On geodetic sets formed by boundary vertices

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Abstract

Let G be a finite simple connected graph. A vertex v is a boundary vertex of G if there exists a vertex u such that no neighbor of v is further away from u than v. We obtain a number of properties involving different types of boundary vertices: peripheral, contour and eccentric vertices. Before showing that one of the main results in [3] does not hold for one of the cases, we establish a realization theorem that not only corrects the mentioned wrong statement but also improves it.

Given $S \subseteq V(G)$, its geodetic closure $I[S]$ is the set of all vertices lying on some shortest path joining two vertices of S. We prove that the boundary vertex set $\partial(G)$ of any graph G is geodetic, that is, $I[\partial(G)] = V(G)$. A vertex v belongs to the contour $Ct(G)$ of G if no neighbor of v has an eccentricity greater than v. We present some sufficient conditions to guarantee the geodeticity of either the contour $Ct(G)$ or its geodetic closure $I[Ct(G)]$.

Key words: Boundary, contour, eccentricity, geodesic convexity, geodetic set, periphery.

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1 Introduction

Usual Euclidean convexity can be extended, as an abstract structure, to the vertex set of a graph in a natural fashion, just by considering shortest paths between vertices [4]. This fact lies into a more general theory, the so called *abstract convexity* [10], that allows to translate typical convex concepts to different environments. As an example, a vertex subset S of a graph G is said to be *convex* if it contains all the shortest paths connecting any pair of vertices in S [8]. Instead of shortest paths, other path classes can be placed in this definition, such as chordless paths [4,5] or triangle-free paths [2], giving rise to interesting graph convexity structures.

We consider only finite, simple, connected graphs. For undefined basic concepts we refer the reader to introductory graph theoretical literature, e.g., [9]. Given vertices u, v in a graph $G = (V, E)$ we let $d_G(u, v)$ denote the distance between u and v in G. When there is no confusion, subscripts will be omitted. A $u - v$ path ρ is called a $u - v$ *geodesic* if it is a shortest $u - v$ path, that is, if $|E(\rho)| = d(u, v)$. The *geodetic interval* $I[u, v]$ is the set of vertices of all $u - v$ geodesics. For $S \subseteq V$, the *geodetic closure* $I[S]$ of S is the union of all geodesic closed intervals $I[u, v]$ over all pairs $u, v \in S$, i.e., $I[S] = \bigcup_{u,v \in S} I[u, v]$.

A (finite) *graph convexity* space is a pair $(G, \mathcal{C})$, formed by a finite connected graph $G = (V, E)$ and a family $\mathcal{C}$ of subsets of V (each such set called a *convex set*) which is closed under intersection, which contains both V and the empty set, and such that every convex set induces a connected subgraph of G.

In this paper, we consider only the so-called *geodesic convexity* $\mathcal{C}_g$ defined as follows. A vertex set $W \subseteq V$ is called *geodesically convex* (or simply *convex*) if $I[W] = W$. Given a set $A \subseteq V$, the smallest convex set containing is denoted $[A]$ and is called the *convex hull* of A. A non-empty set $A \subseteq V$ is called a *hull set* if $[A] = V$, and it is said to be *geodetic* if moreover $I[A] = V$.

Given a graph convexity space $(G, \mathcal{C})$ and a convex set $W \subseteq V(G)$, a vertex $v \in W$ is called an *extreme vertex* of W if the set $W \setminus \{v\}$ is also convex. The convexity $\mathcal{C}$ is called a *convex geometry* if it satisfies the so-called *Krein-Milman* property: *Every convex set is the convex hull of its extreme vertices.* Certainly, this condition can be seen as a rebuilding method, that allows us to recover any convex set from its extreme points, by means of the convex hull operator. Under this point of view, the interest of any similar property is that a small subset of any convex set keeps all the information of the whole set. For computational purposes, this fact represents a kind of store saving.

A graph is called *Ptolemaic* if it is distance-hereditary and chordal, that is, if every chordless path is a geodesic and every cycle of length strictly greater than three possesses a chord. In [4], Farber and Jamison proved that the geodesic convexity $\mathcal{C}_g$ of a graph G is a convex geometry if and only if G is Ptolemaic. Thus, we could think of extending this property in two different ways. On the one hand, recovering convex sets on wider graph classes, and on the other hand, using an operator simpler than the convex hull one $[\,]$, such as for example the geodetic closure operator I. In both cases, finding new vertex sets playing a similar role to that of extreme vertices is necessary.

Concerning the first mentioned extension of the Krein-Milman property, Cáceres et alt. [1] obtained a similar property to this one, valid for every graph, by considering, instead of the extreme vertices, the so-called *contour vertices* (see Subsection 2.1). As for the second generalization, consisting in using the geodetic interval operator I, a number of results have been recently obtained [1,6,7]. For example, it has been proved that in the class of distance-hereditary graphs, every convex set is the geodetic closure of its contour vertices [1,7].

The rest of this paper is organized as follows. In Section 2, we focus our attention on several types of *boundary vertices* [3]: extreme, peripheral, contour and eccentric vertices, obtaining a number of basic structural properties. In addition, we show that one of the main results in [3] does not hold, and establish a realization theorem that not only corrects the mentioned wrong statement but also improves it. Finally, in Section 3, we approach the problem of finding geodetic sets consisting of boundary vertices, proving that the boundary vertex set $\partial(G)$ of any graph G is geodetic and presenting some sufficient conditions to guarantee the geodeticity of either the contour $Ct(G)$ or its geodetic closure $I[Ct(G)]$.

2 Boundary vertices

2.1 Definitions and basic properties

Let $G = (V, E)$ be a connected graph and $u, v \in V$. The vertex v is said to be a *boundary vertex* of u if no neighbor of v is further away from u than v [3]. A vertex v is called a *boundary vertex of* G if it is the boundary vertex of some vertex $u \in V$.

Definition 1 [3] *The boundary $\partial(G)$ of G is the set of all of its boundary vertices:*

$$\partial(G) = \{v \in V \mid \exists u \in V \ s.t.\ \forall w \in N(v) : d(u, w) \leq d(u, v)\}.$$

Given a vertex set $W \subseteq V$, the *eccentricity* in W of a vertex $u \in W$ is defined as $ecc_W(u) = \max\{d(u, v) \mid v \in W\}$. In particular, $ecc_G(u) = ecc(u) =$

$\max\{d(u,v) \mid v \in V\}$. Given $u, v \in V$, the vertex v is called an *eccentric vertex* of u if no vertex in V is further away from u than v, that is, if $d(u,v) = ecc(u)$. A vertex v is called a *eccentric vertex of* G if it is the eccentric vertex of some vertex $u \in V$.

Definition 2 *The eccentricity $Ecc(G)$ of G is the set of all of its eccentric vertices:*

$$Ecc(G) = \{v \in V \mid \exists u \in V \text{ s.t. } ecc(u) = d(u,v)\}.$$

In a similar way, we can define the eccentricity of any proper subset W of V:

$$Ecc(W) = \{v \in V \mid \exists u \in W \text{ s.t. } ecc(u) = d(u,v)\}.$$

A vertex $v \in V$ is called a *peripheral vertex* of G if no vertex in V has an eccentricity greater than $ecc(v)$, that is, if the eccentricity of v is exactly equal to the diameter $D(G)$ of G.

Definition 3 *The periphery $Per(G)$ of G is the set all of its peripheral vertices:*

$$Per(G) = \{v \in V \mid ecc(u) \leq ecc(v), \forall u \in V\} = \{v \in V \mid ecc(v) = D(G)\}.$$

A vertex $v \in V$ is called a *contour vertex* of G if no neighbor vertex of v has an eccentricity greater than $ecc(v)$.

Definition 4 [1] *The contour $Ct(G)$ of G is the set all of its contour vertices:*

$$Ct(G) = \{v \in V \mid ecc(u) \leq ecc(v), \forall u \in N(v)\}.$$

Finally, a vertex $u \in V$ is called *simplicial* if the subgraph induced by its neighborhood, $G[N(v)]$, is a clique. Notice that a vertex is simplicial if and only if it is an extreme vertex of G.

Definition 5 *The extreme set $Ext(G)$ of G is the set of all its simplicial vertices:*

$$Ext(G) = \{v \in V \mid G[N(v)] \text{ is a clique}\}.$$

As a direct consequence of these definitions, the following properties are immediately derived.

Proposition 6 *Let $G = (V,E)$ be a connected graph. Then, the following statements hold (see Figure 1).*

(1) $Ext(G) \subseteq Ct(G)$.

(2) $Per(G) \subseteq Ct(G) \cap Ecc(G)$.
(3) $Ecc(G) \cup Ct(G) \subseteq \partial(G)$.

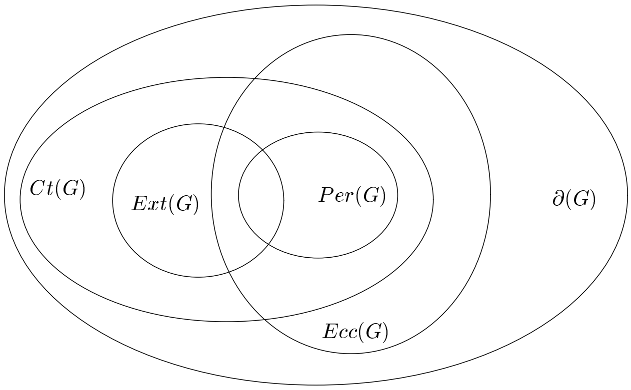


Figure 1. Boundary-type sets.

Next, we present a number of additional properties involving these boundary vertex sets.

Lemma 7 *Let $G = (V, E)$ be a connected graph and $x \in V \setminus Ct(G)$. Then, there exists a geodesic $\rho(x) = x_0x_1x_2 \cdots x_r$ such that $ecc(x_i) = ecc(x_{i-1}) + 1$, $i = 1, \ldots, r$ and $x_r \in Ct(G)$.*

PROOF. Since the eccentricities of two adjacent vertices differ by at most one unit, if x is not a contour vertex, then there exists a vertex $y \in V$, adjacent to x, such that its eccentricity satisfies $ecc(y) = ecc(x) + 1$. This fact implies the existence of a path $\rho(x) = x_0x_1x_2 \cdots x_r$, such that $x = x_0$, $x_i \notin Ct(G)$ for $i \in \{0, \ldots, r-1\}$, $x_r \in Ct(G)$, and $ecc(x_i) = ecc(x_{i-1}) + 1 = l + i$ for $i \in \{1, \ldots, r\}$, where $l = ecc(x)$. Moreover, $\rho(x)$ is a shortest $x - x_r$ path, since otherwise, the eccentricity of x_r would be less than $l + r$.

Proposition 8 *Let $G = (V, E)$ be a connected graph.*

(1) *If $Ct(G) = Per(G)$, then $I[Ct(G)] = V$.*
(2) *If $|Per(G)| = |Ct(G)| = 2$, then either $|\partial(G)| = 2$ or $|\partial(G)| \geq 4$.*
(3) *If $|Ecc(G)| = |Per(G)| + 1$, then $|\partial(G)| > |Ecc(G)|$.*
(4) *If $|Ecc(G)| > |Per(G)|$, then $|\partial(G)| \geq |Per(G)| + 2$.*

PROOF.

(1) Let x be a vertex of $V(G) \setminus Ct(G)$. According to Lemma 7, there exist a vertex $x_r \in Ct(G)$ and a $x - x_r$ geodesic $\rho(x)$ of length r such that $ecc(x_r) = l + r$, where $l = ecc(x)$. But $x_r \in Ct(G) = Per(G)$ implies that $ecc(x_r) = D$ and $D = l + r$. Thus, there exists a vertex $z \in Per(G)$ such that $D = d(z, x_r) \leq d(z, x) + d(x, x_r) \leq ecc(x) + r = l + r = D$, that is,

$d(z,x_r) = d(z,x) + d(x,x_r)$. Hence, x is on a shortest path between the vertices $z, x_r \in Per(G) = Ct(G)$.

(2) Suppose that $|\partial(G)| = 3$, that is, $Per(G) = Ct(G) = \{a,b\}$ and $\partial(G) = \{a,b,c\}$. According to the previous item, the vertex c lies on some $a-b$ geodesic P. Let W be the set of all vertices of which c is a boundary vertex. Notice that $W \cap V(P) = \emptyset$. Take a vertex $y \in W$ satisfying $d(y,c) = \max_{x\in W} d(x,c) = h$. Certainly, $y \notin \partial(G)$, since $y \notin V(P)$ and $|\partial(G)| = 3$. In particular, y is not a boundary vertex of the vertex c, i.e., there exists a neighbor z of y such that $d(c,z) = h+1$ (see Figure 2). Notice that $z \notin V(P)$ since c is a boundary vertex of y. If w is an arbitrary neighbor of the vertex c, then $d(z,w) \leq d(z,y) + d(y,w) \leq 1+h = d(z,c)$. Hence, we have proved that z is a boundary vertex of c, which is a contradiction.

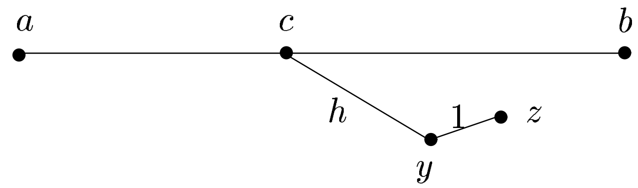


Figure 2.

(3) Let $x \in Ecc(G) \setminus Per(G)$. Take the set $W = \{y \in V \mid d(y,x) = ecc(y)\}$. Notice that $W \cap Per(G) = \emptyset$, since $x \notin Per(G)$. Observe also that $W \cap Ecc(G) = \emptyset$, since $Ecc(G) = Per(G) \cup \{x\}$. Consider a vertex $z \in W$ such that $ecc(z) = \max_{y\in W} ecc(y)$. In order to prove that z is a boundary vertex of x, let us suppose, on the contrary, that there exists a vertex $w \in N(z)$ such that $d(w,x) = d(z,x)+1$. It means both that $ecc(w) = ecc(z)+1$ and $w \in W$, which is a contradiction. Hence, $z \in \partial(G)$ and we are done.

(4) This result is a corollary of the previous one since $Per(G) \subset Ecc(G) \subseteq \partial(G)$.

2.2 A realization theorem

As a direct consequence of Propositions 6 and 8, we obtain the following result.

Corollary 9 *Let G be a nontrivial connected graph such that $|Per(G)| = a$, $|Ct(G)| = b$, $|Ecc(G)| = c$ and $|\partial(G)| = d$. Then,*

$$\begin{cases} 2 \le a \le b \le d, \\ 2 \le a \le c \le d, \\ (a,b,c,d) \ne (2,2,2,3), \ [*] \\ (a,b,c,d) \ne (a,b,a+1,a+1). \ [**] \end{cases}$$

In [3], Chartrand, Erwin, Johns and Zhang included the following realization theorem:

Theorem 10 [3] *For each triple a, c, d of integers with $2 \le a \le c \le d$, there is a connected graph G such that $Per(G)$ has order a, $Ecc(G)$ has order c, and $\partial(G)$ has order d.*

Notice that, as a consequence of the constraint [**] displayed in Corollary 9, it is immediately derived that this result is false for the case $a+1=c=d$.

At this point, we ask ourselves the following question: *Are there further restrictions concerning the cardinalities of the sets $Per(G)$, $Ct(G)$, $Ecc(G)$ and $\partial(G)$*? Next, we present a realization theorem showing the answer to be negative. Before showing it, we first give out five lemmas. We omit the proofs as they are rather straightforward.

Lemma 11 *Let $G=(V,E)$ be a connected graph, $x \in V$ and $\lambda \ge 1$. Let $\widehat{G}$ be the graph obtained from G by replacing the vertex x by a complete graph K_λ and joining every vertex of K_λ to every neighbor of x in G. Then,*

(1) for every vertex $x_i \in V(K_\lambda)$, $ecc_{\widehat{G}}(x_i) = ecc_G(x)$,
(2) for every vertex $y \in \widehat{G} \setminus V(K_\lambda)$, $ecc_{\widehat{G}}(y) = ecc_G(y)$.

Lemma 12 *Let G_1, G_2 and G_3 be the graphs illustrated in Figures 3, 4(a) and 4(c) respectively. Then,*

(1) $Per(G_1) = \{1,7\}$, $Ecc(G_1) = \{1,7,10\}$, $Ct(G_1) = \{1,7,10,11\}$, $\partial(G_1) = \{1,4,7,10,11\}$.
(2) $Per(G_2) = Ct(G_2) = Ecc(G_2) = \{1,5\}$, $\partial(G_2) = \{1,3,5,6\}$.
(3) $Per(G_3) = Ct(G_3) = Ecc(G_3) = \{1,4,6\}$, $\partial(G_3) = \{1,2,4,6\}$.

Lemma 13 *Let G be the graph illustrated in Figure 3, and $r \ge 1$, $s \ge 1$, $t \ge 1$, $u \ge 1$. Let $\widehat{G}$ be the graph obtained from G by replacing the vertices 1, 10, 11 and 4 by K_r, K_s, K_t and K_u, respectively, as shown in Lemma 11. Then,*

$$|Per(\widehat{G})| = r+1, \quad |Ecc(\widehat{G})| = r+s+1,$$
$$|Ct(\widehat{G})| = r+s+t+1, \quad |\partial(\widehat{G})| = r+s+t+u+1.$$

Lemma 14 *Let G be the graph illustrated in Figure 4(a), and $r \ge 1$. Let $\widehat{G}$*

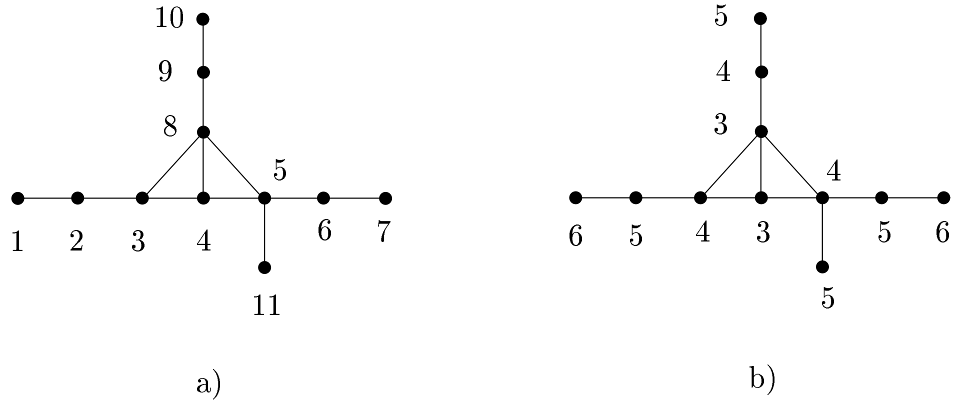


Figure 3. a) Vertex labelling, b) eccentricities.

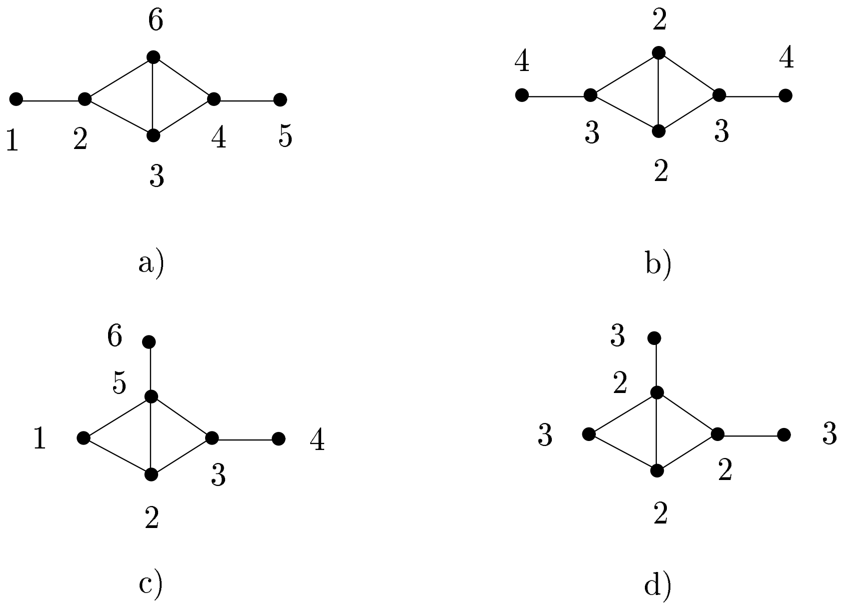


Figure 4. a) and c) Vertex labelling, b) and d) eccentricities.

be the graph obtained from G by replacing the vertex 6 by K_r, as shown in Lemma 11. Then,

$$|Per(\widehat{G})| = |Ct(\widehat{G})| = |Ecc(\widehat{G})| = 2, \quad |\partial(\widehat{G})| = r + 3.$$

Lemma 15 *Let G be the graph illustrated in Figure 4(c), and $r \geq 1$, $s \geq 1$. Let $\widehat{G}$ be the graph obtained from G by replacing the vertices 1 and 2 by K_r and K_s, respectively, as shown in Lemma 11. Then,*

$$|Per(\widehat{G})| = |Ct(\widehat{G})| = |Ecc(\widehat{G})| = r + 2, \quad |\partial(\widehat{G})| = r + s + 2.$$

Now we can show, as promised, our realization theorem, that not only corrects the *mistake* noticed in [3], but also essentially improves and completely solves the posed question.

Theorem 16 *Let* $(a, b, c, d) \in \mathbb{Z}^4$ *integers satisfying the constraints displayed in Corollary 9. Then, there exists a connected graph* $G = (V, E)$ *satisfying:*

$$|Per(G)| = a, \quad |Ct(G)| = b, \quad |Ecc(G)| = c, \quad |\partial(G)| = d.$$

PROOF. Consider the list of all possible cases (see Table 1).

Table 1: List of all possible cases in the proof of the realization theorem.

(i)	$2 \le a = b = c = d$	(ii)	$2 \le a < c < b < d$
(iii)	$2 \le a < b < c < d$	(iv)	$2 \le a = b < c < d$
(v)	$2 \le a < b = c < d$	(vi)	$2 \le a < b < c = d$
(vii)	$2 \le a = c < b < d$	(viii)	$2 \le a < c < b = d$
(ix)	$2 \le a < b = c = d$ s.t. [$**$]	(x)	$2 \le a = b < c = d$ s.t. [$**$]
(xi)	$2 \le a = b = c < d$ s.t. [$*$]	(xii)	$2 \le a = c < b = d$

(i) The complete graph K_a satisfies the desired properties.

The proof of the remaining cases is similar and based on the following procedure:

(1) Consider the *fitting* graph G in Figure 5.
(2) Replace in G a vertex $v_1 \in Per(G)$ by the complete graph K_{a-h} (see Lema 11), where: $h = |Per(G)| - 1$.
(3) If $a < b \le c$, replace in G a vertex $v_2 \in Ct(G) \setminus Per(G)$ by the complete graph K_{b-a-h}, where: $h = |Ct(G)| - |Per(G)| - 1$. If $a < c < b$, replace in G a vertex $v_2 \in Ecc(G) \setminus Per(G)$ by the complete graph K_{c-a-h}, where $h = |Ecc(G)| - |Per(G)| - 1$.
(4) If $b < c$, replace in G a vertex $v_3 \in Ecc(G) \setminus Ct(G)$ by the complete graph K_{c-b-h}, where $h = |Ecc(G)| - |Ct(G)| - 1$. If $c < b$, replace in G a vertex $v_3 \in Ct(G) \setminus Ecc(G)$ by the complete graph K_{b-c-h}, where $h = |Ct(G)| - |Ecc(G)| - 1$.
(5) If $b \le c < d$, replace in G a vertex $v_4 \in \partial(G) \setminus Ecc(G)$ by the complete graph K_{d-c-h}, where $h = |\partial(G)| - |Ecc(G)| - 1$. If $c < b < d$, replace in G a vertex $v_4 \in \partial(G) \setminus Ct(G)$ by the complete graph K_{d-b-h}, where $h = |\partial(G)| - |Ct(G)| - 1$.

For example, if $(a, b, c, d) = (21, 24, 24, 35)$, then (1) the *fitting* graph is (v), since in both cases: $a < b = c < d$; (2) a vertex $v_1 \in Per(G)$ is replaced by the complete graph K_{20}; (3) a vertex $v_2 \in Ct(G) \setminus Per(G)$ by the complete graph K_3; and (5) a vertex $v_4 \in \partial(G) \setminus Ecc(G)$ is replaced by the complete graph K_{11}.

For the sake of clarity, we show the complete proof for two cases.

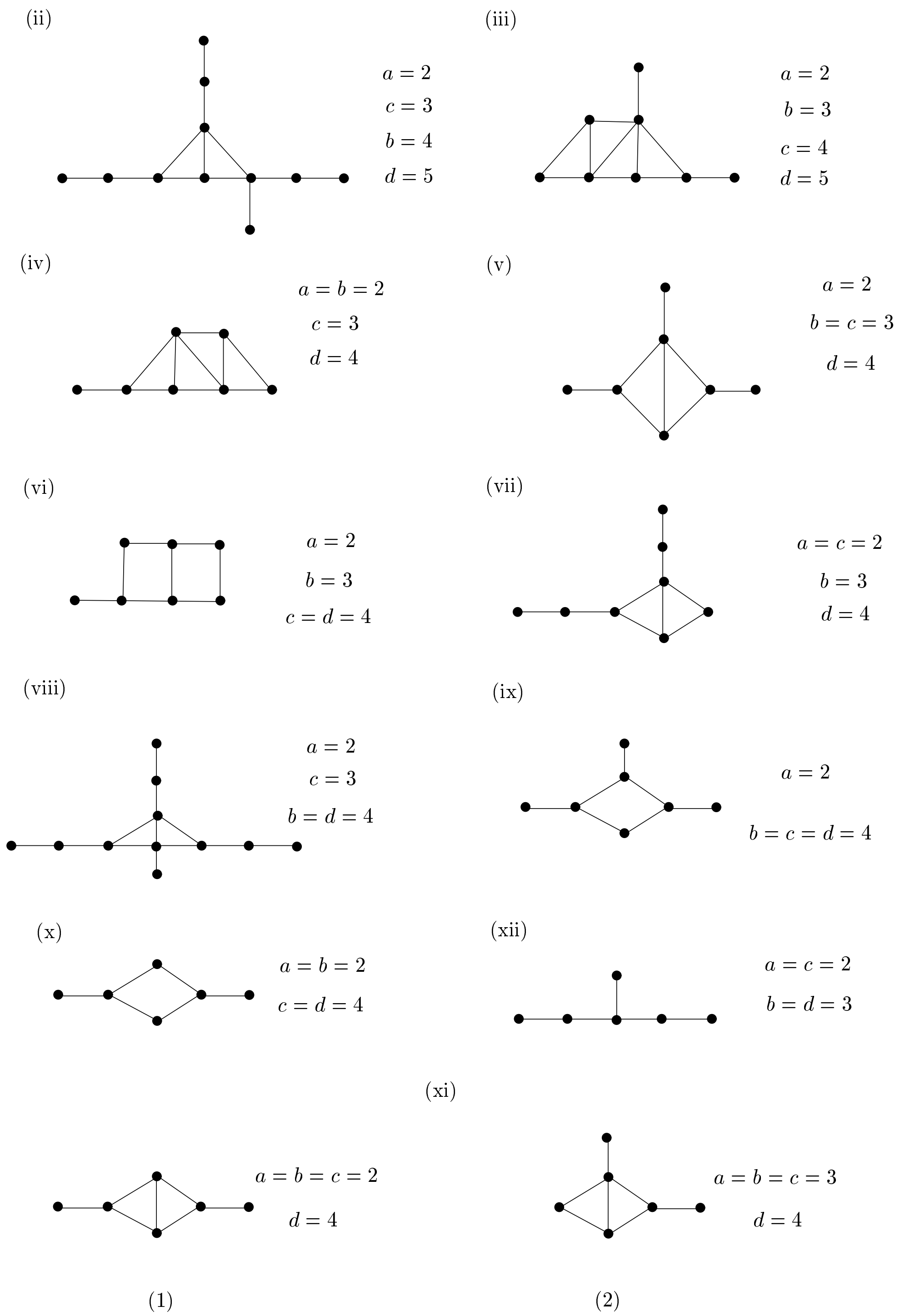


Figure 5. For each graph G, $a = |Per(G)|$, $b = |Ct(G)|$, $c = |Ecc(G)|$ and $d = |\partial(G)|$. Notice that, in all theses cases, either $Ct(G) \subseteq Ecc(G)$, or $Ecc(G) \subseteq Ct(G)$.

(ii) The graph $\widehat{G}$ described in Lemma 13 satisfies the desired conditions just by taking:

$$r = a - 1, \quad s = c - a, \quad t = b - c, \quad u = d - b.$$

(xi)(1) If $a = 2$, the graph $\widehat{G}$ described in Lemma 14 satisfies the desired conditions just by taking $r = d - 3$.

(xi)(2) If $a \geq 3$, the graph $\widehat{G}$ described in Lemma 15 satisfies the desired conditions just by taking $r = a - 2$, and $s = d - a$.

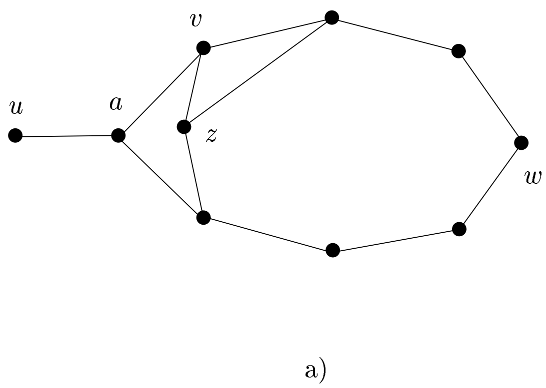
Remark 17 *It remains an open question whether a similar result can be stated by also considering the extreme set, without imposing additional non-trivial constraints.*

3 Boundary vertex sets as geodetic sets

In [1], Cáceres et alt. proved the following statement:

Theorem 18 [1] *Let $G = (V, E)$ be a connected graph and $W \subseteq V$ a convex set. Then, W is the convex hull of its contour vertices.*

As was pointed out in the same paper (see also [7]), the contour of a graph needs not to be geodetic. For example, in Figure 6, we illustrate two graphs whose contour set is $\{u, v, w\}$ and $I[\{u, v, w\}] = V \setminus \{z\}$. As for the eccentricity, it is rather easy to design a graph G such that $I[Ecc(G)] \subsetneq V(G)$ (see Figure 5(xii)).



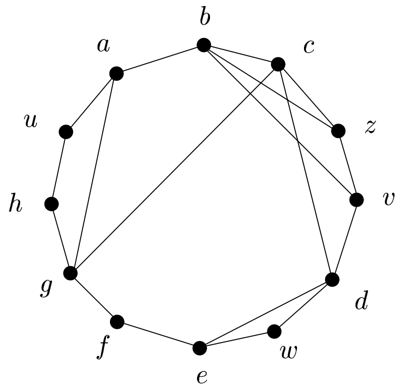


Figure 6. Two graphs whose contour is not geodetic.

Next, we prove that the boundary of every connected graph is geodetic. As a matter of fact, we present the following stronger result.

Theorem 19 *The so-called expanded contour $\Omega(G) = Ct(G) \cup Ecc(Ct(G))$ of every connected graph $G = (V, E)$ is geodetic.*

PROOF. Let x be a vertex of $V(G) \smallsetminus \Omega(G)$. Since $x \notin Ct(G)$, according to Lemma 7, there exist a vertex $x_r \in Ct(G)$ and a $x - x_r$ geodesic $\rho(x)$ of length r such that $ecc(x_r) = ecc(x) + r$. Let y_r be an eccentric vertex of x_r, i.e., such that $d(y_r, x_r) = ecc(x_r)$. Then,

$$ecc(x) + r = ecc(x_r) = d(y_r, x_r) \leq \underbrace{d(y_r, x)}_{\leq ecc(x)} + \underbrace{d(x, x_r)}_{=r} \leq ecc(x) + r$$

and hence we conclude that the inequalities in the formula above are all of them equalities, which means that the vertex x lies in a shortest path joining $x_r \in Ct(G) \subset \Omega(G)$ and $y_r \in Ecc_G(Ct(G)) \subset \Omega(G)$.

Corollary 20 *The boundary $\partial(G)$ of every connected graph $G = (V, E)$ is geodetic.*

3.1 The geodetic closure of the contour

We have seen that, in general, the contour $Ct(G)$ of a graph G needs not to be geodetic. But, *what about its geodetic closure $I[Ct(G)]$*?

To begin with, we have investigated whether, for every graph G, $\partial(G) \subseteq I[Ct(G)]$. Notice that, according to Corollary 20, the above fact would allow us to prove the geodeticity of the geodetic closure of the contour. The following remark shows this approach to be wrong.

Remark 21 *Let G be the graph illustrated in Figure 6(a). Then, it is straightforward to check that*

$$Ct(G) = \{u, v, w\}, \quad I[Ct(G)] = V(G) \setminus \{z\}, \quad \partial(G) = V(G) \setminus \{a\}.$$

From Theorem 19, we obtain the following direct consequence.

Corollary 22 *Let G be a connected graph. If $Ecc_G(Ct(G)) \subseteq I[Ct(G)]$, then $I^2[Ct(G)] = V$. That is, $I[Ct(G)]$ is a geodetic set.*

Starting from this fact, we are currently trying to prove that either, for every graph G, $\Omega(G) \subseteq I[Ct(G)]$ or else find a counterexample.

Remark 23 *Consider the graph G illustrated in Figure 6(b). It is rather simple to obtain the following results:*

$$Ct(G) = \{u, v, w\}, \quad Ecc(G) = \{b, z, v, w, e, f, h, u\}, \quad \partial(G) = Ecc(G),$$

$$Ecc(Ct(G)) = \{u, w, h\}, \quad \Omega(G) = \{u, v, w, h\}, \quad I[Ct(G)] = V(G) \setminus \{z\}.$$

Although the graph above satisfies some interesting and not very common properties such as: $Ecc_G(Ct(G)) \not\subseteq Ct(G)$ and $Ecc(G) \not\subseteq I[Ct(G)]$, it is also true that $\Omega(G) \subseteq I[Ct(G)]$. As a matter of fact, we know of no example of a graph G having an expanded contour $\Omega(G)$ not contained in the geodetic closure of its contour $I[Ct(G)]$.

Remark 24 *We know of no example of a graph G having a contour $Ct(G)$ such that its geodetic closure be not geodetic. We leave it as an open problem as to whether $I^2[Ct(G)] = V(G)$ for every connected graph G.*

Let $G = (V, E)$ be a connected graph and let $W \in V$ be a set of vertices. The *geodetic iteration number* $gin(W)$ of W is defined as the minimum integer $k \geq 1$ such that $I^k[W] = [W]$, where $I^k[W] = I[I^{k-1}[W]]$. For example, geodetic sets are those whose iteration number is equal to 1. Next, we show a number of conditions under which the contour of a graph is either a geodetic set or at least $I^2[Ct(G)] = V$.

Proposition 25 *Let $G = (V, E)$ be a connected graph. If $|Ct(G)| = 2$, then $Ct(G)$ is a geodetic set.*

PROOF. If $|Ct(G)| = 2$, then $Ct(G) = Per(G)$. Therefore, according to Proposition 8 (1), the contour $Ct(G)$ must be geodetic.

Theorem 26 *Let $G = (V, E)$ be a connected graph such that*

$$Ct(G) \setminus Per(G) = \{y_1, \ldots, y_k\}$$

and $ecc(y_i) = ecc(y_j)$, for each $i, j = 1, \ldots k$. Then, $I^2[Ct(G)] = V$.

PROOF. Suppose that $Per(G) \subsetneq Ct(G)$ since otherwise by Proposition 8 (1) $I[Ct(G)] = V$ and we are done. If $Ct(G) = \{x_1, \ldots, x_h, y_1, \ldots, y_k\}$ and $Per(G) = \{x_1, \ldots, x_h\}$, then $ecc(x_i) = D$, $i = 1, \ldots, h$, $ecc(y_j) = d$, $j = i, \ldots, k$ and $d < D$, where D is the diameter of G.

Notice that

$$Ecc(Ct(G)) = Per(G) \cup Ecc(\{y_1\}) \cup \cdots \cup Ecc(\{y_k\}),$$

as $Ecc(Per(G)) = Per(G)$. Take $v \in Ct(G) \setminus Per(G)$. Let w be an eccentric vertex of v, that is, $d(w, v) = ecc(v)$. Next, we prove that $w \in I[Ct(G)]$.

Assume that $w \notin Ct(G)$ since otherwise we are done. Then, by Lemma 7, there exists a geodesic $w_0 = w, w_1, \ldots, w_r$ such that $ecc(w_i) = ecc(w_{i-1}) + 1$,

$i = 1 \dots r$ and $w_r \in Ct(G)$. Hence, $w_r \in Per(G)$, since $ecc(v) \leq ecc(w) < ecc(w_r)$. Let z be an eccentric vertex of w_r. Note that $z \in Per(G)$ and

$$D = d(z, w_r) \leq \underbrace{d(z, w)}_{\leq ecc(w)} + \underbrace{d(w, w_r)}_{r = ecc(w_r) - ecc(w) = D - ecc(w)} \leq D$$

Thus, w lies in a geodesic joining z and w_r, and $\{z, w_r\} \subseteq Per(G) \subseteq Ct(G)$. In other words, $w \in I[Ct(G)]$. We conclude that $Ecc_G(Ct(G)) \subset I[Ct(G)]$ and by Corollary 22, $I^2[Ct(G)] = V$.

As particular cases of the above theorem the following corollaries are immediately derived.

Corollary 27 *Let $G = (V, E)$ be a connected graph such that $|Ct(G)| = |Per(G)| + 1$, then $I^2[Ct(G)] = V$.*

Corollary 28 *Let $G = (V, E)$ be a connected graph such that $|Ct(G)| = 3$, then $I^2[Ct(G)] = V$.*

PROOF. If $|Ct(G)| = 3$, then $|Per(G)| = 2$ or $|Per(G)| = 3$. In the first case, $|Ct(G)| = |Per(G)| + 1$ and we apply Corollary 27. In the second case, $Ct(G) = Per(G)$ and by Proposition 25 we deduce $I[Ct(G)] = V$.

3.2 The k-iterated geodetic closure

Having in mind Theorem 26, this subsection examines the geodeticity of the set $I^k[Ct(G)]$ for some $k \geq 2$. To begin with, we need to introduce the following definition.

Definition 29 *An integer sequence $(d_1, d_2, \dots, d_s)$ satisfying*

$$d_1 > d_2 > d > \cdots > d_s$$

is called the eccentricity sequence *of a vertex subset W of a connected graph $G = (V, E)$ if*

$$\{k \in \mathbb{N} : \ k = ecc(v), \ for\ some\ v \in W\} = \{d_1, d_2, \dots, d_s\}.$$

Moreover, the integer s is called the size of the sequence.

Proposition 30 *Let $(d_1, d_2, \dots, d_s)$ be the eccentricity sequence of the contour of a connected graph $G = (V, E)$. Let $x \in Ct(G)$ and $i \in \{1, \dots, s\}$ such that $ecc(x) = d_i$. Then, $Ecc(\{x\}) \subseteq I^{i-1}[Ct(G)]$.*

PROOF. We proceed by induction on i. First, if $i = 1$, then $ecc(x) = d_1 = D$, which means that $x \in Per(G)$. Hence, $Ecc(\{x\}) \subseteq Ct(G) = I^0[Ct(G)]$ since $Ecc(Per(G)) = Per(G)$.

Take $i \in \{2, \ldots, s\}$ and assume (as Inductive Hypothesis) that, for every vertex $z \in Ct(G)$ such that $ecc(z) = d_j$ with $j \in \{1, \ldots, i-1\}$, $Ecc(\{z\}) \subseteq I^{j-1}[Ct(G)]$. Let $x \in Ct(G)$ such that $ecc(x) = d_i$. Take $y \in Ecc(\{x\})$, i.e., such that $d(x, y) = d_i$.

Suppose that $y \notin Ct(G)$ as otherwise we are done. According to Lemma 7, there exists a geodesic $y = y_0 y_1 \cdots y_r$ such that $ecc(y_j) = ecc(y_{j-1}) + 1$, $j = 1, \ldots, r$ and $y_r \in Ct(G)$. If $z \in Ecc(\{y_r\})$, then $y \in I[z, y_r]$ since

$$ecc(y) + r = ecc(y_r) = d(z, y_r) \leq \underbrace{d(z, y)}_{\leq ecc(y)} + \underbrace{d(y, y_r)}_{=r} \leq ecc(y) + r.$$

Moreover, it is clear that $d_i = ecc(x) \leq ecc(y) < ecc(y_r)$. Hence, we obtain that $ecc(y_r) = d_j$ for some $j < i$, which, according to the Inductive Hypothesis, means that $z \in I^{j-1}[Ct(G)] \subseteq I^{i-2}[Ct(G)]$. This fact, along with the statements $y \in I[z, y_r]$ and $y_r \in Ct(G)$ allows us to conclude that $y \in I^{i-1}[Ct(G)]$ and we are done.

As a consequence of this proposition and Theorem 19, we immediately obtain the following results.

Corollary 31 *Let $G = (V, E)$ be a connected graph whose contour has an eccentricity sequence of size s. Then,*

(1) $Ecc(Ct(G)) \subseteq I^{s-1}[Ct(G)]$,
(2) $\Omega(G) \subseteq I^{s-1}[Ct(G)]$,
(3) $I^s[Ct(G)] = V$.

This properties along with the known fact that in every graph G of diameter D there are at most $\lfloor D/2 \rfloor + 1$ different eccentricities, allow us to derive the following theorem.

Theorem 32 *Let $G = (V, E)$ be a connected graph of diameter D whose contour has an eccentricity sequence of size s. Then, $I^k[Ct(G)] = V$, where $k = gin(Ct(G)) \leq \min\left\{|Ct(G)| - 1, \frac{D}{2} + 1, s\right\}$.*

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