RIGID PLATE FRAMEWORKS

Problems concerning the rigidity of frameworks are of interest in many contexts, ranging from architecture and engineering (the design of bridges, towers and other structures) to recreational mathematics. In this paper we deal with rigidity problems that lie somewhere between the serious and frivolous. Their solution requires both constructive ingenuity as well as some knowledge of the theory of rigidity.

We begin by confining ourselves to the plane, and instead of the more usual rods we use polygonal plates pivoted at their vertices. More precisely, by a plate framework we mean a set of plates, which are pairwise congruent regular n-gons ($n \geq 3$), such that:

1. the number of plates is finite;
2. no two plates coincide;
3. each vertex of every polygonal plate is a pivot;
4. every pivot is a vertex of precisely two plates;
5. no two pivots coincide.

Les charpentes de plaques rigides

Les problèmes touchant la rigidité des charpentes suscitent l'intérêt dans divers contextes allant de l'architecture et du génie (la conception de ponts, de tours et d'autres structures) aux mathématiques récréatives. Dans cet article, nous traitons de problèmes de rigidité qui se situent quelque part entre le sérieux et le frivole. Leurs solutions exigent à la fois de l'ingéniosité pour les constructions et une certaine connaissance de la théorie de la rigidité.

Pour commencer, nous nous limitons au plan et, au lieu des habituelles tiges, nous utilisons des plaques polygonales articulées à leurs sommets. Plus précisément, par une charpente de plaques nous entendons un ensemble de plaques qui sont des n-gones ($n \geq 3$) réguliers congruents par paire, tel que:

1. le nombre de plaques est fini ;
2. jamais deux plaques ne coïncident  ;
3. chaque sommet de chaque plaque polygonale est une articulation ;
4. chaque articulation est un sommet d'exactement deux
Condition ii is introduced to exclude trivial examples, such as that consisting of two \( n \)-gonal plates superimposed and pivoted together at their \( n \) vertices. Note that the plates are permitted to (partially) overlap. Plate frameworks formed the topic of the unsolved part of Advanced Problem #6367, published in the American Mathematical Monthly [2], which asked for rigid structures of this kind.

In Figure 1 we give two examples of plate frameworks. The first, which uses \( 2n \) \( n \)-gons (in the figure \( n = 5 \); see [2] for this construction) is not rigid; that is, the angles at which the polygons meet are not uniquely determined. The second, which uses 12 squares \( (n = 4) \) is rigid; it is the only solution to the original problem that was ever published [2]. The fact that one of these frameworks is rigid and the other is not is far from obvious, and we invite
the reader to convince himself here, and in later examples, that our statements concerning rigidity are correct.

We now establish the following surprising result: there exist infinitely many rigid plate frameworks in the plane using as plates either equilateral triangles \((n = 3)\), or else squares \((n = 4)\). On the other hand, we know of no rigid plate frameworks with \(n\)-gonal plates \((n \geq 5)\), and we conjecture that such plate frameworks exist.

To begin with, let us consider rigid plate frameworks with triangular plates. In Figure 2 we show such a framework with 42 plates. If each triangular plate in Figure 2 is replaced by a triplet of plates such as those in the “triangle” ABC we obtain a rigid plate framework with \(3 \times 42 = 126\) triangular plates. This process may be repeated \(n\) times, \(n = 1, 2, \ldots\), which produces an infinite sequence of rigid plate frameworks using \(3^n \times 42\) triangular plates. This demonstrates the first part of our assertion. The frameworks that have been described are not the only rigid plate frameworks composed of triangular plates; several other constructions are possible.

For square plates, we use a different approach. The framework in Figure 3 can be described as being constructed from 12 squares built on the sides of a regular star polygon with 12 sides \(12/(5,1)\), so that each one contains the central part of the star. In a different context, this arrangement, as well as the arrangement of squares shown in Figure 1b, have been used.

Pour les plaques carrées, nous utilisons une approche différente. La charpente de la figure 3 peut être décrite comme constituée de 12 carrés construits sur les côtés d’un polygon étoilé régulier à 12 côtés \(12/(5,1)\), de sorte que chacun d’eux renferme la partie centrale de l’étoile. Dans un contexte différent, cet arrangement, ainsi que l’arrangement de carrés montré à la figure 1b, ont été utilisés.
repeated \( n \) times, \( n = 1, 2, \ldots \), leading to an infinite sequence of rigid plate frameworks, with \( 3^n \times 42 \) triangular plates. Thus the first part of our assertion is proved. The rigid plate frameworks just described are not the only ones with triangular plates — many other constructions are possible.

For square plates we use a different approach. The framework in Figure 3 may be described as consisting of 12 squares constructed on the sides of a regular 12-sided star polygon \((12/5)\), so that each contains the central part of the star. In a different context, both this and the arrangement of squares shown in Figure 1b were described in [3].

Starting from these two frameworks, infinitely many others can be constructed by the process of addition described on p. 58 of [3]; examples of frameworks obtained in this way are shown in Figure 4. To add two rigid plate frameworks we place them in such a position that some squares of one coincide with some squares of the other, and then delete all the duplicated squares. If the choices are suitably made, the resulting collection is also rigid and is a plate framework because each vertex belongs to exactly two squares. In Figure 4a we show the result of adding two copies of the framework in Figure 1b — here the frameworks are positioned so that two duplicated squares are deleted, and the resulting framework contains 20 square plates. In Figure 4b we show the result of adding two of the frameworks shown in Figure 3. Again two duplicated squares are deleted. Sometimes the process of addition is not possible since it leads to collections of squares that violate the definition of a plate framework. For example, it is not possible to add the framework in Figure 1b to that in Figure 3 since this will lead to a point (the center of one of the external 12-gons) which is a vertex of four squares. However, in the case of the framework shown in Figure 3, any number of copies can be added and in some cases a few copies of the framework in Figure 1b may also be included! Thus we obtain an infinite number of rigid plate décrits en [3].

À partir de ces deux charpentes, il est possible d’en construire une infinité d’autres par le procédé d’addition décrit à la page 58 de [3] ; des exemples de charpentes obtenues de cette manière sont montrés à la figure 4. Pour additionner deux charpentes de plaque rigides, on les place dans une position telle que certains carrés de l’une coïncident avec certains carrés de l’autre, et on supprime alors tous les carrés doubles. Si les choix sont faits convenablement, l’assemblage qui en résulte est rigide lui aussi et constitue une charpente de plaques puisque chaque sommet appartient à exactement deux carrés. La figure 4a montre le résultat de l’addition de deux copies de la charpente de la figure 1b; ces charpentes y sont disposées de façon à supprimer deux carrés doubles, et la charpente qui en résulte renferme 20 plaques carrées. La figure 4b montre le résultat de l’addition de deux des charpentes montrées à la figure 3. Ici aussi on supprime deux carrés doubles. Il est quelquefois impossible d’appliquer le procédé d’addition parce qu’il produit des assemblages de carrés qui violent la définition d’une charpente de plaques. Par exemple, il n’est pas possible d’additionner la charpente de la figure 1b à celle de la figure 3 puisqu’il en resulterait un point (le centre d’un des 12-gones externes) qui serait le sommet de quatre carrés. Toutefois, dans le cas de la charpente montrée à la figure 3, on peut additionner n’importe quel nombre de copies et, dans certains cas, on peut aussi leur ajouter quelques copies de la charpente de la figure 1b ! On obtient ainsi un nombre infini de charpentes de plaques rigides et à plaques carrées, ce qui complète la démonstration de notre assertion.

On observera que, dans les charpentes des figures 3 et 4b, certaines paires de carrés comportent des arêtes qui se chevauchent en partie et que certaines paires ont deux sommets (et une arête) en commun. Ces possibilités ne sont pas exclues par la définition d’une charpente de plaques. Si toutefois on souhaite les éviter, il
frameworks with square plates, completing the proof of our assertion.

It will be observed that in the frameworks of Figures 3 and 4 some pairs of squares have partially overlapping edges and some pairs share two vertices (and an edge). These possibilities are not excluded by the definition of a plate framework. If, however, we wish to avoid them, then a variant of the addition procedure still enables one to construct an infinite number of rigid plate frameworks with

existe une variante du procédé d'addition qui permet quand même de construire un nombre infini de charpentes de plaques rigides et à plaques carrées. Deux exemples particulièrement symétriques, constitués de 28 et de 36 carrés, sont montrés à la figure 5. Un examen de ces diagrammes devrait mettre en évidence la façon de construire d'autres charpentes semblables. Il serait intéressant de caractériser les cas où les charpentes de plaques obtenues sont rigides.

Parmi les exemples qui viennent d'être décrits, ceux à plaques triangulaires et ceux à plaques carrées diffèrent de façons significatives. Par exemple, ceux qui impliquent des plaques triangulaires ne sont rigides que lorsque les plaques sont enfermées dans le plan, alors que les charpentes qui impliquent des carrés semblent être rigides même en trois dimensions. (S'il n'est pas très difficile de démontrer la rigidité des charpentes des figures 1b et 3 dans l'espace à trois dimensions, il ne semble pas facile de trouver une démonstration générale.) Aussi les charpentes de plaques rigides et à plaques triangulaires de nos exemples sont constituées de triangles qui ne se chevauchent pas. Mais, d'autre part, en étudiant les points extrêmes des enveloppes convexes, il est facile de montrer que dans les cas des charpentes qui utilisent des plaques carrées (ou, en fait, des plaques n-gonales où n ≥ 4) un certain chevauchement doit nécessairement se produire.

FIGURE 4
Two rigid frameworks obtained by the addition process described in the text. Deux charpentes rigides obtenues par le procédé d'addition décrit dans le texte.
square plates. Two particularly symmetric examples, with 28 and 36 squares, are shown in Figure 5. From an examination of these diagrams the method of constructing other such frameworks should be apparent. It would be interesting to characterize the cases in which rigid plate frameworks are obtained.

The examples with triangular and square plates just described differ in significant ways. For example, those involving triangular...
plates are only rigid when the plates are confined to the plane, but
the frameworks involving squares seem to be rigid in three dimen-
sional space as well. (It is not very hard to prove the rigidity in
3-dimensional space of the frameworks in Figures 1b and 3, but a general
proof appears elusive.) Also, our examples of rigid plate frame-
works with triangular plates consist of triangles which do not
overlap. On the other hand, it is easy to show, by considering the
extreme points of the convex hull, that in the case of frameworks
which use square plates (or, indeed, \( n \)-gonal plates with \( n \geq 4 \))
some overlapping necessarily occurs.

There are analogous problems in 3-dimensional space, using the
same definition of a plate framework as in the plane. It now be-
comes possible to construct rigid plate frameworks with non-
overlapping \( n \)-gonal plates for \( n = 3, 4, 5, 6, 8 \) or 10. For \( n = 3 \) we
choose four edge-disjoint faces of an octahedron; for \( n = 4 \), the six
square faces of a cuboctahedron; for \( n = 5 \), the twelve pentagonal
faces of an icosidodecahedron; for \( n = 6 \), the four \{eight, twenty\}
hexagonal faces of a truncated tetrahedron \{octahedron, icosahed-
ron\}; for \( n = 8 \), the six octagonal faces of a truncated cube; and
for \( n = 10 \), the twelve decagonal faces of a truncated dodecahe-
dron. (See [1] for descriptions and drawings of the Archimedean
solids mentioned above.) The rigidity of these plate frameworks
follows from Cauchy’s famous rigidity theorem (see [5]), since the
plates by themselves provide as much rigidity as the whole set of
faces of the polyhedron. The process of addition described above
can be applied to these three-dimensional frameworks, and so we
obtain an infinite number of other plate frameworks which seem
to be rigid; some of them are not infinitesimally rigid. However, we
do not know of any general criteria that guarantee rigidity in such
situations. Moreover, we do not know of any rigid plate frame-
works in three-dimensional space that use \( n \)-gonal plates for a
value of \( n \) other than those given above \( (3, 4, 5, 6, 8 \) and 10).
The concept of a plate frameworks can be extended to three-dimensional space in another way. Let us define a **solid framework** as a collection of pairwise congruent regular solids (polyhedra), no two of which are coincident and which are pivoted at their vertices; each vertex of every one of the solids is a pivot, and every pivot is a vertex of precisely two polyhedra. Examples of a trivial nature are easy to construct: from any of the rigid plate frameworks with square plates in the plane we can construct a solid framework with cubes by simply placing cubes on top of each square plate. In fact, apart from such trivial examples we do not know with certainty of any rigid plate frameworks, but we have two candidates. One is the family of five cubes inscribed in a Platonic (that is, regular pentagonal) dodecahedron (see, for example, [J, Figure 167], [4, page 36]). The other candidate is the family of ten regular tetrahedra inscribed in the regular pentagonal dodecahedron (see, for example, [1, Figure 177], [6, page 45]). If this second family is indeed rigid, then it should be possible to construct further examples consisting of tetrahedra by using the *addition* process described above. But we have no information or results with which to formulate any conjectures concerning the existence of other kinds of non-trivial rigid solid frameworks.

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