Evolution and dynamics of a matter creation model

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ABSTRACT

In a flat Friedmann–Lemaître–Robertson–Walker (FLRW) geometry, we consider the expansion of the universe powered by the gravitationally induced 'adiabatic' matter creation. To demonstrate how matter creation works well with the expanding universe, we have considered a general creation rate and analysed this rate in the framework of dynamical analysis. The dynamical analysis hints the presence of a non-singular universe (without the big bang singularity) with two successive accelerated phases, one at the very early phase of the universe (i.e. inflation), and the other one describes the current accelerating universe, where this early, late accelerated phases are associated with an unstable fixed point (i.e. repeller) and a stable fixed point (attractor), respectively. We have described this phenomena by analytic solutions of the Hubble function and the scale factor of the FLRW universe. Using Jacobi last multiplier method, we have found a Lagrangian for this matter creation rate describing this scenario of the universe. To match with our early physics results, we introduce an equivalent dynamics driven by a single scalar field, discuss the associated observable parameters and compare them with the latest *Planck* data sets. Finally, introducing the teleparallel modified gravity, we have established an equivalent gravitational theory in the framework of matter creation.

Key words: cosmological parameters – dark energy – early Universe – inflation.

1 INTRODUCTION

No doubt, cosmology is one of the biggest and fascinating topics in science. However, at the late 1990s, a dramatic change appeared in its history when it was discovered that the universe is going through a phase of accelerated expansion (Riess et al. 1998; Perlmutter et al. 1999). After that, several independent observations (de Bernardis et al. 2000; Percival et al. 2001; Spergel et al. 2003, 2007; Tegmark et al. 2004; Eisenstein et al. 2005; Komatsu et al. 2011) confirmed this accelerating expansion. As a result, comprehending this late-accelerating phase has become an attracting research field in modern cosmology since the end of 1990s. There are mainly two distinct approaches we use in order to describe this accelerating phase. First of all, if we consider that gravity is correctly described by Einstein's theory, then there must have some matter component with large negative pressure entitled 'dark energy' with equation of state (EoS) '$w < -1/3$', in order to start this acceleration. As a result, cosmologists brought back the presence of a non-zero cosmological constant Λ (EoS: $w = -1$) which fuels this current acceleration. Subsequently, 'Λ cold dark matter' (ΛCDM) was proposed to describe the current accelerating phase, and it was

found that the model agrees with a large number of astronomical data. However, Λ-cosmology has two fundamental problems. Observations demand that a very small energy density of Λ is enough to power this accelerating universe, whereas the predictions from quantum theory of fields claim that its energy density should be so large, leading to a discrepancy between them of the order of 10^{121}. This is known as cosmological constant problem (Weinberg 1989). On the other hand, it is not understandable 'why did our Universe begin to accelerate just now ($z \sim 1$) where both the matter and the cosmological constant evolve differently with the evolution of the universe' – known as the cosmic coincidence problem (Zlatev, Wang & Steinhardt 1999). As a result, some alternatives to ΛCDM were proposed, such as quintessence, K-essence, phantom, tachyons and others (for a review of dark energy candidates, see Copeland, Sami & Tsujikawa 2006; Amendola & Tsujikawa 2010). Also, it has been argued that modifications in the Einstein gravity can describe the current acceleration (the models are sometimes called as geometrical dark energy; Nojiri & Odintsov 2007; De Felice & Tsujikawa 2010; Sotiriou & Faraoni 2010).

However, besides these two distinct approaches, very recently, another alternative to describe the current accelerating universe has attracted special attention. The approach is the gravitationally induced 'adiabatic' matter creation, a non-equilibrium thermodynamical process. Long time ago, during 1960–1980, Parker and his collaborators (Parker 1968, 1969, 1970; Ford & Parker 1977; Birrell

& Davies 1980, Birrell & Davies 1982), and in Russia Zeldovich and others (Zeldovich & Starobinsky 1972, 1977; Grib, Levitskii & Mostepanenko 1974; Grib, Mamaev & Mostepanenko 1976; Grib, Mamaev & Mostepanenko 1994), were investigating on the material content of the universe. Following Schrodinger's ideas presented in Schrodinger (1939), they proposed that, as the universe is expanding, the gravitational field of this expanding universe is acting on the quantum vacuum, which results in a continuous creation of radiation and matter particles, and the produced particles have their mass, momentum and the energy. The idea was really fascinating, and even today it is, as we do not know how the universe came into its present position after qualifying its previous stages. On the other hand, while dealing with this matter creation process, there is another point which we need to address. It is a real mystery that the ratio of baryon to entropy in our Universe is approximately 9.2×10^{-11} (Oikonomou 2016), and it still remains an unsolved problem why this baryon-to-entropy ratio exists in our Universe. However, we have an answer from the Sakharov criterion (Sakharov 1967), which states that the baryon asymmetry in our Universe can occur if the thermodynamical processes in our expanding universe are non-equilibrium in nature, that means matter creation can take place. So, it is fine that we have strong motivation behind the material content of our Universe. Now, the main question is how the particle productions play an effective role in the evolution of the universe. It was Prigogine and his group (Prigogine et al. 1989) who thought that, since the Einstein's field equations are the background equations to understand the evolution of the universe, there must be some way out to calculate the evolution equations. And hence the conservation equation gets modified as

$$N^{\mu}_{;\mu} \equiv n_{,\mu}u^{\mu} + \Theta n = n\Gamma \Longleftrightarrow N_{,\mu}u^{\mu} = \Gamma N, \qquad (1)$$

where Γ stands for the rate of change of the particle number in a physical volume V containing N number of particles, $N^{\mu} = nu^{\mu}$ represents particle flow vector, u^{μ} is the usual particle four velocity, $n = N/V$ is the particle number density and $\Theta = u^{\mu}_{;\mu}$ denotes the fluid expansion. The new quantity Γ has a special meaning. It is the rate of the produced particles, and the most interesting thing is that it is completely unknown to us. But, we have one constraint over Γ, which comes from the validity of the generalized second law of thermodynamics leading to $\Gamma \geq 0$. However, still we have an open question about the nature of created particles by this gravitational field. One may ask what kind of particles are created by the gravitational field and what are their physical properties. We cannot properly say, but there are some justifications over this puzzle. It has been shown that the kind of particles created by this process are much limited by the local gravity constraints (Ellis et al. 1989; Hagiwara et al. 2002; Peebles & Ratra 2003), and practically radiation has no effect or impact on the late-time accelerated expansion of the universe, whereas dark matter is one of the dominant sources after the unknown 'dark energy' component. In what follows, we may assume that the produced particles by this gravitational field are simply the cold dark matter particles. Following this motivation, it has been argued that the models for different particle creation rates can mimic ΛCDM cosmology (Steigman, Santos & Lima 2009; Lima, Jesus & Oliveira 2010; Fabris, Pacheco & Piattella 2014; Lima et al. 2014; Chakraborty, Pan & Saha 2015). In particular, the constant matter creation rate can explain the big bang singularity, as well as intermediate phases ending at the final de Sitter regime (Haro & Pan 2015). Further, recently, Nunes and Pavón (2015) showed that the matter creation models can explain the phantom behaviour of our Universe (Xia, Li & Zhang 2013; Cheng & Huang 2014; Planck Collaboration XVI 2014b; Rest et al. 2014; Shafer

& Huterer 2014) without invoking any phantom fields (Caldwell 2002). Subsequently, the cosmological consequences of the matter creation models realizing this phantom behaviour have been investigated (Nunes & Pan 2016). Moreover, particle productions in modified gravity theories have attracted several authors at recent time, for instance, through a non-minimal curvature-matter coupling in modified theories of gravity theories, particle productions by the gravitational field have been discussed (Harko et al. 2015). Also, in the context of $f(R)$ gravity, the aspects of particle productions have been investigated (Capozziello, Luongo & Paolella 2016).

On the other hand, particle production scenario took a novel attempt in order to explain the early accelerated expansion (known as inflation). In the background of the particle creation process, using the energy-momentum tensor of the created particles and their creation rate (Zeldovich & Starobinsky 1972, 1977), inflation as a result of this phenomenon was first investigated in Gurovich & Starobinsky (1979). However, it appeared that such a model with a small number of non-conformal fields cannot produce a sufficiently low curvature during inflation and a graceful exit from it. Soon after that, a viable inflationary model was proposed in Starobinsky (1980), where dissipation and creation of particles occurred just after the end of inflation. However, in the same context, it has been discussed earlier that the particle creation of light non-minimally coupled scalar fields due to the changing geometry of a space–time could drive the early inflationary phase (Sahni & Habib 1998). Also, quantum particle productions in Einstein–Cartan–Sciama–Kibble theory of gravity could also result in an inflationary scenario (Desai & Popławski 2016). Furthermore, very recently, a connection between early and late accelerated universes by the mechanism of particle productions has been pointed out by Nunes (2016).

In the present work, we have considered a generalized matter creation model in order to produce a clear image about the matter creation models as a third alternative for current accelerating universe aiming to realize the early physics and its compatibility with the current astronomical data, as well as the stability of the matter creation models. Hence, we explicitly wrote down the Friedmann and Raychaudhuri equations in the framework of matter creation. The field equations form an autonomous system of differential equations, where the Friedmann equation constrains the dynamics of the universe and the Raychaudhuri equation essentially describes its evolution. Now, considering the Raychaudhuri equation for the matter creation model, we have found the fixed points of the model which are the functions of the model parameters. As the model parameters are simply real numbers, so we have divided the whole phase space into several sub-phase spaces, which opens some new possibilities to understand the possible dynamics of the universe with respect to the behaviour of the fixed points. The fixed points analysis provides a non-singular model of our Universe with two successive accelerating phases, one at very early evolution of the universe which is unstable in nature, and the other one is the present accelerating phase which is stable in nature. We have presented an analytic description for this said evolution of the universe. Further, we apply the Jacobi last multiplier method in matter creation which eventually provides an equivalent Lagrangian for this creation mechanism. Moreover, as we are also interested to investigate the early physics scenario extracted from matter creation models, so we introduced a scalar field dynamics, where we found that it is possible to find an analytic scalar field solution mimicking the evolution of the universe. Then we have introduced a modification to the Einstein's gravitational theory, namely $f(T)$, the teleparallel equivalent of General Relativity (TEGR), where we have established that a perfect fluid in addition to matter creation can lead

us to an exact expression for $f(T)$ which can be considered as an equivalent gravitational theory for this dynamical description.

The above discussions can be seen in a flowchart as follows: perfect fluid in $f(T)$ gravity $\Longleftrightarrow$ matter creation + perfect fluid $\Longleftrightarrow$ scalar field dynamics. Next, we introduce the cosmology of decaying vacuum energy and its equivalence with gravitationally induced matter creation, which essentially tells us that there is a one-to-one correspondence between these models. But, we observed that the equivalence not always gives a one-to-one correspondence. The paper is organized as follows.

In Section 2, we derived the field equations for matter creation in the flat Friedmann–Lemaître–Robertson–Walker (FLRW) space–time. Then introducing a generalized model of matter creation in Section 3, we have analysed its dynamical stability and analytic solutions in the Section 3.1, and further, we have introduced Jacobi last multiplier in Section 3.2 and discussed the cosmological features. Section 4 contains an equivalent field theoretic description for the present model and its corresponding early physics scenario is given in Section 4.1. Furthermore, we have provided a short description on $f(T)$ gravity in the framework of matter creation in Section 5. Finally, in Section 6, we have summarized our results.

We note that, throughout the text, we have used matter creation and particle creation synonymously.

2 THE FIELD EQUATIONS IN MATTER CREATION

At this stage, it has been verified that our Universe is perfectly homogeneous and isotropic on the largest scale, and this information gives us a space–time metric known as FLRW metric:

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right], \quad (2)$$

where $a(t)$ is the scale factor of the universe, and the curvature scalar $k = 0, +1, -1$ stands for flat, closed and open universes, respectively. Furthermore from the cosmological data (Planck Collaboration XXII 2014a) the value of the spatial curvature is very close to zero; hence, we set $k = 0$.

For the comoving observer, $u^\mu = \delta_t^\mu$, in which $u^\mu u_\mu = -1$, and for the line element (equation 2), the fluid expansion becomes $\Theta = 3H$, where $H = \dot{a}/a$ is the Hubble parameter. Hence, the conservation equation (1) becomes

$$N^\mu_{;\mu} \equiv n_{,\mu} u^\mu + 3Hn = n\Gamma, \quad (3)$$

where now the comoving volume is $V = a^3$. Clearly, $\Gamma > 0$ indicates the creation of particles while $\Gamma < 0$ stands for particle annihilation.

From Gibb's equation it follows (Prigogine et al. 1989; Zimdahl 1996, 2000)

$$T ds = d \left(\frac{\rho}{n} \right) + p d \left(\frac{1}{n} \right), \quad (4)$$

and with the use of equation (3), we have

$$nT\dot{s} = \dot{\rho} + 3H \left(1 - \frac{\Gamma}{3H} \right) (\rho + p), \quad (5)$$

where T indicates the fluid temperature, and 's' is the specific entropy (i.e. entropy per particle). Now, by assuming that the creation happens under 'adiabatic' conditions (see for instance Barrow 1990; Calvão, Lima & Waga 1992), the specific entropy does not change, i.e. $\dot{s} = 0$, and from equation (5) one obtains the conservation equation

$$\dot{\rho} + 3H (\rho + p) = \Gamma (\rho + p). \quad (6)$$

Then from conservation equation (6) and taking the derivative of the Friedmann equation, which is nothing else but the first Friedmann's equation (in the units $8\pi G = 1$)

$$3H^2 = \rho, \quad (7)$$

one gets the Raychaudhuri equation

$$\dot{H} = -\frac{1}{2} \left(1 - \frac{\Gamma}{3H} \right) (\rho + p), \quad (8)$$

where for a perfect fluid with a lineal EoS of the form $p = (\gamma - 1)\rho$, that is the case we will consider throughout the paper, the latter becomes

$$\dot{H} = -\frac{3\gamma}{2} H^2 \left(1 - \frac{\Gamma}{3H} \right). \quad (9)$$

Thus, the cosmological scenario can be described after we specify the particle creation rate Γ, and the EoS γ. We see that under the condition $\Gamma \ll 3H$, we have the standard Raychaudhuri equation without any particle creation process. Further, if one specifies the EoS γ to be constant, the standard evolution equation $a \propto t^{2/3\gamma}$ is retrieved. So, the mechanism of particle creation deviates from the standard physical laws, but can be recovered under the condition $\frac{\Gamma}{3H} \ll 1$. However, the deceleration parameter, q, a measurement of state of acceleration/deceleration of the universe, is defined as

$$q \equiv - \left(1 + \frac{\dot{H}}{H^2} \right) = -1 + \frac{3\gamma}{2} \left(1 - \frac{\Gamma}{3H} \right). \quad (10)$$

Further, the effective EoS parameter is given by

$$\omega_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2} = -1 + \gamma \left(1 - \frac{\Gamma}{3H} \right), \quad (11)$$

which represents quintessence era for $\Gamma < 3H$, and phantom era for $\Gamma > 3H$. Also, $\Gamma = 3H$ indicates the cosmological constant, i.e.

Perfect fluid + $(\Gamma = 3H) \equiv$ cosmological constant.

An equivalent way to see the derivation of the field equations (7) and (8) is to consider the energy-momentum tensor in the Einstein field equations as a total energy-momentum tensor $T^{(\text{eff})}_{\mu\nu} = T^{(\gamma)}_{\mu\nu} + T^{(c)}_{\mu\nu}$, where $T^{(\gamma)}_{\mu\nu}$ is the energy-momentum tensor for the fluid with EoS parameter, $p = (\gamma - 1)\rho$, i.e.

$$T^{(\gamma)}_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \quad (12)$$

and $T^{(c)}_{\mu\nu}$ is the energy-momentum tensor which corresponds to the matter creation term. Hence, $T^{(c)}_{\mu\nu}$ has the following form

$$T^{(c)}_{\mu\nu} = P_c \left(g_{\mu\nu} + u_\mu u_\nu \right); \quad (13)$$

the latter energy-momentum tensor provides us with the matter creation pressure (Graef, Costa & Lima 2014). Therefore, the Einstein field equations are

$$G_{\mu\nu} = T^{(\gamma)}_{\mu\nu} + T^{(c)}_{\mu\nu}. \quad (14)$$

Since the two fluids are interacting, the Bianchi identity gives

$$g^{\nu\sigma} \left(T^{(\gamma)}_{\mu\nu} + T^{(c)}_{\mu\nu} \right)_{;\sigma} = 0, \quad (15)$$

or equivalently[1]

$$\dot{\rho} + 3H (\rho + p + P_c) = 0, \quad (16)$$

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[1] Recall that for a Killing vector field X, of the metric tensor $g_{\mu\nu}$, i.e. $L_X g_{\mu\nu} = 0$, holds $L_X G_{\mu\nu} = 0$; consequently we have that ρ, p and P_c are functions of 't' only.

where with the use of Gibb's equation (5), we find that

$$P_c = -\frac{\Gamma}{3H}(\rho + p),$$ (17)

or

$$P_c = -\frac{\gamma}{3H}\Gamma\rho.$$ (18)

Since $\rho > 0$, and for $H > 0$, i.e. $\dot{a} > 0$, from the latter we have that $P_c < 0$, when $\Gamma > 0$, and $P_c > 0$ when $\Gamma < 0$. Furthermore, from equation (14), we find the following system

$$3H^2 = \rho,$$ (19)

$$2\dot{H} + 3H^2 = -p - P_c,$$ (20)

where if we substitute equations (18) and (19) in equation (20), we derive the Raychaudhuri equation (9).

In the present model, the cosmic history is characterized by the fundamental physical quantities, namely the expansion rate H and the energy density which can define in a natural way the gravitational creation rate Γ. From a thermodynamic notion, Γ should be greater than H in the very early universe to consider the created radiation as a thermalized heat bath. So, the simplest choice of Γ should be $\Gamma \propto H^2$ (Abramo & Lima 1996; Gunzig, Maartens & Nesteruk 1998) (i.e. $\Gamma \propto \rho$) at the very early epoch. The corresponding cosmological solution (Lima & Germano 1992; Zimdahl 1996, 2000; Lima, Basilakos & Costa 2012) shows a smooth transition from inflationary stage to radiation phase, and for this 'adiabatic' production of relativistic particles, the energy density scales as $\rho_r \sim T^4$ (blackbody radiation; for details see Lima, Basilakos & Costa 2012). Further, $\Gamma \propto H$ (Pan & Chakraborty 2015) explains the decelerated matter-dominated era, and $\Gamma \propto 1/H$ has some accelerating feature of the universe (Pan & Chakraborty 2015).

Motivated by the above studies, a more generalized particle creation rate, $\Gamma = \Gamma_0 + lH^2 + mH + n/H$ was considered in order to explain the whole cosmic evolution (Chakraborty, Pan & Saha 2014). Later on, it was established in Haro & Pan (2015) that $\Gamma = \Gamma_0$, a constant, can predict the initial big bang singularity, subsequent intermediate phases and finally describes the late de Sitter phase. Further, it has been noticed that the effective EoS of the cosmic substratum could go beyond '-1' without introducing any kind of phantom fields (Nunes & Pavón 2015; Nunes & Pan 2016). So, Γ plays an important role in elucidating the cosmic evolution. Thus, it is clear that we can produce any arbitrary Γ as a function of H from which we can develop the dynamics of the universe analytically (if possible) or numerically (if analytic solutions are not found). But, the dynamics could be stable or unstable which may lead to some discrepancies in the dynamical behaviour of the model.

Keeping all these in mind, the present paper aims to study a generalized model for matter creation in order to study their viability to describe the current accelerating phase of the universe, and also to check their limit of extension to trace back the early physics scenario as well.

3 COSMOLOGICAL SOLUTIONS:

In this section, we will study the solutions of the Raychaudhuri equation (9) for the following matter creation rate:

$$\Gamma(H) = -\Gamma_0 + mH + n/H,$$ (21)

where we have chosen the negative sign in Γ_0 for convenience. Note that the choice (equation 21) is a generalized one which could cover different matter creation rate, for instance, $\Gamma \propto H$, $\Gamma = $ constant,

$\Gamma \propto 1/H$ and some other combinations. However, in that case, the dynamical equation becomes

$$\dot{H} = -\frac{\gamma}{2}\left((3 - m)H^2 + \Gamma_0 H - n\right).$$ (22)

Since the equation (9) or equivalently equation (22) is a one-dimensional first-order differential equation, the dynamics is obtained from the study of its critical points (or fixed points).

The fixed points of the equation (9) are obtained by $\dot{H} = 0$. Thus, if $H = H_*$ be the fixed point of equation (9), then

$$\dot{H} = 0 \implies H_* = 0, \text{ or, } \Gamma(H_*) = 3H_*.$$ (23)

Now, at the fixed points, in which $H_* \neq 0$, the FLRW metric (equation 2) describes a de Sitter universe.

Let $\dot{H} = F(H)$ be the general form of equation (9). Now, if at the fixed point, $\frac{dF(H_*)}{dH} < 0$, then the fixed point is asymptotically stable (attractor), and on the other hand, if we have $\frac{dF(H_*)}{dH} > 0$, then the fixed point is unstable in nature (repeller). The repeller point is suitable for early universe, since it can describe the inflationary epoch, whereas the attractor point is stable for late-time accelerating phase.

For the simplest case in which the particle creation rate is $\Gamma = n/H$ with $n > 0$, solving equation (9) for the fixed points, we have $H_* = \pm\sqrt{\frac{n}{3}}$. Now, for the above choice for Γ, one has $F(H) = -\frac{3\gamma}{2}\left(H^2 - \frac{n}{3}\right)$ and thus $\frac{dF(\pm\sqrt{\frac{n}{3}})}{dH} = \mp\gamma\sqrt{3n}$ which means that $\sqrt{\frac{n}{3}}$ is an attractor and $-\sqrt{\frac{n}{3}}$ is a repeller.

If $\Gamma(H)$ is a polynomial function of H, then the fixed point condition (equation 23) for $H_* \neq 0$ is a polynomial equation which has as many solutions (not necessary real solutions) as is the higher power of the polynomial $\Gamma(H_*) = 3H_*$.

Hence, for equation (21), we have the following second-order polynomial equation

$$F(H_*) \equiv (m - 3)H_*^2 - \Gamma_0 H_* + n = 0$$ (24)

where in order to find two critical points, as many as the inflationary phases of the universe, we are interested in the case when $m \neq 3$ and $n \neq \frac{\Gamma_0^2}{4(m-3)}$.

3.1 Dynamical study

For our model, the matter creation rate is $\Gamma(H) = -\Gamma_0 + mH + n/H$. Now, solving equation (24) for our model, the critical points are found to be

$$H_\pm = \frac{\Gamma_0}{2(m-3)}\left(1 \pm \sqrt{1 + \frac{4(3-m)n}{\Gamma_0^2}}\right).$$

To perform the dynamical analysis, we start with the case $\Gamma_0 > 0$, then we have to divide the plane (m, n) into six different regions:

(i) $\Omega_1 = \{(m, n): m - 3 < 0, n \geq 0\}$, where $H_+ < 0$ and $H_- > 0$. H_+ is a repeller and H_- an attractor.

(ii) $\Omega_2 = \{(m, n): m - 3 > 0, n \geq 0, \frac{4(3-m)n}{\Gamma_0^2} > -1\}$, where $H_+ > H_- > 0$. H_+ is a repeller and H_- an attractor.

(iii) $\Omega_3 = \{(m, n): m - 3 > 0, n > 0, \frac{4(3-m)n}{\Gamma_0^2} < -1\}$, where $H_\pm$ are complex numbers. $\dot{H}$ is always positive.

(iv) $\Omega_4 = \{(m, n): m - 3 < 0, n < 0, \frac{4(3-m)n}{\Gamma_0^2} < -1\}$, where $H_\pm$ are complex numbers. $\dot{H}$ is always negative.

(v) $\Omega_5 = \{(m, n): m - 3 < 0, n < 0, \frac{4(3-m)n}{\Gamma_0^2} > -1\}$, where $H_+ < H_- < 0$. H_+ is a repeller and H_- an attractor.

(vi) $\Omega_6 = \{(m, n): m - 3 > 0, n < 0\}$, where $H_+ > 0$ and $H_- < 0$. H_+ is a repeller and H_- an attractor.

On the other hand, for $\Gamma_0 < 0$, we have

(i) $\Omega_7 = \{(m, n): m - 3 < 0, n \geq 0\}$, where $H_+ > 0$ and $H_- < 0$. H_+ is an attractor and H_- a repeller.

(ii) $\Omega_8 = \{(m, n) : m - 3 > 0, n \geq 0, \frac{4(3-m)n}{\Gamma_0^2} > -1\}$, where $H_+ < H_- < 0$. H_+ is an attractor and H_- a repeller.

(iii) $\Omega_9 = \{(m, n) : m - 3 > 0, n > 0, \frac{4(3-m)n}{\Gamma_0^2} < -1\}$, where $H_\pm$ are complex numbers. $\dot{H}$ is always positive.

(iv) $\Omega_{10} = \{(m, n) : m - 3 < 0, n < 0, \frac{4(3-m)n}{\Gamma_0^2} < -1\}$, where $H_\pm$ are complex numbers. $\dot{H}$ is always negative.

(v) $\Omega_{11} = \{(m, n) : m - 3 < 0, n < 0, \frac{4(3-m)n}{\Gamma_0^2} > -1\}$, where $H_+ > H_- > 0$. H_+ is an attractor and H_- a repeller.

(vi) $\Omega_{12} = \{(m, n): m - 3 > 0, n < 0\}$, where $H_+ < 0$ and $H_- > 0$. H_+ is an attractor and H_- a repeller.

The case $m = 3$ is special, in the sense that there is only one critical point given[2] by $H_- = \frac{n}{\Gamma_0}$ which is always an attractor for $\Gamma_0 > 0$ and a repeller for $\Gamma_0 < 0$.

To have a non-singular universe (without the big bang singularity) with an accelerated phase both at early and late times, one possibility is to have two critical points $H_+ > H_- > 0$, where H_+ was a repeller and H_- must be an attractor. If so, in principle, when the universe leaves H_+, realizing the inflationary phase, and when it comes asymptotically to H_-, it enters into the current accelerated phase. Of course, the viability of the background has to be checked dealing with cosmological perturbations and comparing the theoretical predictions with the observational ones.

For our model, this only happens in the region Ω_2, and when $m = 3$ with $\Gamma_0 > 0$, that is in the region of the space parameters given by

$$W = \{(\Gamma_0, m, n) : \ \Gamma_0 > 0, \ m \geq 3, \ n \geq 0, \ \frac{4(3-m)n}{\Gamma_0^2} > -1\}. \tag{25}$$

Note that, in the case $m = 3$, we have $H_+ = +\infty$, but the universe is not singular, because in that case the Raychaudhuri equation becomes $\dot{H} = -\frac{\gamma}{2}(\Gamma_0 H - n)$. For large values of H, this equation is approximately $\dot{H} = -\frac{\gamma}{2}\Gamma_0 H$, the solution of which is given by $H(t) = H_0 e^{-\frac{\gamma}{2}\Gamma_0(t-t_0)}$. Therefore, H only diverges when $t = -\infty$, that is, there are no singularities at finite time.

For the parameters that belong to W, the solution of the Raychaudhuri equation is given by

$$H(t) = \frac{\Gamma_0}{2(m-3)} - \frac{\omega}{2(m-3)} \tanh\left(\frac{\gamma}{4}\omega(t-t_0)\right), \tag{26}$$

for $m > 3$, where $\omega = \sqrt{\Gamma_0^2 + 4(3-m)n}$.

For the completeness of our analysis, for $m = 3$, we have that

$$H(t) = \Gamma_0 e^{-\frac{\gamma\Gamma_0}{2}(t-t_0)} + \frac{n}{\Gamma_0}. \tag{27}$$

Last but not least, when $m \neq 3$ and $n = \frac{\Gamma_0^2}{4(m-3)}$, where $H_+ = H_-$, that is, equation (22) admits one fixed point, we find the following analytical solution for the Hubble function

$$H(t) = \frac{\Gamma_0}{2(m-3)} - \frac{1}{\gamma(m-3)}\frac{1}{(t-t_0)}, \tag{28}$$

in which for $m > 3$, in order to have $H(t) > 0$, we have $t \in (-\infty, t_0)$.

Note that this last solution, when the values of the parameters belong to W, depicts a phantom universe that starts at the critical point and ends in a big rip singularity at $t = t_0$.

From equations (26)–(28), we can find the solution of the scale factor. Hence, from equation (26) we have

$$a(t) = a_0 \exp\left[\frac{\Gamma_0}{2(m-3)}(t-t_0) - \frac{2}{(m-3)\gamma}\ln\left(\cosh\left(\frac{\gamma}{4}\omega(t-t_0)\right)\right)\right]. \tag{29}$$

Furthermore, from equation (27) we have

$$a(t) = a_0 \exp\left[-\frac{2}{\gamma}(e^{-\gamma\frac{\Gamma_0}{2}(t-t_0)} - 1) + \frac{n}{\Gamma_0}(t-t_0)\right]. \tag{30}$$

Finally, from the case $m \neq 3$, and $n = \frac{\Gamma_0^2}{4(m-3)}$, the scale factor becomes

$$a(t) = a_0 \exp\left[\frac{\Gamma_0}{2(m-3)}(t-t_0)\right]\left(\frac{t}{t_0}\right)^{-\frac{2}{\gamma(m-3)}}, \tag{31}$$

in which for $-\frac{2}{\gamma(m-3)} = \frac{1}{3}$, the last solution describes also the two-scalar field cosmological model in which the scalar fields are interacting in their kinetic parts (Paliathanasis & Tsamparlis 2014), where it has been shown that the model fits the cosmological data in a similar way as the Λ-cosmology.

Now, with the use of equation (11), it is possible to determine the effective EoS parameter. Therefore, we have

$$\omega_{\text{eff}} = -1 + \frac{(m-3)}{3}\omega^2\gamma\left(\Gamma_0\cosh\left(\frac{\gamma}{4}\omega(t-t_0)\right) - \omega\sinh\left(\frac{\gamma}{4}\omega(t-t_0)\right)\right)^{-2}, \tag{32}$$

or,

$$\omega_{\text{eff}} = -1 + \frac{\gamma}{3}\frac{e^{-\frac{\gamma}{2}\Gamma_0(t-t_0)}}{\left(e^{-\frac{\gamma}{2}\Gamma_0(t-t_0)} + \frac{n}{\Gamma_0^2}\right)^2}, \tag{33}$$

and

$$\omega_{\text{eff}} = -1 - \frac{4(m-3)\gamma}{3(\Gamma_0\gamma(t-t_0)-2)^2}, \tag{34}$$

for the solutions (26), (27) and (28), respectively.

Consider now the initial condition that at $t = t_1$, $\omega_{\text{eff}}(t_0) = \gamma - 1$. From the latter, we can define a constraint equation between the free parameters of the model, i.e. $\{\Gamma_0, m, n\}$. Without any loss of generality, let us say that $t_1 = t_0$, which is possible since the model is autonomous and invariant under time translations.

Hence, from equation (32), we find the condition

$$\Gamma_0^2 = \frac{m-3}{3}\omega^2. \tag{35}$$

Fig. 1 shows the evolution of the effective EoS parameter (equation 32) for a set of parameters $(\Gamma_0, m, n) \in W$, describing the early and late de Sitter phases of the universe, where we have shown its evolution for three different values of γ, namely $\gamma = 4/3$, 1 and 1.03.

3.2 Particle creation rate from Jacobi last multiplier

Equation (9) is a first-order differential equation for the Hubble function $H(t)$ or a second-order differential equation for the scale factor $a(t)$. Applying in equation (9) the transformation
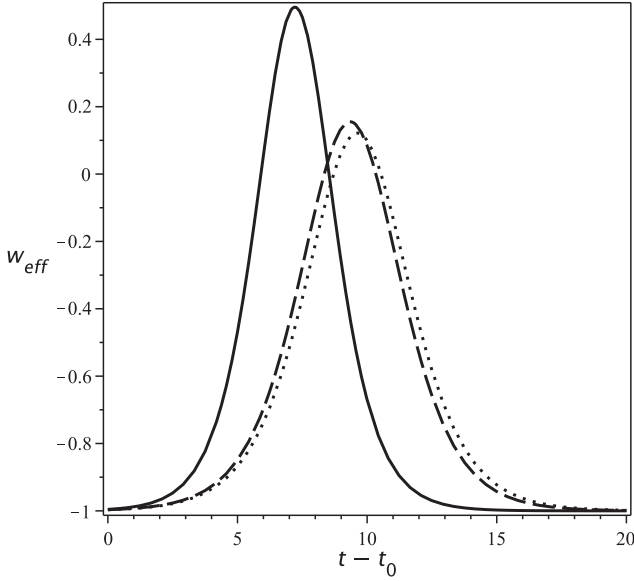
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[2] When $m = 3$, equation (24) is a linear equation which admits only one real solution.

Figure 1. Qualitative behaviour of the effective EoS for a set of parameters $(\Gamma_0, m, n) \in W$. The solid, doted and dashed curves have been plotted for $\gamma = 4/3$, 1 and 1.03.

$a(t) = \exp(\mathcal{N}(t))$, i.e. $H = \dot{\mathcal{N}}$, we have the second-order differential equation

$$\ddot{\mathcal{N}} = -\frac{3\gamma}{2}\dot{\mathcal{N}}^2 \left(1 - \frac{\Gamma(\dot{\mathcal{N}})}{3\dot{\mathcal{N}}}\right) \tag{36}$$

which is of the form $\ddot{x} = F(t, x, \dot{x})$. One would like to have a geometric method to construct the unknown function $\Gamma(\dot{\mathcal{N}})$, such as the application of group-invariant transformations in scalar field cosmology or in modified theories of gravity.[3] In this approach, we would like to solve the inverse problem, i.e. to construct a Lagrangian function for equation (36) by using the method of Jacobi last multiplier. For one-dimensional second-order differential equations, if there exists a function $M(t, x, \dot{x})$, which satisfy the following condition

$$\frac{\mathrm{d}}{\mathrm{d}t}(\ln M) + \frac{\partial F}{\partial \dot{x}} = 0, \tag{37}$$

then for the second-order differential equation $\ddot{x} = F(t, x, \dot{x})$, a Lagrangian can be constructed (Nucci & Tamizhmani 2010). For equation (36), we have that $F = F(\dot{x}) = F(\dot{\mathcal{N}})$; therefore, condition (37) gives that

$$\frac{\partial}{\partial t}(\ln M) + \dot{x}\frac{\partial}{\partial x}(\ln M) + F\frac{\partial}{\partial \dot{x}}(\ln M) = -\frac{\partial F}{\partial \dot{x}}. \tag{38}$$

Then, since for our model we have $F(\dot{x}) = -\frac{\gamma}{2}\left((3-m)\dot{x}^2 + \Gamma_0\dot{x} - n\right)$, we can deduce that $\frac{\partial}{\partial t}(\ln M) = \frac{\gamma\Gamma_0}{2}$, $\frac{\partial}{\partial x}\ln(M) = \gamma(3-m)$ and $\frac{\partial}{\partial \dot{x}}(\ln M) = 0$, that is,

$$M(t, x) = \mathrm{e}^{\gamma(3-m)x + \frac{\gamma\Gamma_0}{2}t}. \tag{39}$$

Finally, using that the Lagrangian is determined by the relation

$$\frac{\partial^2 L}{\partial \dot{x}^2} = M, \tag{40}$$

[3] For instance, see Paliathanasis et al. (2015), Vakili (2014) and Dimakis, Christodoulakis & Terzis (2014) and references therein.

after comparing with equation (36) one gets the following Lagrangian for our model

$$L(\mathcal{N}, \dot{\mathcal{N}}, t) = \mathrm{e}^{\gamma(3-m)\mathcal{N} + \frac{\gamma}{2}\Gamma_0 t}\left(\frac{1}{2}\dot{\mathcal{N}}^2 + \frac{n}{2(3-m)}\right). \tag{41}$$

On the other hand, someone can start with special forms of the Lagrange multiplier and from condition (37) to determine the creation rate. For instance, consider that $M = M(x) = M(\mathcal{N})$; hence, equation (37) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(M) = -\frac{1}{\dot{x}}\frac{\partial F}{\partial \dot{x}}; \tag{42}$$

therefore, the l.h.s. of the latter equation is constant, i.e. $\frac{\partial}{\partial x}\ln(M) = \gamma(3-m)$, and

$$\Gamma(H) = mH + \frac{n}{H}. \tag{43}$$

This is a particular case the one we considered above, i.e. it is expression (21) for $\Gamma_0 = 0$. Hence, the analytical solution of equation (36) is

$$a = a_0\left[\sinh\left(\frac{\gamma}{2}\sqrt{n(3-m)}(t-t_0)\right)\right]^{\frac{2}{\gamma(3-m)}} \tag{44}$$

for $n \neq 0$, or

$$a(t) = a_0((t-t_0))^{\frac{2}{\gamma(3-m)}}, \tag{45}$$

for $n = 0$. Finally, the Lagrangian function for equation (36) which follows from the Lagrange multiplier M (equation 38) is

$$L(\mathcal{N}, \dot{\mathcal{N}}) = \frac{\exp(\gamma(m-3)\mathcal{N})}{2}\left(\dot{\mathcal{N}}^2 - \frac{n}{(m-3)}\right). \tag{46}$$

The latter is an autonomous Lagrangian and the Hamiltonian function is a conservation law, that is

$$I_0 = \mathrm{e}^{\gamma(m-3)\mathcal{N}}\left(\dot{\mathcal{N}}^2 + n\right) \tag{47}$$

hence

$$\frac{H^2}{H_0^2} = \Omega_{m0}a^{(3-m)\gamma} + \Omega_\Lambda, \tag{48}$$

where $\Omega_{m0} = I_0 H_0^2$, and $\Omega_{\Lambda 0} = -nH_0^2$, which describes a universe with cosmological constant and a perfect fluid $\bar{p} = (\bar{\gamma} - 1)\bar{\rho}$, in which $\bar{\gamma} = \frac{(m-3)}{3}\gamma$. We can see that when $m = 6, \gamma = 1$ or $(3 - m)\gamma = -3$, Λ-cosmology is recovered; furthermore, $|n| = \rho_\Lambda$. Recall that such an analytical solution has been found recently for a Brans–Dicke cosmological model, in which the term $(m - 3)\gamma$ is related with the Brans–Dicke parameter (Paliathanasis et al. 2016). In particular, we found that

$$m(\gamma) = 3 + \frac{1}{\gamma}\frac{3\omega_{\mathrm{BD}} + 4}{3\omega_{\mathrm{BD}} + 1}. \tag{49}$$

As far as the Hubble function (48) is concerned, we can see that the power of the scale factor a can be written as $(3 - m)\gamma = -\bar{m}(m, \gamma)$, that is, the independent parameters that we have to determine are H_0^2, Ω_{m0} and $\bar{m}$. In order to constrain the cosmological parameters, joint likelihood analysis using the Type Ia supernova (SNIa) data set of Union 2.1 (Suzuki et al. 2012), the 6dF, SDSS and WiggleZ baryon acoustic oscillation (BAO) data (Percival et al. 2010; Blake et al. 2011, and the 21 one Hubble data of (Farooq, Mania & Ratra 2013) has been performed. Further, in order to reduce the number of the free variables to two, we select to use the present value of the Hubble function, i.e. $H_0 = 69.6$ km s^{-1} Mpc^{-1} (Bennet

Table 1. The overall statistical results (using SNIa+BAO) for the ΛCDM and Brans–Dicke models, respectively. Notice that in our analysis we use equation (48). In the last three columns, we present the number of free parameters and the goodness-of-fit statistics.

Data	Ω_{m0}	w_Λ	$\bar{m}$	$\chi^2_{\min}$
SNIa	0.28	−1	3.0	561.73
SNIa & BAO	$0.29^{+0.04}_{-0.035}$	−1	$2.93^{+0.24}_{-0.244}$	564.29
SNIa & BAO & $H(z)$	$0.29^{+0.03}_{-0.03}$	−1	$2.99^{+0.22}_{-0.21}$	577.81
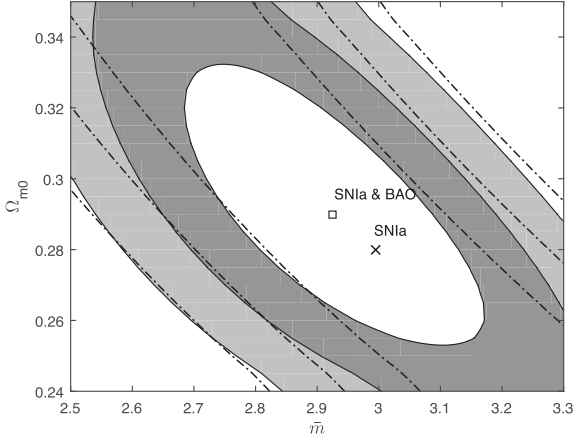
Figure 2. Confidence levels of 1σ, 2σ and 3σ for the best-fitting parameters for the cosmological tests by using the (A) SNIa, (B) SNIa & BAO data. With '×' are denoted the best-fitting parameters for the test A and with a box the best-fitting parameters for the test B.
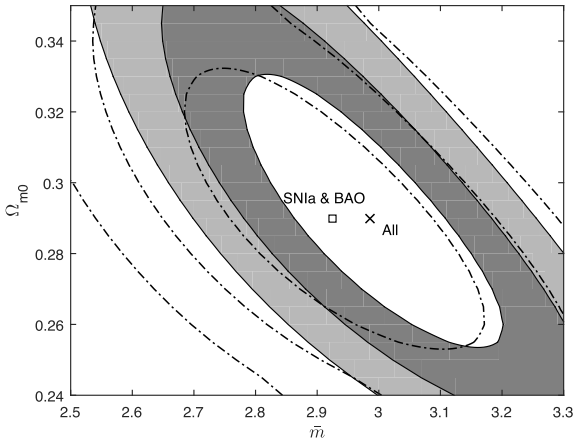

Figure 3. Confidence levels of 1σ, 2σ and 3σ for the best-fitting parameters for the cosmological tests by using the (B) SNIa & BAO and (C) SNIa & BAO & $H(z)$ data. With '×' are denoted the best parameters for the test C while with box the best-fitting parameters for the test B.

et al. 2014). Hence, the likelihood function depends on the values of the parameter $\{\Omega_{m0}, \bar{m}(m, \gamma)\}$, and it is given as follows:

$$\mathcal{L}(\Omega_{m0}, \bar{m}(m, \gamma)) = \mathcal{L}_{\text{SNIa}} \times \mathcal{L}_{\text{BAO}} \times \mathcal{L}_{H(z)}, \quad (50)$$

where $L_A \propto e^{-\chi^2_A/2}$, that is $\chi^2 = \chi^2_{\text{SNIa}} + \chi^2_{\text{BAO}} + \chi^2_{H(z)}$. The results are given in Table 1. In Figs 2 and 3, we give the confidence levels $1\sigma, 2\sigma, 3\sigma$ for the best-fitting values. Specifically, Fig. 2 compares the constraints SNIa versus SNIa+BAO data while Fig. 3 compares the SNIa+BAO versus SNIa+BAO+$H(z)$.

Furthermore, we note that for the relation $m = \frac{1}{\gamma}(3 + \bar{m})$, for a specific value of γ, we can determine m from Table 1, and the constants n, I_0 and m.

We conclude that the application of the Jacobi last multiplier gives a function $\Gamma(H)$, which includes the terms which explain the decelerated matter-dominated era, and the acceleration features of the universe. However, one may study the group-invariant transformations of equation (36), and from the requirement that the equation (36) is invariant under a specific algebra, the particle creation rate, Γ, might be determined. This would be geometric selection rule; however, this analysis is not in the scope of this work. In the following sections, we study the relation of the particle creation rate with some other cosmological theories.

4 EQUIVALENCE WITH THE DYNAMICS DRIVEN BY A SINGLE SCALAR FIELD

To check the viability of the models, one has to verify if they support the observational data, relative to inflation, provided by the *Planck* team. However, it is not clear at all how hydrodynamical perturbations (Mukahanov, Feldman & Brandenberger 1992) could provide viable theoretical data, i.e. that fit well with current observational ones, because during the inflationary period one has $p \cong -\rho$, and thus, the square of the velocity of sound, which appears in the Mukhanov–Sasaki equation (Mukhanov 1985; Sasaki 1986), could be approximately $c_s^2 \equiv \frac{\dot{p}}{\dot{\rho}} \cong -1$, which is negative, leading to a Jeans instability for modes well inside the Hubble radius. However, for a universe filled by a scalar field, this problem does not exist because in that case one always has $c_s^2 = 1$. This is an essential reason why we try to mimic the dynamics of an open system, where matter creation is allowed, obtained in the previous section by a scalar field φ with potential $V(\varphi)$. To do that, we use the energy density, namely ρ_φ, and pressure, namely p_φ, of the scalar field given by

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \quad (51)$$

$$p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi). \quad (52)$$

To show the equivalence with our system as described in equation (8) with EoS $p = (\gamma - 1)\rho$, we perform the replacement

$$\rho \longrightarrow \rho_\varphi, \quad p - \frac{\gamma\Gamma}{3H}\rho \longrightarrow p_\varphi, \quad (53)$$

and the Friedmann and Raychaudhuri equations will become

$$3H^2 = \rho_\varphi, \quad 2\dot{H} = -\dot{\varphi}^2. \quad (54)$$

Note that equation (54) uses the equations of General Relativity (GR) for a single scalar field; this means that we are dealing with the equivalence with an open system and the one driven by a single scalar field in the context of GR.

Using the above two equations, we see that the effective EoS parameter is

$$\omega_{\text{eff}} = -1 + \gamma\left(1 - \frac{\Gamma}{3H}\right) = \omega_\varphi = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)}. \quad (55)$$

Note that the Raychaudhuri equation (54) tells us that $\dot{H} < 0$, which means from equation (11) that $\omega_{\text{eff}} > -1$, and thus, one has $\Gamma < 3H$.

On the other hand, from the Friedmann and Raychaudhuri equations, one easily obtains

$$\dot{\varphi} = \sqrt{-2\dot{H}} = \sqrt{3\gamma H^2 \left(1 - \frac{\Gamma}{3H}\right)}, \tag{56}$$

and

$$V(\varphi) = \frac{3H^2}{2}\left((2-\gamma) + \frac{\gamma\Gamma}{3H}\right). \tag{57}$$

The first step is to integrate equation (56). Performing the change of variable $dt = \frac{dH}{\dot{H}}$, we will obtain

$$\varphi = -\int \sqrt{-\left(\frac{2}{\dot{H}}\right)}\, dH = -\frac{2}{\sqrt{\gamma}}\int \frac{dH}{\sqrt{3H^2 - \Gamma H}}. \tag{58}$$

In the particular case $\Gamma = -\Gamma_0 + mH + n/H$, one has

$$\varphi = -\frac{2}{\sqrt{\gamma}}\int \frac{dH}{\sqrt{(3-m)H^2 + \Gamma_0 H - n}}. \tag{59}$$

This integral could be solved analytically in the region W, giving

$$\varphi = \frac{2}{\sqrt{(m-3)\gamma}}\arcsin\left(\frac{m-3}{\omega}\left(\frac{\Gamma_0}{m-3} - 2H\right)\right), \tag{60}$$

when $m > 3$, and

$$\varphi = -\frac{4}{\sqrt{\gamma\Gamma_0}}\sqrt{\Gamma_0 H - n}, \tag{61}$$

for $m = 3$.

Conversely,

$$H = \frac{1}{2(m-3)}\left[\Gamma_0 - \omega\sin\left(\frac{\sqrt{(m-3)\gamma}}{2}\varphi\right)\right], \quad \text{when } m > 3, \tag{62}$$

and

$$H = \frac{n}{\Gamma_0} + \frac{\gamma\Gamma_0}{16}\varphi^2, \quad \text{when } m = 3. \tag{63}$$

On the other hand, for our model, the potential (57) is given by

$$V(\varphi) = \frac{1}{2}\left((6 + (m-3)\gamma)H^2 - \gamma\Gamma_0 H + \gamma n\right), \tag{64}$$

then, inserting in it the values of H given by equations (62) and (63), one obtains the corresponding potentials. In fact, in the case equation (62), one gets

$$V(\varphi) = \frac{3}{4(m-3)^2}\left[\Gamma_0 - \omega\sin\left(\frac{\sqrt{(m-3)\gamma}}{2}\varphi\right)\right]^2$$
$$- \frac{\gamma\omega^2}{8(m-3)}\cos^2\left(\frac{\sqrt{(m-3)\gamma}}{2}\varphi\right), \tag{65}$$

and for equation (63)

$$V(\varphi) = \frac{\gamma^2\Gamma_0^2}{256}\varphi^4 + \frac{\gamma}{8}\left(3n - \frac{\gamma\Gamma_0^2}{4}\right)\varphi^2 + \frac{3n^2}{\Gamma_0^2}. \tag{66}$$

The following remark is in order: in the context of GR driven by a scalar field, the backgrounds (26) and (27), which now have to be understood as mere solutions of the Raychaudhuri equation when the universe is filled by a scalar field and not as solutions of an open system, are not viable because they do not contain a mechanism to reheat the universe, because the potential has a minimum when the universe reaches the de Sitter solution H_-, which depicts the current cosmic acceleration, but it is clear that, in order to match with the hot Friedmann universe, it has to reheat at higher scales. Then,

the simplest solution is to introduce a sudden phase transition that breaks the adiabaticity, and thus, particles could be produced in an enough amount to thermalize the universe (Peebles & Vilekin 1999).

4.1 A viable model

What we choose is a continuous transition at some scale H_E, of the rate of particle production Γ, of the form

$$\Gamma = \begin{cases} -\Gamma_0 + 3H + \frac{\Gamma_0^2}{12H} & \text{for} \quad H > H_E \\ \Gamma_1 & \text{for } H_E > H > \bar{H}_- \end{cases} \tag{67}$$

where $0 < \Gamma_1 \ll \Gamma_0$ and $\bar{H}_- = \frac{\Gamma_1}{3}$. The continuity requires

$$H_E = \frac{\Gamma_0 + \Gamma_1}{6}\left(1 + \sqrt{1 - \frac{\Gamma_0^2}{(\Gamma_0 + \Gamma_1)^2}}\right) \cong \frac{\Gamma_0}{6}. \tag{68}$$

Moreover, we will assume that universe has a deflationary phase, which can be mimicked by a stiff fluid, at the transition phase, since at that moment one has

$$\omega_{\text{eff}} = -1 + \gamma\left(1 - \frac{\Gamma_1}{H_E}\right) \cong -1 + \gamma \tag{69}$$

and one has to choose $\gamma = 2$, i.e. the EoS must be $p = \rho$.

Now, to check the viability, we have to study the model at early times. We start with the slow roll parameters (Bassett, Tsujikawa & Wands 2006)

$$\epsilon = -\frac{\dot{H}}{H^2}, \quad \eta = 2\epsilon - \frac{\dot{\epsilon}}{2H\epsilon}, \tag{70}$$

which allow us to calculate the spectral index (n_s), its running (α_s) and the ratio of tensor to scalar perturbations (r) given by

$$n_s - 1 = -6\epsilon + 2\eta, \quad \alpha_s = \frac{H\dot{n}_s}{H^2 + \dot{H}}, \quad r = 16\epsilon. \tag{71}$$

At early times, i.e. when $H > H_E$, introducing the notation $x \equiv \frac{\Gamma_0}{H}$, since for our model the Raychaudhuri equation is

$$\dot{H} = -\Gamma_0 H + \frac{\Gamma_0^2}{12}, \tag{72}$$

one will have

$$\epsilon = x\left(1 - \frac{x}{12}\right), \quad \eta = \epsilon + \frac{x}{2}, \tag{73}$$

and as a consequence,

$$n_s - 1 = -3x + \frac{x^2}{3}. \tag{74}$$

From recent *Planck*+WP 2013 data (see table 5 of Planck Collaboration XXII 2014a), the spectral index at 1σ confidence level (C.L.) is $n_s = 0.9583 \pm 0.0081$, which means that $1 - n_s \cong 5 \times 10^{-2}$. Therefore, we can apply the results obtained in Haro & Pan (2015).

Since

$$x = \frac{9}{2}\left(1 - \sqrt{1 - \frac{4(1-n_s)}{27}}\right) \tag{75}$$

at 2σ C.L., one has $0.0085 \leq x \leq 0.0193$, and thus, $0.1344 \leq r = 16\epsilon \leq 0.3072$. Since *Planck*+WP 2013 data provide the constraint $r \leq 0.25$, at 95.5 per cent C.L., then when $0.0085 \leq x \leq 0.0156$, the spectral index belongs to the one-dimensional marginalized 95.5 per cent C.L., and also $r \leq 0.25$, at 95.5 per cent C.L.

For the running at 1σ C.L., *Planck*+WP 2013 data give $\alpha_s = -0.021 \pm 0.012$, and our background leads to the theoretical value

$\alpha_s \cong -\frac{3x\epsilon}{1-\epsilon} \cong -3x^2$. Consequently, at the scales we are dealing with, $-7 \times 10^{-4} \leq \alpha_s \leq -2 \times 10^{-4}$, and thus, the running also belongs to the one-dimensional marginalized 95.5 per cent C.L.

Note also that we have the relation $w_{\rm eff}(H) = -1 + \frac{2}{3}\epsilon$. Therefore, if we assume that the slow roll ends when $\epsilon = 1$, and let $H_{\rm end}$ be the value of the Hubble parameter when the slow roll ends, then the slow roll will end when $w_{\rm eff}(H_{\rm end}) = -\frac{1}{3}$, i.e. when the universe will start to decelerate.

On the other hand, the number of e-folds from observable scales exiting the Hubble radius to the end of inflation, namely $N(H)$, could be calculated using the formula $N(H) = -\int_{H_{\rm end}}^{H} \frac{H}{\dot{H}} dH$, leading to

$$N(x) = \frac{1}{x} - \frac{1}{x_{\rm end}} + \frac{1}{12} \ln\left(\frac{12-x}{12-x_{\rm end}} \frac{x_{\rm end}}{x}\right), \tag{76}$$

where $x_{\rm end} = 6(1 - \sqrt{2/3}) \cong 1.1010$ is the value of the parameter x when inflation ends. For our values of x that allow to fit well with the theoretical value of the spectral index, its running and the tensor/scalar ratio with their observable values, we will obtain $64 \leq N \leq 117$.

The value of Γ_0 could be established taking into account the theoretical (Bassett et al. 2006) and the observational (Bunn, Liddle & White 1996) value of the power spectrum

$$\mathcal{P} \cong \frac{H^2}{8\pi^2\epsilon} = \frac{\Gamma_0^2}{18\pi^2\epsilon x^2} = \frac{4\Gamma_0^2}{9\pi m_{\rm pl}^2 \epsilon x^2} \cong 2 \times 10^{-9}, \tag{77}$$

where we have explicitly introduced the Planck mass, which in our units is $m_{\rm pl} = \sqrt{8\pi}$. Using the values of x in the range [0.0085, 0.0156], we can conclude that

$$9 \times 10^{-7} m_{\rm pl} \leq \Gamma_0 \leq 2 \times 10^{-7} m_{\rm pl}. \tag{78}$$

4.1.1 Particle production and reheating

We will study the production of massless particles nearly conformally coupled with gravity due to the phase transition in our model. To simplify our reasoning, we will choose $\Gamma_1 = 0$, and then $H(t_E) = \frac{\Gamma_0}{6}$; thus, after the transition, the universe is exactly in a deflationary phase if we choose $\gamma = 2$.

The energy density of the produced particles will be given by (Birrell & Davies 1982)

$$\rho_\chi = \frac{1}{(2\pi a)^3 a} \int_0^\infty k|\beta_k|^2 d^3k, \tag{79}$$

where the β-Bogoliubov coefficient is given by (Zeldovich & Starobinsky 1977; Birrell & Davies 1982)

$$\beta_k \cong \frac{{\rm i}(\xi - \frac{1}{6})}{2k} \int_{-\infty}^{\infty} {\rm e}^{-2ik\tau} a^2(\tau) R(\tau) d\tau, \tag{80}$$

where $R = 6(\dot{H} + 2H^2)$ is the scalar curvature, τ the conformal time and ξ the coupling constant. This integral is convergent because at early and late time $a^2(\tau)R(\tau)$ converges to zero fast enough. It is not difficult to show, integrating twice by parts, that $\beta_k \sim \mathcal{O}(k^{-3})$ (this is due to the fact that $\dot{H}$ is continuous during the phase transition) and, as we will see, this means that the energy density of produced particles is not ultraviolet divergent. Moreover, $\beta_k = (1 - 6\xi)f(\frac{k}{a_E\Gamma_0})$, where f is some function.

Then, taking for instance $1 - 6\xi \sim 10^{-1}$, the energy density of the produced particles is of the order of

$$\rho_\chi \sim 10^{-2}\Gamma_0^4 \left(\frac{a_E}{a}\right)^4 \frac{1}{2\pi^2} \int_0^\infty s^3 f^2(s) ds \sim 10^{-2}\mathcal{M}\Gamma_0^4 \left(\frac{a_E}{a}\right)^4, \tag{81}$$

where we have introduced the notation $\mathcal{M} \equiv \frac{1}{2\pi^2} \int_0^\infty s^3 f^2(s) ds$.

Since the sudden transition occurs at $H_E \cong \frac{\Gamma_0}{6} \sim 10^{-7} m_{\rm pl} \sim 10^{12}$ GeV (the same result was obtained in formula 15 of Peebles & Vilenkin 1999), one can deduce that the universe preheats, due to the gravitational particle production, at scales

$$\rho = \frac{3H_E^2 m_{\rm pl}^2}{8\pi} \sim 10^{-17} \rho_{\rm pl}, \tag{82}$$

where $\rho_{\rm pl} = m_{\rm pl}^4$ is the Planck energy density. On the other hand, at the transition time, the energy density of the produced particles is of the order of

$$\rho_\chi \sim 10^{-30}\mathcal{M}\rho_{\rm pl}, \tag{83}$$

which is smaller than the energy density of the background.

After the phase transition, first of all, these particles will interact exchanging gauge bosons and constituting a relativistic plasma that thermalizes the universe (Spokoiny 1993; Peebles & Vilekin 1999) before the universe was radiation dominated. Moreover, in our model, the background is in a deflationary stage, meaning that its energy density decays as a^{-6}, and the energy density of the produced particles decreases as a^{-4}. Then, eventually the energy density of the produced particles will dominate and the universe will become radiation dominated and matches with the standard hot Friedmann universe. The universe will expand and cool becoming the particles no-relativistic, and thus, the universe enters into a matter-dominated regime, essential for the growth of cosmological perturbations, and only at very late time, when the Hubble parameter is of the same order as Γ_1, the field takes back its role to start the cosmic acceleration.

The reheating temperature, namely T_R, is defined as the temperature of the universe when the energy density of the background and the one of the produced particles are of the same order ($\rho \sim \rho_\chi$). Since $\rho_\chi \sim 10^{-2}\mathcal{M}\Gamma_0^4 \left(\frac{a_E}{a}\right)^4$ and $\rho = \frac{3H_E^2 m_{\rm pl}^2}{8\pi} \sim 10^{-3}\Gamma_0^2 m_{\rm pl}^2 \left(\frac{a_E}{a}\right)^6$, one obtains $\frac{a_E}{a(t_R)} \sim \sqrt{\mathcal{M}}\frac{\Gamma_0}{m_{\rm pl}}$, and therefore,

$$T_R \sim \rho_\chi^{1/4}(t_R) \sim \mathcal{M}^{\frac{3}{4}}\frac{\Gamma_0^2}{m_{\rm pl}} \sim 10^5 \mathcal{M}^{\frac{3}{4}} \text{ GeV}. \tag{84}$$

This reheating temperature is below the GUT scale 10^{16} GeV, which means that the GUT symmetries are not restored preventing a second monopole production stage. Moreover, this guarantees the standard successes with nucleosynthesis, because it requires a reheating temperature below 10^9 GeV (Allahverdi et al. 2010).

Finally, to obtain the temperature when the equilibrium is reached, we will follow the thermalization process depicted in Spokoiny (1993, see also Peebles & Vilekin 1999), where it is assumed that the interactions between the produced particles are due to gauge bosons, and one might estimate the interaction rate as $\Gamma \sim \alpha^2 T_{\rm eq}$. Then, since thermal equilibrium is achieved when $\Gamma \sim H(t_{\rm eq}) \sim H_E \left(\frac{a_E}{a_{\rm eq}}\right)^3$ (recall that, in our model, this process is produced in the deflationary phase where $\rho \sim a^{-6}$), and $T_{\rm eq} \sim 10^{-\frac{1}{2}}\mathcal{M}^{\frac{1}{4}} H_E \frac{a_E}{a_{\rm eq}}$, when the equilibrium is reached one has $\frac{a_E}{a_{\rm eq}} \sim 10^{-\frac{1}{4}}\alpha\mathcal{M}^{\frac{1}{8}}$, and thus, $T_{\rm eq} \sim 10^{-\frac{3}{4}}\mathcal{M}^{\frac{3}{8}}\alpha H_E$. Therefore, one obtains

$$T_{\rm eq} \sim 10^{-8}\mathcal{M}^{\frac{3}{8}}\alpha m_{\rm pl} \sim 10^{11}\mathcal{M}^{\frac{3}{8}}\alpha \text{ GeV}. \tag{85}$$

And choosing as usual $\alpha \sim (10^{-2}-10^{-1})$ (Spokoiny 1993; Peebles & Vilekin 1999), one obtains the following equilibrium temperature:

$$T_{\rm eq} \sim (10^9-10^{10})\mathcal{M}^{\frac{3}{8}} \text{ GeV}. \tag{86}$$

5 $f(T)$ GRAVITY AND PARTICLE CREATION RATE

$f(T)$ gravity has recently gained a lot of attention. The essential properties of this modified theory of gravity are based on the rather old formulation of the TEGR (Einstein 1928; Hayashi & Shirafuji 1979; Maluf 1994; Arcos & Pereira 2004; Unzicker & Case 2005). In particular, one utilizes the curvature-less Weitzenböck connection in which the corresponding dynamical fields are the four linearly independent *vierbeins* rather than the torsion-less Levi-Civita connection of the classical GR. A natural generalization of TEGR gravity is $f(T)$ gravity which is based on the fact that we allow the gravitational action integral to be a function of T (Ferraro & Fiorini 2007; Bengochea & Ferraro 2009; Linder 2010), in a similar way as $f(R)$ Einstein–Hilbert action. However, $f(T)$ gravity does not coincide with $f(R)$ extension, but it rather consists of a different class of modified gravity. It is interesting to mention that the torsion tensor includes only products of first derivatives of the vierbeins, giving rise to second-order field differential equations in contrast with the $f(R)$ gravity that provides fourth-order equations.

Consider the unholonomic frame e_i, in which $g(e_i, e_j) = e_i.e_j = \eta_{ij}$, where η_{ij} is the Lorentz metric in canonical form; we have $g_{\mu\nu}(x) = \eta_{ij} h^i_\mu(x) h^j_\nu(x)$, where $e^i(x) = h^i_\mu(x) dx^i$ is the dual basis. The non-null torsion tensor which flows from the Weitzenböck connection is defined as

$$T^\beta_{\mu\nu} = \hat{\Gamma}^\beta_{\nu\mu} - \hat{\Gamma}^\beta_{\mu\nu} = h^\beta_i (\partial_\mu h^i_\nu - \partial_\nu h^i_\mu), \tag{87}$$

and the action integral of the gravitation field equations in $f(T)$ gravity is assumed to be

$$\mathcal{A}_T = \int d^4 x \, |e| \, f(T) + \int d^4 x \, |e| \, L_m, \tag{88}$$

where $e = \det(e^i_\mu \cdot e^i_\nu) = \sqrt{-g}$.

The scalar T is given from the following expression,

$$T = S_\beta{}^{\mu\nu} T^\beta{}_{\mu\nu}, \tag{89}$$

where

$$S_\beta{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_\beta + \delta^\mu_\beta T^{\theta\nu}{}_\theta - \delta^\nu_\beta T^{\theta\mu}{}_\theta), \tag{90}$$

and $K^{\mu\nu}{}_\beta$ is the contorsion tensor

$$K^{\mu\nu}{}_\beta = -\frac{1}{2}(T^{\mu\nu}{}_\beta - T^{\nu\mu}{}_\beta - T_\beta{}^{\mu\nu}), \tag{91}$$

which equals the difference of the Levi-Civita connection in the holonomic and the unholonomic frame. We note that, in the special case where $f(T) = \frac{T}{2}$, the gravitational field equations are that of GR (Li, Miao & Miao 2011; Haro & Amorós 2013).

For the spatially flat FLRW space–time (equation 2) with a perfect fluid $\bar{p} = (\gamma - 1) \bar{\rho}$ minimally coupled to gravity, and for the vierbeins given by the diagonal tensor,

$$h^i_\mu(t) = \text{diag}(1, a(t), a(t), a(t)), \tag{92}$$

the modified Friedmann's equation is (Basilakos et al. 2013; Nesseris et al. 2013)

$$12H^2 f' + f = \bar{\rho}, \tag{93}$$

while the modified Raychaudhuri equation is as follows:

$$48H^2 \dot{H} f'' - 4(\dot{H} + 3H^2) f' - f = \bar{p}, \tag{94}$$

where $f'(T) = \frac{df(T)}{dT}$ and $T = -6H^2$. Finally, for the perfect fluid from the Bianchi identity, it follows that $\dot{\bar{\rho}} + 3H(\bar{\rho} + \bar{p}) = 0$. Obviously, the extra terms which arise from the function $f(T)$ can be

seen as an extra fluid. In this work, we are interested in the evolution of the total fluid.

Now, with the use of equation (93), equation (94) becomes

$$\dot{H} = -\frac{3\gamma}{2} \left(\frac{4H^2 f' + \frac{f}{3}}{2f' - 24H^2 f''} \right), \tag{95}$$

which is a first-order differential equation on H, since $f(T) = f\left(\sqrt{\frac{1}{6}|T|}\right) = f(H)$. It is easy to see that equation (95) is the same in comparison with equation (9) and provides the same solution if and only if

$$\frac{4H^2 f' + \frac{f}{3}}{2f' - 24H^2 f''} = H^2 \left(1 - \frac{\Gamma}{3H}\right), \tag{96}$$

or equivalently,

$$H^2 \left(1 - \frac{\Gamma}{3H}\right) \left(\frac{d^2 f}{dH^2}\right) - 2\left(H \left(\frac{df}{dH}\right) - f\right) = 0. \tag{97}$$

The latter is a linear non-autonomous second-order differential equation. For example, when the particle creation rate is $\Gamma(H) = mH$, then from equation (97) we have the solution

$$f(T) = f_0 \sqrt{|T|} + f_1 T^{\frac{3}{3-m}} \tag{98}$$

while for $\Gamma(H)$, given by equation (21), $f(T)$ function is given in terms of the Legendre polynomials. On the other hand, starting from a known $f(T)$ model, the solution of the algebraic equation (97) provides us with the function $\Gamma(H)$.

Here, we would like to remark that the evolution of the perfect fluid, with energy density $\bar{\rho}$, will be different from that of the matter creation model with energy density ρ. However, the total fluid, i.e. the fluid $\bar{\rho}$, and the fluid components which correspond to $f(T)$ gravity provide us with an effective fluid which has the same evolution with the fluid ρ, of the previous sections when equation (97) holds.

However, as far as equation (98) is concerned, since equation (95) provides us with the same scalar factor with equation (9) for $\Gamma(H) = mH$, or because only the r.h.s. of equation (96) depends only on m, then we can say that the constants f_0, f_1 are not essential, while m is related with the power of the power-law solution of the scale factor and specifically for $m \neq 3$, it holds that $a(t) = a_0 t^p$, $p = \frac{2}{3-m}$ (Basilakos et al. 2013). Of course, the equivalence between these two theories is only on the level that they can provide the same scale factor, which is possible since the two theories have exactly the same degree of freedom, in contrast to $f(R)$-gravity which has more degrees of freedom.

6 SUMMARY AND DISCUSSIONS

In the present work, we have addressed several issues concerning the expanding universe powered by adiabatic matter creations. In general, for any cosmological model, the dynamical analysis plays a very important role related to its stability issues. As matter creation models are phenomenological and the literature contains a variety of models, so a generalized model could be a better choice to start with for any study in any context. Hence, in the present work, we have taken a generalized matter creation model as $\Gamma = -\Gamma_0 + mH + n/H$ (where Γ_0, m, n are real numbers). Then solving the evolution equation described by the Raychaudhuri equation, the model gives 'two' fixed points, one of which is unstable or repeller in nature (represented by H_+) describing the early inflationary phase of the universe, and the other one is a stable or attractor fixed point (represented by H_-) leading to the present accelerated expansion of

the universe asymptotically which is of de Sitter type. In addition to this, the model depicts a non-singular universe. That means it had no big bang singularity in the past. Further, we have shown that it is possible to find the analytic solutions for such a scenario. Hence, we found a model of a non-singular universe describing two successive accelerated expansions of the universe at early and present times. We then applied the Jacobi last multiplier method in our framework, and found a Lagrangian which can be taken as an equivalent description to realize such a scenario as we found from the dynamical analysis of the present matter creation model. Also, we have shown that, under a simple condition, Jacobi last multiplier can give rise to a Lagrangian (see equation 46) which predicts a model of our Universe constituting a cosmological constant and a perfect fluid, which can be realized as a ΛCDM model under certain choice of the parameters involved (see Section 3.2). Moreover, we found that the analytic solution for this Lagrangian (equation 46) is an equivalent character with the Brans–Dicke cosmology. Now, performing a joint analysis of SNIa and baryon BAO sets, we constrained the density parameters of the model and hence, the Brans–Dicke parameter.

Now, in order to survey the predicted early accelerated expansion without big bang singularity as produced by our matter creation model, we introduced an equivalent field theoretic description governed by a single scalar field, for the dynamics of the universe supervised by the matter creation mechanism. The prescription established a relation between these two approaches where we were able to produce a complete analytic structure of the field theory, that means it is possible to get explicit analytic expressions for φ and $V(\varphi)$. Further, introducing the slow roll parameters for this scalar field model, we have calculated the spectral index, its running and the ratio of tensor to the scalar perturbations, and finally compared with the latest *Planck* data sets (Planck Collaboration XXII 2014a, see table 5) which stay in 95.9 per cent C.L. Also, we have shown that it is possible to give a bound on the constant Γ_0 of the matter creation rate that allows us to calculate approximately the reheating and thermalization temperature of the universe.

After that, we have introduced the effects of the teleparallel gravity $f(T)$ in the matter creation model, and shown that it is possible to establish an exact functional form of $f(T)$ for matter creation models.

Finally, one thing is clear that the present work keeps itself in the domain of cosmology, more specifically in the accelerating cosmology which is a certain plight to understand the evolution of the universe. It seems reasonable that the matter creation mechanism can be studied in several contexts, such as one can construct an equivalent cosmological theory, for instance, using its equivalence with decaying vacuum models as $\Lambda(H) = \Gamma H$, established in Graef et al. (2014), one may find the corresponding cosmological evolutions driven by decaying vacuum models, and further one may analyse its effect on astrophysical objects, namely stellar evolution (specifically, in wormhole configuration), gravitational collapse and structure formation of the universe, which can be considered for future works.

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