B \in P$ and a polynomial p such that for all x , $x \in A$ if and only if $(\exists y_1)_p \cdots (Q_k y_k)_p (\langle x, y_1, \dots, y_k \rangle \in B)$. A similar characterization of the classes Π_k^P , $k \geq 1$, also holds.

It is clear that for every set A and every integer $i \geq 0$, $\Delta_i^P(A) \subseteq \Sigma_i^P(A) \cap \Pi_i^P(A) \subseteq \Sigma_i^P(A) \cup \Pi_i^P(A) \subseteq \Delta_{i+1}^P(A)$. Also, for every set A and every integer $i \geq 0$, $\Delta_{i+1}^P(A) = P(\Pi_i^P(A))$ and $\Sigma_{i+1}^P(A) = \text{NP}(\Sigma_i^P(A))$. For any A and $i \geq 1$, if $\Sigma_i^P(A) = \Pi_i^P(A)$ or $\Delta_i^P(A) = \Sigma_i^P(A)$ or $\Delta_i^P(A) = \Pi_i^P(A)$, then the polynomial-time hierarchy relative to A contains only finitely many different classes.

It is known [3] that there is a set B such that $\Sigma_1^P(B) \subsetneq \Sigma_2^P(B) \subsetneq \Sigma_3^P(B)$ but it is not known whether there is any set C such that the polynomial-time hierarchy relative to C is infinite.

Clearly, for every set A , $\text{PH}(A) = \bigcup_{i \geq 0} \Pi_i^P(A)$ and $\text{PH}(A) = \bigcup_{i \geq 0} \Delta_i^P(A)$. For the most part the definition of $\text{PH}(A)$ as $\bigcup_{i \geq 0} \Sigma_i^P(A)$ is the most useful here.

3. The decomposition of PH. There are a number of results that give conditions on NP under which the polynomial-time hierarchy collapses to Σ_n^P for some n , that

is, $PH = \Sigma_n^P$. For example, Mahaney [13] has shown that $P = NP$ if and only if there is a sparse set S that is $\leq_m^P$ -complete for NP , and Long [10] has shown that if there is a sparse set S such that $NP \subseteq P(S)$ and S is in Δ_2^P , then $PH = \Delta_2^P$. Schöning [15] developed a decomposition theory for NP that generalizes many of these results.

DEFINITION 3.1 [15]. For each integer $n > 0$, let $L_n = \{A \in NP \mid \Sigma_n^P(A) \subseteq \Sigma_n^P\}$ and let $H_n = \{A \in NP \mid \Sigma_{n+1}^P \subseteq \Sigma_n^P(A)\}$. Let $LH = \bigcup_{n>0} L_n$ and $HH = \bigcup_{n>0} H_n$.

PROPOSITION 3.2 [15]. (a) *The polynomial-time hierarchy has only finitely many levels if and only if $LH \cap HH \neq \emptyset$. Also, $LH \cap HH \neq \emptyset$ if and only if for some $n > 0$, $L_n = H_n = LH = HH = NP$.*

(b) *The polynomial-time hierarchy has infinitely many levels if and only if $NP - (LH \cup HH) \neq \emptyset$.*

One of the contributions of the present paper is to extend the techniques of [15] to develop a decomposition of the class PH that is similar to the decomposition of NP . In this section we describe this decomposition. In § 4 we study some of the conditions that cause the polynomial-time hierarchy to collapse, conditions that can be defined on classes in this decomposition.

DEFINITION 3.3. For every $i, j > 0$, define $L(i, j) = \{A \mid \Sigma_i^P(A) \subseteq \Sigma_j^P\}$. For every $k, l > 0$, define $H(k, l) = \{A \mid \Sigma_k^P \subseteq \Sigma_l^P(A)\}$.

Notice that for each $n > 0$, $L_n = L(n, n) \cap NP$ and $H_n = H(n+1, n) \cap NP$. We call sets "low" if they lie in $LH = \bigcup_{n>0} L_n$ and "high" if they lie in $HH = \bigcup_{n>0} H_n$, and we use the term "generalized lowness" to refer to properties of sets in $\bigcup_{i,j} L(i, j)$ and the term "generalized highness" to refer to properties of sets in $\bigcup_{k,l} H(k, l)$.

Consider generalized lowness. Clearly, if $A \in L(i, j)$, then $A \in \Sigma_j^P$ so that $\bigcup_{i,j} L(i, j) \subseteq PH$. On the other hand, it makes no sense to take $i > j$ when considering $L(i, j)$ since for $i > j$, $L(i, j) = \emptyset$ if $\Sigma_j^P \neq \Sigma_{j+1}^P$ and $L(i, j) = PH$ if $\Sigma_j^P = \Sigma_{j+1}^P$. Thus, we consider classes $L(i, j)$ only when $i \leq j$.

Which lowness properties are significant? Let $A \in \Sigma_n^P$. Then for any i , $\Sigma_i^P(A) \subseteq \Sigma_i^P(\Sigma_n^P) = \Sigma_{i+n}^P$ so that if $j \geq i + n$, we have $A \in L(i, j)$ trivially. Thus, if $A \in \Sigma_n^P$, then the lowness property $A \in L(i, j)$ has significance only if $i \leq j < i + n$.

There are some obvious inclusion relations.

PROPOSITION 3.4. *For all i, j , $L(i, j) \subseteq L(i+1, j+1) \cap L(i, j+1) \cap L(i-1, j)$.*

Consider generalized highness. Clearly, if $k \leq l$, then every set A satisfies $\Sigma_k^P \subseteq \Sigma_l^P(A)$ so that $H(k, l) = 2^{\Sigma^*}$. Thus, we consider classes $H(k, l)$ only when $k > l$.

Which highness properties are significant? If A is $\leq_T^P$ -complete for Σ_n^P , then $\Sigma_l^P(A) = \Sigma_{l+n}^P$ so that for $k > l + n$, $A \in H(k, l)$ implies that the polynomial-time hierarchy collapses. More precisely, if $k > l + n$, then $\Sigma_n^P \cap H(k, l) = \emptyset$ if $\Sigma_{l+n}^P \neq \Sigma_{l+n+1}^P$ and $H(k, l) = H(l+n, l)$ if $\Sigma_{l+n}^P = \Sigma_{l+n+1}^P$. Thus, if $A \in \Sigma_n^P$, then the highness property $A \in H(k, l)$ has significance only if $l < k \leq l + n$.

There are some obvious inclusion relations.

PROPOSITION 3.5. *For all i, j , $H(i, j) \subseteq H(i+1, j+1) \cap H(i, j+1) \cap H(i-1, j)$.*

There are some characterizations of certain of these classes. The proofs are left to the reader.

PROPOSITION 3.6. (a) *For every j , $L(1, j) = \Sigma_j^P \cap \Pi_j^P$.*

(b) *For every k , $H(k+1, 1) = \{A \mid A \text{ is } \leq_T^{SN} \text{-hard for } \Sigma_k^P\}$.*

Schöning [15] showed that the polynomial-time hierarchy collapses if and only if there exist m and n such that $L_m \cap H_n \neq \emptyset$. This is a consequence of the following fact.

THEOREM 3.7. *If $L(i, j) \cap H(k, l) \neq \emptyset$ and $j + l < i + k$, then the polynomial-time hierarchy collapses to $\Sigma_{\max\{i, l\} + j - i}^P$.*

Proof. Let $A \in L(i, j) \cap H(k, l)$. Since $A \in L(i, j)$, $\Sigma_i^P(A) \subseteq \Sigma_j^P$ so that $\Sigma_{\max\{i, l\}}^P(A) \subseteq \Sigma_{\max\{i, l\}+j-i}^P$. Since $A \in H(k, l)$, $\Sigma_k^P \subseteq \Sigma_l^P(A)$ so that $\Sigma_{k+\max\{i, l\}-l}^P \subseteq \Sigma_{\max\{i, l\}}^P(A)$. Thus, $\Sigma_{k+\max\{i, l\}-l}^P \subseteq \Sigma_{\max\{i, l\}+j-i}^P$ which implies that $\text{PH} = \Sigma_{\max\{i, l\}+j-i}^P$ provided that $k-l > j-i$ or, equivalently, $j+l < i+k$. $\square$

Consider the proof of Theorem 3.7. The crucial parameter in the argument is the difference $j-i$ or the difference $k-l$. One might define a set A to be n -low if there exist i and j such that both $n = j-i$ and also $A \in L(i, j)$. Similarly, one might define a set A to be n -high if there exist k and l such that both $n = k-l$ and also $A \in H(k, l)$. Then Theorem 3.7 can be restated as follows: For every $n \geq 0$, no set can be both $(n+1)$ -high and also n -low unless the polynomial-time hierarchy collapses.

The notion of " A is n -low" may be interpreted as setting an upper bound on the amount of information that can be encoded in A : the set A has the power of at most n alternating quantifiers or of the composition of at most n applications of the $\text{NP}(\)$ operator. Similarly, the notion of " A is n -high" can be interpreted as setting a lower bound on the amount of information that can be encoded in A : the set A has the power of at least n alternating quantifiers, or of the composition of at least n applications of the $\text{NP}(\)$ operator. Thus, Theorem 3.7 can be interpreted as saying that no set in PH can have the power of at most n alternating quantifiers and also have the power of at least $n+1$ alternating quantifiers (unless PH collapses to level n , i.e., $\text{PH} = \Sigma_n^P$).

4. Generalized and extended lowness and sparse oracles. The property of being generalized low asserts that the usefulness of a set as an oracle set is quite restricted. If $A \in L(i, j)$ where $i \leq j$, then A does not have the power of more than $j-i$ alternating quantifiers since $\Sigma_i^P(A) \subseteq \Sigma_j^P$. What type of sets has this property? Ko and Schöning [9] studied sets in NP that have this property. As noted in § 3, any generalized low set is in PH. We want to exhibit sets in PH that have this property.

Consider tally sets in the polynomial-time hierarchy. Notice that for any tally set T there is a polynomial-time oracle machine that on input 0^n will enumerate relative to T the set of strings in T of length at most n . Suppose that T is a tally set in Σ_k^P . By first enumerating the appropriate initial segment of T and then simulating a Σ_{k+1}^P oracle machine, one can show that $\Sigma_{k+1}^P(T) = \Sigma_{k+1}^P$. Hence, for every k , if T is a tally set in Σ_k^P , then $T \in L(k+1, k+1)$.

In a similar way (e.g., see [10]) it can be shown that if S is a sparse set and $S \in \Sigma_k^P$, then $S \in L(k+1, k+1)$. We can show that if S has polynomial-size circuits and $S \in \Sigma_k^P$, then $S \in L(k+2, k+2)$. However we will establish a more general theorem.

THEOREM 4.1. *For any set A , if there is a sparse set S and integers n, k with $n > k$ such that $A \in \Sigma_k^P(S)$ and $A \in \Sigma_n^P$, then A is in $L(n-k+2, n+2)$.*

Proof. Since $A \in \Sigma_k^P(S)$, there are a polynomial p and a deterministic polynomial time-bounded oracle machine M_1 such that for all inputs $x, x \in A$ if and only if $(\exists y_1)_p \cdots (Q_k y_k)_p \langle x, y_1, \dots, y_k \rangle \in L(M_1, S)$. Let $B = \{ \langle x, c(T) \rangle \mid T \text{ is a finite set and } (\exists y_1)_p \cdots (Q_k y_k)_p \langle x, y_1, \dots, y_k \rangle \in L(M_1, T) \}$. Clearly, B is in Σ_k^P . Since S is sparse, there is a polynomial q such that for all n , $\|S_{\leq n}\| \leq q(n)$.

Let $i > 0$ and consider any $C \in \Sigma_i^P(A)$. There are a polynomial r and a deterministic polynomial time-bounded oracle machine M_2 such that for all $u, u \in C$ if and only if $(\exists z_1)_r \cdots (Q_i z_i)_r \langle u, z_1, \dots, z_i \rangle \in L(M_2, A)$. There is a polynomial s (depending on p and the time bound of M_1) such that for all $u, u \in C$ if and only if there is a finite set T whose elements are of size at most $s(|u|)$ and $\|T\| \leq q(s(|u|))$ such that (i) $\langle x, c(T) \rangle \in B \Leftrightarrow x \in A$, and (ii) $(\exists z_1)_r \cdots (Q_i z_i)_r \langle u, z_1, \dots, z_i \rangle \in L(M_2, \{w \mid \langle w, c(T) \rangle \in B\})$. The predicate described by (i) is in Π_{n+1}^P because $A \in \Sigma_n^P$, $B \in \Sigma_k^P$ and $k < n$. The predicate described by (ii) is in $\Sigma_i^P(B) \subseteq \Sigma_i^P(\Sigma_k^P) = \Sigma_{i+k}^P$.

Thus, membership in C is described by a predicate of the form $\exists(\Pi_{n+1}^P \wedge \Sigma_{i+k}^P)$ where the existential quantifier is polynomially bounded so that C is in $\Sigma_{\max(n+2, i+k)}^P$. Hence, C is in Σ_{n+2}^P if we choose $i = n - k + 2$. Hence we have shown that $\Sigma_{n-k+2}^P(A) \subseteq \Sigma_{n+2}^P$ so that $A \in L(n - k + 2, n + 2)$. $\square$

Observe that $A \in L(n - k + 2, n + 2)$ for a set $A \in \Sigma_n^P$ is a "significant" lowness condition as described in § 3 since $n - k + 2 \leq n + 2 < 2n - k + 2$ since $k < n$. Also, notice that in the case $k = 0$ and $u = 1$, we have $A \in \text{NP}$ and $A \in \text{P}(S)$ so that $A \in L(3, 3)$. Since $\text{NP} \cap L(3, 3) = L_3$, we see that sets in NP with polynomial-size circuits are in L_3 . This was first proved by Ko and Schöning [9].

COROLLARY 4.2. *Let A be a set that is $\leq_T^{\text{SN}}$ -complete for Σ_n^P . Suppose that for some sparse set S and some $k < n$, $A \in \Sigma_k^P(S)$. Then A is in $L(n - k + 2, n + 2)$ and the polynomial-time hierarchy collapses to Σ_{n+2}^P .*

Proof. Since A is $\leq_T^{\text{SN}}$ -complete for Σ_n^P , $\Sigma_{n+1}^P = \Sigma_1^P(A)$ so that $A \in H(n + 1, 1)$. Also, A is in $\Sigma_n^P \cap \Sigma_k^P(S)$ and $k < n$ so that A is in $L(n - k + 2, n + 2)$ by Theorem 4.1. Thus, $A \in L(n - k + 2, n + 2) \cap H(n + 1, 1)$ so that the polynomial-time hierarchy collapses to Σ_{n+2}^P by Theorem 3.7. $\square$

Notice that in both Theorem 4.1 and Corollary 4.2, the condition that S be sparse could be replaced by the condition that S have polynomial-size circuits and the conclusions will still hold. A set that is in a certain level of the polynomial-time hierarchy relativized to a sparse set might be called "pseudo-sparse." This notion encompasses such properties as a set being in "almost polynomial time" (APT) [14], a set having Σ_k^P -circuits [9], a set having small generators [16], [19], and other properties that can be translated to being recognized relative to a sparse set, that is, properties that can be expressed by membership in a set in $\cup \{\text{PH}(S) \mid S \text{ is sparse}\}$.

From one viewpoint the notion of generalized lowness should not force sets to be in PH but rather should apply to all sets with some particular property, e.g., to all sparse sets. This requires a change in the definitions. One way to accomplish this is as follows.

DEFINITION 4.3. For each $n > 0$, let $\text{EL}_n = \{A \mid \Sigma_n^P(A) \subseteq \Sigma_{n-1}^P(A \oplus \text{SAT})\}$ and let $\text{EH}_n = \{A \mid \Sigma_n^P(A \oplus \text{SAT}) \subseteq \Sigma_n^P(A)\}$. Let $\text{EL} = \cup_n \text{EL}_n$ and $\text{EH} = \cup_n \text{EH}_n$. A set is *extended low* if it is in EL and is *extended high* if it is in EH.

Let K be any set that is $\leq_m^P$ -complete for PSPACE. Clearly, for all n , $\Sigma_n^P(K) = \Sigma_{n-1}^P(K \oplus \text{SAT}) = \Sigma_n^P(K \oplus \text{SAT}) = \text{PSPACE}$. Thus, K is in $\text{EL} \cap \text{EH}$, so EL and EH are not disjoint. However, it is easy to see that $\text{EL} \cap \text{EH} \cap \text{PH}$ is empty if and only if the polynomial-time hierarchy is infinite.

THEOREM 4.4. *If S is sparse, then $S \in \text{EL}_3$.*

Proof. Let $B \in \Sigma_3^P(S)$. We must show that $B \in \Sigma_2^P(S \oplus \text{SAT})$. Since $B \in \Sigma_3^P(S)$, there is a polynomial p and a deterministic polynomial time-bounded oracle machine M such that for all strings x , $x \in B$ if and only if $(\exists y_1)_p (\forall y_2)_p (\exists y_3)_p \langle x, y_1, y_2, y_3 \rangle \in L(M, S)$. Since S is sparse, there is a polynomial q such that for all n , $\|S_{\leq n}\| \leq q(n)$. Thus, there is a polynomial r such that for all strings x , $x \in B$ if and only if there exists a set T of size at most $r(|x|)$ such that (i) for all u with $|u| \leq r(|x|)$, $u \in T$ if and only if $u \in S$, and (ii) $(\exists y_1)_r (\forall y_2)_r (\exists y_3)_r \langle x, y_1, y_2, y_3 \rangle \in L(M, T)$. The predicate described by (i) is in $\Pi_1^P(S)$ and the predicate described by (ii) is in $\Sigma_3^P = \Sigma_2^P(\text{SAT})$. Thus, membership in B is described by a predicate of the form $\exists(\Pi_1^P(S) \wedge \Sigma_2^P(\text{SAT}))$ so that B is in $\Sigma_2^P(S \oplus \text{SAT})$. Hence, $\Sigma_3^P(S) \subseteq \Sigma_2^P(S \oplus \text{SAT})$ so $S \in \text{EL}_3$. $\square$

Recall that a set A has polynomial-size circuits if and only if there is a sparse set S such that $A \in \text{P}(S)$. The proof of Theorem 4.4 can be trivially modified to yield the following fact.

COROLLARY 4.5. *If A has polynomial-size circuits, then A is in EL_3 .*

The notions of extended lowness and extended highness appear to be natural extensions of the basic idea of Schöning that was developed in [15]. It is easy to see that for every n , $EL_n \cap NP = L_n$ and $EH_n \cap NP = H_n$.

5. Generalized highness and sparse sets. The property of being generalized high asserts that using such a set as an oracle set is an advantage. If $A \in H(k, l)$ where $k > l$, then A has the power of at least $k - l$ alternating quantifiers since $\Sigma_k^P \subseteq \Sigma_l^P(A)$. What type of sets have this property? Schöning [15] showed that certain types of complete sets for NP have this property and it is easy to see that similar sets in each of the classes Σ_i^P , $i > 0$, have this property.

It is known [10], [13] that if a sparse set is hard for NP with respect to $\leq_T^P$ or even weaker reducibilities such as $\leq_T^{SN}$, then the polynomial-time hierarchy collapses. Being hard for any of the classes Σ_i^P , $i > 0$, represents a certain "highness." Thus, we consider the situation in which a sparse set is generalized high. We need the following notion.

DEFINITION 5.1. A set A is *self-reducible* if there exists a deterministic polynomial time-bounded oracle machine M such that (i) on input of size n , M queries the oracle only about strings of length at most $n - 1$ and (ii) $L(M, A) = A$.

This definition captures the essential idea of the seemingly more general notions due to Ko [8] and Meyer and Paterson [14]. Notice that there are $\geq_m^P$ -complete sets for NP that are self-reducible, and that at each stage Σ_k^P of the polynomial-time hierarchy, the $\leq_m^P$ -complete set for Σ_k^P described by Wrathall [18] is self-reducible.

Condition (ii) of the definition of self-reducible set may be interpreted as saying that the set is the unique fixed point specified by the oracle machine. We say this formally as follows.

LEMMA 5.2. Let A be a self-reducible set and let M witness A 's self-reducibility. For any set B and any n , if $L(M, B) \subseteq B_{\leq n}$, then $A_{\leq n} = B_{\leq n}$.

Now we have our results.

LEMMA 5.3. Let A be a self-reducible set. Suppose that for some $k \geq 0$ and some sparse set S , $A \in \Sigma_k^P(S)$. Then $A \in L(2, k + 2)$, that is, $\Sigma_2^P(A) \subseteq \Sigma_{k+2}^P$.

Proof. Let M_1 witness A 's reducibility so that M_1 is a deterministic polynomial time-bounded oracle machine. Since $A \in \Sigma_k^P(S)$ there are a polynomial p and a deterministic polynomial time-bounded oracle machine M_2 such that for all x , $x \in A$ if and only if $(\exists y_1)_p \cdots (Qy_k)_p \langle x, y_1, \dots, y_k \rangle \in L(M_2, S)$. For each finite set T , define the set A_T as follows: $A_T = \{x | (\exists y_1)_p \cdots (Qy_k)_p \langle x, y_1, \dots, y_k \rangle \in L(M_2, T)\}$. Then for each T , $A_T \in \Sigma_k^P(T) = \Sigma_k^P$.

Let L be an arbitrary set in $\Sigma_2^P(A)$. Then there are a polynomial q and a deterministic polynomial time-bounded machine M_3 such that for all x , $x \in L$ if and only if $(\exists v)_q (\forall w)_q \langle x, v, w \rangle \in L(M_3, A)$.

Now consider membership in L . For some polynomial r and all x , $x \in L$ if and only if there is a set T of size at most $r(|x|)$ such that

- (i) for all u , $u \in L(M_1, A_T)$ if and only if $u \in A_T$, and
- (ii) $(\exists v)_q (\forall w)_q \langle x, v, w \rangle \in L(M_3, A_T)$.

The predicate described by (i) implies that M_3 accepts a suitably small input string relative to A_T if and only if it accepts that input string relative to A ; this follows from Lemma 5.2. Now the predicate described by (i) is in Π_{k+1}^P and the predicate described by (ii) is in Σ_{k+2}^P since $A_T \in \Sigma_k^P$, so that membership in L is described by a predicate of the form $\exists(\Pi_{k+1}^P \wedge \Sigma_{k+2}^P)$. Thus, L is in Σ_{k+2}^P .

Since L was chosen arbitrarily in $\Sigma_2^P(A)$, this yields $\Sigma_2^P(A) \subseteq \Sigma_{k+2}^P$ as desired. $\square$

THEOREM 5.4. *If there exists a sparse set that is generalized high, then the polynomial-time hierarchy collapses. That is, if there is a sparse set S such that for some k, l with $k > l$, $S \in H(k, l)$, then $\Sigma_{k+2}^P = \Pi_{k+2}^P = \text{PH}$.*

Proof. Suppose that $\Sigma_k^P \subseteq \Sigma_l^P(S)$ where S is sparse and $k > l$. Let A be a set that is both self-reducible and also $\leq_m^P$ -complete for Σ_k^P . Since A is self-reducible and $A \in \Sigma_k^P \subseteq \Sigma_l^P(S)$, we have $\Sigma_2^P(A) \subseteq \Sigma_{l+2}^P$ by Lemma 5.3. Since A is $\leq_m^P$ -complete for Σ_k^P , $\Sigma_2^P(A) \subseteq \Sigma_{l+2}^P$ implies $\Sigma_{k+2}^P \subseteq \Sigma_{l+2}^P$. Since $k+2 > l+2$, $\Pi_{l+2}^P \subseteq \Sigma_{k+2}^P$ so that $\Sigma_{k+2}^P \subseteq \Sigma_{l+2}^P$ implies $\Pi_{l+2}^P \subseteq \Sigma_{l+2}^P$. Hence, $\text{PH} = \Sigma_{l+2}^P = \Pi_{l+2}^P$. $\square$

THEOREM 5.5. *If there exists a sparse set S such that S is extended high, then the polynomial-time hierarchy collapses.*

Proof. Let S be sparse. If S is in EH_n , then $\Sigma_n^P(S \oplus \text{SAT}) \subseteq \Sigma_n(S)$. Let A be any set that is $\leq_m^P$ -complete for Σ_{n+1}^P and is self-reducible. Thus, $\Sigma_2^P(A) = \Sigma_{n+3}^P$. Now $A \in \Sigma_{n+1}^P \subseteq \Sigma_n^P(S \oplus \text{SAT}) \subseteq \Sigma_n^P(S)$ so by Lemma 5.3, $\Sigma_2^P(A) \subseteq \Sigma_{n+2}^P$. Hence, $\Sigma_{n+3}^P \subseteq \Sigma_{n+2}^P$ so $\Sigma_{n+3}^P = \Sigma_{n+2}^P = \text{PH}$. $\square$

Clearly, the notion of "sparse set" can be replaced by the notion of "set with polynomial-size circuit" in both Theorem 5.4 and Theorem 5.5 and the conclusions will not change.

Theorem 5.4 generalizes a number of results that assert conditions that force the polynomial-time hierarchy to collapse. Some of these conditions are as follows:

- (a) there is a sparse set S such that $\text{SAT} \leq_m^P S$ [13];
- (b) there is a sparse or co-sparse set S that is $\leq_T^P$ -complete for NP [10], [13];
- (c) NP has polynomial-size circuits [7];
- (d) $\text{NP} \subseteq \text{APT}$ [14];
- (e) $\text{NP} = \text{R}$ [1];
- (f) there are a sparse set S and an integer k such that $\text{PH} \subseteq \Sigma_k^P(S)$;
- (g) NP is p -selective [8];
- (h) $\text{NP} \subseteq \text{BPP}$ [1].

Theorem 5.4 generalizes results of Karp and Lipton [7] and of Yap [19]. The interested reader might compare the proofs of Lemma 5.3 and Theorem 5.4 with those in [7] and [19].

6. Main result. In Theorem 5.4 we showed that the existence of a sparse set that is generalized high causes the polynomial-time hierarchy to collapse. Clearly, if the polynomial-time hierarchy does collapse, then every sparse set in the polynomial-time hierarchy is " n -high" for sufficiently large n and so is generalized high. Thus, we have the first of our two main theorems.

THEOREM 6.1. *Either every sparse set in PH is generalized high or no sparse set in PH is generalized high.*

In Theorem 5.5 we showed that the existence of a sparse set that is extended high causes the polynomial-time hierarchy to collapse. But it is conceivable that one sparse set S_1 might be extended high while another sparse set S_2 is not extended high. However, this cannot be the case.

PROPOSITION 6.2 [12]. *If the polynomial-time hierarchy collapses, then for every sparse set S , the polynomial-time hierarchy relative to S collapses. That is, if for some $k \geq 2$, $\Sigma_k^P = \Pi_k^P$, then for every sparse set S , $\Sigma_k^P(S) = \Pi_k^P(S)$.*

Proof. Let S be sparse. Suppose that for some $k \geq 2$, $\Sigma_k^P = \Pi_k^P$. It suffices to show that if $A \in \Pi_k^P(S)$, then $A \in \Sigma_k^P(S)$. Since $A \in \Pi_k^P(S)$, there is a polynomial p and a deterministic polynomial time-bounded oracle machine M such that for every x , $x \in A$ if and only if $(\forall y_1)_p (\exists y_2)_p \cdots (Q_k y_k)_p \langle x, y_1, \dots, y_k \rangle \in L(M, S)$. Hence, $x \in A$ if and only if $(\forall y_1)_p (\exists y_2)_p \cdots (Q_k y_k)_p \langle x, y_1, \dots, y_k \rangle \in L(M, S_{\leq q(|x|)})$ for some appropriate

polynomial q . Let

$$B = \{(x, c(T)) \mid T \text{ is a finite set and}$$

$$(\forall y_1)_p (\exists y_2)_p \cdots (Q_k y_k)_p \langle x, y_1, y_2, \dots, y_k \rangle \in L(M, T)\}.$$

From the quantifier characterization of PH, it is clear that $B \in \Pi_k^P = \Sigma_k^P = \Sigma_k^P$. Using B , membership in A can be expressed as follows: $x \in A$ if and only if there exists a finite set T such that (i) $(\forall u)_q [u \in T \text{ if and only if } u \in S]$ and (ii) $\langle x, c(T) \rangle \in B$. This predicate has the form $\exists (C \wedge B)$ where $C \in \Pi_1^P(S)$ and $B \in \Pi_k^P = \Sigma_k^P$. Since $k \geq 2$, this yields $A \in \Sigma_k^P(S)$. $\square$

Thus, if the polynomial-time hierarchy collapses, then for every sparse set S there is an integer $k > 0$ such that $\text{PH}(S) = \Sigma_k^P(S)$. Since $\text{NP} \subseteq \text{PH} \subseteq \text{PH}(S)$, this means that $\Sigma_k^P(S \oplus \text{SAT}) = \Sigma_k^P(S)$ so that S is extended high. Thus, we have our other result.

THEOREM 6.3. *Either every sparse set is extended high or no sparse set is extended high.*

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