ENUMERATION AND LIMIT LAWS
OF DISSECTIONS ON A CYLINDER

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ABSTRACT. We compute the generating function for triangulations on a cylinder, with the restriction that all vertices belong to its boundary and that the intersection of a pair of different faces is either empty, a vertex or an edge. We generalize these results to maps with either constant ($\{k\}$-dissections) or unrestricted (unrestricted dissections) face degree. We apply singularity analysis to the resulting generating functions to obtain asymptotic estimates for their coefficients, as well as limit distributions for natural parameters.

1. INTRODUCTION

The enumeration of triangulations on a labelled disc is one of the first non-trivial problems in enumerative combinatorics. This question gives rise to the well-known Catalan numbers, which appear in different contexts of discrete mathematics [14]. This counting problem can be generalized in the context of map enumeration as follows: let $\mathbb{S}$ be a surface with boundary. We say that a triangular map on $\mathbb{S}$ (i.e., faces have degree 3) is a *simplicial decomposition* if each face is incident with 3 vertices and the intersection of a pair of different faces is either empty, a vertex or an edge. One can easily check that simpliciality is equivalent to the non-existence of neither loops nor multiple edges. Under these assumptions, which is the number of simplicial decompositions of $\mathbb{S}$ with the restriction that all vertices lie on the boundary of $\mathbb{S}$? Observe that this general problem covers the enumeration of triangulations on a labelled disc, as a disc is a surface with boundary. For other surfaces with boundary some work has been done: the Möbius band was first studied in [3], and the picture was completed in [12] (see also [7, 8]). In the present work we study the next step on this problem: the study of simplicial decompositions on a cylinder, without internal vertices.

Our techniques let us study general families of maps on a cylinder, with all vertices on the boundary. We say that a map on a cylinder is a *dissection* if each face of degree k is incident with exactly k vertices, and if the intersection of two different faces is either empty, a single vertex or an edge. In this work we obtain the enumeration of dissections on a cylinder where faces have degree k (also called $\{k\}$-*dissections*) and also the enumeration of dissections where the degree of each face is unrestricted (*unrestricted dissections*). In particular, simplicial decompositions are $\{3\}$-dissections. Examples of a simplicial decomposition, a $\{4\}$-dissection and an unrestricted dissection are shown in Figure 1.
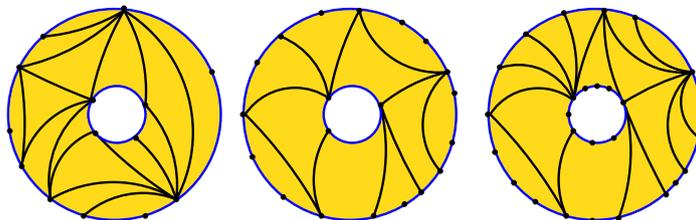

FIGURE 1. A simplicial decomposition, a $\{4\}$-dissection and an unrestricted dissection.

The main contribution of this paper is the method used to get this enumeration: in previous works the main tool is the decomposition induced by the root (combinatorial surgery arguments), method which is reminiscent to the seminal works of Tutte on map enumeration [15, 16]. In our study we introduce a composition scheme, which arises from a bijective characterization of the combinatorial families under study. These bijective techniques makes the analysis more transparent than the one made using root decompositions (see [13] for a combinatorial study using bijective tools). In particular, we avoid the long inclusion-exclusion argument used in [12].

Once we obtain the exact enumeration for this families using the associated generating functions, we get asymptotic estimates for the coefficients. Asymptotic results for families of rooted maps on arbitrary surfaces with boundary are obtained in [1]. In this work the authors consider a unique edge-root on one of the boundary components. The main difference in the present work with respect to [1] is that in our analysis each boundary is rooted, hence the maps under study carry a pair of roots. The main analytic tool in this part is the transfer of singularities [5], which provides a systematic method to translate analytic properties of a generating function into asymptotic estimates of their coefficients. More concretely, let $h_n^{\{k+1\}}$ denote the number of $\{k+1\}$-dissections on a cylinder with n faces, such that all the vertices belong to the boundary. Our main result in this part states that

$$h_n^{\{k+1\}} \sim_{n\to\infty} \frac{(k-1)^2}{16} \cdot n \cdot \rho_{k+1}^{-n},$$

where $\rho_{k+1} = (k-1)^{k-1}/k^k$ is the radius of convergence of the generating function of $\{k+1\}$-dissections on a labelled disc. In particular, we observe that the exponential growth of the coefficients depends on the *type* of the dissection (that is, in the allowed degrees for the faces).

We also study parameters on a random $\{k\}$-dissection with a fixed number of vertices. The main technical problem in this part is that we cannot apply general theorems as the quasi-powers theorem [9]: we need to make a case-by-case analysis in order to find the limit distribution of the corresponding random variable. Using generating functions we are able to extract factorial moments of the random variables under study, from which we deduce by the method of moments the existence of a limiting distribution. Some additional work must be done in order to characterize this limit. In this point, the Laplace transform is the main tool in order to get the expression of the density probability function in terms of its factorial moments. More concretely, let $\mathbf{h}_n$ be a simplicial decomposition on a cylinder (with all its vertices on the boundary) chosen uniformly at random among all simplicial decompositions on a cylinder with n vertices. Denote by $\mathbf{Z}_n$ the size of the core of $\mathbf{h}_n$ (see Section 8.2 for a proper definition) and by $\mathbf{W}_n$ the number of vertices of $\mathbf{h}_n$ on one of the boundary components. Let $\mathbf{Z}$, $\mathbf{W}$ be random variables with density probability functions

$$f_{\mathbf{Z}}(t) = t \cdot \operatorname{erfc}\left(\frac{t}{2}\right) \cdot \mathbb{I}_{[0,\infty[}(t), \ f_{\mathbf{W}}(t) = \frac{8}{\pi}\sqrt{t - t^2}\mathbb{I}_{[0,1]}(t).$$

Where $\mathbb{I}_{[0,\infty[}(t)$ (resp. $\mathbb{I}_{[0,1]}(t)$) is the characteristic function of the set $[0,\infty[$ (resp. $[0,1]$), and $\operatorname{erfc}(t)$ is the complementary error function. Our main result in this part states that $\mathbf{Z}_n/\sqrt{n} \to \mathbf{Z}$ and $\mathbf{W}_n/n \to \mathbf{W}$ in distribution.

To conclude we show that the framework presented in this paper explains previous related works. In particular, as a direct consequence of our results, we are able to generalize results of Gao, Xiao and Gang [8] on the exact enumeration of simplicial decompositions on a cylinder, with the difference that there exists a unique root-edge on one of the boundary components.

Outline of the paper. We start in Section 2 recalling the necessary background and setting our terminology. We continue studying simplicial decompositions in Section 3. In order to obtain the enumeration of $\{k\}$-dissections and unrestricted dissections, we need to study a combinatorial class (fundamental dissections) which is introduced in Section 4. Ideas used in Section 3 are refined in Section 5 and Section 6 in order to obtain the generating functions for $\{k\}$-dissections and unrestricted dissections. In Section 7 we study the asymptotic enumeration

of these families. We use analytical tools to study the distribution of parameters in Section 8. In Appendix A we introduce technical lemmas related to the integration of formal power series, which are used in the preceding sections. In Appendix B we present results, without the proofs, on dissections on a cylinder with a single root. In our approach these results are simple consequences of the ideas developed in the previous sections.

2. Preliminaries

In this section we introduce the basic notions of the symbolic method, which provides a direct way to translate combinatorial conditions into equations. We introduce also the basic definitions in the map framework. We apply the symbolic method to get the enumeration of certain families of maps defined on a disc. To conclude, we introduce generalities about the notation used in the rest of the work.

2.1. The symbolic method.
A common technique to deal with enumerative problems is the language of generating functions. We use the methodology introduced by Flajolet and Sedgewick in the context of *analytic combinatorics* (see [6]).

Let $\mathcal{A}$ be a set of objects, and let $|\cdot|$ be an application from $\mathcal{A}$ to $\mathbb{N}$. We say that $|a|$ is the *size* of a. A pair $(\mathcal{A}, |\cdot|)$ is called a *combinatorial class*. We restrict ourselves to *admissible* combinatorial classes, which means that for every n the number of elements in $\mathcal{A}$ with size n (this set is denoted by $\mathcal{A}(n)$) is finite. Under this assumption, we define the formal power series $A(z) = \sum_{a \in \mathcal{A}} z^{|a|} = \sum_{n=0}^{\infty} a_n z^n$. That is, a_n is the number of elements in $\mathcal{A}$ with size n, and conversely we write $[z^n]A(z) = a_n$. We say that $A(z)$ is the *generating function* (or GF for short) associated to the combinatorial class $(\mathcal{A}, |\cdot|)$.

The *symbolic method* is a tool that provides systematic rules to translate set conditions between combinatorial classes into equations between generating functions. We introduce the basic classes and combinatorial constructions, as well as their translation into the GF language. The *neutral* class $\mathcal{E}$ is made of a single object of size 0, and its generating function is $e(z) = 1$. The *atomic* class $\mathcal{Z}$ is made of a single object of size 1, and its associated GF is $Z(z) = z$. The *union* $\mathcal{A} \cup \mathcal{B}$ of two classes $\mathcal{A}$ and $\mathcal{B}$ refers to the disjoint union of classes (and the corresponding induced size). The *cartesian product* $\mathcal{A} \times \mathcal{B}$ of two classes $\mathcal{A}$ and $\mathcal{B}$ is the set of pairs (a, b) where $a \in \mathcal{A}$, and $b \in \mathcal{B}$. The size of (a, b) is the sum of the sizes of a and b. The *sequence* $\mathrm{Seq}(\mathcal{A})$ of a set $\mathcal{A}$ corresponds with the set $\mathcal{E} \cup \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) \cup \ldots$. The size of an element $(a_1, \ldots, a_s)$ in $\mathrm{Seq}(\mathcal{A})$ is the sum of sizes of the elements a_i. The *pointing* operator over a class $\mathcal{A}$ works in the following way: for each element $a \in \mathcal{A}$, such that $|a| = n$, the pointing operator distinguish one of the n atoms that compounds a. Finally, the *substitution* of $\mathcal{B}$ in the class $\mathcal{A}$ consists in substituting each atom of every element of $\mathcal{A}$ by an element of $\mathcal{B}$. In Table 1 all this constructions (and their translation into equations) are shown.

Construction		GF
Union	$\mathcal{A} \cup \mathcal{B}$	$A(z) + B(z)$
Product	$\mathcal{A} \times \mathcal{B}$	$A(z) \cdot B(z)$
Sequence	$\mathrm{Seq}(\mathcal{A})$	$\frac{1}{1 - A(z)}$
Pointing	$\mathcal{A}^{\bullet}$	$z \frac{\partial}{\partial z} A(z)$
Substitution	$\mathcal{A} \circ \mathcal{B}$	$A(B(z))$

Table 1. Translation of combinatorial specifications into equations.

2.2. Maps and dissections.
Our reference for maps is the monograph of Lando and Zvonkin [11]. A *map* on a surface $\mathbb{S}$ is a subdivision of $\mathbb{S}$ into 0-dimensional sets (*vertices* of the map), 1-dimensional contractible sets (*edges* of the map) and 2-dimensional contractible open sets (*faces* of the map). The degree of a face (resp. vertex) is the number of vertices (resp. faces) with

multiplicity incident with it. For a map M on $\mathbb{S}$, We denote by $v(M)$, $e(M)$ and $f(M)$ the set of vertices, edges and faces of a map M. The value $|f(M)| + |v(M)| - |e(M)|$ coincides with the Euler characteristic of $\mathbb{S}$. In the particular case of the cylinder, this value is equal to 0. Maps are considered up to orientation preserving homeomorphisms of the underlying surface, preserving the combinatorial structure of the map (incidences between vertices, edges and faces).

An edge of a map has two *ends* (incidence with a vertex) and either one or two *sides* (incidence with a face). A map is *rooted* if an end and a side of an edge are distinguished as the *root-end* and *root-side* respectively. The vertex, edge and face defining these incidences are the *root-vertex*, *root-edge* and *root-face*, respectively. Rooted maps are considered up to homeomorphism preserving the root-end and root-side.

Let $\Delta \subset \mathbb{N} - \{1, 2\}$. A map is a Δ-*dissection* if all faces have degree in Δ, such that any face of degree k is incident with k distinct vertices, and the intersection of any two faces is either empty, a vertex or an edge. It is easy to see that triangular maps are dissections if and only if they have neither loops nor multiple edges. These maps are called *simplicial decompositions*. In this work we study dissections on a cylinder in which the degree of the faces is either constant or unrestricted. These families of dissections are called $\{k\}$-*dissections* (where $k > 2$ is the face degree) and *unrestricted dissections*, respectively. Additionally, our dissections carry a pair of roots, one on each boundary component. These pair of roots are distinguishable.

2.3. Enumeration of dissections on a disc.

The symbolic method can be applied to obtain the enumeration of the number of triangulations on a disc without internal vertices (also called planar triangulations on a disc). This is a well studied problem and the solution is known since Euler's time. The reader can consult [4] for more constructions on a disc with all the vertices on the boundary.

Consider a disc with n vertices on its boundary, which are labelled in counter clockwise order (equivalently, we can take the edge $\overline{12}$ as the root of the map). We consider edges that join vertices on the boundary on a disc. Let c_n^{Δ} be the number of Δ-dissections on a disc with n faces, and we denote by $C_{\Delta}(z) = \sum_{n>0} c_n^{\Delta} z^n$ the associated generating function, where z marks faces. In the special case $\Delta = \{3\}$ we denote by $C(z)$ the corresponding generating function. The symbolic method applies in this case, giving the relation $C(z) = 1 + zC(z)^2$. The solution of this equation with positive expansion is the *Catalan function* $C(z) = (1 - \sqrt{1 - 4z})/(2z)$. Developing the term $\sqrt{1 - 4z}$ as a series around $z = 0$ we obtain the explicit expression for the *Catalan numbers* $[z^n]C(z) = c_n = \frac{1}{n+1}\binom{2n}{n}$.

In the case of $\{k + 1\}$-dissections, we denote $c_n^{\{k+1\}}$ the number of dissections on disc with n faces such that all faces have degree k. Denote by $C_{k+1}(z) = \sum_{n>0} c_n^{\{k+1\}} z^n$ the corresponding generating function, where z marks faces. We obtain in this particular situation that $C_{k+1}(z)$ satisfies the equation $C_{k+1}(z) = 1 + zC_{k+1}^k(z)$. This equation can we written also in terms of vertices: by the Euler relation on a disc, a $\{k + 1\}$-dissection with n faces has $(k - 1)n + 2$ vertices. Consequently, the equation $C_{k+1}(z) = 1 + zC_{k+1}^k(z)$ is translated into $C_{k+1}(x) = x^2 + x^{1-k} C_{k+1}^k(x)$, where the x marks vertices.

To conclude, if $\Delta = \mathbb{N} - \{1, 2\}$, the degree of each face is unrestricted. We denote by $d_{n,m}$ the number of unrestricted dissections on a disc with m vertices and n faces. Define also $D(u, x) = \sum_{m,n>0} d_{m,n} u^n x^m$ where x marks vertices and u marks faces. As it is shown in [4], $D(u, x)$ satisfies the implicit equation $(1 + u)D(u, x)^2 - x(1 + x)D(u, x) + x^3 = 0$, whose solution is

$$D(u, x) = \frac{x}{2(1+u)}\left(1 + x - \sqrt{1 - 2x + x^2 - 4xu}\right).$$

2.4. Terminology and additional definitions.

We introduce some notation used in the rest of this paper. Let $\mathcal{H}$ be a cylinder. Let $\mathbb{S}_1^1$ and $\mathbb{S}_2^1$ be the disjoint circles which are the connected components of the boundary of $\mathcal{H}$. We represent graphically a cylinder by drawing $\mathbb{S}_2^1$ inside $\mathbb{S}_1^1$: a cylinder is the region defined by this pair of circles. We say that $\mathbb{S}_1^1$ is the *external circle* of $\mathcal{H}$ and $\mathbb{S}_2^1$ the *internal circle* of $\mathcal{H}$. Vertices on the external circle are called *external vertices* and vertices on the internal circle are called *internal vertices*. We label external vertices with $1, 2, \ldots$ in counterclockwise order; similarly for internal vertices, using labels $1', 2', \ldots$. Observe that this

labelling gives the same information as marking both an internal and an external vertex (the one with labels 1 and $1'$, respectively). This is equivalent to rooting the map at the edge with end vertices $1, 2$ and $1', 2'$, respectively. We use all this conventions in order to deal with the roots on the boundary of the surface. We also say that a face incident with a root is a *root face*.

Let e be an edge whose endpoints belong to the boundary of $\mathcal{H}$. We say that e is an *ordinary edge* if its two endpoints belong to the same circle (either $\mathbb{S}_1^1$ or $\mathbb{S}_2^1$). In particular, we say that e is a *boundary edge* if e belongs to the boundary. We call e a *transversal edge* if it joins an internal vertex with an external vertex.

The conventions used for generating functions in this work is the following: variable x marks vertices, variable z marks faces. If we need to refine the analysis, variable x would mark internal vertices and variable y would mark external vertices. Additionally we use variables u, v and w when we deal with an extra parameter. We also assume from now on that all vertices of the dissections under study belong to the boundary.

3. Simplicial decompositions

In this section we deduce the generating function of simplicial decompositions on a cylinder $\mathcal{H}$ with all the vertices on its boundary. The method developed here is refined later in order to get generating functions of $\{k\}$-dissections and unrestricted dissections. Recall that a simplicial decomposition of $\mathcal{H}$ is a triangular map on $\mathcal{H}$ with neither loops nor multiple edges. We also assume that all the vertices belong to the boundary of $\mathcal{H}$. In general, a triangular map on $\mathcal{H}$ may not verify the simplicial condition. In Figure 2 a simplicial decomposition on a cylinder and a triangular map which is not simplicial are shown.
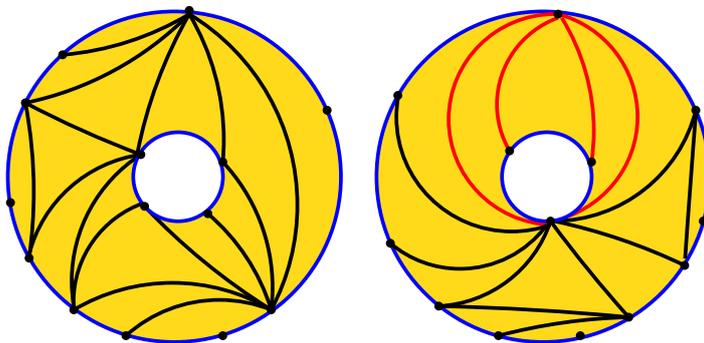

FIGURE 2. A simplicial decomposition, and a map which is not simplicial (due to the red edges)

In terms of enumeration, recall also that we are considering rooted maps, with a root-edge on each border of the boundary. Due to the simplicial condition the number of vertices on each boundary component is at least 3 (multiple edges and loops are not allowed). Consequently, the smallest simplicial decomposition on a cylinder have 3 vertices on each boundary component.

We classify triangles according to their edges. A triangle has either three ordinary edges or two transversal edges. We call this triangles *ordinary triangles* and *transversal triangles*, respectively. We say that a simplicial decomposition is *transversal* if it is compounded only by transversal triangles. Denote by $m_{r,s}$ the number of transversal simplicial decompositions with r internal vertices and s external vertices. Let $M(x, y) = \sum_{r,s>2} m_{r,s} x^r y^s$ be the associated generating function, where x (resp. y) marks internal (resp. external) vertices. In the next lemma we find an explicit expression for $M(x, y)$.

Lemma 1. *The generating function for transversal simplicial decompositions on a cylinder is*

$$(1) \qquad M(x,y) \;=\; xy\frac{-1+x+y-3xy+2x^4y^4}{(x-1)^3(y-1)^3} + \frac{18x^3y^3}{(x-1)^2(y-1)^2} +$$

$$6x^3y^3\frac{x^2+y^2-xy^2-x^2y}{(x-1)^3(y-1)^3} + \frac{xy}{(x+y-1)^2}$$

$$=\; 21x^3y^3 + (48x^4y^3 + 48x^3y^4) + (90x^5y^3 + 124x^4y^4 + 90x^3y^5) + \dots,$$

where x, y mark the number of internal and external vertices, respectively.

Proof. Take r labelled vertices on the internal circle and s vertices on the external circle of a cylinder $\mathcal{H}$. We obtain in this lemma the exact enumeration of the number of transversal simplicial decompositions on a cylinder with r internal vertices and s external vertices.

Denote by $1', 2', \dots, r'$ and $1, 2, \dots, s$ the vertices of the simplicial decomposition. Consider the edge $\overline{12}$, and let x' be the third vertex in the unique triangle containing $\overline{12}$. The resulting topological space obtained from $\mathcal{H}$ by removing the interior of $\triangle 12x'$ and the edge $\overline{12}$, without erasing vertices 1 and 2, is a rectangle with two identified extremal vertices. Its four corners are vertices 1, 2 and two copies of x' (see Figure 3). The points in the internal circle lie on the side defined by $\overline{x'x'}$, and points on the external circle lie on the side defined by the vertices 1 and 2, as shown in Figure 3.
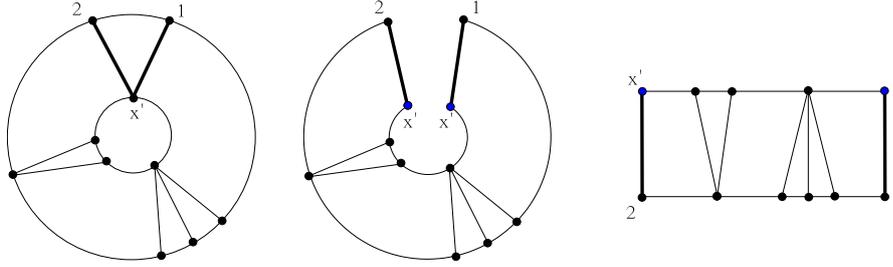

FIGURE 3. The topological space after erasing the interior of $\triangle 12x'$.

The initial question has been translated into an enumerative problem on a rectangle with some restrictions: there are $s \geq 3$ vertices on the edge defined by the extremal vertices 1, 2, and $r+1 \geq 4$ vertices on the edge defined by the two copies of x', including in both cases the corners. We call these edges the *upper boundary* and *lower boundary* of the rectangle, respectively. We can suppose that $x' = 1'$ and, by symmetry, multiply by r the resulting number of simplicial decompositions.

We associate to every triangular map a binary word of length $r + s - 1$ (the total number of vertices minus 2) as follows: if the corresponding triangle has its boundary edge on the upper boundary, we write a 0, otherwise we write a 1. Then the binary word is obtained by writing this sequence from left to right. It is clear then that every binary word has exactly s zeroes, because this is the number of boundary edges in the upper boundary. This is shown in the left hand side of Figure 4. Conversely, we can always construct a triangular map from a binary word of this type. As a conclusion, there is a bijection between the number of triangular maps of the rectangle, and the number of binary words of length $r + s - 1$ with r zeros. This number is $\binom{r+s-1}{r}$.

Let us count now the number of forbidden configurations (i.e. the ones which do not satisfy the simplicial condition). A map is not a simplicial decomposition if there is a multiple edge of the form $\overline{1'a}$. This choice can be done in s ways, depending on the choice of the vertex a. An example is shown in the right hand side of Figure 4.
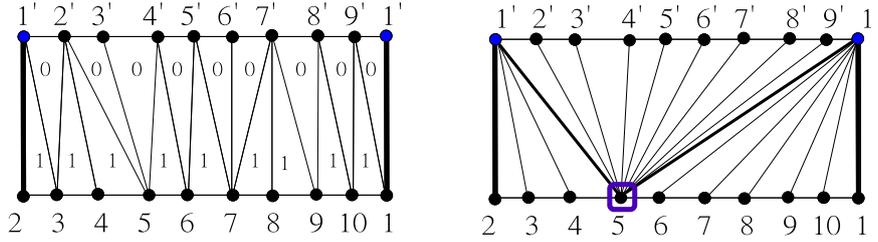
FIGURE 4. On the left, a valid triangulation of the band, with the corresponding encoding 101100101001101010. On the right, a forbidden triangulation.

Thus we obtain the number $r\left(\binom{r+s-1}{r} - s\right)$, which gives the following GF

$$M(x,y) = \sum_{r,s \geq 3} \left(r\binom{r+s-1}{r} - rs\right) x^r y^s.$$

Writing $s = k - r$ (k is the number of triangles in the simplicial decomposition) we have

$$M(x,y) = \sum_{k \geq 6} y^k \sum_{r=3}^{k-3} \left(r\binom{k-1}{r} - r(k-r)\right) \left(\frac{x}{y}\right)^r.$$

A simple computation gives (1) as claimed. $\qquad \square$

Observe that, due to the symmetry of a cylinder, one has that $M(x,y) = M(y,x)$ (which can be also observed from Expression (1)). Denote by $h_{n,m,s}$ the number of simplicial decompositions on $\mathcal{H}$ with n internal vertices, m external vertices and s transversal triangles. Let $H(u,x,y) = \sum_{n,m,s>2} h_{n,m,s} x^n y^m u^s$ be the associated generating function, where x, y mark internal and external vertices, respectively, and u marks transversal triangles. The following theorem gives the expression of $H(u,x,y)$ in terms of $M(x,y)$ and the Catalan function $C(z)$ (see Subsection 2.3).

Theorem 2. *The generating function for simplicial decompositions on a cylinder is*

$$(2) \qquad H(u,x,y) = \frac{1-xC(x)}{1-2xC(x)} \frac{1-yC(y)}{1-2yC(y)} M\left(uxC(x), uyC(y)\right),$$

where $M(x,y)$ is the generating function for transversal simplicial decompositions defined in Lemma 1, x, y mark the number of internal and external vertices respectively, u marks transversal triangles and $C(z)$ is the Catalan function.

Proof. Denote by $H_0(w,v,x,y)$ the generating function of simplicial decompositions on a cylinder $\mathcal{H}$, where w (resp. v) is an additional parameter which marks transversal triangles with exactly one external (resp. internal) vertex. We call these transversal triangles of *type w* and of *type v*, respectively. The purpose of this theorem is obtaining an expression for $H(u,x,y) = H_0(u,u,x,y)$. The strategy used consists on showing that there is a bijection between a pair of families of simplicial decompositions with additional marked vertices. For a given simplicial decomposition $\mathbf{h}_{n,m}$ with n internal vertices and m external vertices, the set of transversal triangles of $\mathbf{h}_{n,m}$ is a transversal simplicial decomposition called the *core* of $\mathbf{h}_{n,m}$.

Consider the class of simplicial decompositions on $\mathcal{H}$ where a triangle of type w and a triangle of type v are pointed. Observe that pointing a triangle of type w (resp. v) is equivalent to pointing the unique external (resp. internal) vertex of the marked triangle. Consequently, these pointing faces induces the roots (in the internal and the external circle) on the subjacent core of each simplicial decomposition. Hence, each simplicial decomposition on $\mathcal{H}$ with a pair of pointed triangles (of type w and of type v) carry four pointed vertices: vertices 1 and $1'$ (the initial roots-vertices of the map), and two corresponding to the core of each simplicial decomposition.

The associated GF can be obtained just applying the pointing operator on triangles on type w and v getting the expression

$$(3) \qquad wv\frac{\partial}{\partial v}\frac{\partial}{\partial w}H_0(w,v,x,y).$$

This combinatorial class can be obtained in the opposite way, departing from a transversal simplicial decomposition (which makes the role of the core of the simplicial decomposition the we will construct). Consider a transversal simplicial decomposition, and attach to each boundary edge a planar triangulation on a disc. One can make this operation by pasting to the corresponding boundary edge the edge-root of the corresponding planar simplicial decomposition. After this pasting procedure, we mark an internal and an external vertex. This combinatorial procedure is translated in the generating function language as follows: substitution of each boundary edge by a planar simplicial decomposition on a disc is translated into the substitution $x \leftarrow wxC(x)$ and $y \leftarrow vyC(y)$ on $M(x,y)$ (the number of boundary edges on the internal (resp. external) circle is equal tot the number of internal (resp. external) vertices). Later, we point an internal and an external vertex. Hence the resulting generating function is the following:

$$(4) \qquad xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}M(wxC(x),vyC(y)).$$

Observe that the rooting on the transversal simplicial decomposition brings the information on which triangles of type w and v are pointed, and the final pointing over vertices provides the roots of the simplicial decomposition. Consequently, we have constructed the same combinatorial family using two different arguments, hence the corresponding generating functions are equal. Matching Equation (3) and Equation (4) we obtain

$$vw\frac{\partial}{\partial v}\frac{\partial}{\partial w}H_0(w,v,x,y) = xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}M(wxC(x),vyC(y)).$$

We can apply Lemma 21 from Appendix A to this last expression for $r = 2$: initial conditions are satisfied because every simplicial decomposition has triangles of type w and of type v. In other words, $H_0(0,v,x,y) = H_0(u,0,x,y) = 0$, and $M(0,y) = M(x,0) = 0$. Therefore, applying Lemma 21 we get

$$H_0(w,v,x,y) = \frac{1-xC(x)}{1-2xC(x)}\frac{1-yC(y)}{1-2yC(y)}M(wxC(x),vyC(y)),$$

from which we obtain (2) by writing $w = v = u$ (i.e., $H(u,x,y) = H_0(u,u,x,y)$). $\qquad\square$

The GF for the number of simplicial decompositions on a cylinder in terms of the number of triangles is derived directly from the previous result. Let h_n be the number of simplicial decompositions on a cylinder with n faces. Let $H(z) = \sum_{n\geq 6} h_n z^n$ be the corresponding GF.

Corollary 3. *The GF of simplicial decompositions on a cylinder, counted by the number of faces, is*

$$\begin{aligned} H(z) &= \frac{-8z^5 + 18z^4 - 52z^3 + 20z^2 + 2z - 1}{z(1-4z)^2}C(z) + \frac{8z^5 - 2z^4 + 33z^3 - 20z^2 - z + 1}{z(1-4z)^2} \\ &= 21z^6 + 264z^7 + 2134z^8 + 14108z^9 + \dots, \end{aligned}$$

where $C(z)$ is the Catalan function.

Proof. Write $x = y = z$ and $u = v = 1$ in the function obtained in Theorem 2. Recall also that, by the Euler's relation, the number of faces in a simplicial decomposition on a cylinder is equal to the total number of vertices. $\qquad\square$

4. Fundamental dissections

In order to obtain formulas for $\{k\}$-dissections and unrestricted dissections, we need to deal with a special class of maps which are, in some sense, the basic pieces in order to construct more involved dissections. This is the goal of this section. The arguments are quite similar (but more involved) to the ones developed in Section 3.

We say that a quadrangle with two vertices on each boundary component of a cylinder is called a *transversal quadrangle*. A *fundamental dissection* is a map on a cylinder whose faces are transversal triangles and quadrangles, where points on the internal circle *are not labelled*. Observe that in a fundamental dissections quadrangles necessary have two boundary edges and triangles have one boundary edge. To obtain the corresponding generating function, we use the variable Z to mark transversal quadrangles and the variable Y (resp. X) to mark transversal triangles whose unique boundary edge lies on the internal (resp. external) circle. An example of a fundamental dissection is shown in Figure 5.
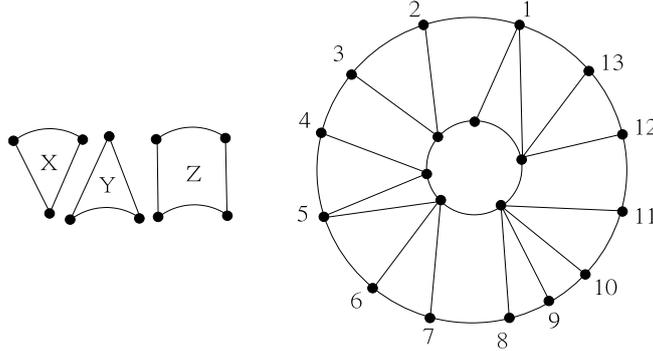

FIGURE 5. On the left, a pair of transversal triangles and a transversal quadrangle. On the right, a fundamental dissection on a cylinder. labelled.

The next lemma shows the enumeration of this family.

Lemma 4. *The generating function associated to fundamental dissections is*
$$J(X,Y,Z) = ZJ_1(X,Y,Z) + XJ_2(X,Y,Z) = ZJ_1 + XJ_2,$$
where

$$(5) \qquad \begin{aligned} J_1 &= \frac{1}{1-X-Y-Z} - \frac{1}{1-Y} - \frac{X+Z}{(1-Y)^2} - \frac{1}{1-X} - \frac{Z+Y}{(1-X)^2} + \\ &\quad 1 + Y + Z + X + 2YX, \\ J_2 &= J_1(X,Y,Z) - \frac{(Z+Y)^2}{(1-X)^3} + Y^2 + 2ZY + 3XY^2 - \\ &\quad \frac{X^2}{(1-X)^2}\frac{Y^3}{1-Y} - \frac{2X^2}{1-X}\frac{Y^3}{1-Y}. \end{aligned}$$

Proof. We obtain a combinatorial decomposition on a cylinder $\mathcal{H}$ in terms of the root face (the unique face incident with the edge $\overline{12}$). The root face is contained either on a transversal quadrangle (of type Z) or on a transversal triangle (of type X). The strategy is based on applying topological surgery on the root face and enumerate the resulting topological surface. The first case is encapsulated into the GF $J1$ and the second case into $J2$, hence the final result ZJ_1+XJ_2. We consider these two cases separately.

First, suppose that $\overline{12}$ belongs to a transversal quadrangle. Cutting $\mathcal{H}$ through the pair of transversal edges of this quadrangle, we get a rectangle. Vertices of the external circle are located on one of the edges of this rectangle, and vertices on the internal circle are located on the opposite side. We use the same terminology as in the case of simplicial decompositions, upper and lower boundary, respectively. Transversal quadrangles and transversal triangles are translated in this rectangle by quadrangles and triangles with a pair and a single edge from the upper to the lower boundary. We also call these type of faces *transversal* (as in the case of a cylinder). Under these assumptions, the number of fundamental dissections on a cylinder with a root of type Z is equal to the number of dissections into transversal triangles and transversal quadrangles of this

rectangle. This contribution is encapsulated in the generating function $J_1(X, Y, Z)$. In order to get the enumeration, we count the total number of maps (dissections or not) and we delete the forbidden ones (the ones which do not provide a fundamental dissection once we paste the root quadrangle). It is obvious that in this case the forbidden configurations are those where:

  (1) The number of vertices on the upper boundary of the rectangle is smaller than three.
  (2) The number of vertices on the lower boundary of the rectangle is smaller than three.

These configurations are forbidden since a dissection on a cylinder has at least three internal vertices and three external vertices, and the resulting rectangle is constructed by cutting the root face, hence the number of vertices on each boundary is at least three.

Let us get the GF for this family of maps on a rectangle. An arbitrary map on a rectangle (recall that all the vertices are on the boundary of the rectangle) is simply a sequence of elements on the set $\{X, Y, Z\}$. From this sequence, we subtract maps from which we do not deduce dissections on a cylinder (by pasting the root face). This argument is done using an inclusion-exclusion argument on GFs. The possible cases are shown in Figure 6. On the left of this figure, an arbitrary decomposition is shown. On the right, all forbidden conditions. Below, pairs of forbidden conditions. A shaded face represents either a transversal fundamental quadrangle or a transversal triangle, and a white face is a sequence of transversal quadrangles and transversal triangles.
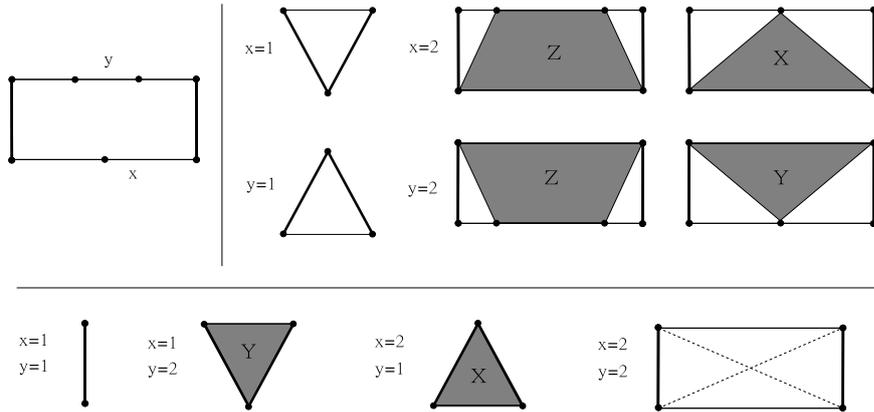

FIGURE 6. Forbidden configurations if the root is a fundamental quadrangle.

Table 2 shows the translation between these specifications and the corresponding GFs.

Summing all the previous contributions with the corresponding sign given by the inclusion-exclusion theorem, we obtain the first generating function stated in Equation (5):

$$J_1(X, Y, Z) = \frac{1}{1 - X - Y - Z} - \frac{1}{1 - Y} - \frac{X + Z}{(1 - Y)^2} - \frac{1}{1 - X} - \frac{Z + Y}{(1 - X)^2} + 1 + Y + Z + X + 2XY.$$

Let us consider now the case where $\overline{12}$ belongs to a transversal triangle (Case J_2). We cut the cylinder through the transversal edges of the root triangle. We obtain also a rectangle, but with two corners identified. We use the same terminology here as for the computation of J_1. Fundamental dissections of $\mathcal{H}$ whose root-face is a transversal triangle are in bijection with maps on a rectangle (with a pair of identified corners) with restrictions (to be discussed later). As in the previous case, we count the total number of maps (dissections or not) on a rectangle with a pair of identified corners, and then we erase those which do not derive a fundamental dissection on a cylinder once we paste the root triangle. In this case, allowed maps satisfies the following conditions:

x	y	Sign	Structure	GF
		$+$	$\operatorname{Seq}(\{X, Y, Z\})$	$1/(1 - X - Y - Z)$
1		$-$	$\operatorname{Seq}(\{Y\})$	$1/(1 - Y)$
2		$-$	$\operatorname{Seq}(\{Y\}) \times \{Z, X\} \times \mathcal{S}\mathrm{eq}\{Y\}$	$(X + Z)/(1 - Y)^2$
	1	$-$	$\operatorname{Seq}(\{X\})$	$1/(1 - X)$
	2	$-$	$\operatorname{Seq}(\{X\}) \times \{Z, Y\} \times \mathcal{S}\mathrm{eq}\{X\}$	$(Y + Z)/(1 - X)^2$
1	1	$+$	$\emptyset$	1
1	2	$+$	$\{Y\}$	Y
2	1	$+$	$\{X\}$	X
2	2	$+$	$\{Z\} \cup \{X\} \times \{Y\} \cup \{Y\} \times \{X\}$	$Z + 2XY$

TABLE 2. Translation into GFs of the restrictions introduced in Figure 6.

(1) The number of vertices on the upper boundary of the rectangle is greater or equal than four. Observe that when we erase a triangle of type X, we are doubling a vertex (to be compatible with the initial fundamental dissection on the cylinder).
(2) The number of vertices on the lower boundary of the rectangle is greater or equal than three (we do not delete any vertex when we cut the initial root-face)
(3) If the intersection of two different faces of the map is non-empty, then the intersection is a vertex or an edge (due to the existence of a pair of identified corners, this case must be also considered now).

Observe that all the forbidden cases in the previous situation (the edge $\overline{12}$ belongs to a transversal quadrangle) are also forbidden here. So we must subtract from $J_1(X, Y, Z)$ configurations where $y = 3$ and the forbidden cases which appear as a consequence of the existence of a pair of identified corners. Those are shown in Figure 7. In the first row, shaded faces represent either transversal triangles (Y) or fundamental quadrangles (Z). In the second row, the possible configurations for $x = 2$, $y = 3$ are shown. The third row shows forbidden decompositions coming from the condition of incompatibility with the initial transversal triangle (pasting this configuration with the initial transversal triangle gives a decomposition which is not cellular). Table 3 translates these conditions into GFs.
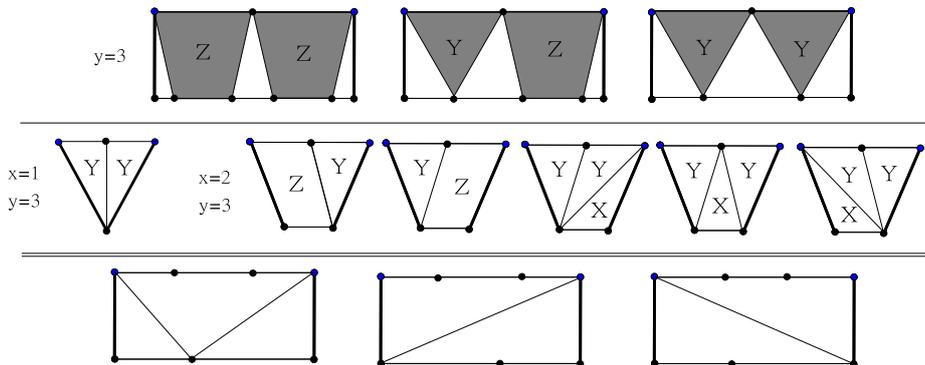

FIGURE 7. Forbidden configurations when the root is a transversal quadrangle.

x	y	Sign	Structure	GF
	3	$-$	$\operatorname{Seq}\left(\{X\}\right)^3 \times \{Z,Y\}^2$	$(Z+Y)^2/(1-X)^3$
1	3	$+$	$\{Y\} \times \{Y\}$	Y^2
2	3	$+$	shown in Figure 7	$2YZ + 3XY^2$
		$-$	$\operatorname{Seq}\left(\{X\}\right)^2 \times \{Y\}^3 \times \operatorname{Seq}\left(\{Y\}\right)$	$X^2Y^3/((1-X)^2(1-Y))$
		$-$	$\{Y\}^2 \times \operatorname{Seq}\left(\{Y\}\right) \times \{X\}^2 \times \operatorname{Seq}\left(\{X\}\right)$	$X^2Y^3/((1-X)(1-Y))$
		$-$	$\{Y\}^2 \times \operatorname{Seq}\left(\{Y\}\right) \times \{X\}^2 \times \operatorname{Seq}\left(\{X\}\right)$	$X^2Y^3/((1-X)(1-Y))$

TABLE 3. Translation into GFs of the restrictions introduced in Figure 7.

Summing up each contribution with the corresponding sign due to the inclusion-exclusion argument, gives the second term in Equation (5):

$$
\begin{aligned}
J_2(X,Y,Z) \;=\; & J_1(X,Y,Z) - \frac{(Z+Y)^2}{(1-X)^3} + Y^2 + 2ZY + 3XY^2 - \\
& \frac{X^2}{(1-X)^2}\frac{Y^3}{1-Y} - \frac{2X^2}{1-X}\frac{Y^3}{1-Y}.
\end{aligned}
$$

$\square$

5. $\{k+1\}$-DISSECTIONS

In this section we study $\{k+1\}$-dissections on a cylinder without interior vertices. The case $k=2$ corresponds to simplicial decompositions on a cylinder, which has been studied in Section 3.

As we have done for simplicial decompositions, we start enumerating transversal families. A face is said to be *transversal* if it has exactly two transversal edges. A *transversal $\{k+1\}$-dissection* on a cylinder is a dissection where all faces are transversal. It is obvious that a transversal dissection arises from either a transversal triangle or a transversal quadrangle by subdividing their boundary edges. We use the variable u to mark transversal faces which come from a transversal triangle of type X, and v to mark transversal faces which arise from a transversal triangle of type Y (recall the notation introduced in Section 4). We use variable z_s to mark transversal faces with $k+1$ vertices which arise from a transversal quadrangle adding s vertices on its external boundary. Therefore, z_s has $s+2$ external vertices and $k+1-s-2 = k-s-1$ internal vertices. As an example, in Figure 8 all possible transversal hexagons (with the corresponding variable which codifies them) are shown.
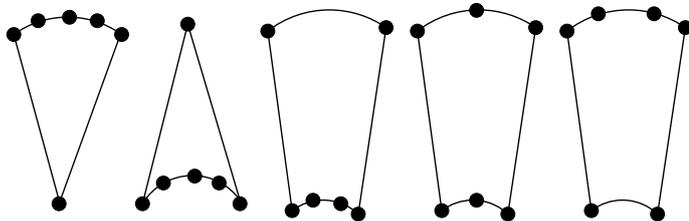
FIGURE 8. Transversal hexagons. The first and the second one are generated from triangles of type X and Y (variables u and v respectively). The others (z_0, z_1 and z_2, respectively) are generated from fundamental quadrangles (Z).

We start computing the generating functions associated to transversal $\{k+1\}$-dissections. The main point in the following proposition is the use of the GF associated to fundamental dissections.

Proposition 5. *Let $K_1^{\{k+1\}}(u, v, z_0, \ldots, z_{k-3}, x, y)$ and $K_2^{\{k+1\}}(u, v, z_0, \ldots, z_{k-3}, x, y)$ be the GFs defined by the relations*

$$K_1^{\{k+1\}} = xyJ_1\left(y^{k-1}u, x^{k-1}v, xy\left(\sum_{s=0}^{k-3} z_s y^s x^{k-3-s}\right)\right),$$

$$K_2^{\{k+1\}} = yJ_2\left(y^{k-1}u, x^{k-1}v, xy\left(\sum_{s=0}^{k-3} z_s y^s x^{k-3-s}\right)\right),$$

where J_1 and J_2 are defined in Lemma 4. Then, the generating function of transversal $\{k+1\}$-dissections on a cylinder is $M_0^{\{k+1\}}(u, v, z_0, \ldots, z_{k-3}, x, y) = M_0^{\{k+1\}}$,

$$M_0^{\{k+1\}} = x\frac{\partial}{\partial x}\left(\left(\sum_{s=0}^{k-3}(s+1)z_s y^s x^{k-3-s}\right)K_1^{\{k+1\}} + u(k-1)y^{k-2}K_2^{\{k+1\}}\right),$$

where x marks internal vertices and y marks external vertices.

Proof. The starting point are fundamental dissections, which are enumerated using the variables X, Y and Z (recall Section 4). The construction works in the following way: we need to substitute each face of a fundamental dissection by a transversal face (in order to get faces of degree $k+1$), and consider then the resulting number of vertices. We also need to put a mark on the internal circle, in order to induce the cyclic enumeration on the internal circle.

More specifically, in order to construct $\{k+1\}$-dissections we apply three consecutive steps:

(1) Substitution of every transversal triangle and every transversal quadrangle on fundamental dissections by a transversal face of degree $k+1$, counted by their number of vertices.

(2) Pointing process on the external circle, in which we erase the initial root and we produce the new one. Observe that it is necessary to erase the initial root because in a transversal $\{k+1\}$-dissection vertex 1 could not belong to the subjacent fundamental dissection.

(3) Pointing process on the internal circle. Recall that the internal circle of a fundamental decomposition is not labelled, hence we need to apply a pointing process in order to obtain the vertex $1'$.

Let us start with step (1). Consider the combinatorial families which are counted by J_1 and J_2. We must substitute each variable X, Y and Z in order to deal with vertices (instead of faces). Observe that we construct a transversal face with degree $k+1$ from a transversal triangle of type u adding $k-2$ external vertices. Additionally, an extra factor y appears in order to take into account one of the two vertices of the initial triangle of type u (in all cases we consider vertices in counter anticlockwise order around the external circle). Hence, we apply the substitution $X = y \cdot y^{k-2}u$. Similar arguments are made for Y, writing $Y = x \cdot x^{k-2}v$. In the case of Z, we must distribute $k-3$ vertices between $\mathbb{S}_1^1$ and $\mathbb{S}_2^1$. In all these cases we consider a pair of vertices on each boundary (hence, the factor xy). Resuming, we write $Z = xy\cdot\sum_{s=0}^{k-3} z_s y^s x^{k-3-s}$. Finally, there is either a single internal vertex (in family J_1) or a pair of vertices on each boundary (in family J_2) which are not considered yet in order to reconstruct the initial cylinder. Consequently, we multiply J_1 and J_2 by xy and y, respectively.

Step (2) consists in marking an external vertex to get vertex 1 on the external circle. This vertex is induced by the root face in the fundamental dissection (which has $\overline{12}$ as an edge in the initial fundamental dissection). If the root face in the subjacent fundamental dissection is a fundamental quadrangle, and we are adding s vertices on the external boundary, we have $(s+1)$ ways to choose the vertex whose label is 1. Hence, we get the sum $\sum_{s=0}^{k-3}(s+1)z_s y^s x^{k-3-s}$. Observe that we are taking into account extremal vertices of the root face. Otherwise, we have a transversal triangle of type u, so we must add $k-2$ vertices on the external boundary, and we have and additional term of the form $u(k-1)y^{k-2}$ (we have $k-1$ possibilities in this case to choose vertex 1).

To conclude, in step (3) we apply the pointing operator $x\frac{\partial}{\partial x}$ to obtain the vertex $1'$ on the internal circle. Since we are considering labellings in counter anticlockwise order, the position of vertex $1'$ determines the labels of the remaining vertices on the internal circle.   □

To obtain the final enumeration of the family of $\{k+1\}$-dissections, we only need a simplified version of the GF stated in the previous proposition. This is what is shown in the following corollary:

Corollary 6. *Let $L_1^{\{k+1\}}(u,x,y)$ and $L_2^{\{k+1\}}(u,x,y)$ be defined by*

$$L_1^{\{k+1\}}(u,x,y) = K_1^{\{k+1\}}(u,u,\ldots,u,x,y),$$
$$L_2^{\{k+1\}}(u,x,y) = K_2^{\{k+1\}}(u,u,\ldots,u,x,y),$$

where $K_1^{\{k+1\}}$ and $K_2^{\{k+1\}}$ are defined in Proposition 5. Then the generating function for transversal $\{k+1\}$-dissections on a cylinder is

$$M_1^{\{k+1\}}(u,x,y) = ux\frac{\partial}{\partial x}\left(\left(\frac{\partial}{\partial y}\frac{x^{k-1}-y^{k-1}}{x-y}\right)L_1^{\{k+1\}}(u,x,y) + u(k-1)y^{k-2}L_2^{\{k+1\}}(u,x,y)\right).$$

where x,y mark internal and external vertices and u marks transversal faces.

Proof. Substitute the set of variables $v, z_0, z_1, \ldots, z_{k-3}$ by u in Proposition 5.   □

Finally, setting $y = x$ we obtain the GF in terms of the total number of vertices. In this case, the expression is explicit:

Corollary 7. *The generating function of transversal $\{k+1\}$-dissections on a cylinder is*

$$M^{\{k+1\}}(u,x) = \frac{(k-1)f_k(u,x)}{12(1-kux^{k-1})^2(1-ux^{k-1})^4}$$

where x marks vertices, u marks transversal faces and $f_k(u,x)$ is

$$\begin{aligned}
f_k(u,x) &= \left(2x^{3k-3}(4k-3)(k-2)^3\right)u^3 - \\
&\quad \left(x^{4k-4}(k-2)^2(5k^3-43k^2-42k+66)\right)u^4 - \\
&\quad \left(4x^{5k-5}(k-2)(6k^4+11k^3-86k^2-13k+72)\right)u^5 + \\
&\quad \left(2x^{6k-6}(22k^5-122k^4-83k^3+677k^2-206k-258)\right)u^6 - \\
&\quad \left(2x^{7k-7}(10k^5-166k^4+243k^3+426k^2-392k-96)\right)u^7 + \\
&\quad \left(x^{8k-8}(5k^5-215k^4+636k^3-20k^2-360k-24)\right)u^8 + \\
&\quad \left(4x^{9k-9}k(13k^3-62k^2+36k+12)\right)u^9 + \left(24x^{10k-10}(k-1)k^2\right)u^{10}.
\end{aligned}$$

Proof. Write $y = x$ in the result of Corollary 6, and after simplifying the expressions, we get the result as claimed.   □

Observe that the exponents of the variable x in the series $M^{\{k+1\}}(u,x)$ are multiples of $k-1$, because the total number of vertices is multiple of $k-1$ (due to Euler's relation). To conclude this section, we obtain a closed formula for the number of $\{k+1\}$-dissections on a cylinder in terms of the GF for transversal dissections. Denote by $h_{n,m,r}^{\{k+1\}}$ the number of $\{k+1\}$-dissections on a cylinder with n internal vertices, m external vertices and r transversal faces. Denote by $H^{\{k+1\}}(u,x,y) = \sum_{n,m,r\geq 3} h_{n,m,r}^{\{k+1\}} x^n y^m u^r$.

Theorem 8. *The generating function of $\{k+1\}$-dissections on a cylinder is*

$$\begin{aligned}
H^{\{k+1\}}(u,x,y) &= \frac{1-(xC_{k+1}(x^{k-1}))^{k-1}}{1-k(xC_{k+1}(x^{k-1}))^{k-1}}\frac{1-(yC_{k+1}(y^{k-1}))^{k-1}}{1-k(yC_{k+1}(y^{k-1}))^{k-1}} \times \\
&\quad M_1^{\{k+1\}}\left(u, xC_{k+1}(x^{k-1}), yC_{k+1}(y^{k-1})\right).
\end{aligned}$$

(6)

where x,y mark internal and external vertices, respectively, u marks transversal faces of degree $k+1$, $M_1^{\{k+1\}}(u,x,y)$ is defined in Corollary 6 and $C_{k+1}(z)$ is the generating function for planar $\{k+1\}$-dissections (recall Subsection 2.3).

Proof. Denote by $H_0^{\{k+1\}}(u, v, z_0, z_1, \ldots, z_{k-3}, x, y)$ the GF of $\{k+1\}$-dissection on a cylinder, where x, y marks internal and external vertices, respectively, and the rest of the variables take into account the type of transversal face in the dissection. The argument is the same as in simplicial decompositions: we construct in two different ways the set of $\{k+1\}$-dissections with a pair of pointed vertices belonging to the core of each element.

For a $\{k+1\}$-dissection, we consider a pair of pointed vertices (one on each boundary) on the induced transversal $\{k+1\}$-dissection (namely, the core of the initial $\{k+1\}$-dissection). We consider then the family of $\{k+1\}$-dissection on a cylinder with a pair of labelled vertices (one of them external and the other internal) which belong to the core. This family can be defined in two different ways: either departing from the transversal family of $\{k+1\}$-dissections and pasting on every boundary edge a planar $\{k+1\}$-dissection, or pointing the corresponding vertices on the general $\{k+1\}$-dissection (see the full argument for simplicial decompositions).

Let r, s be the variables used to mark external and internal vertices on the core of the $\{k+1\}$-dissections. For instance, variable u must be substitute in $H_0^{\{k+1\}}$ by ur^{k-1} in order to deal with these vertices. The combinatorial argument used in simplicial decompositions gives the following identity between generating functions in the case of $\{k+1\}$-dissections

$$rs\frac{\partial}{\partial r}\frac{\partial}{\partial s}H_0^{\{k+1\}}(ur^{k-1}, vs^{k-1}, z_0 rs^{k-2}, z_1 r^2 s^{k-3}, \ldots, z_{k-3} r^{k-2}s, x, y)$$
$$= xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}M_0^{\{k+1\}}\left(u, v, z_0, z_1, \ldots, z_{k-3}, rxC_{k+1}(x^{k-1}), syC_{k+1}(y^{k-1})\right),$$

where $M_0^{\{k+1\}}$ is the GF defined in Proposition 5 and $C_{k+1}(z)$ is the generating function for $\{k+1\}$-dissections on a disc (recall its properties in Subsection 2.3). Applying Lemma 22 from Appendix A we reduce the previous equation

$$H_0^{\{k+1\}}(ur^{k-1}, vs^{k-1}, z_0 rs^{k-2}, z_1 r^2 s^{k-3}, \ldots, z_{k-3} r^{k-2}s, x, y)$$
$$= \frac{1}{C_{k+1}(x^{k-1})}\frac{1}{C_{k+1}(y^{k-1})}\left(\frac{\partial}{\partial x}xC_{k+1}(x^{k-1})\right)\left(\frac{\partial}{\partial y}yC_{k+1}(y^{k-1})\right)\times$$
$$M_0^{\{k+1\}}\left(u, v, z_0, z_1, \ldots, z_{k-3}, rxC_{k+1}(x^{k-1}), syC_{k+1}(y^{k-1})\right).$$

Setting $z_0 = z_1 = \cdots = z_{k-3} = v = u$ and $r = s = 1$ on $H_0^{\{k+1\}}$ we get the function $H^{\{k+1\}}(u, x, y)$, which can be expressed as

$$H^{\{k+1\}}(u, x, y)$$
$$= \frac{1}{C_{k+1}(x^{k-1})}\frac{1}{C_{k+1}(y^{k-1})}\left(\frac{\partial}{\partial x}xC_{k+1}(x^{k-1})\right)\left(\frac{\partial}{\partial y}yC_{k+1}(y^{k-1})\right)\times$$
$$M_1^{\{k+1\}}\left(u, xC_{k+1}(x^{k-1}), yC_{k+1}(y^{k-1})\right).$$

Using the property $C_{k+1}(x^{k-1}) = 1 + x^{k-1}C_{k+1}(x^{k-1})^k$ from Subsection 2.3, we obtain Expression (6) as it is stated. $\qquad\square$

To conclude, if we set $y = x$ and $u = 1$ in Equation (6) (using the expression for $M^{\{k+1\}}$ as stated in Corollary 7) we obtain the generating function in terms of the total number of vertices: let $h_m^{\{k+1\}}$ the number of $\{k+1\}$-dissections on a cylinder with m vertices, and denote by $H^{\{k+1\}}(x) = \sum_{m\geq 6} h_m^{\{k+1\}}x^m$ the corresponding generating function. Then, from the last result we get the following expression:

$$(7) \qquad H^{\{k+1\}}(x) = \left(\frac{1 - x^{k-1}C_{k+1}(x^{k-1})^{k-1}}{1 - kx^{k-1}C_{k+1}(x^{k-1})^{k-1}}\right)^2 M^{\{k+1\}}\left(xC_{k+1}(x^{k-1})\right),$$

For $k = 3$ we obtain the generating function for dissections into quadrangles, which has an expansion of the form

$$H^{\{4\}}(x) = 3x^6 + 112x^8 + 1902x^{10} + 23396x^{12} + 243698x^{14} + 2299064x^{16} + \ldots,$$

and for $k = 4$, we get

$$H^{\{5\}}(x) = 52x^9 + 1874x^{12} + 37448x^{15} + 586001x^{18} + 8048356x^{21} + \ldots$$

All expressions can be written in terms of x^{k-1}: $M^{\{k+1\}}(u, x)$ can be written in terms of x^{k-1} (see the corresponding expression in Corollary 7), hence the GF $M^{\{k+1\}}(u, xC_{k+1}(x^{k-1}))$ can be written in terms of

$$x^{k-1}C_{k+1}(x^{k-1})^{k-1} = zC_{k+1}(z)^{k-1} = (C_{k+1}(z) - 1)/C_{k+1}(z).$$

Writing $x^{k-1} = z$, we obtain the the enumeration of dissections in terms of the number of faces in the dissection. This parameter is the one used in Section 7 in order to obtain asymptotic estimates for the coefficients.

6. UNRESTRICTED DISSECTIONS

A similar strategy used for the enumeration of $\{k+1\}$-dissections can be adapted to obtain the GF for dissections on a cylinder where the degree of each face is arbitrary. By an *unrestricted dissection* on a cylinder $\mathcal{H}$ we mean a dissection where all vertices lie on the boundary of $\mathcal{H}$, and the degree of each face is unrestricted. Recall that a transversal face has exactly 2 transversal edges. As in the previous section, we first obtain the GF for transversal unrestricted dissections, and then we attach planar unrestricted dissections at boundary edges in order to build unrestricted dissections in full generality.

In what follows, variable $z_{i,j}$ marks transversal faces with i external vertices and j internal vertices. As before, x marks internal vertices and y marks external vertices. We write $\mathbf{Z}$ for the infinite vector $(z_{1,2}, z_{2,1}, z_{2,2}, z_{2,3}, \ldots)$. We omit the proofs of the forthcoming results, because they are a straightforward modification of the results in the previous section. We start with transversal unrestricted dissections.

Proposition 9. *Let K_1^D and K_2^D be the formal power series in a set of infinite variables*

$$K_1^D = xyJ_1\left(\sum_{i=2}^{\infty} z_{i,1}y^{i-1}, \sum_{j=2}^{\infty} z_{1,j}x^{j-1}, \sum_{i=2}^{\infty}\sum_{j=2}^{\infty} z_{i,j}y^{i-1}x^{j-1}\right),$$

$$K_2^D = yJ_2\left(\sum_{i=2}^{\infty} z_{i,1}y^{i-1}, \sum_{j=2}^{\infty} z_{1,j}x^{j-1}, \sum_{i=2}^{\infty}\sum_{j=2}^{\infty} z_{i,j}y^{i-1}x^{j-1}\right),$$

where J_1 and J_2 are defined in Lemma 4. The GF of transversal unrestricted dissections on a cylinder is

$$M_0^D(\mathbf{Z}, x, y) = x\frac{\partial}{\partial x}\left(\left(\sum_{j=2}^{\infty}\sum_{i=2}^{\infty}(i-1)y^{i-2}x^{j-2}z_{i,j}\right)K_1^D + \sum_{i=2}^{\infty}(i-1)y^{i-2}z_{i,1}K_2^D\right).$$

The previous result is not useful in practical terms, because infinitely many variables appear in the expressions. The next corollary is a simple consequence of the previous proposition, and the expressions obtained are simpler.

Corollary 10. *Let L_1^D and L_2^D be*

$$L_1^D = K_1^D(u, u, u, \ldots, x, y) = xyJ_1\left(u\frac{y}{1-y}, u\frac{x}{1-x}, u\frac{xy}{(1-x)(1-y)}\right),$$

$$L_2^D = K_2^D(u, u, u, \ldots, x, y) = yJ_2\left(u\frac{y}{1-y}, u\frac{x}{1-x}, u\frac{xy}{(1-x)(1-y)}\right).$$

The generating function of transversal unrestricted dissections on a cylinder is

$$M_1^D(u, x, y) = \frac{ux}{(1-y)^2}\frac{\partial}{\partial x}\left(\frac{1}{1-x}L_1^D + L_2^D\right),$$

where x marks internal vertices, y marks external vertices and u marks transversal faces.

The final step consists in setting $y = x$ in the previous corollary.

Corollary 11. *The generating function of transversal unrestricted dissections is*

$$M^D(u,x) = \frac{g(u,x)}{(1 - 2x + x^2 - 2ux + ux^2)^2(1-x)^6(-1+x+ux)^4},$$

where

$$
\begin{aligned}
g(u,x) \;=\; & 3x^6(z-1)^6u^3 + x^6(25x^2 - 54x + 27)(x-1)^4u^4 + \\
& 2x^6(43x^3 - 131x^2 + 117x - 27)(x-1)^3u^5 + \\
& x^6(151x^4 - 568x^3 + 684x^2 - 276x + 21)(x-1)^2u^6 + \\
& x^7(x-1)(145x^4 - 633x^3 + 924x^2 - 500x + 72)u^7 + \\
& 2x^8(38x^4 - 186x^3 + 312x^2 - 204x + 41)u^8 + \\
& 4x^9(x-2)(5x^2 - 12x + 5)u^9 + 2x^{10}(x-2)^2u^{10},
\end{aligned}
$$

x marks vertices and u transversal faces.

We are now ready to obtain the GF for unrestricted dissections on a cylinder. The main idea is the same as in Theorems 2 and 8: we define consider the family of unrestricted dissections on a cylinder with a pair of marked vertices (each on each boundary component) which belongs to the core of the unrestricted dissection. The difference in this case is the introduction of a variable w which takes into account the number of faces. In the next statement, $D(w,x)$ denotes the GF of dissections of a disc, as it is stated in Subsection 2.3.

Theorem 12. *The generating function of unrestricted dissection on a cylinder is*

$$
\begin{aligned}
(8) \qquad H^D(u,w,x,y) \;=\; & \frac{x^2y^2}{D(w,x)D(w,y)} \frac{\partial}{\partial x}\left(\frac{D(w,x)}{x}\right) \frac{\partial}{\partial y}\left(\frac{D(w,y)}{y}\right) \cdot \\
& M_1^D\left(uw, \frac{D(w,x)}{x}, \frac{D(w,y)}{y}\right),
\end{aligned}
$$

where $M_1^D(u,x,y)$ is as defined in Corollary 10, x,y mark internal and external vertices respectively, u marks transversal faces and w marks faces.

If we set $y = x$ and $u = 1$, then $H^D(w,x) = H^D(1,w,x,x)$ is the GF of unrestricted dissections on a cylinder in terms of vertices and faces. The explicit expression for this generating function can be obtained in terms of the formula for M^D obtained in Corollary 11:

$$
\begin{aligned}
H^D(w,x) \;=\; & (3w^3 + 27w^4 + 54w^5 + 21w^6)x^6 + \\
& (24w^3 + 264w^4 + 792w^5 + 840w^6 + 264w^7)x^7 + \\
& (108w^3 + 1438w^4 + 5932w^5 + 10422w^6 + 8000w^7 + 2134w^8)x^8 + \dots
\end{aligned}
$$

In particular, the sequence of coefficients of the form $w^r x^r$ is the same as the one obtained in Corollary 3: recall that simplicial decompositions maximizes the number of faces for a fixed number of vertices in an unrestricted dissection.

7. Asymptotic enumeration

In this section we obtain asymptotic estimates for the number of dissections studied in previous sections. It turns out that they invariably satisfy a general formula of the form $c \cdot n \cdot \rho_\Delta^{-n}(1 + o(n^{-1}))$, where c is a constant (depending on Δ) and ρ_Δ is the radius of convergence of the corresponding family of dissections on a disc. As it is shown in [1], this fact is general: the exponent of the subexponential term depends only on the genus of the surface (that is, on the *topology* of the surface), and the exponential term depends only on the *allowed degrees* of the dissection (that is, on the set of degrees Δ).

The plan for this section is the following: we start introducing the main results in singularity analysis for generating functions, including the Transfer Theorem for singularity analysis. Later, we apply this results to deduce the asymptotic estimates.

7.1. **Singularity analysis.** The study of the growth of the coefficients of a generating function can be obtained by considering it as a complex function which is analytic around the origin. This is the main idea of analytic combinatorics. The growth behavior of the coefficients depends only on the smallest positive singularity of the GF. Its *location* provides the *exponential growth* of the coefficients, and its *behavior* gives the *subexponential growth* of the coefficients. The basic results in this area are the so-called *Transfer Theorems for singularity analysis*. These results allows us to deduce asymptotic estimates of an analytic function using its asymptotic expansion near its dominant singularity.

More concretely, for $R > \rho > 0$ and $0 < \phi < \pi/2$, let $\Delta_\rho(\phi, R)$ be the set $\{z \in \mathbb{C} : |z| < R, z \neq \rho, |\mathrm{Arg}(z - \rho)| > \phi\}$. We call a set of this type a *dented domain* or a *domain dented* at ρ. Let $A(z)$ and $B(z)$ be GFs whose smallest singularity is the real number ρ. We write $A(z) \sim_{z \to \rho} B(z)$ if $\lim_{z \to \rho} A(z)/B(z) = 1$. We obtain the asymptotic expansion of $[z^n]A(z)$ by *transfering* the behavior of $A(z)$ around its singularity from a simpler function $B(z)$, from which we know the asymptotic behavior of their coefficients. This is the main idea of the so-called Transfer Theorems developed by Flajolet and Odlyzko [5]. In our work we use a mixture of Theorems VI.1 and VI.3 from [6]:

Theorem 13 (Transfer Theorem). *If $A(z)$ is analytic in a dented domain $\Delta = \Delta_\rho(\phi, R)$, where ρ is the smallest singularity of $A(z)$, and*

$$A(z) \underset{z \in \Delta, z \to \rho}{\sim} c \cdot \left(1 - \frac{z}{\rho}\right)^{-\alpha} + o\left(\left(1 - \frac{z}{\rho}\right)^{-\alpha}\right),$$

for $\alpha \notin \{0, -1, -2, \dots\}$, then

$$(9) \qquad a_n = c \cdot \frac{n^{\alpha-1}}{\Gamma(\alpha)} \cdot \rho^{-n} \left(1 + o(n^{-1})\right),$$

where Γ is the Gamma function: $\Gamma(u) = \int_0^\infty t^{u-1} e^{-t} dt$.

We also use the following notation: for a pair of series of real numbers $(a_n)_{n>0}$ and $(b_n)_{n>0}$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$, we write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

7.2. **Asymptotic estimates.** For $k > 1$, we write ρ_{k+1} for the radius of convergence of the generating function $C_{k+1}(z)$ of $\{k+1\}$-dissections on a disc, and ρ_D for the radius of convergence of the generating function $D(1, x)$ associated to unrestricted dissections on a disc (recall that z and x mark faces and vertices, respectively). Using the theory of Lagrange inversion formulas, one can prove that $\rho_{k+1} = (k-1)^{k-1}/k^k$ and $\rho_D = 3 - 2\sqrt{2}$ [4]. The main result in this section is the following:

Proposition 14. *Let $h_n^{\{k+1\}}$ be the number of $\{k+1\}$-dissections on a cylinder with n faces. Let h_m^D be the number of unrestricted dissections on a cylinder with m vertices. Then, the following asymptotic estimates hold:*

$$(10) \qquad h_n^{\{k+1\}} \sim \frac{(k-1)^2}{16} \cdot n \cdot \rho_{k+1}^{-n}, \qquad h_m^D \sim \frac{1}{16} \cdot m \cdot \rho_D^{-m},$$

where $\rho_{k+1} = (k-1)^{k-1}/k^k$ and $\rho_D = 3 - 2\sqrt{2}$ are the radius of convergence of the generating functions of $\{k+1\}$-dissections and of unrestricted dissections on a disc, respectively.

Proof. The analysis is made over the expressions obtained in Corollary 3 and Equations (7), (8). In the case of $\{k+1\}$-dissections, observe that Equation (7) is written in terms of x^{k-1}, where x mark vertices. Using Euler's relation we can write $z = x^{k-1}$ in order to consider the enumeration in terms of faces.

We argue for $\{k+1\}$-dissections, but the same arguments work for unrestricted dissections. In all cases under study, the generating function can be expressed in the following form:

$$F\left(C_{k+1}(z)\right) G_{k+1}(z, C_{k+1}(z)),$$

where F and G_{k+1} are rational functions. In all cases, one can check easily that both functions $F(C_{k+1}(z))$ and $G_{k+1}(z, C_{k+1}(z))$ are analytic in a neighborhood of each complex number z_0 such that $|z_0| < \rho_{k+1}$ (the denominators of the rational functions F and G_{k+1} do not vanish). Hence, its smallest singularity is located at $z = \rho_{k+1}$.

In order to obtain explicit expressions, we write $Z = \sqrt{1 - z/\rho_\Delta}$ and $X = \sqrt{1 - x/\rho_\Delta}$, (consequently $z = \rho_\Delta(1 - Z^2)$ and similarly for X). We consider the singular expansion of the generating function of dissections on a disc, and how this singular expansion is transformed by the effect of the composition scheme. Expanding the expressions of H, $H^{\{k+1\}}$ and H^D in terms of these news variables, we get:

(1) Simplicial decompositions: $H(Z) \sim_{z \to \rho} \frac{1}{16} Z^{-4} + o(Z^{-4})$.

(2) $\{k+1\}$-dissections: $H^{\{k+1\}}(Z) \sim_{z \to \rho_{k+1}} \frac{(k-1)^2}{16} Z^{-4} + o(Z^{-4})$.

(3) Unrestricted dissections: $H^D(1, X) \sim_{z \to \rho_D} \frac{1}{16} X^{-4} + o(X^{-4})$.

In unrestricted dissections, the term 1 in the generating function $H^D(1, X)$ means that we take $u = 1$ (i.e., we not consider the parameter marking faces). Applying now the Equation (9), the estimates in Equation (10) hold. $\qquad\square$

In table 4 this asymptotic is compared with the corresponding families on the Möbius band (computed in [12]).

Family	Möbius band	Cylinder
$\{k+1\}$ dissections	$\frac{k-1}{4}\rho_{k+1}^{-n}$	$\frac{(k-1)^2}{16}n\rho_{k+1}^{-n}$
Unrestricted dissections	$\frac{1}{4}\rho_D^{-n}$	$\frac{1}{16}n\rho_D^{-n}$

TABLE 4. Asymptotic enumeration for the Möbius band and the cylinder. For the value of $k = 2$ in $\{k+1\}$-dissections we obtain the asymptotic for simplicial decompositions.

8. Limit laws

In this section we study two statistical parameters which arise naturally in the previous families: the size of the core on a uniformly random distributed dissection with n faces, and the number of vertices on the external circle on a uniformly random distributed simplicial decomposition with n faces. To get the limit law for these parameters, we apply the Method of Moments, together with properties of the *Laplace transform*.

8.1. **Probability in discrete enumeration.** This subsection reminds the probabilistic tools which are used, joint with the Method of Moments and a reminder of the properties of the Laplace transform.

8.1.1. *The analysis of random variables.* Probability is introduced in the framework of analytic combinatorics in the following way. Let $(\mathcal{A}, |\cdot|)$ be an admissible combinatorial class, and let $\chi : \mathcal{A} \to \mathbb{N}$ be a parameter. We define the bivariate generating function $A(u, z) = \sum_{a \in \mathcal{A}} u^{\chi(a)} z^{|a|} = \sum_{n,m=0}^{\infty} a_{m,n} u^m z^n$. In particular $A(1, z) = A(z)$, and $\sum_{m=0}^{\infty} a_{m,n} = a_n$. For each value of n, the parameter χ defines a random variable $\mathbf{X}_n$ over the set of elements of $\mathcal{A}$ with size n, with discrete probability density function $\mathbf{p}(\mathbf{X}_n = m) = a_{m,n}/a_n$. This probabilities can be encapsulated into the *probability generating function*:

$$p_n(u) = \frac{[z^n]A(u, z)}{[z^n]A(1, z)}.$$

In order to study this function, we deal with the derivatives of $A(u, z)$ with respect to u, which is the way to obtain the moments of the random variable $\mathbf{X}_n$. The notation we use in this section is the following: $\mathbb{E}[\mathbf{X}]$ is the *expectation* of $\mathbf{X}$. The rth *ordinary moment* (or shortly, the rth moment) of $\mathbf{X}$ is $\mathbb{E}[\mathbf{X}^r]$, and the rth *factorial moment* is $\mathbb{E}[(\mathbf{X})_r] = \mathbb{E}[\mathbf{X}(\mathbf{X}-1)\ldots(\mathbf{X}-r+1)]$.

Using the identity $x^k = \sum_{j=0}^{k} S(j,k)(x)_j$, where the $S(j,k)$ are Stirling numbers of the second kind and $(x)_j = x(x-1)\ldots(x-j+1)$ it follows easily that when n tends to infinity, $\mathbb{E}\left[(\mathbf{X}_n)^k\right] \sim \mathbb{E}\left[(\mathbf{X}_n)_k\right]$.

We are concerned only with *convergence in distribution* (or *convergence in law*): for a sequence of random variables $(\mathbf{X}_n)_{n>0}$, such that each of them has a probability density function $f_{\mathbf{X}_n}(x)$, we say that the sequence tends in distribution (or in law) to a random variable $\mathbf{X}$, if the sequence of distribution probability functions $(F_{\mathbf{X}_n}(x))_{n>0}$ converges pointwise to the distribution function $F_{\mathbf{X}}(x)$ of $\mathbf{X}$. We denote this fact writing $\mathbf{X}_n \overset{d}{\to} \mathbf{X}$.

8.1.2. *The method of moments.* In this framework, the *Method of Moments* [2] provides a way to assure convergence in law using only the ordinary moments of the sequence of random variables. Even more, this method provides a direct way to calculate the limit law using its moments. More concretely, the version we use in this paper is the following one:

Lemma 15 (Method of Moments). *Let $(\boldsymbol{X}_n)_{n>0}$ and $\boldsymbol{X}$ be real random variables satisfying:*

  (A) *there exists $R > 0$ such that $\dfrac{R^r}{r!}\mathbb{E}\left[\boldsymbol{X}^r\right] \to 0$, as $r \to \infty$,*
  (B) *for all $r \in \mathbb{N}$, $\mathbb{E}\left[\boldsymbol{X}_n^r\right] \to \mathbb{E}\left[\boldsymbol{X}^r\right]$, as $n \to \infty$.*

Then $\boldsymbol{X}_n \overset{d}{\to} \boldsymbol{X}$.

Point (A) in Lemma 15 implies that the distribution of the random variable $\mathbf{X}$ is determined by its moments. We need also the following modification of the Method of Moments, which is based on the properties of Stirling numbers. The proof of the following result can be found in [1]:

Lemma 16. *Let $\mathcal{A}$ be a combinatorial class and let $\boldsymbol{U}_n$ be the random variable associated to a parameter $U_n : \mathcal{A}(n) \to \mathbb{N}$. Let $A(u,z)$ be the corresponding GF. Denote by $\theta : \mathbb{N} \to \mathbb{N}$ a function such that $\theta(n) \to +\infty$ when n tends to ∞. If a random variable $\boldsymbol{X}$ satisfies Condition (A) in Lemma 15 and*

  (B') *for all $r \in \mathbb{N}$, $\dfrac{[z^n]\,\frac{\partial^r}{\partial u^r}A(u,z)\big|_{u=1}}{\theta(n)^r[z^n]A(1,z)} \to \mathbb{E}\left[\boldsymbol{X}^r\right]$, as $n \to \infty$,*

then the rescaled random variables $\boldsymbol{X}_n = \dfrac{\boldsymbol{U}_n}{\theta(n)} \overset{d}{\to} \boldsymbol{X}$.

8.1.3. *Laplace Transform.* Recall that given a piecewise continuous function $f : [0,\infty) \to \mathbb{C}$, the *Laplace transform* of f, $\mathcal{L}(f(t)) = F(s)$ is $F(s) = \int_0^\infty f(t)e^{-ts}dt$. The Laplace transform can be exploited in our context in the following way: let $\mathbf{Y}$ be a random variable with density function $f_{\mathbf{Y}}(t)$. Then the Laplace transform of $f_{\mathbf{Y}}(t)$, $\mathcal{L}(f_{\mathbf{Y}}(t)) = F_{\mathbf{Y}}(s)$ is equal to

$$F_{\mathbf{Y}}(s) = \int_0^\infty f_{\mathbf{Y}}(t)e^{-ts}dt = \sum_{r=0}^\infty \mathbb{E}[\mathbf{Y}^r]\frac{1}{r!}(-s)^r.$$

If there exists $\rho > 0$ such that $\mathbb{E}[\mathbf{Y}^r] < \rho^r \cdot r!$, then the Laplace transform is analytic in a neighborhood of the origin, and the density probability function $f_{\mathbf{Y}}(t)$ is uniquely determined by its moments. In fact, the use of Laplace transform is the key tool in the Method of Moments.

To deduce limit laws we use the following strategy: let $(\mathbf{Y}_n)_{n>0}$ be a sequence of random variables. From the GFs obtained in the previous sections, we compute the rth factorial moments of $(\mathbf{Y}_n)_{n>0}$, from which we deduce the rth factorial moment of the limit random variable $\mathbf{Y}$. By the Method of Moments we prove that $\mathbf{Y}$ is characterized by its moments. Then we compute the moment generating function of $\mathbf{Y}$, which corresponds to $F_{\mathbf{Y}}(s) = \mathcal{L}(f_{\mathbf{Y}}(t))$, and finally we apply the inverse Laplace transform to recover $f_{\mathbf{Y}}(t)$. Computing the inverse Laplace transform is not always simple, but one can use the properties of this operator recalled in Table 5. The indicator function of a set of real numbers A is denoted by $\mathbb{I}_A(t)$. All functions considered here are defined on the set of positive real numbers.

Function	Laplace transform
$a_1 \cdot f_1(t) + a_2 \cdot f_2(t)$	$a_1 \cdot F_1(s) + a_2 \cdot F_2(s)$
$t^n \cdot f(t)$	$(-1)^n \cdot F^{(n)}(s)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0^-) - \cdots - f^{(n-1)}(0^-)$
$f(at)$	$1/|a| \cdot F(s/a)$

TABLE 5. Properties of the Laplace transform.

8.2. **The size of the core in a dissection.** Recall that the core of a dissection is the set of transversal faces of the dissection, and the size of the core is its number of transversal faces. In order to study this parameter, we introduce an auxiliar random variable $\mathbf{Y}$ with density probability function

$$(11) \qquad f_{\mathbf{Y}}(t) = \frac{1}{\sqrt{\pi}} e^{-t^2/4} \mathbb{I}_{[0,\infty[}(t).$$

The random variable $\mathbf{Y}$ is characterized by its moments, which have the following form:

$$\mathbb{E}[\mathbf{Y}^r] = \frac{2^r}{\sqrt{\pi}} \Gamma\left(\frac{1+r}{2}\right) = \frac{\Gamma(1+r)}{\Gamma(1+r/2)} = \frac{r!}{\Gamma(1+r/2)}.$$

In this previous formula we have used *Gauss duplication formula* $\Gamma(r) \cdot \Gamma(1/2+r) = 2^{1-2r}\sqrt{\pi} \cdot \Gamma(2r)$. We also need the *complementary error function*, defined by

$$\mathrm{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-r^2} dr.$$

Properties and applications of this function in many different areas can be found in [10]. The important fact here is that $\mathrm{erfc}(t)' = -\frac{2}{\sqrt{\pi}} e^{-t^2}$, which relates the function $\mathrm{erfc}(t)$ with the density probability function of $\mathbf{Y}$ (up to a linear transform of the variable).

8.2.1. *The size of the core in simplicial decompositions.* The generating function we want to study is

$$(12) \qquad H(u,z) = H_0(u,u,z,z) = \left(\frac{1-zC(z)}{1-2zC(z)}\right)^2 M(uzC(z), uzC(z)),$$

which is obtained directly from Theorem 2. Recall that $M(x,y)$ is the rational function defined in Lemma 1, and $C(z)$ is the Catalan function. Write $\mathbf{Z}_n$ for the discrete random variable equals to the size of the core on a uniformly distributed random simplicial decomposition with n faces. The following theorem characterizes the limit law of the sequence $(\mathbf{Z}_n)_{n>0}$:

Theorem 17. *Let $\mathbf{Z}$ be a random variable with density function*

$$f_{\mathbf{Z}}(t) = t \cdot \mathrm{erfc}\left(\frac{t}{2}\right) \cdot \mathbb{I}_{[0,\infty[}(t).$$

Then, $\mathbf{Z}_n/\sqrt{n} \xrightarrow{d} \mathbf{Z}$.

Proof. We compute the moments of $\mathbf{Z}_n$. Consider the expression for $H(u,z)$, which is deduced from Equation (12). Near the point $(u,z) = (1,1/4)$, $H(u,z)$ has a singular expansion of the form

$$H(u,z) \sim_{(u,z)\to(1,1/4)} \frac{1}{16} \frac{1}{1-4z} \frac{1}{\left(1 - u\left(1 - \sqrt{1-4z}\right)\right)^2}.$$

Factorial moments can be computed from the previous expression

$$\sum_{n\geq 0} H(n)\mathbb{E}[(\mathbf{Z}_n)_r]z^n = \frac{\partial^r H(1,z)}{\partial u^r} \sim_{z\to 1/4} \frac{1}{16} \frac{(1+r)!}{(1-4z)^{r/2+2}}.$$

These functions satisfy the condition of the Transfer Theorem (with dominant singularity at $z = 1/4$) for each value of r, hence we can extract coefficients using Expression (9). Using the asymptotic estimate $H(n) \sim n \cdot 4^{n-2}$ obtained in Section 7, we obtain the following estimate for the rth factorial moment of $\mathbf{Z}_n$:

$$\mathbb{E}[(\mathbf{Z}_n)_r] \sim \frac{(1+r)! \cdot n^{r/2}}{\Gamma(2+r/2)} = \frac{2+2r}{2+r} \frac{n^{r/2}}{\Gamma(1+r/2)} r!.$$

As we noticed in Section 8.1.1, the same estimate holds for the ordinary moment $\mathbb{E}[\mathbf{Z}_n^r]$. This can be written as

$$\mathbb{E}\left[\left(\frac{\mathbf{Z}_n}{\sqrt{n}}\right)^r\right] \sim \frac{2+2r}{2+r} \frac{r!}{\Gamma(1+r/2)}.$$

These values are bounded by $r!$, and by the Method of Moments in the version of Lemma 16, they determine uniquely a limit law $\mathbf{Z}$, such that $\mathbf{Z}_n/\sqrt{n} \xrightarrow{d} \mathbf{Z}$. The next step consists of characterizing $\mathbf{Z}$. At this point, general properties of the Laplace transform are used. Consider the Laplace transform of the density probability function of $\mathbf{Z}$:

$$F_{\mathbf{Z}}(s) = \sum_{r=0}^{\infty} \frac{2+2r}{2+r} \frac{(-s)^r}{\Gamma(1+r/2)} = 2\sum_{r=0}^{\infty} \frac{(-s)^r}{\Gamma(1+r/2)} - 2\sum_{r=0}^{\infty} \frac{1}{r+2} \frac{(-s)^r}{\Gamma(1+r/2)},$$

which can be written in terms of the Laplace transform of $f_{\mathbf{Y}}(t)$ (Equation (11)) in the form:

$$F_{\mathbf{Z}}(s) = 2F_{\mathbf{Y}}(s) - 2G(s),$$

where $G(s) = \sum_{r=0}^{\infty} \frac{1}{2+r} \frac{1}{\Gamma(1+r/2)} (-s)^r$ is the Laplace transform of $g(t)$. By linearity (see Table 5), $f_{\mathbf{Z}}(t) = 2f_{\mathbf{Y}}(t) - 2g(t)$. We only need to find $g(t)$ to conclude this discussion. The function $G(s)$ satisfies

$$F_{\mathbf{Y}}(s) = \frac{1}{s} \frac{\partial}{\partial s}\left(s^2 G(s)\right) = 2G(s) + sG'(s).$$

Applying the inverse Laplace transform, we obtain the differential equation

$$\frac{1}{\sqrt{\pi}} e^{-t^2/4} \mathbb{I}_{[0,\infty[}(t) = tg'(t) - g(t),$$

with the initial condition $\int_0^{\infty} g(t)dt = 1/2$ (since $2g(t) = 2f_{\mathbf{Y}}(t) - f_{\mathbf{Z}}(t)$). This differential equation can be solved writing $g(t) = th(t)$ and solving by parts the resulting integral expression for $h(t)$. The unique solution to this equation satisfying the previous initial conditions is

$$g(t) = \frac{1}{\sqrt{\pi}} e^{-t^2/4} \mathbb{I}_{[0,\infty[}(t) - \frac{t}{2}\mathrm{erfc}\left(\frac{t}{2}\right) \cdot \mathbb{I}_{[0,\infty[}(t),$$

and $f_{\mathbf{Z}}(t) = 2f_{\mathbf{Y}}(t) - 2g(t) = t \cdot \mathrm{erfc}\left(\frac{t}{2}\right) \cdot \mathbb{I}_{[0,\infty[}(t)$, as claimed.   □

8.2.2. *Size of the core for $\{k+1\}$-dissections and unrestricted dissections.* The previous analysis can be extended to all classes of dissections studied in the previous sections. Let $\mathbf{Z}_{k+1,n}$ the random variable equals to the number of transversal faces in a uniformly random distributed $\{k+1\}$-dissection on a cylinder.

Theorem 18. *Let $\mathbf{Z}_{k+1}$ be a random variable with density function*

$$f_{\mathbf{Z}_{k+1}}(t) = \frac{t}{a_{k+1}^2} \mathrm{erfc}\left(\frac{t}{2a_{k+1}}\right) \cdot \mathbb{I}_{[0,\infty[}(t),$$

where $a_{k+1} = k(k-1)^{-2}\sqrt{(k-1)^3/(2k)}$. Then $\mathbf{Z}_{k+1,n}/\sqrt{n} \xrightarrow{d} \mathbf{Z}_{k+1}$.

Proof. By Theorem 8, $H^{\{k+1\}}(u,x) = H^{\{k+1\}}(u,u,x,x)$. Recall that $H^{\{k+1\}}(u,x)$ can be written as $\sum_{r=0}^{\infty} a_r(u)x^{(3+r)(k-1)}$, where r is the number of $(k+1)$−gons in the dissection. We study the new function $H^{\{k+1\}}(u,x^{k-1})$ which is the GF of dissections in terms of faces instead of

vertices. Using $C_{k+1}(z) = 1 + zC_{k+1}(z)^k$ we write $zC_{k+1}(z)^{k-1} = (C_{k+1}(z) - 1)/C_{k+1}(z)$. Near the singular point $(u, z) = (1, \rho_{k+1})$, $H^{\{k+1\}}(u, z)$ admits a singular expression of the form

$$H^{\{k+1\}}(u, z) \sim \frac{k(k-1)}{12(C_{k+1}(z) - k(C_{k+1}(z) - 1))^2} \cdot$$
$$\frac{k(k+1)u((C_{k+1}(z) - 1)^2) + 2C_{k+1}(z)(C_{k+1}(z) - 1)}{(C_{k+1}(z) - uk(C_{k+1}(z) - 1))^2}.$$

From the previous expression, we compute the rth factorial moment of $\mathbf{Z}_{k+1,n}$,

$$\frac{\partial^r H^{\{k+1\}}(1, z)}{\partial u^r} \sim \frac{1}{4}\left(\frac{k}{k-1}\right)^{r+2} \frac{(k-1)^2(1+r)!}{(C_{k+1}(z) - k(C_{k+1}(z) - 1))^{r+4}}.$$

Writing $l_{k+1} = \sqrt{2k/(k-1)^3}$ and extracting the nth coefficient we obtain

$$H^{\{k+1\}}(n)\mathbb{E}[(\mathbf{Z}_{k+1,n})_r] \sim \frac{1}{4}\left(\frac{k}{k-1}\right)^{r+2} \frac{1}{l_{k+1}^{r+4}(k-1)^{r+2}}[z^n]\frac{(1+r)!}{(1 - z/\rho_{k+1})^{r/2+2}}$$
$$\sim \frac{1}{4}\frac{k^{r+2}}{(k-1)^{2r+4}}\frac{(1+r)!}{l_{k+1}^{r+4}\Gamma(2+r/2)}n^{1+r/2}\rho_k^n.$$

From Section 7 we have the estimate $H^{\{k+1\}}(n) \sim (k-1)^2/16 \cdot n \cdot \rho_{k+1}^n$. Using this estimate and the asymptotic equality between factorial moments and ordinary moments, we get

$$\mathbb{E}\left[\left(\frac{\mathbf{Z}_{k+1,n}}{\sqrt{n}}\right)^r\right] \sim \left(\frac{k}{(k-1)^2 l_{k+1}}\right)^r \frac{(1+r)!}{\Gamma(2+r/2)}.$$

From now on, write $a_{k+1} = \frac{k}{(k-1)^2 l_{k+1}}$. The procedure now is the same as in Theorem 17. The moment generating function of $(\mathbf{Z}_{k+1,n}/\sqrt{n})^r$ is

$$F_{\mathbf{Z}_{k+1}}(s) = \sum_{r=0}^{\infty} a_{k+1}^r \frac{(r+1)!}{\Gamma(2+r/2)\,r!}(-s)^r = F_{\mathbf{Z}}(a_{k+1}s),$$

where $F_{\mathbf{Z}}(s)$ is the Laplace transform studied in Theorem 17. Applying the forth property in Table 5 we conclude the proof. $\qquad\square$

Observe also that if $k = 2$, then $a_{k+1} = 1$ and we recover the result for simplicial decompositions in Theorem 17. The analysis of the problem for the case of unrestricted dissections is similar as the one made in the case of $(k + 1)$-gons. We state the result without the proof. As before, let $\mathbf{Z}_{D,n}$ let the random variable equals to the number of transversal faces in a random uniformly distributed dissection with n vertices.

Theorem 19. *Let $\mathbf{Z}_D$ be a random variable with density function*

$$f_{\mathbf{Z}_D}(t) = \frac{t}{a_D^2}\operatorname{erfc}\left(\frac{t}{2a_D}\right) \cdot \mathbb{I}_{[0,\infty[}(t),$$

where $a_D = (2^{7/4}(\sqrt{2} - 1))^{-1}$. Then $\mathbf{Z}_{D,n}/\sqrt{n} \xrightarrow{d} \to \mathbf{Z}_D$.

Notice that the limit laws obtained so far have all the same shape, and are variants (up to a scale factor) of the distribution defined in Theorem 17.

8.3. Distribution of vertices on a simplicial decomposition.

Consider the following problem: given a random uniformly distributed simplicial decomposition on a cylinder with n vertices, how many of them lie on the external circle? In this section we show that this parameter has a very simple limit law.

Let $\mathbf{W}_n$ be the random variable defined on simplicial decomposition on a cylinder with n vertices, equals to the number of vertices that lie in the external circle. We show that the rescaled sequence $(\mathbf{W}_n/n)_{n>0}$ tends to a limit law $\mathbf{W}$ with density $8/\pi\sqrt{t - t^2}$ for $t \geq 0$.

The GF we need to study in order to compute moments is

$$h(u, z) = H_0(1, 1, uz, z),$$

where $H_0(u, v, x, y)$ is stated in Theorem 2: we set $u = v = 1$ since we are not interested in the associated parameters, and setting $x = uz$, $y = z$, now z marks vertices (both internal and external), and u marks external vertices.

Theorem 20. *Let* $\boldsymbol{W}$ *be a random variable with density function*

$$f_{\boldsymbol{W}}(t) = \frac{8}{\pi} \sqrt{t - t^2} \mathbb{I}_{[0,1]}(t).$$

Then $\boldsymbol{W}_n/n \xrightarrow{d} \boldsymbol{W}$.

Proof. One easily shows that near the point $(u, z) = (1, 1/4)$, $h(u, z)$ can be written in the form:

$$h(u, z) \sim_{(u,z) \to (1,1/4)} \frac{1}{4\sqrt{1 - 4z} \cdot \sqrt{1 - 4uz} \cdot \left(\sqrt{1 - 4z} + \sqrt{1 - 4uz}\right)^2}.$$

It is more convenient to write this expression as

$$\frac{1}{4\sqrt{1 - 4z}^3} \left(\frac{1}{\sqrt{1 - 4uz}} - \frac{1}{\sqrt{1 - 4z} + \sqrt{1 - 4uz}} - \frac{\sqrt{1 - 4uz}}{\left(\sqrt{1 - 4z} + \sqrt{1 - 4uz}\right)^2} \right).$$

The dominant term is the third one. We compute derivatives, and obtain a closed form for $H(n)\mathbb{E}\left[(\boldsymbol{W}_n)_r\right]$. We apply the Transfer Theorem (Theorem 13), and obtain the following estimate

$$H(n)\mathbb{E}\left[(\boldsymbol{W}_n)_r\right] \sim \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{3}{2} + r\right)}{\Gamma(3 + r)} n^{r+1} \cdot 4^{n-1}.$$

Using $H(n) \sim n4^{n-2}$, and the asymptotic equality between factorial moments and ordinary moments we obtain the estimate

$$\mathbb{E}\left[\left(\frac{\boldsymbol{W}_n}{n}\right)^r\right] \sim \frac{4}{\sqrt{\pi}} \frac{\Gamma\left(\frac{3}{2} + r\right)}{\Gamma(3 + r)}.$$

These values are bounded by $4/\sqrt{\pi}$, and the Method of Moments in the version of Lemma 15 guarantees that $\mathbf{W}_n/n$ tends in distribution to a random variable $\mathbf{W}$, which is characterized by its moments. The moment generating function is equal to

$$F_{\mathbf{W}}(s) = \frac{4}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{3}{2} + r\right)}{\Gamma(3 + r)\Gamma(1 + r)} (-s)^r.$$

Using the Gauss duplication formula $\Gamma(r) \cdot \Gamma(1/2 + r) = 2^{1-2r}\sqrt{\pi} \cdot \Gamma(2r)$ and recalling that for a positive integer n, $\Gamma(n) = (n-1)!$ we obtain a simplified form for $F_{\mathbf{W}}(s)$:

$$F_{\mathbf{W}}(s) = \sum_{r=0}^{\infty} \binom{2 + 2r}{r} \frac{1}{(1 + r)!} \left(\frac{-s}{4}\right)^r.$$

Checking this sum with `Maple`, this function can be written in terms of Bessel functions of first kind (see [10] for more details on these families of functions) as

$$F_{\mathbf{W}}(s) = \frac{-4}{s} \cdot I_1(-s/2) \cdot e^{-s/2}.$$

Computing the inverse Laplace transform of $F_{\mathbf{W}}(s)$ (using `Maple`) gives directly the density probability function claimed in the statement of the theorem. $\square$

Appendix A. Integration lemmas

In order to obtain the generating function associated to a family of dissections on a cylinder (either simplicial, $\{k+1\}$-dissections or unrestricted dissections), we need to differenciate an initial generating function with respect to a set of variables, and integrate the resulting expression with respect to a set of variables (disjoint with the initial one). In this appendix we encapsulate this technical operation to make the proofs of the forthcoming sections more transparent.

All the generating functions are considered at a formal power series level: operations (sum, product, . . .) are done term by term. In particular, for a formal power series $A(z) = \sum_{n\geq 0} a_n z^n$, the derivative of $A(z)$ is $\frac{d}{dz}A(z) = \sum_{n\geq 1} n a_n z^{n-1}$, and a primitive of $A(z)$ is denoted by $\int_0^z A(s)ds = -a_0 + \sum_{n\geq 0} \frac{1}{n+1} a_n z^{n+1}$.

Let us set our notation. Let r be a non-negative integer. Write $(x_1, x_2, \ldots, x_r) = \overline{x}_r$, $(u_1, u_2, \ldots, u_r) = \overline{u}_r$ and $(X_1, X_2, \ldots, X_r) = \overline{X}_r$. All these variables are considered to be distinct. The operator $\frac{\partial^r}{\partial x_1 \ldots \partial x_r}$ is written in the form $\frac{\partial^r}{\partial \overline{x}_r}$, and a similar definition applies if we substitute $\overline{x}_r$ by either $\overline{u}_r$ or $\overline{X}_r$.

Lemma 21. *Consider the formal power series* $H(u_1, \ldots, u_r, x_1, x_2, \ldots, x_r) = H(\overline{u}_r, \overline{x}_r)$, *defined in terms of a formal power series* M *by the equation*

$$(13) \qquad u_1 \ldots u_r \frac{\partial^r}{\partial \overline{u}_r} H(\overline{u}_r, \overline{x}_r) = x_1 \ldots x_r \frac{\partial^r}{\partial \overline{x}_r} M(u_1 h_1(x_1), \ldots, u_r h_r(x_r)),$$

for some formal power series h_i, $i = 1, \ldots, r$. *If the formal power series* H *and* M *satisfy the set of initial conditions* $H(0, u_2, \ldots, u_r, \overline{x}_r) = H(u_1, 0, \ldots, u_r, \overline{x}_r) = \cdots = H(u_1, u_2, \ldots, 0, \overline{x}_r) = 0$, *and also* $M(0, u_2, \ldots, u_r) = M(u_1, 0, \ldots, u_r) = \cdots = M(u_1, u_2, \ldots, 0) = 0$, *then*

$$H(\overline{u}_r, \overline{x}_r) = \left(\prod_{i=1}^r \frac{x_i}{h_i(x_i)} \frac{\partial}{\partial x_i} h_i(x_i) \right) M(u_1 h_1(x_1), \ldots, u_r h_r(x_r)).$$

Proof. We start developing the right hand side of Equation (13). Denote by $\frac{\partial M}{\partial X_i}$ the derivative of M with respect to its ith variable. We express Equation (13) in the following way:

$$u_1 \ldots u_r \frac{\partial^r}{\partial \overline{u}_r} H(\overline{u}_r, \overline{x}_r) = \left(\prod_{i=1}^r x_i \frac{\partial}{\partial x_i} u_i h_i(x_i) \right) \frac{\partial^r M}{\partial \overline{X}_r} \bigg|_{X_i = u_i h_i(x_i)}.$$

Canceling the term $u_1 \ldots u_r$ in both sides of the previous equation, and integrating the resulting expression with respect to u_r we get

$$(14) \qquad \frac{\partial^{r-1}}{\partial \overline{u}_{r-1}} H(\overline{u}_r, \overline{x}_r) + F(\overline{u}_{r-1}, \overline{x}_r) = \left(\prod_{i=1}^r x_i \frac{\partial h_i(x_i)}{\partial x_i} \right) \frac{\partial^{r-1}}{\partial \overline{X}_{r-1}} \left(\int_0^{u_r} \frac{\partial M}{\partial X_r} \bigg|_{X_i = s_i h_i(x_i)} ds_r \right),$$

for a function F not depending on u_r. Applying the change of variables $X_r = s_r h_r(x_r)$, $dX_r = h_r(x_r)ds_r$, the previous integral is

$$(15) \qquad \frac{1}{h_r(x_r)} \left(M\left(u_1 h_1(x_1), \ldots, u_r h_r(x_r) \right) - M\left(u_1 h_1(x_1), \ldots, u_{r-1} h_{r-1}(x_{r-1}), 0 \right) \right).$$

By the initial conditions of M, the term $M(u_1 h_1(x_1), \ldots, u_{r-1} h_{r-1}(x_{r-1}), 0)$ is equal to 0. Taking $u_r = 0$ in Equation (14) (joint with Equation (15)), and using the hypothesis for the formal power series M, we deduce that F is identically equal to 0. Hence,

$$\frac{\partial^{r-1}}{\partial \overline{u}_{r-1}} H(\overline{u}_r, \overline{x}_r) = \frac{1}{h_r(x_r)} \left(\prod_{i=1}^r x_i \frac{\partial}{\partial x_i} h_i(x_i) \right) M(u_1 h_1(x_1), \ldots, u_r h_r(x_r)).$$

Proceeding in the same way for the remaining $r-1$ variables u_i, $i = 1, \ldots, r-1$, we get the proof of the lemma. $\qquad \square$

The next lemma is a variation of the previous one, and it is used to get compact expressions in Sections 5 and 6. The proof is another application of the chain rule, but in this case we deal with infinitely many variables.

Lemma 22. *Let $\{f_{i,j}(r,s)\}_{i,j\in\mathbb{N}}$ be a set of formal power series in the variables r and s. Let $\overline{z}$ be the infinite vector $(z_{1,1}, z_{1,2}, z_{2,2}, \dots)$, and let $G(r,s,\overline{z})$ be*

$$(f_{1,1}(r,s)z_{1,1}, f_{1,2}(r,s)z_{1,2}, f_{2,2}(r,s)z_{2,2}, f_{1,3}(r,s)z_{1,3}, f_{2,3}(r,s)z_{2,3}, \dots).$$

Suppose that the formal power series $H(G(r,s,\overline{z}),x,y)$ is defined by the equation

$$(16) \qquad rs\frac{\partial^2}{\partial r\partial s}H(G(r,s,\overline{z}),x,y) = xy\frac{\partial^2}{\partial x\partial y}M(\overline{z}, rg_1(x), sg_2(y)),$$

where M, g_1, g_2 are formal power series satisfying the conditions $M(\overline{z}, 0, y) = M(\overline{z}, x, 0) = 0$, and $H(G(0,s,\overline{z}),x,y) = H(G(r,0,\overline{z}),x,y) = 0$. Then $H(G(r,s,\overline{z}),x,y)$ admits the following expression:

$$H(G(r,s,\overline{z}),x,y) = \frac{xy}{g_1(x)g_2(y)}\frac{\partial g_1(x)}{\partial x}\frac{\partial g_2(y)}{\partial y}M(\overline{z}, rg_1(x), sg_2(y)).$$

Proof. The proof is obtained using the same arguments as in the proof of Lemma 21. Denote by M_{12} the derivative of M with respect to the last two variables. Equation (16) can be written then in the form:

$$\frac{\partial^2}{\partial r\partial s}H(G(r,s,\overline{z}),x,y) = xy\frac{\partial g_1(x)}{\partial x}\frac{\partial g_2(y)}{\partial y}M_{12}(\overline{z}, rg_1(x), sg_2(y)).$$

Integrating this expression (some partial steps are omitted) with respect to r gives

$$\frac{\partial}{\partial s}H(G(r,s,\overline{z}),x,y) - \frac{\partial}{\partial s}H(G(0,s,\overline{z}),x,y) = \frac{xy}{g_1(x)}\frac{\partial g_1(x)}{\partial x}\frac{\partial g_2(y)}{\partial y}M_2(\overline{z}, rg_1(x), sg_2(y)) -$$
$$\frac{xy}{g_1(x)}\frac{\partial g_1(x)}{\partial x}\frac{\partial g_2(y)}{\partial y}M_2(\overline{z}, 0, sg_2(y)).$$

Applying the initial conditions for M and H as it is done in Lemma 21 simplifies the previous expression, getting

$$\frac{\partial}{\partial s}H(G(r,s,\overline{z}),x,y) = \frac{xy}{g_1(x)}\frac{\partial g_1(x)}{\partial x}\frac{\partial g_2(y)}{\partial y}\left(M_2(\overline{z}, rg_1(x), sg_2(y))\right),$$

To conclude we apply the same integration argument to the variable s. $\square$

Appendix B. A related problem

Instead of taking two independent labellings on $\mathbb{S}_1^1$ and $\mathbb{S}_2^1$, we can also count dissections where only vertices on $\mathbb{S}_1^1$ (the external circle) are labeled. In this section we introduce results for these families without proofs. These results are straightforward adaptations of the arguments used in the previous sections. With these results, we complete the picture introduced in [8], in which the enumeration of simplicial decompositions on a cylinder is obtained.

The main idea in the enumeration of these families is the same as in previous sections: we obtain the generating function for transversal dissections, and then we make a convenient substitution to obtain the generating function for the general family. Let $\Delta \subseteq \mathbb{N} - \{1,2\}$. Denote by $\overrightarrow{M}^\Delta(u,x,y)$ the GF of transversal Δ-dissections on a cylinder, where vertices on the internal circle are not labelled; as in the previous sections, u marks transversal faces, and x, y mark the number of internal and external vertices, respectively. The main relation between $\overrightarrow{M}^\Delta(u,x,y)$ and $M^\Delta(u,x,y)$ (recall that M^Δ is the GF for transversal Δ-dissections, which has been deduced in previous Sections for several choices of Δ) is

$$x\frac{\partial}{\partial x}\overrightarrow{M}^\Delta(u,x,y) = M^\Delta(u,x,y),$$

which means that a transversal Δ-dissection is obtained from $\overrightarrow{M}^\Delta(u,x,y)$ by pointing one internal vertex (in other words, pointing the vertex whose label is $1'$). This equation is translated into

$$(17) \qquad\qquad \overrightarrow{M}^\Delta(u,x,y) = \int_0^x \frac{M^\Delta(u,r,y)}{r}dr.$$

These equalities are valid for all type of dissections (i.e., for every choice of Δ). The same argument used in Proposition 2 (with a modification of the integration lemma used on it) shows that the GF for dissections (simplicial decompositions, $\{k+1\}$-dissections and unrestricted dissections) can be written using the expression derived from Equation (17). The argument is the same as in Theorems 2 and 8. The proof for other families is straightforward. The only difference is that we do not consider labels on the internal circle. For the case of $\{k+1\}$-dissections the GF considered is $\overrightarrow{H}^{\{k+1\}}(u,x)$, which is equal to

$$\overrightarrow{H}^{\{k+1\}}(u,x) = \frac{1}{C_{k+1}(x^{k-1})} \left(\frac{\partial}{\partial x} x C_{k+1}(x^{k-1}) \right) \overrightarrow{M}^{\{k+1\}}(u, x C_{k+1}(x^{k-1}), x C_{k+1}(x^{k-1})).$$

For unrestricted dissections the expression obtained is

$$\overrightarrow{H}^{D}(u,x) = \frac{1}{D(u,x)} \left(\frac{\partial}{\partial x} x D(u,x) \right) \overrightarrow{M}^{D} \left(u, \frac{1}{x} D(u,x), \frac{1}{x} D(u,x) \right),$$

in all cases, x marks vertices and u marks transversal faces. For the case of simplicial decompositions, we obtain (recall that z is used to mark faces)

$$\begin{aligned}
\overrightarrow{H}(z) &= \frac{-2z^4 - z^3 - 2z^2 + 7z - 2}{z(1-4z)} + \frac{3z^4 + 2z^3 + 7z^2 - 9z + 2}{z(1-4z)} C(z) \\
&= 7z^6 + 77z^7 + 555z^8 + 3318z^9 + \ldots,
\end{aligned}$$

which was first obtained in [8]. In our approach, this result is a simple consequence of all the previous computations. For $\{k+1\}$-dissections, the GF of transversal dissections is

$$\overrightarrow{M}^{\{k+1\}}(u,x) = \frac{p_k(u,x)}{2(-1 + ux^{k-1})^4(-1 + kux^{k-1})},$$

where $p_k(u,x)$ is

$$\begin{aligned}
p_k(u,x) = \quad & (k-1)(2-k)^3 x^{3k-3} u^3 - \\
& 2(k-1)(4+k)(k-2)^2 x^{4k-4} u^4 + \\
& 6(k-1)(k-2)(-2k-5+k^2) x^{5k-5} u^5 - \\
& 2(k-1)(27 + 18k - 18k^2 + 2k^3) x^{6k-6} u^6 + \\
& (k-1)(20 + 48k - 24k^2 + k^3) x^{7k-7} u^7 + \\
& 2(k-1)(-1 - 10k + 3k^2) x^{8k-8} u^8 + 2(k-1)k x^{9k-9} u^9.
\end{aligned}$$

For unrestricted dissections the expression we obtain is

$$\overrightarrow{M}^{D}(u,x) = \frac{-d(u,x)}{(1 - 2x + x^2 + u(x^2 - 2x))(x-1)^5(-1 + x + xu)^3},$$

where $d(u,x)$ is

$$\begin{aligned}
d(u,x) = \quad & (x^6(x-1)^3) u^3 + (x^6(-9 + 7x)(x-1)^2) u^4 + \\
& (3x^6(x-1)(-3 + 2x)(-2 + 3x)) u^5 + \\
& (x^6(-7 + 41x - 54x^2 + 19x^3)) u^6 + \\
& (x^7(7 - 19x + 8x^2)) u^7 + (x^8(x-2)) u^8.
\end{aligned}$$

As in Section 7, we can also obtain asymptotic estimates for the coefficients. This results are summarized in Table 6. For simplicial decompositions and $\{k+1\}$-dissections, n is referred to the number of faces, and for unrestricted dissections to the number of vertices. Recall the definitions of X and Z stated in Section 7.

In all cases we miss the subexponential term which appears in all expressions in Section 7. As it is shown in [1], this subexponential term does not depend on the choice of Δ.

Family	Singular expansion	Asymptotic growth
Simplicial decompositions	$\frac{1}{4}Z^{-2}$	$\frac{1}{4}4^n$
$\{k+1\}$-dissections	$\frac{k-1}{4}Z^{-2}$	$\frac{k-1}{4}\rho_{k+1}^{-n}$
Unrestricted dissections	$\frac{1}{4}X^{-2}$	$\frac{1}{4}\rho_D^{-n}$

TABLE 6. Asymptotic behavior and estimates for those families: simplicial decompositions, $\{k+1\}$-dissections and unrestricted dissections.

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