Characterizing (ℓ, m)-Walk-Regular Graphs *

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Abstract

A graph Γ with diameter D and $d+1$ distinct eigenvalues is said to be (ℓ, m)-*walk-regular*, for some integers $\ell \in [0, d]$ and $m \in [0, D]$, $\ell \geq m$, if the number of walks of length $i \in [0, \ell]$ between any pair of vertices at distance $j \in [0, m]$ depends only on the values of i and j. In this paper we study some algebraic and combinatorial characterizations of (ℓ, m)-walk-regularity based on the so-called predistance polynomials and the preintersection numbers.

Keywords: Distance-regular graph; Walk-regular graph; Adjacency matrix; Spectrum; Predistance polynomial; Preintersection number.

MSC 2000: 05C50 Graphs and matrices; 05E30 Association schemes, etc..

1 Introduction

Throughout this paper, $\Gamma = (V, E)$ denotes a simple and connected graph with order $n = |V|$, diameter D, adjacency matrix $\boldsymbol{A}$, and spectrum $\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, where $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ and the superscripts stand for the multiplicities $m_i = m(\lambda_i)$. Recall that the diameter is always smaller than the number of distinct eigenvalues, $D \leq d$; see e.g. [1]. Let $Z = \prod_{i=0}^{d}(x - \lambda_i)$ be the minimal polynomial of $\boldsymbol{A}$. The vector space $\mathbb{R}_d[x]$ of real polynomials of degree at most d is isomorphic to $\mathbb{R}[x]/(Z)$. For every $0 \leq i \leq d$, let us consider the Lagrange interpolating polynomials $\lambda_i^* = \frac{1}{\phi_i} \prod_{j=0, i \neq j}^{d}(x - \lambda_i)$, where

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$\phi_i = \prod_{j=0,j\neq i}^{d}(\lambda_i - \lambda_j)$, satisfying $\lambda_i^*(\lambda_j) = \delta_{ij}$. Then the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}_i = \mathrm{Ker}(\boldsymbol{A} - \lambda_i \boldsymbol{I})$ are given by the matrices $\boldsymbol{E}_i = \lambda_i^*(\boldsymbol{A})$, called the (*principal*) *idempotents* of $\boldsymbol{A}$.

Given the graph Γ, with distance matrices $\boldsymbol{A}_0(=\boldsymbol{I})$, $\boldsymbol{A}_1(=\boldsymbol{A})$, ..., $\boldsymbol{A}_D$, consider the algebras

$$\mathcal{A}_\ell = \mathbb{R}_\ell[\boldsymbol{A}] = \mathrm{span}\{\boldsymbol{A}^0, \boldsymbol{A}^1, \ldots, \boldsymbol{A}^\ell\} \qquad \text{and} \qquad \mathcal{D}_m = \mathrm{span}\{\boldsymbol{A}_0, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m\}$$

for some integers $\ell \leq d$ and $m \leq D$. Note that the adjacency (or Bose-Mesner) algebra of Γ, $\mathcal{A} = \mathbb{R}[\boldsymbol{A}]$, is just $\mathcal{A}_d$. Then, Γ is distance-regular if and only if

$$\mathcal{A}_d = \mathcal{D}_D,$$

(in particular, equating dimensions, $D = d$) which is equivalent to the invariance of the number of walks of length $i \geq 0$ between vertices at a given distance j, $0 \leq j \leq d$ (see e.g. [1, 12]).

Similarly, Γ is walk-regular if and only if

$$\mathcal{A}_d \circ \mathcal{D}_0 = \mathcal{D}_0$$

or, what is the same, the number of closed walks of length $i \geq 0$ rooted at any given vertex is a constant (see e.g. [9, 8]).

Inspired by these definitions, the authors [4] introduced a generalization of both distance-regularity and walk-regularity, which we called m-walk-regularity. For a given integer m, $0 \leq m \leq D$, we say that Γ is m-walk-regular when the number of walks of length i between vertices u and v depends only on the distance between these vertices, provided that $\mathrm{dist}(u, v) = j \leq m$. Thus, in terms of the above algebras, this corresponds to:

$$\mathcal{A}_d \circ \mathcal{D}_m = \mathcal{D}_m.$$

In this paper we generalize the above definitions by introducing the concept of (ℓ, m)-walk-regularity. For some integers ℓ, m satisfying $\ell \leq d$ and $m \leq D$, $\ell \geq m$, we say that a graph Γ is (ℓ, m)-*walk-regular* if the number of walks of length $i \leq \ell$ between any pair of vertices u, v at distance $j \leq m$ does not depend on such vertices but depends only on i, j. Therefore, Γ is (ℓ, m)-walk-regular if and only if

$$\mathcal{A}_\ell \circ \mathcal{D}_m = \mathcal{D}_m.$$

In fact, from the known results in the literature and the results of this paper we have the equivalences:

- (d, D)-walk-regular graph $\equiv$ distance-regular graph [1, 2]
- (m, m)-walk-regular graph $\equiv$ partially m-distance-regular graph [6, 11]

- $(d, 0)$-walk-regular graph $\equiv$ walk-regular graph [9, 8]
- $(m, 0)$-walk-regular graph $\equiv$ partially m-walk-regular graph [6, 3]
- (d, m)-walk-regular graph $\equiv$ m-walk-regular graph [4]

Our algebraic characterizations of (ℓ, m)-walk-regular graphs are mainly based on the concepts of predistance polynomial and preintersection number defined as follows. From the spectrum of Γ, consider the following scalar product:

$$\langle p, q \rangle = \frac{1}{n} \operatorname{sum}(p(\boldsymbol{A}) \circ q(\boldsymbol{A})) = \frac{1}{n} \operatorname{tr}(p(\boldsymbol{A}) q(\boldsymbol{A})) = \frac{1}{n} \sum_{k=0}^{d} m_k p(\lambda_k) q(\lambda_k), \tag{1}$$

where $\circ$ stands for the Hadamard—entrywise—product of matrices and $\operatorname{sum}(\cdot)$ denotes the sum of the entries of the corresponding matrix. Then, the *predistance polynomials* $p_0, p_1, \ldots, p_d$, are the orthogonal polynomials with respect to such a product, normalized in such a way that $\|p_k\|^2 = p_k(\lambda_0)$; see [7]. Furthermore, we define the *preintersection numbers* ξ_{ij}^k as the Fourier coefficients of $p_i p_j$ in terms of the basis $\{p_k\}_{0 \le k \le d}$; that is:

$$\xi_{ij}^k = \frac{\langle p_i p_j, p_k \rangle}{\|p_k\|^2} = \frac{1}{n p_k(\lambda_0)} \sum_{h=0}^{d} m(\lambda_h) p_i(\lambda_h) p_j(\lambda_h) p_k(\lambda_h). \tag{2}$$

As expected, when Γ is distance-regular, the predistance polynomials and the preintersection numbers become, respectively, the distance polynomials, giving the distance matrices, $p_k(\boldsymbol{A}) = \boldsymbol{A}_k$, $0 \le k \le D$, and the intersection numbers $p_{ij}^k = |\Gamma_i(u) \cap \Gamma_j(v)|$, $\operatorname{dist}(u, v) = k$, of Γ.

2 Some characterizations

Now we are ready to give some characterizations of (ℓ, m)-walk-regularity. In the following lemma we first give some immediate results.

Lemma 2.1 *Let Γ be a graph with diameter D, adjacency matrix $\boldsymbol{A}$ and $d+1$ distinct eigenvalues. Let $\boldsymbol{S}_m = \boldsymbol{A}_0 + \boldsymbol{A}_1 + \cdots + \boldsymbol{A}_m$ for some $m \le D$. Then, for any polynomial $p \in \mathbb{R}_\ell[x]$ and $m \le D$, $\ell \le d$, $m \le \ell$, the following statements are equivalent:*

(*a*) *$p(\boldsymbol{A}) \circ \boldsymbol{S}_m \in \mathcal{D}_m$.*

(*b*) *$p(\boldsymbol{A}) \circ \boldsymbol{A}_j \in \mathcal{D}_m$, for every $0 \le j \le m$.*

(*c*) *For each $j = 0, 1, \ldots, m$ there exists $\zeta_j(p) \in \mathbb{R}$ such that $p(\boldsymbol{A}) \circ \boldsymbol{A}_j = \zeta_j(p) \boldsymbol{A}_j$.*

(*d*) *If p has degree $i \le m$, then $p(\boldsymbol{A}) = \sum_{j=0}^{i} \zeta_j(p) \boldsymbol{A}_j$.* □

Moreover, by linearity it is clear that, if any of the above conditions holds for a basis of $\mathbb{R}_\ell[x]$, then it also holds for any polynomial of degree at most ℓ. In some of the following results, we use different basis of such a space of polynomials.

Theorem 2.2 *Let $\Gamma = (V, E)$ be a graph with diameter D, adjacency matrix $\boldsymbol{A}$ having $d+1$ distinct eigenvalues, and predistance polynomials $p_0, p_1, \ldots, p_d$. Let $\boldsymbol{S}_m = \boldsymbol{A}_0 + \boldsymbol{A}_1 + \cdots + \boldsymbol{A}_m$ for some $m \le D$. Then, the following statements are equivalent:*

- *(a)* *Γ is (ℓ, m)-walk-regular.*
- *(b)* $\mathcal{A}_\ell \circ \boldsymbol{S}_m = \mathcal{D}_m$.
- *(c)* $p_i(\boldsymbol{A}) \circ \boldsymbol{S}_m = \boldsymbol{A}_i \circ \boldsymbol{S}_m \qquad (0 \le i \le \ell)$.

Proof. First note that (a) and (b) are equivalent because $\{1, x, x^2, \ldots, x^\ell\}$ is a basis of $\mathbb{R}_\ell[x]$ and, in the equality $\boldsymbol{A}^i \circ \boldsymbol{A}_j = \zeta_j(x^i)\boldsymbol{A}_j$, the coefficient $\zeta_j(x^i)$ is the number of walks of length i between vertices at distance j, $0 \le j \le m$.

Assume that (c) holds. Then, $p_i(\boldsymbol{A}) \circ \boldsymbol{S}_m = \sum_{j=0}^m \boldsymbol{A}_i \circ \boldsymbol{A}_j = \sum_{j=0}^m \delta_{ij}\boldsymbol{A}_j \in \mathcal{D}_m$ for any $0 \le i \le \ell$. Therefore, we have $\mathcal{A}_\ell \circ \boldsymbol{S}_m \subset \mathcal{D}_m$ since the predistance polynomials $\{p_0, p_1, \ldots, p_\ell\}$ are a basis of the space $\mathbb{R}_\ell[x]$. Then (b) follows from $\dim(\mathcal{A}_\ell \circ \boldsymbol{S}_m) \ge \dim \mathcal{D}_m = m$.

Now note that statement (c) can be split into:
(c_1) $p_i(\boldsymbol{A}) = \boldsymbol{A}_i \quad (0 \le i \le m)$;
(c_2) $p_i(\boldsymbol{A}) \circ \boldsymbol{A}_j = \boldsymbol{0} \quad (m+1 \le i \le \ell\,,\ 0 \le j \le m)$.

$(b) \Rightarrow (c_1)$: Let $0 \le i \le m$. The matrix $\boldsymbol{S}_m$ operates on the matrices $p_i(\boldsymbol{A})$ as a unit for the Hadamard product. Then, $p_i(\boldsymbol{A}) = p_i(\boldsymbol{A}) \circ \boldsymbol{S}_m \in \mathcal{D}_m$. Thus, there exist constants $\alpha_{i0}, \alpha_{i1}, \ldots, \alpha_{ii}$ such that $p_i(\boldsymbol{A}) = \sum_{j=0}^i \alpha_{ij}\boldsymbol{A}_j$. Hence, as $\alpha_{ii} \neq 0$ and $\{p_i(\boldsymbol{A})\}_{0 \le i \le m}$ and $\{\boldsymbol{A}_j\}_{0 \le j \le m}$ are orthogonal basis of the same space, we get $p_i(\boldsymbol{A}) = \alpha_{ii}\boldsymbol{A}_i$. Then, Γ_i is a regular graph with degree $\frac{p_i(\lambda_0)}{\alpha_{ii}}$. Consequently, from

$$p_i(\lambda_0) = \|p_i(\boldsymbol{A})\|^2 = \alpha_{ii}^2 \|\boldsymbol{A}_i\|^2 = \alpha_{ii}^2 \frac{p_i(\lambda_0)}{\alpha_{ii}} = \alpha_{ii} p_i(\lambda_0),$$

we get $\alpha_{ii} = 1$ and $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$, as claimed.

$(b) \Rightarrow (c_2)$: Consider the matrix $\boldsymbol{N} = \boldsymbol{J} - \sum_{j=0}^m \boldsymbol{A}_j$ which has zeros in the entries corresponding to the pairs of vertices at distance at most m. Let $p \in \mathbb{R}_\ell[x]$. By Lema 2.1(c), for each $j = 0, 1, \ldots, m$, there exists $\xi_j(p)$ such that $p(\boldsymbol{A}) \circ \boldsymbol{A}_j = \xi_j(p)\boldsymbol{A}_j$ or, what is the same, $p(\boldsymbol{A}) \circ p_j(\boldsymbol{A}) = \xi_j(p)p_j(\boldsymbol{A})$. Thus, $\xi_j(p)$ is the Fourier coefficient of $p(\boldsymbol{A}) \circ p_j(\boldsymbol{A})$ in terms of $p_j(\boldsymbol{A})$:

$$\xi_j(p) = \frac{\langle p(\boldsymbol{A}) \circ p_j(\boldsymbol{A}), p_j(\boldsymbol{A})\rangle}{\|p_j(\boldsymbol{A})\|^2} = \frac{\langle p(\boldsymbol{A}), p_j(\boldsymbol{A})\rangle}{p_j(\lambda_0)},$$

and we get:

$$\begin{aligned} p(\boldsymbol{A}) = p(\boldsymbol{A}) \circ \boldsymbol{J} &= \sum_{j=0}^{m} p(\boldsymbol{A}) \circ \boldsymbol{A}_j + p(\boldsymbol{A}) \circ \boldsymbol{N} \\ &= \sum_{j=0}^{m} \xi_j(p) \boldsymbol{A}_j + \boldsymbol{N}(p) = \left(\sum_{j=0}^{m} \xi_j(p) p_j \right) (\boldsymbol{A}) + \boldsymbol{N}(p), \end{aligned}$$

where $\boldsymbol{N}(p) = p(\boldsymbol{A}) \circ \boldsymbol{N}$ has also null entries if the corresponding pairs of vertices are at distance at most m. Therefore,

$$\left(p - \sum_{j=0}^{m} \frac{\langle p(\boldsymbol{A}), p_j(\boldsymbol{A}) \rangle}{p_j(\lambda_0)} p_j \right) (\boldsymbol{A}) = \boldsymbol{N}(p) \tag{3}$$

so that, if we take $p = p_{m+1}, p_{m+2}, \ldots, p_\ell$ in (3), the sumatory is null and we obtain $p_i(\boldsymbol{A}) = \boldsymbol{N}(p_i)$ for $i = m+1, m+2, \ldots, \ell$, which proves (c_2). □

From these characterizations, the role of the preintersection numbers is made clear in the next proposition.

Proposition 2.3 *Let* Γ *be an* (ℓ, m)*-walk-regular graph with predistance polynomials* $p_0, p_1, \ldots, p_d$ *and preintersection numbers* ξ_{ij}^k *given by* (2). *Then, for any* $i, j, k \leq m$ *we have:*

(a) $\boldsymbol{A}_i \boldsymbol{A}_j \circ \boldsymbol{S}_m \in \mathcal{D}_m \qquad (i + j \leq \ell)$.

(b) If $i + j \leq \ell$, *the preintersection numbers* ξ_{ij}^k *coincide with the intersection numbers* $p_{ij}^k = |\Gamma_i(u) \cap \Gamma_j(v)|$ *for any vertices* u, v *at distance* k.

(c) If $i + j \geq \ell + 1$, *the preintersection numbers* ξ_{ij}^k *become the mean* $\overline{p}_{ij}^k$ *of the values* $p_{ij}^k(u, v) = |\Gamma_i(u) \cap \Gamma_j(v)|$ *for any vertices* u, v *at distance* k.

Proof. (a) Assume that $i, j \leq m$ and $i + j \leq \ell$. Then, from the Fourier decomposition of $p_i p_j$ in terms of $p_0, p_1, \ldots, p_{i+j}$ and using Theorem 2.2(c), we get:

$$\begin{aligned} \boldsymbol{A}_i \boldsymbol{A}_j \circ \boldsymbol{S}_m = p_i(\boldsymbol{A}) p_j(\boldsymbol{A}) \circ \boldsymbol{S}_m &= \sum_{k=0}^{i+j} \xi_{ij}^k p_k(\boldsymbol{A}) \circ \boldsymbol{S}_m \\ &= \sum_{k=0}^{i+j} \xi_{ij}^k \boldsymbol{A}_k \circ \boldsymbol{S}_m = \sum_{k=0}^{m} \xi_{ij}^k \boldsymbol{A}_k, \end{aligned} \tag{4}$$

so that $\boldsymbol{A}_i \boldsymbol{A}_j \circ \boldsymbol{S}_m \in \mathcal{D}_m$, as claimed.

(*b*) For any two vertices u, v at distance $k \leq m$ we have, by (4), $\xi_{ij}^k = (\boldsymbol{A}_i \boldsymbol{A}_j)_{uv} = |\Gamma_i(u) \cap \Gamma_j(v)| = p_{ij}^k$.

(*c*) Let δ_k be the average degree of Γ_k (that is, the k-th distance graph, with adjacency matrix $\boldsymbol{A}_k$). Then, when $i, j, k \leq m$ and $i + j \geq \ell + 1$, we get:

$$\xi_{ij}^k = \frac{\langle p_i p_j, p_k \rangle}{\|p_k\|^2} = \frac{1}{\|\boldsymbol{A}_k\|^2} \langle \boldsymbol{A}_i \boldsymbol{A}_j, \boldsymbol{A}_k \rangle$$
$$= \frac{1}{n\delta_k} \sum_{\mathrm{dist}(u,v)=k} (\boldsymbol{A}_i \boldsymbol{A}_j)_{uv} (\boldsymbol{A}_k)_{uv} = \frac{1}{n\delta_k} \sum_{\mathrm{dist}(u,v)=k} p_{ij}^k(u,v) = \overline{p}_{ij}^k.$$

This completes the proof. □

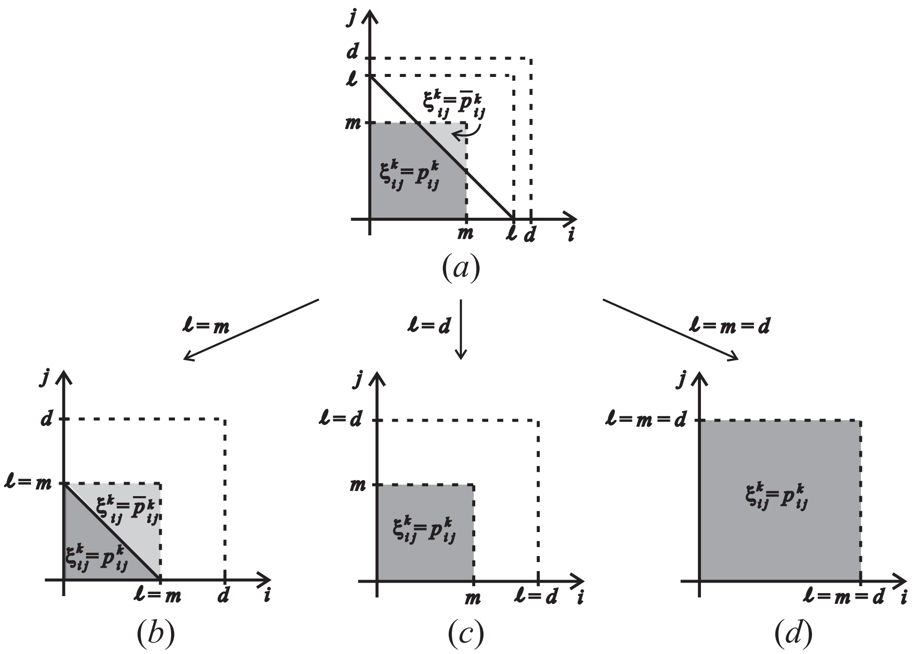

Figure 1: The meaning of the preintersection numbers.

In the general case, the meaning of the preintersection numbers is depicted in Fig.1(*a*) as a function of i, j (always assuming $k \leq m$). In Fig.1 we also show the situation in the following "extreme" cases:

- ($\ell = m$): When Γ is an (m, m)-walk-regular graph, Theorem 2.2(*c*) yields $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$, $0 \leq i \leq m$, which is a known characterization of *partially m-distance-regular graphs*; see e.g. [11, 3]. In this case, $\xi_{ij}^k = p_{i,j}^k$ provided that $i + j \leq m$ (and $i, j, k \leq m$); see Fig. 1(*b*).

- ($\ell = d$): In the case of (d,m)-walk-regular graphs, called m-walk regular in [4], the condition $i+j \leq d$ is irrelevant (since $\dim \mathcal{A} = d$) and we are in the situation showed in Fig. 1(c).

  Moreover, in this case we also have the following characterization in terms of the idempotents:

  (f) $\boldsymbol{E}_i \circ \boldsymbol{S}_m \subset \mathcal{D}_m \quad (0 \leq i \leq d)$.

  (We then say that Γ is m-spectrally regular.) This is because the interpolating polynomials $\lambda_0^\star, \lambda_1^\star, \ldots \lambda_d^\star$ are a basis of $\mathbb{R}_d[x]$ and, in the equality $\lambda_i^\star(\boldsymbol{A}) \circ \boldsymbol{A}_j = \zeta_j(\lambda_i^\star)\boldsymbol{A}_j$, the coefficient $\zeta_j(\lambda_i^\star)$ is the so-called *crossed local multiplicity* of λ_i, $m_{ji}(\lambda_i) = (\boldsymbol{E}_i)_{uv}$, between any two vertices u, v at distance j.

- ($\ell = d, m = D$): This corresponds to distance-regular graphs where, as it has been already commented, the predistance polynomials and preintersection numbers coincide, respectively, with the standard concepts of distance polynomial and intersection number; see Fig. 1(d).

Notice that, if $m \leq \ell \leq \min\{2m, d\}$, the region where $\xi_{ij}^k = p_{i,j}^k$ is univocally determined by the parameters ℓ, m, and viceversa. This suggests the following characterization in terms of the (pre)intersection numbers.

Theorem 2.4 *A graph Γ with $d+1$ distinct eigenvalues is (ℓ, m)-walk-regular with $m \leq \ell \leq \min\{2m, d\}$ if and only if there exist the intersection numbers $p_{i,j}^k (= \xi_{ij}^k)$ for any $i, j, k \leq m$ and $i + j \leq \ell$.*

Proof. The necessity has been already proved. To prove sufficiency, the existence of such intersection numbers can be described as in (4); that is,

$$\boldsymbol{A}_i\boldsymbol{A}_j \circ \boldsymbol{S}_m = \sum_{k=0}^{i+j} p_{ij}^k \boldsymbol{A}_k \circ \boldsymbol{S}_m \qquad (i, j, k \leq m, \ i + j \leq \ell). \tag{5}$$

Let us show that this implies the condition in Theorem 2.2(c); that is, $\boldsymbol{A}_i \circ \boldsymbol{S}_m = p_i(\boldsymbol{A}) \circ \boldsymbol{S}_m$ for $i \leq \ell$. We use induction. First, the result clearly holds for $\boldsymbol{A}_0 = \boldsymbol{I}$ and $\boldsymbol{A}_1 = \boldsymbol{A}$ since $p_0 = 1$ and $p_1 = x$. Now, assume that $\boldsymbol{A}_i \circ \boldsymbol{S}_m = p_i(\boldsymbol{A}) \circ \boldsymbol{S}_m$ for every $i = 0, 1, \ldots, r-1$, $1 \leq r-1 < \ell$. Then, taking any integers $s, t \leq m$ such that $s + t = r$, Eq. (5) yields:

$$\begin{aligned}
\boldsymbol{A}_r \circ \boldsymbol{S}_m &= \frac{1}{p_{st}^r}\left(\boldsymbol{A}_s\boldsymbol{A}_t \circ \boldsymbol{S}_m - \sum_{k=0}^{r-1} p_{st}^k \boldsymbol{A}_k \circ \boldsymbol{S}_m\right) \\
&= \frac{1}{p_{st}^r}\left(p_s(\boldsymbol{A})p_t(\boldsymbol{A}) - \sum_{k=0}^{r-1} p_{st}^k p_k(\boldsymbol{A})\right) \circ \boldsymbol{S}_m \\
&= \frac{1}{p_{st}^r}\left(\sum_{k=0}^{r} p_{st}^k p_k(\boldsymbol{A}) - \sum_{k=0}^{r-1} p_{st}^k p_k(\boldsymbol{A})\right) \circ \boldsymbol{S}_m = p_r(\boldsymbol{A}) \circ \boldsymbol{S}_m,
\end{aligned}$$

which, by induction, proves the result. $\square$

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