STABLE MODEL CATEGORIES AND COHOMOLOGICAL DESCENT

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ABSTRACT. We prove that the subcategory of fibrant objects of a stable simplicial model category is a cohomological descent category, in the sense of Guillén and Navarro.

1. INTRODUCTION

Let k be a field of characteristic zero, $Sm(k)$ the category of smooth schemes over k and $Sch(k)$ the category of separeted and finite type schemes over k. Guillén and Navarro have proved in [5] an extension result for cohomological functors defined on $Sm(k)$ to cohomological functors defined on $Sch(k)$. Classical cohomogical functors take values in the category of graded abelian groups or in the category of abelian chain complexes. In order to apply their main result to non-abelian situations, such as the rational homotoy type of C-schemes or to motives of singular varieties, they introduced the notion of (cohomological) descent category as a good class of categories in which cohomology theories take values.

The descent categories arised as a higher category variation of Vedier triangulated categories, since its definition take into account all finte diagrams over them, and cover also some non abelian and non stable settings.

Depending on the variance of the functor to extend in the descent theorem, there are two dual notions of descent category: homological and cohomological. Here we refer only to the cohomological case.

A descent category is, essentially, a triple $(\mathcal{D}, E, \mathbf{s})$ given by a cartesian category $\mathcal{D}$ with initial object 0, a saturated class of morphisms E of $\mathcal{D}$, called *weak equivalences*, and for every cubical type $\square$ a functor

$$\mathbf{s}_\square : (\square, \mathcal{D}) \to \mathcal{D},$$

called *simple*, natural in $\square$ in a precise sense, which satisfyies the following properties:

1.*Multiplicativity.* The simple of an object X considered as a diagram is isomorphic to X, and for every pair $(\mathbf{X}, \mathbf{Y})$ of $\square$-diagrams there is an isomorphism $\mathbf{s}_\square(\mathbf{X} \times \mathbf{Y}) \to \mathbf{s}_\square \mathbf{X} \times \mathbf{s}_\square \mathbf{Y}$.

2.*Factorisation.* For every $\square \times \square'$-diagram $\mathbf{X} = (\mathbf{X}_{\alpha\beta})$ exists an isomorphism $\mu : \mathbf{s}_{\alpha\beta} \mathbf{X}_{\alpha\beta} \to \mathbf{s}_\alpha \mathbf{s}_\beta \mathbf{X}_{\alpha\beta}$.

3.*Exactness.* Let $f : \mathbf{X} \to \mathbf{Y}$ be a morphism of $\square$-diagrams. If it is a pointwise weak equivalence, then the morphism $\mathbf{s}_\square f : \mathbf{s}_\square \mathbf{X} \to \mathbf{s}_\square \mathbf{Y}$ is a weak equivalence.

4.*Aciclicity criterium.* A morphism $f : X_0 \to X_1$ is a weak equivalence if and only if the simple of the $\square_1$-diagram

$$0 \to X_0 \xleftarrow{f} X_1$$

is acyclic, that is, is weakly equivalent to the final object of $\mathcal{D}$.

One problem that appears is to have sufficiently many examples of descent categories. The aim of this note is to prove that the subcategory of fibrant objects of a simplicial stable model category is a cohomological descent category. As a corollary, we will have that the category of fibrant spectra is a cohomological descent category. This result will have applications in algebraic K-theory (see [9]).

We have restricted our attention to simplicial model categories, where there is defined an homotopy limit functor. This should not be an important restriction, as it is proved in [11] that every cofibrantly generated, proper, stable model category is in fact Quillen equivalent to a simplicial model category.

2. Recollections and notations

2.1.   Model categories were introduced by Quillen in [10]. We use the definition adopted by Hirschhorn in [6], 7.1, which has stronger conditions than Quillen's (all small limits and colimits are required to exist, and we also need functorial factorisations in axiom five).

Given $\mathcal{M}$ a simplicial model category, tensoring an object X by a simplicial set K will be denoted by $X \otimes K$, and $F(K, X)$ will be the cotensorisation (usually denoted X^K).

By adjointness properties (cf.[3], II, 2.1 and 2.2), $F(K, -)$ preserves limits and $F(-, X)$ converts colimits to limits. We also have $F(K \times L, X) \cong F(K, F(L, X))$ (see [6], 9.1.11).

2.2.   A model category is *pointed* if the initial and final objects coincide. In this case the initial and final object is denoted $*$. Let $*$ also denote the one point simplicial set. There are isomorphisms $F(*, X) = X$ and $F(K, *) = *$.

An object X in a pointed model category is *acyclic* if the natural morphism $* \to X$ is a weak equivalence. (Equivalently, if the morphism $X \to *$ is a weak equivalence.)

2.3.   If $\mathcal{C}$ is a small category, $B\mathcal{C}$ will denote the classifying space or nerve of $\mathcal{C}$. We will use the notations and definitions of [6] concerning classifying spaces, overcategories, homotopy limits and homotopy cofinal functors.

2.4.   The exactness property of the homotopy limit will require pointwise fibrant diagrams. Fibrant objects are closed by cotensoring and by taking finite products. They are also closed by taking homotopy limits, as we will state later.

2.5.   The homotopy category of a model category is obtained by inverting the weak equivalences. We remark that weak equivalences are *saturated*: a morphism $f : X \to Y$ is a weak equivalence if, and only if, it is an isomorphism in the homotopy category (see [6], 8.3.10).

2.6.   Recall that the homotopy category of a pointed model category supports a suspension functor Σ with a right adjoint loop functor Ω. A *stable model category* is a pointed model category where the functors Ω and Σ in the homotopy category are inverse equivalences. Stability will be required to obtain the aciclicity criterium, the key point being that the homotopy theory of a stable model category is triangulated.

3. Homotopy limits

In this section we recall the definition the homotopy limit of a diagram in a simplicial model category. Then we will recall its main properties.

3.1.   Given $\mathcal{C}$ a small category and $\mathcal{M}$ a category, a $\mathcal{C}$-*diagram of* $\mathcal{M}$ is a functor $\mathcal{C} \to \mathcal{M}$. We will also call it a *codiagram of type* $\mathcal{C}$. We call it codiagram to be consistent with the terminology used in [5] where a diagram of type $\mathcal{C}$, or $\mathcal{C}$-object of $\mathcal{M}$, is a functor $\mathcal{C}^{op} \to \mathcal{M}$.

If X is an object of $\mathcal{M}$, $\mathcal{C} \times X$ will denote the constant codiagram of type $\mathcal{C}$ (with all morphisms equal to the identity of X).

Definition 3.1. Let $\mathbf{X}$ be $\mathcal{C}$-diagram of a simplicial model category $\mathcal{M}$. The *homotopy limit* of $\mathbf{X}$, $\operatorname{holim} \mathbf{X}$, is the equaliser of the morphisms

$$\prod_{\alpha \in Ob(\mathcal{C})} F(B(\mathcal{C} \downarrow \alpha), \mathbf{X}_\alpha) \quad \overset{\phi}{\underset{\psi}{\rightrightarrows}} \quad \prod_{(\sigma : \alpha \to \alpha') \in \mathcal{C}} F(B(\mathcal{C} \downarrow \alpha), \mathbf{X}_{\alpha'})$$

where the projection of ϕ in the factor $\sigma : \alpha \to \alpha'$ is the composition of the natural projection from the product with the morphism

$$\sigma_*^{1_{B(\mathcal{C} \downarrow \alpha)}} : F(B(\mathcal{C} \downarrow \alpha), \mathbf{X}_\alpha) \to F(B(\mathcal{C} \downarrow \alpha), \mathbf{X}_{\alpha'})$$

and the projection of ψ to the factor $\sigma : \alpha \to \alpha'$ is the composition of the natural projection from the product with the morphism

$$F(B(\sigma_*), 1_{\mathbf{X}_{\alpha'}}) : F(B(\mathcal{C} \downarrow \alpha'), \mathbf{X}_{\alpha'}) \to F(B(\mathcal{C} \downarrow \alpha), \mathbf{X}_{\alpha'}),$$

where $\sigma_* : (\mathcal{C} \downarrow \alpha) \to (\mathcal{C} \downarrow \alpha')$.

Example 3.2. If $\mathbf{X}$ is a constant $\mathcal{C}$-diagram in an object X of a simplicial model category $\mathcal{M}$, then $\operatorname{holim} \mathbf{X} = F(B\mathcal{C}, X)$, as is easily seen from [2], XI, 2.3.

3.2.   We recall the basic properties of homotopy limits in a simplicial model category $\mathcal{M}$. Observe that homotopy invariance and cofinality require pointwise fibrant diagrams.

3.2.1.   The homotopy limit of a pointwise fibrant diagram is a fibrant object (cf. [6], 18.5.2).

3.2.2.   The homotopy limit is an end. The homotopy limit of a $\mathcal{C}$-diagram is the end of the functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{M}$, $(\alpha, \alpha') \mapsto F(B(\mathcal{C} \downarrow \alpha), \mathbf{X}_{\alpha'})$. We can write

$$\operatorname{holim} \mathbf{X} = \int_\alpha F(B(\mathcal{C} \downarrow \alpha), \mathbf{X}_\alpha)$$

with the notation of [8] (cf. [6], 18.3.2 and 18.3.6). Therefore end properties as Fubini hold (see [8]).

3.2.3.   The homotopy limit is functorial with respect to both variables, and is particularly well behaved with respect to pointwise fibrant diagrams:

i) If $\mathbf{X}$ and $\mathbf{Y}$ are $\mathcal{C}$-diagrams, a morphism $f : \mathbf{X} \to \mathbf{Y}$ induces a morphism

$$\operatorname{holim} f : \operatorname{holim} \mathbf{X} \to \operatorname{holim} \mathbf{Y}.$$

If $\mathbf{X}$ and $\mathbf{Y}$ are pointwise fibrant and f is a pointwise weak equivalence, then $\operatorname{holim} f$ is a weak equivalence of fibrant objects. This is the *homotopy invariance* property of holim ([6], 18.5.3).
      If $\mathbf{X}$ and $\mathbf{Y}$ are pointwise fibrant and f is a pointwise fibration, then $\operatorname{holim} f$ is a fibration ([6], 18.5.1).

ii) Assume that $F : \mathcal{C} \to \mathcal{D}$ is a functor between small categories and $\mathbf{X}$ a $\mathcal{D}$-diagram. There is induced a $\mathcal{C}$-diagram $F^*\mathbf{X}$. Then there is a natural morphism

$$\operatorname*{holim}_{\mathcal{D}} \mathbf{X} \to \operatorname*{holim}_{\mathcal{C}} F^*\mathbf{X}$$

induced by the morphisms $F_* : B(\mathcal{C} \downarrow \alpha) \to B(\mathcal{D} \downarrow F\alpha)$ (cf. [6], 19.1.8).

If the functor F is homotopy left cofinal and $\mathbf{X}$ is pointwise fibrant, the natural morphism $\operatorname{holim}_{\mathcal{D}} \mathbf{X} \to \operatorname{holim}_{\mathcal{C}} F^* \mathbf{X}$ is a weak equivalence. This is the *cofinality* theorem of holim ([6], 19.6.7b).

3.2.4.   The homotopy limit considered as a functor from the category of $\mathcal{C}$-diagrams to $\mathcal{M}$ preserves limits. This property is deduced as in [13], lemma 5.11, where is stated for spectra.

3.3. **Homotopy fibre sequences.** We define the the *homotopy fibre* of a morphism $f : X \to Y$ in a pointed simplicial model category $\mathcal{M}$ as $Ff = \operatorname{holim}(X \to Y \leftarrow *)$; the corresponding limit defines the fibre.

The homotopy category of a stable model category is triangulated ([7], 7.1.6). The cofibre and fibre sequences of [10], 1.3, coincide up to sign (cf. [7], 7.1.11) and define the distinguished triangles.

We remark that the morphisms between fibre sequences must respect the action over the fibre (see [10], 1.3).

Lemma 3.3 (Five lemma). *Let $\mathcal{M}$ be a stable simplicial model category. Given a morphism of fibre sequences or homotopy fibre sequences*

$$
\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
\downarrow{a} & & \downarrow{b} & & \downarrow{c} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z'
\end{array}
$$

if two of the morphisms a, b and c are weak equivalences, then the third is also a weak equivalence.

Proof. The morphism of fibre sequences gives a morphism of triangles in the homotopy category. By saturation the result follows from the analog property of triangulated categories (see [7], 6.5.3b). □

Lemma 3.4 (Aciclicity criterium). *Let $\mathcal{M}$ be a stable simplicial model category. A morphism $f : X \to Y$ is a weak equivalence if and only if the homotopy fibre Ff is acyclic.*

Proof. By saturation and passing to the homotopy category, the result follows from the following property of triangulated categories: If

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X$$

is a distinguished triangle in a triangulated category, then g is an isomorphism if and only if $X \cong 0$. □

As in [13], lemma 5.12, we deduce from the properties of holim above that the homotopy limit preserves fibre sequences:

Proposition 3.5. *Let $\mathbf{X} \to \mathbf{Y}$ be a morphism of pointwise fibrant $\mathcal{C}$-diagrams in a simplicial model category $\mathcal{M}$. Suppose that for all $\alpha \in \mathcal{C}$, $\mathbf{X}_\alpha \to \mathbf{Y}_\alpha$ is a fibration with fibre $\mathbf{F}_\alpha$. Then $\operatorname{holim} \mathbf{F}$ is the fibre of the fibration $\operatorname{holim} \mathbf{X} \to \operatorname{holim} \mathbf{Y}$. Therefore, holim preserves fibre sequences, and also homotopy fibre sequences of fibrant objects.* □

4. Cubical diagrams

In this section we define the simple or realization functor of a codiagram by means of the homotopy limit. This functor is defined over the category $Codiag_\Pi \mathcal{M}$ of cubical codiagrams with variable type in the category Π. We begin by recalling the definition of the category Π of cubical types (cf. [5], 1.1.1). After defining the category of codiagrams we define the simple functor and then we eplain how to extend the definition to augmented cubical codiagrams.

4.1. **Cubical diagrams.** Associate to a non-empty set S the set of non-empty subsets, ordered by inclusion: that defines the category $\square_S$. We denote $\square_n$ the set $\square_{\{0,1,\dots,n\}}$.

Given S and T two finite sets, any injective map $u : S \to T$ defines a functor $\square_u : \square_S \to \square_T$.

Associate to a family $S = (S_i)_{i \in I}$, with I a finite set, the cartesian product $\Pi_{i \in I}\square_{S_i}$ with the product order. Write $\square_S = \Pi_{i \in I}\square_{S_i}$. Define $dimS = \sum_{i \in I} dimS_i$.

The objects of the category Π are the families $(S_i)_{i \in I}$ of non-empty finite sets, with I finite variable. Given $S = (S_i)_{i \in I}$ and $T = (T_i)_{j \in J}$, a morphism $u : S \to T$ of Π is an injective map $u : \Pi_i S_i \to \Pi_j T_j$ such that, for every $\alpha = (\alpha_i) \in \square_S$, exists $\beta = (\beta_j) \in \square_T$ such that $u(\Pi\alpha_i) = \Pi\beta_j$.

Now we turn to augmented cubical diagrams. Associate to a finite set S, possibly empty, the set $\square_S^+$ of subsets of S, ordered by inclusion. Observe that $\square_n^+$ is a $(n + 1)$-cube.

For example, $\square_2$ and $\square_2^+$ are represented by the diagrams

and
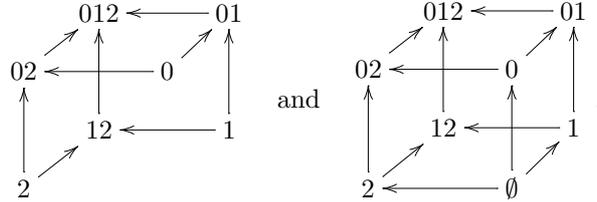
where identities and morphisms composition of two are not represented.

4.2. **The category $Codiag_\Pi \mathcal{M}$.** The category $Codiag_\Pi \mathcal{D}$ of codiagrams is analogous to the category $Diag_\Pi \mathcal{D}$ of diagrams of [5], 1.2.1.

Given $\delta : \square \to \square'$ a morphism of Π, there is an induced inverse image functor

$$\delta^* : (\square', \mathcal{M}) \to (\square, \mathcal{M})$$

defined by $F \mapsto \delta^*(F) := F \circ \delta$.

The category $Codiag_\Pi \mathcal{M}$ of cubical codiagrams of objects of $\mathcal{M}$ is defined as follows. An object is a pair $(X, \square)$, where $\square \in Ob\Pi$ and $X : \square \to \mathcal{M}$ is a codiagram. A morphism $(\mathbf{Y}, \square') \to (\mathbf{X}, \square)$ is a pair (a, δ) where $\delta : \square \to \square'$ is a morphism of Π and $a : \delta^*\mathbf{Y} \to \mathbf{X}$ is a natural transformation of functors of $(\square, \mathcal{M})$.

4.3. **Simple functor of a codiagram.** Let $\mathcal{M}_f$ be the category of fibrant objects of a simplicial model category. Let $\square$ be an object of Π. The *simple functor*

$$\mathbf{s} : Codiag_\Pi \mathcal{M}_f \to \mathcal{M}_f$$

associates to a $\square$-diagram $\mathbf{X}$ the object defined by the homotopy limit $\mathbf{s}_\square(\mathbf{X}) = \mathrm{holim}\,\mathbf{X}$.

If we have $(\mathbf{Y}, \square') \to (\mathbf{X}, \square)$ a morphism in $Codiag_\Pi \mathcal{M}_f$ given by $\delta : \square \to \square'$ and $a : \delta^* \mathbf{Y} \to \mathbf{X}$, the functorial properties of the homotopy limit allow us to define the composition

$$\mathbf{s}_{\square'} \mathbf{Y} = \operatornamewithlimits{holim}_{\square'} \mathbf{Y} \to \operatornamewithlimits{holim}_\square \delta^* \mathbf{Y} \to \operatornamewithlimits{holim}_\square \mathbf{X} = \mathbf{s}_\square \mathbf{X},$$

which gives the covariance of the functor.

A diagram $\mathbf{X}$ is *acyclic* if the object $\mathbf{sX}$ is acyclic.

4.4. Simple of an augmented cubical codiagram.

In this subsection we explain how to extend the simple functor to augmented cubical diagrams. Then we prove a variation in the required generality of a property of acyclic cubes of spaces and spectra, following [14], 1.1 (see also [4], 1.1).

The functor $\mathbf{s}$ is extended to augmented cubical diagrams using the cone construction (see [5], 1.4.3). Thus, if $f : X \to Y$ is a $\square_0^+$-diagram, its simple is the homotopy fibre of f.

Given a $\square_n^+$-diagram $\mathbf{X}^+$, view it as a morphism of two $\square_{n-1}^+$ diagrams, $f : \mathbf{X}_0^+ \to \mathbf{X}_1^+$. The simple object associated to $\mathbf{X}^+$ is obtained as the simple of the $\square_{n-1}^+$-diagram which in each degree α has the homotopy fibre of f_α.

The acyclic cubes are also called *homotopy cartesian* cubes.

Proposition 4.1. *Let $\mathcal{M}_f$ be the category of fibrant objects of a stable simplicial model category. An augmented diagram $\mathbf{X}^+ : \square_n^+ \to \mathcal{M}_f$ is acyclic if, and only if, the natural morphism $\mathbf{X}_0 \to \operatornamewithlimits{holim}_{\square_n} \mathbf{X}$ is a weak equivalence.*

Before proving the proposition we need some technical lemmas:

Lemma 4.2. *The functor $f : \square_n \to \square_{n-1}^+$ defined by $(i,j) \mapsto (j)$ is homotopy left cofinal.*

Proof. Given $\alpha = (1,j) \in \square_n$, it is easy to see that the category $(f \downarrow f(\alpha))$ is isomorphic to the category $(\square_n \downarrow \alpha)$ and thus contractile. $\square$

Lemma 4.3.   a) *The functor $f : \square_1 \times \square_{n-1} \to \square_n$ defined by $f((0,1),k) = (0,k)$, $f((1,1),k) = (1,k)$ and $f((1,0),k) = (1,0,\dots,0)$ is homotopy left cofinal.*

   b) *Given a diagram $\mathbf{E} : \square_n \to \mathcal{M}$ such that $\mathbf{E}_{1,k}$ is acyclic for every $k \in \square_{n-1}$, then there is a weak equivalence $\operatornamewithlimits{holim}_{\square_n} \mathbf{E} \to \operatornamewithlimits{holim}_{0 \times \square_{n-1}} \mathbf{E}$.*

Proof. It is easy to see that the required categories are contractile: the category $(f \downarrow (1,0,\dots,0))$ is isomorphic to $\square_{n-1}$, the category $(f \downarrow (0,k))$ is isomorphic to $(\square_{n-1} \downarrow (k))$ and the category $(f \downarrow (1,k))$ is isomorphic to $(\square_n \downarrow (1,k))$.

Part b) is deduced by cofinality, Fubini and homotopy invariance ($\mathbf{E}_{1,k}$ is acyclic):

$$\operatornamewithlimits{holim}_{\square_n} \mathbf{E} \xrightarrow{\simeq} \operatornamewithlimits{holim}_{\square_1 \times \square_{n-1}} f^* \mathbf{E} \cong \operatornamewithlimits{holim}_{j \in \square_1} \operatornamewithlimits{holim}_{k \in \square_{n-1}} \mathbf{E}_{f(j,k)} \xrightarrow{\simeq}$$

$$\xrightarrow{\simeq} \operatornamewithlimits{holim}(* \to * \leftarrow \operatornamewithlimits{holim}_{k \in \square_{n-1}} \mathbf{E}_{0,k}) \cong \operatornamewithlimits{holim}_{0 \times \square_{n-1}} \mathbf{E}$$

$$\square$$

Proof of the proposition. The case $n = 0$ is the aciclicity criterium (lemma 3.4).

Define $\mathbf{Y} : \square_n^+ \to \mathcal{M}_f$ by $\mathbf{Y}_{0,j} = \mathbf{Y}_{1,j} = \mathbf{X}_{1,j}$, where $j \in \square_{n-1}^+$. Consider the functor $f : \square_n \to 1 \times \square_{n-1}^+$ defined by $(i,j) \mapsto (1,j)$. By the lemma above this functor is homotopy left cofinal.

By the cofinality theorem and as $\square_{n-1}^+$ has an initial object we have:

$$\operatornamewithlimits{holim}_{\square_n} \mathbf{Y} = \operatornamewithlimits{holim}_{\square_n} f^*(\mathbf{X}_{1,-}) \xleftarrow{\simeq} \operatornamewithlimits{holim}_{1 \times \square_{n-1}^+} \mathbf{X}_{1,-} \xleftarrow{\simeq} \mathbf{X}_{1,0,\dots,0}.$$

Define $\mathbf{E}_{i,j} = \operatorname{hofib}(\mathbf{X}_{i,j} \to \mathbf{Y}_{i,j})$. Observe that the simple of $\mathbf{X}$ is the simple of $\mathbf{E}_{0,-}$.

By definition, $\mathbf{E}_{0,j} \to \mathbf{X}_{0,j} \to \mathbf{X}_{1,j}$ is a homotopy fibre sequence for every $j \in \square_{n-1}^{+}$. We have the following diagram with rows homotopy fibre sequences, for holim preserves them (proposition 3.5):

$$
\begin{array}{ccccc}
\mathbf{E}_0 & \longrightarrow & \mathbf{X}_0 & \longrightarrow & \mathbf{X}_{1,0,\dots,0} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \simeq} \\
\operatorname{holim}_{\square_n} \mathbf{E} & \longrightarrow & \operatorname{holim}_{\square_n} \mathbf{X} & \longrightarrow & \operatorname{holim}_{\square_n} \mathbf{Y}
\end{array}
$$

We want to see that $\mathbf{X} : \square_n^+ \to \mathcal{M}$ is acyclic if and only if $\mathbf{X}_0 \to \operatorname{holim}_{\square_n} \mathbf{X}$ is a weak equivalence. As the simple of $\mathbf{X}$ is the simple of $\mathbf{E}_{0,-}$ and by the five lemma (lemma 3.3) applied to the diagram above, it is enough to see that $\mathbf{E}_{0,-} : \square_{n-1}^+ \to \mathcal{M}$ is acyclic if and only if $\mathbf{E}_0 \to \operatorname{holim}_{\square_n} \mathbf{E}$ is a weak equivalence.

By induction we have the result for every $(n-1)$-cube. In particular we have that $\mathbf{E}_{0,-}$ is acyclic if and only if $\mathbf{E}_0 \to \operatorname{holim}_{0 \times \square_{n-1}} \mathbf{E}$ is a weak equivalence. Therefore it is clear that it is enough to see that $\operatorname{holim}_{0 \times \square_{n-1}} \mathbf{E} \to \operatorname{holim}_{\square_n} \mathbf{E}$ is a weak equivalence. This is b) from the lemma above. $\qquad\square$

5. The main result

A *cohomological descent category* (see [5], 1.5.3, 1.7.1) is given by $(\mathcal{D}, E, \mathbf{s}, \mu, \lambda)$ satisfying the eight properties $(\mathrm{CD1})^{op}$ to $(\mathrm{CD8})^{op}$ below, which are stated there for $\mathcal{D} = \mathcal{M}_f$ and E the class of weak equivalences.

Let $\mathcal{M}$ be a pointed simplicial model category and $\mathcal{M}_f$ the subcategory of fibrant objects. Let $\mathbf{s}$ be the simple functor defined by the homotopy limit.

By the general properties of ends (see [8]), given $\square, \square' \in \Pi$, we have a natural transformation of functors

$$\mu_{\square,\square'} : \mathbf{s}_\square \circ \mathbf{s}_{\square'} \to \mathbf{s}_{\square \times \square'}$$

such that $\mu_{\square,\square'}(\mathbf{X}) : \mathbf{s}_\square \circ \mathbf{s}_{\square'}(\mathbf{X}) \to \mathbf{s}_{\square \times \square'}(\mathbf{X})$ is an isomorphism for every $\square \times \square'$-diagram $\mathbf{X}$. This isomorphism is called Fubini isomorphism.

The morphism $\lambda_{\square_S}(X) : X = F(*, X) \to F(B(\square_S), X)$ is the one induced by the simplicial set morphism $B(\square_S) \to *$.

Theorem 5.1. *The category $\mathcal{M}_f$ of fibrant objects of a stable simplicial model category, with the class E of weak equivalences, the simple functor $\mathbf{s}$ and μ and λ defined above, is a cohomological descent category.*

Proof. See $(\mathrm{CD1})^{op}$ to $(\mathrm{CD8})^{op}$ below. Observe that the stability hypothesis is only used in $(\mathrm{CD8})^{op}$. $\qquad\square$

$(\mathbf{CD1})^{op}$. *$\mathcal{M}_f$ is a cartesian category.*

Proof. Recall that a category is cartesian if it has all finite products and $\mathcal{M}_f$ has them. $\qquad\square$

$(\mathbf{CD2})^{op}$. *The class of weak equivalences is a saturated class of morphisms, stable by products: if $f : X \to X'$ and $g : Y \to Y'$ are weak equivalences, then $f \times g : X \times Y \to X' \times Y'$ is a weak equivalence.*

Proof. In every model category, the class of weak equivalences is saturated (cf. [6], 8.3.10).

The stability by products is seen obtaining $X \times Y$ as a homotopy limit of a discrete diagram and using $(\mathrm{CD5})^{op}$. $\qquad\square$

The direct image of a codiagram is defined dually to the one for diagrams (see [5], 1.2.2). If $\delta : \square \to \square'$ is a morphism of Π, there is a *direct image* functor

$$\delta_* : (\square, \mathcal{M}) \to (\square', \mathcal{M})$$

such that if $\mathbf{X}$ is a $\square$-diagram of $\mathcal{M}$ then $\delta_* \mathbf{X}$ is the $\square'$-diagram defined by

$$(\delta_* \mathbf{X})_\beta = \begin{cases} \mathbf{X}_\alpha & \text{if } \beta = \delta(\alpha), \alpha \in \square \\ * & \text{if } \beta \in \square' \setminus \delta(\square) \end{cases}$$

with the evident morphisms.

(CD3)^{op}. $\mathbf{s} : Codiag_\Pi \mathcal{M}_f \to \mathcal{M}_f$ *is a covariant functor such that if* $\delta : \square \to \square'$ *is a morphism of* Π *and* $\mathbf{X}$ *is a* $\square$*-diagram of* $\mathcal{M}_f$, *the morphism* $\mathbf{s}_{\square'} \delta_* \mathbf{X} \to \mathbf{s}_\square \mathbf{X}$ *is a weak equivalence.*

Proof. The functor $\mathbf{s}$ has been defined in section 4.3.

We may assume that $\square = \square_S = \square_{S_1} \times \cdots \times \square_{S_s}$ and $\square' = \square_T = \square_{T_1} \times \cdots \times \square_{T_t}$ with $t \geq s$, and that the morphism δ is induced by inclusions $S_i \subset T_i$ and constants $\gamma_{s+1} \in T_{s+1}, \ldots, \gamma_t \in T_t$.

Let $\beta = (M_1, \ldots, M_t) \in \square_T$ be a vertex of $\square_T$. There is an arrow of the form $\delta(\alpha) = (N_1, \ldots, N_s, \{\gamma_{s+1}\}, \ldots, \{\gamma_t\}) \to \beta$ if $N_1 \subset M_1, \ldots, N_s \subset M_s, \gamma_{s+1} \in M_{s+1}, \ldots, \gamma_t \in M_t$.

Therefore the category $(\delta \downarrow \beta)$ is identified with the category $\square_{S_1 \cap M1} \times \cdots \times \square_{S_s \cap M_s} \times \square_{\{\gamma_{s+1}\} \cap M_{s+1}} \times \cdots \times \square_{\{\gamma_t\} \cap M_t}$, which is contractile if it is non-empty.

Thus, δ is a homotopy left cofinal functor and by cofinality we have that

$$\underset{\square'}{\text{holim}}(\delta_* \mathbf{X}) \to \underset{\square}{\text{holim}}(\delta^*(\delta_* \mathbf{X})) = \underset{\square}{\text{holim}} \, \mathbf{X}$$

is a weak equivalence. $\qquad \square$

(CD4)^{op}. *For every object* $\square$ *of* Π, *the functor* $\mathbf{s}_\square : (\square, \mathcal{M}_f) \to \mathcal{M}_f$ *is op-monoidal and quasistrict.*

Proof. The Künneth morphism $\sigma = \sigma_\square(\mathbf{X}, \mathbf{Y}) : \mathbf{s}_\square(\mathbf{X} \times \mathbf{Y}) \to \mathbf{s}_\square \mathbf{X} \times \mathbf{s}_\square \mathbf{Y}$ is an isomorphism because holim preserves limits:

$$\mathbf{s}_\square(\mathbf{X} \times \mathbf{Y}) = \text{holim}(\mathbf{X} \times \mathbf{Y}) \cong \text{holim}(\mathbf{X}) \times \text{holim}(\mathbf{Y}) = \mathbf{s}_\square(\mathbf{X}) \times \mathbf{s}_\square(\mathbf{Y})$$

Observe that σ is natural in $(\mathbf{X}, \mathbf{Y})$.

The unit morphism $\sigma_\square^1 : \mathbf{s}_\square(1 \times \square) \to 1$ is clearly an isomorphism: the realization of a constant diagram in the initial object 1 is $F(K, 1) = 1$ (see example 3.2).

Finally, it is clear that σ and σ^1 verify the associativity and unit restrictions and that $\mathbf{s}_\square$ is an op-monoidal functor. $\qquad \square$

(CD5)^{op}. *If* $f : \mathbf{X} \to \mathbf{Y}$ *is a morphism of* $\square$*-diagrams of* $\mathcal{M}_f$ *such that for every* $\alpha \in \square$, f_α *is a weak equivalence, then* $\mathbf{s}_\square f : \mathbf{s}_\square \mathbf{X} \to \mathbf{s}_\square \mathbf{Y}$ *is a weak equivalence of fibrant objects.*

Proof. This is exactly the property of homotopy invariance of the homotopy limit. $\qquad \square$

We introduce now the category $Coreal_\Pi \mathcal{M}$ of corealizations, which is analogous to the category $Real_\Pi \mathcal{M}$ of [5], 1.2.1.

Given $\delta : \square \to \square'$ a morphism of Π, there is an induced direct image functor

$$\delta_* : ((\square, \mathcal{M}), \mathcal{M}) \to ((\square', \mathcal{M}), \mathcal{M})$$

defined by $f \mapsto \delta_*(f) := f \circ \delta^*$.

The category $Coreal_\Pi \mathcal{M}$ of corealizations of cubical codiagrams is defined as follows. An object is a functor $s_\square \in ((\square, \mathcal{M}), \mathcal{M})$. A morphism from $s_{\square'} \in ((\square', \mathcal{M}), \mathcal{M})$ to $s_\square \in ((\square, \mathcal{M}), \mathcal{M})$ is a morphism $\delta : \square \to \square'$ of Π and a natural transformation of functors $s_{\square'} \to \delta_* s_\square$ of $((\square', \mathcal{M}), \mathcal{M})$.

Remark 5.2. The category $Coreal_\Pi \mathcal{M}$ has a structure of monoidal category: Given $\square, \square' \in \Pi$, $\mathbf{s}_\square \in ((\square, \mathcal{M}), \mathcal{M})$, $\mathbf{s}_{\square'} \in ((\square', \mathcal{M}, \mathcal{M}))$, the composition

$$\mathbf{s}_\square \circ \mathbf{s}_{\square'} : ((\square \times \square'), \mathcal{M}) \to \mathcal{M}$$

is defined by

$$\mathbf{s}_\square \circ \mathbf{s}_{\square'}(\mathbf{X}) = \mathbf{s}_\square(\alpha \mapsto \mathbf{s}_{\square'}(\beta \mapsto \mathbf{X}_{\alpha\beta})).$$

The unit object is the evaluation functor $Av : (\square_0, \mathcal{M}) \to \mathcal{M}$.

(CD6)^{op}. $(\mathbf{s}, \mu, \lambda_0) : \Pi \to Coreal_\Pi \mathcal{M}_f$, $\square \mapsto (\mathbf{s}_\square : (\square, \mathcal{M}_f) \to \mathcal{M}_f)$, *is an strict monoidal functor.*

Proof. A (strict) monoidal functor $(\mathbf{s}, \mu, \lambda_0) : (\Pi, \times, \square_0) \to (Codiag_\Pi \mathcal{M}_f, \circ, Av)$ is given by

(i) a functor $\mathbf{s} : \Pi \to Coreal_\Pi \mathcal{M}_f$,
(ii) for every pair $(\square, \square')$ of $\Pi \times \Pi$, a (iso)morphism of $Coreal_\Pi \mathcal{M}_f$ (i.e. a natural transformation of functors)

$$\mu_{\square, \square'} : \mathbf{s}_\square \circ \mathbf{s}_{\square'} \to \mathbf{s}_{\square \times \square'}$$

natural in $(\square, \square')$, and
(iii) a (iso)morphism of $Coreal_\Pi \mathcal{M}_f$

$$\lambda_0 : Av \to \mathbf{s}_{\square_0}$$

compatible with the associativity and unit restrictions.

If we consider $X \in \mathcal{M}_f$ as a $\square_0$-diagram, $\mathbf{s}_{\square_0} X = F(*, X) \cong X$ and $Av(X) = X$. It is clear that we have a natural transformation of functors $\lambda_0 : Av \to \mathbf{s}_{\square_0}$ such that $\lambda_0(X)$ is an isomorphism.

It is easy to see that μ and λ_0 satisfy the associativity and unit restrictions, so we are done. $\square$

Given S a non-empty finite set, $\mathbf{s}_{\square_S}(\square_S \times X) = F(B(\square_S), X) = F(\Delta^S, X)$ (see example 3.2). For $S = \prod_i S_i$, if we set $\Delta^S = \prod_i \Delta^{S_i}$, the equality also holds.

We denote by $i_\square$ the functor $X \mapsto \square \times X$.

(CD7)^{op}. *The morphism λ is a monoidal natural transformation from the functor $G : \square \mapsto id_{\mathcal{M}_f}$ to the op-monoidal functor $H : \square \mapsto \mathbf{s}_\square \circ i_\square$.*

Proof. For every $\square \in \Pi$ and $X \in \mathcal{M}_f$, the morphism $\lambda_\square(X)$ is natural in $\square$ and X, and therefore defines a natural transformation from $G : \Pi^{op} \to (\mathcal{M}_f, \mathcal{M}_f)$, $\square \mapsto id_{\mathcal{M}_f}$, to $H : \Pi^{op} \to (\mathcal{M}_f, \mathcal{M}_f)$, $\square \mapsto \mathbf{s}_\square \circ i_\square$.

The naturality of λ in $\square$ is clear: for every morphism $\square_S \to \square_T$ of Π, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;\;\;id\;\;\;} & X \\ {\scriptstyle \lambda_{\square_S}(X)}\downarrow & & \downarrow{\scriptstyle \lambda_{\square_T}(X)} \\ \mathbf{s}_{\square_T}(i_{\square_T} X) = F(\Delta^T, X) & \longrightarrow & \mathbf{s}_{\square_S}(i_{\square_S} X) = F(\Delta^S, X) \end{array}$$

commutes.

It is also clear that λ is natural in X: if $X \to Y$ is a morphism of $\mathcal{M}_f$,

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow{\scriptstyle \lambda_\square(X)} & & \downarrow{\scriptstyle \lambda_\square(Y)} \\
F(\Delta, X) & \longrightarrow & F(\Delta, Y)
\end{array}
$$

commutes.

We have to see that the natural transformation λ is monoidal. The functors G and H are between monoidal categories: $(\Pi^{op}, \times, \square_0)$ and $((\mathcal{M}_f, \mathcal{M}_f), \circ, id_{\mathcal{M}_f})$.

We observe that given $X \in \mathcal{M}_f$,

$$H(\square \times \square')(X) = \mathbf{s}_{\square \times \square'} \circ i_{\square \times \square'}(X) = \mathbf{s}_{\square \times \square'}(i_{\square \times \square'} X) = F((\Delta \times \Delta'), X)$$

and

$$H(\square) \circ H(\square')(X) = (\mathbf{s}_\square \circ i_\square) \circ (\mathbf{s}_{\square'} \circ i_{\square'})(X) = \mathbf{s}_\square(i_\square[\mathbf{s}_{\square'}(i_{\square'} X)]) =$$
$$= F(\Delta, F(\Delta', X)) \cong F((\Delta \times \Delta'), X),$$

where $\Delta = B\square$ and $\Delta' = B\square'$. With this define the morphism in $(\mathcal{M}_f, \mathcal{M}_f)$

$$H_2(\square, \square') : \mathbf{s}_{\square \times \square'} \circ i_{\square \times \square'} \to (\mathbf{s}_\square \circ i_\square) \circ (\mathbf{s}_{\square'} \circ i_{\square'}).$$

For all $X \in \mathcal{M}_f$ the morphism $\mathbf{s}_{\square_0}(i_{\square_0} X) = F(*, X) \to X$ defines a morphism in $(\mathcal{M}_f, \mathcal{M}_f)$

$$H_0 : \mathbf{s}_{\square_0} \circ i_{\square_0} \to id_{\mathcal{M}_f}.$$

We have that (H, H_2, H_0) is an op-monoidal functor.

Now it is easy to see that the natural transformation λ is monoidal. $\qquad \square$

(CD8)^{op}. *For every diagram $\mathbf{X}$ of type $\square := \square_S$ in a stable simplicial model category, where S is a finite non-empty set and every augmentation $\varepsilon : \mathbf{X}_0 \to \mathbf{X}$, the morphism $\lambda_\varepsilon := \mathbf{s}_\square(\varepsilon) \circ \lambda_\square(\mathbf{X}_0) : X_0 \to \mathbf{s}_\square \mathbf{X}$ is a weak equivalence if and only if the canonical morphism $0 \to \mathbf{s}_{\square^+}\mathbf{X}^+$ is a weak equivalence.*

Proof. This result is exactly the proposition 4.1. $\qquad \square$

Our initial target was to prove this result for the category of fibrant spectra, in the sense of Bousfield and Friedlander ([1]). The result applies to any of the model categories of spectra available, including symmetric and orthogonal spectra.

Example 5.3. See [12] for a list of examples of interesting stable simplicial model categories, including modules over ring spectra, presheaves of spectra, equivariant stable homotopy and motivic stable homotopy of schemes.

Remark 5.4. There are descent categories that are not stable simplicial model categories. For example the category of topological spaces is a descent category with the morphisms that induce isomorphism in homology as saturated class of morphisms (see [5], 1.5.13).

Another example of descent category that does not fit in this setting is the category of complexes in an additive category (see [5], 1.7.7).

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