On Golden Spectral Graphs

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Abstract

The concept of golden spectral graphs is introduced and some of their general properties reported. Golden spectral graphs are those having a golden proportion for the spectral ratios defined on the basis of the spectral gap, spectral spread and the difference between the second largest and the smallest eigenvalue of the adjacency matrix. They are good expanders and display excellent synchronizability. Here we report some new construction methods as well as several of their topological parameters.

Key words: networks, eigenvalues, golden number, expanders, Ramanujan graphs, synchronizability
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1. Introduction

The motivation of this work is given by the necessity of designing highly robust networks having the ability of easily synchronize individual processes taking place at their nodes [13],[7]. Complex networks are graph-theoretic representations of complex systems. The nodes, which represent the entities of the system, are connected by links representing the relationships between such entities [14]. These complex networks are ubiquitous in nature and society, representing molecular and cellular interactions, ecologies, social relations or technological and infrastructural systems [14],[3]. The topological structure of such networks has been extensively studied in recent years. As a consequence some basic architectures have been discovered. The most celebrated of these properties is the fact that many complex networked systems display a heavy-tailed or scale-free degree distribution [1]. That is, the probability of a node having certain degree decays as a power of the degree.

Scale-free networks are quite robust to random failures of nodes but are hypersensitive to the intentional attacks against the most connected nodes [2]. Many other systems, which are called small-world networks, display the property of having short average path length and high clustering [18]. The interplay between the topology of the network and the dynamical processes taking place on it has been of great relevance in recent years. In particular, the study of the synchronizability of individual dynamical processes occurring at the nodes of a given network is of both conceptual and practical interest [4],[16]. The problem is of particular relevance in determining the most efficient topology for communication networks, the optimal topology of social networks to reach consensus, or the performance of neural networks. Both scale-free and small-world networks display better synchronizability than regular graphs [4],[16]. However, it has been observed that networks with strong heterogeneity in the degree distribution are much more difficult to synchronize than random homogeneous networks [15]. Thus, the scale-freeness property of complex networks may be disadvantageous when considering dynamical processes on the networks. Inspired by the necessity of designing robust and highly synchronizable networks some researchers have proposed different strategies to search for such graphs. For instance, Donetti et al. [7], [8] have proposed entangled networks, which are extremely homogeneous regular networks, with long cycles and poor modular structure. The robustness of these graphs is given by the fact that they are good expanders related to Ramanujan graphs. Entangled graphs are obtained by using a numerical optimization algorithm, which excludes the possibility of using mathematical tools to generate infinitely many of such networks. A radically different approach was proposed in a recent paper by one of the current authors [9] who introduced the concept of *golden spectral networks (graphs)*. These graphs are formally defined in a following section. Golden spectral graphs (GSGs) can be built using analytical tools, which allows the construction of infinite series of such graphs. They have been proved to have good expansion properties and high synchronizability. Here we first introduce the concepts and definitions related to GSGs and then we report new bounds for several topological properties of these networks.

2. Preliminary definitions

Let $G = (V, E)$ be a simple connected graph of order $n = |V|$. For any vertex $u \in V$, the degree of u is denoted by $\delta(u)$ and the maximum degree of the vertices of the graph is denoted by Δ. The distance between two vertices is represented by $d(u, v)$, the diameter of G is denoted by $D = \max_{u,v \in V} d(u, v)$, and the mean distance is $\bar{l} = \frac{1}{n(n-1)} \sum\limits_{(u,v)\in V^2} d(u, v)$.

The isoperimetric number of the graph (ref. [11]) stands for

$$i(G) = \min_{|X| \leq \frac{N}{2}} \frac{|\delta X|}{|X|}$$

where X is a subset of vertices and δX is the boundary of X, i.e. the set of edges in G between vertices in X and vertices not in X.

As usual, $\boldsymbol{A}$ stands for the adjacency matrix and its associated spectra is denoted by

$$\mathrm{sp}(\boldsymbol{A}) = \{\lambda_1^{m_1} \geq \lambda_2^{m_2} \geq \ldots \geq \lambda_d^{m_d}\},$$

where m_i stands for the multiplicity of the ith eigenvalue. The difference between the largest and the smallest eigenvalue, $\lambda_1 - \lambda_n$, is known as the spread of the spectrum of G and it is denoted by $S(\boldsymbol{A})$. Besides the difference $\lambda_1 - \lambda_2$ between the two largest eigenvalues is known as the spectral gap. We define the width of the "bulk" part of the spectrum as $\lambda_2 - \lambda_n$. Consider the following ratios:

$$\omega_1(G) = \frac{\lambda_2 - \lambda_n}{\lambda_1 - \lambda_2}, \quad \lambda_1 \neq \lambda_2. \tag{1}$$

$$\omega_2(G) = \frac{\lambda_1 - \lambda_n}{\lambda_2 - \lambda_n}, \quad \lambda_2 \neq \lambda_n. \tag{2}$$

Usually these two spectral ratios are different and they form the basis of the definition of the golden spectral graphs, which is given below.

Definition 1. *A golden spectral graph (GSG) is a graph for which both spectral ratios are identical, that is*

$$\omega_1(G) = \omega_2(G) = \varphi,$$

where φ is the golden section, golden mean or divine proportion.

Indeed, we point out that if one of the ratios is equal to φ, the other one will also be φ, that is $\omega_1(G) = \varphi \Leftrightarrow \omega_2(G) = \varphi$. Alternatively, we can characterize GSGs by using the concepts of spectral gap and spread in the following:

Lemma 2. *A graph G is GS $\Leftrightarrow S(G) = \varphi^2 \, gap(G)$.*

Proof. Let us denote $S(G) = \alpha$ and $gap(G) = \beta$, then $\omega_1(G) = (\alpha-\beta)/\beta$ and $\omega_2(G) = \alpha/(\alpha - \beta)$. If G is a GSG, $\omega_1(G) = \omega_2(G) = \varphi$ and $\omega_1(G)\omega_2(G) = \alpha/\beta = \varphi^2 = \varphi + 1$. Conversely, suppose that $S(G) = (\varphi + 1)gap(G)$, then it is easy to obtain the result $\omega_1(G) = \omega_2(G) = \varphi$. □

Lemma 3. *Let G be a bipartite graph with golden spectra, then*

$$gap(G) = 2\varphi^{-2}\lambda_1.$$

Proof. It is a well known fact that the spectra of a bipartite graph is symmetric. Thus $\lambda_1 = -\lambda_n$ and using Equation (2) we have $\lambda_2 = \frac{2-\varphi}{\varphi}\,\lambda_1$, and hence we get the result. □

Finally, we recall some graph operations that will be used in this paper. Given a graph G we denote by $\overline{G}$ its complement, the graph with the same vertex set and where two vertices are adjacent if and only they are not adjacent in G. The line graph $L(G)$ is the graph having as vertex set the edge set of G, and where two vertices are adjacent in $L(G)$ if the edges they represent have a vertex in common in G. Moreover, given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, their product $G_1 \times G_2 = (V_{1\times2}, E_{1\times2})$ is the graph with vertex set $V_{1\times2} = V_1 \times V_2$, and with two vertices (x_1, x_2) and (y_1, y_2) being adjacent in $G_1 \times G_2$ if and only if x_1 and y_1 were adjacent in G_1 and x_2 and y_2 were adjacent in G_2.

3. Existence of golden spectral graphs

For the sake of self-consistency we state here some of the previous results obtained by one of the current authors ([9]) concerning the existence of GSGs among certain families of graphs. We denote by C_n the n-cycle graph. The following results are given here without proof and the reader is directed to the reference [9] to see them. The first result is concerned to the existence of the smallest GSGs.

Theorem 4. *The smallest GSG is the pentagon C_5.*

The following graph operations can also be found in reference [17]. Let $\boldsymbol{J}_k$ be the all-ones matrix of order k. Then if G is a graph with adjacency matrix $\boldsymbol{A}$ we denote by $G \otimes \boldsymbol{J}_n$ the graph with adjacency matrix $\boldsymbol{A} \otimes \boldsymbol{J}_n$, and by $G \circledast J_n$ the graph with adjacency matrix $(\boldsymbol{A} + \boldsymbol{I}) \otimes \boldsymbol{J}_n - \boldsymbol{I}$.

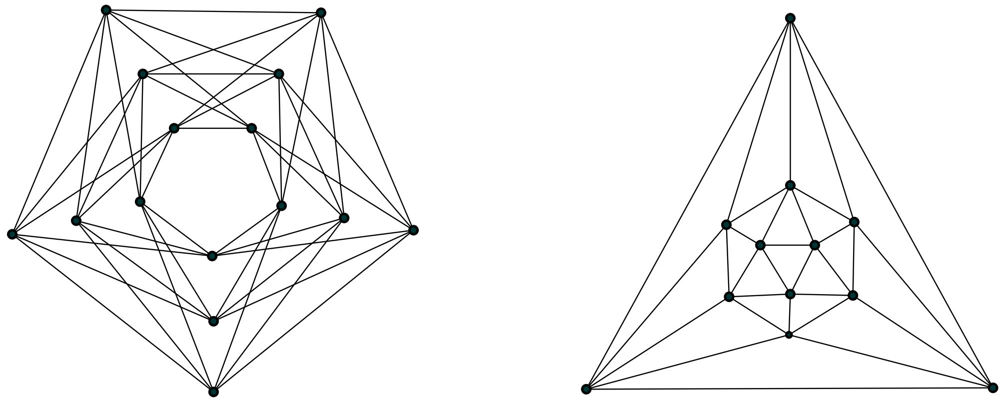

Figure 1: Two examples of GSGs: $C_5 \otimes J_3$ and the icosahedral graph.

Theorem 5. *The graphs* $C_5 \otimes \boldsymbol{J}_k$ *and* $C_5 \circledast J_n$ *are GSG, for any* $k \geq 1$.

Now let $\boldsymbol{C}$ be the $k \times k$ circulant matrix whose elements are $c_{ij} = 1$ if $j = i + 1 (\text{mod } k)$, and $c_{ij} = 0$ otherwise (ref.[17]), then let $\boldsymbol{P}$ be the matrix defined as follows

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{I} & \dots & \boldsymbol{I} \\ \boldsymbol{C} & \boldsymbol{C} & \dots & \boldsymbol{C} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}^{k-1} & \boldsymbol{C}^{k-1} & \dots & \boldsymbol{C}^{k-1} \end{pmatrix}.$$

Let $\boldsymbol{D} = (\boldsymbol{J}_k - \boldsymbol{I}_k) \otimes \boldsymbol{I}_k$, and now consider the graphs having the following adjacency matrices

$$\boldsymbol{A}_3 = \begin{pmatrix} \boldsymbol{D} & \boldsymbol{P} & \boldsymbol{P}^T \\ \boldsymbol{P}^T & \boldsymbol{D} & \boldsymbol{P} \\ \boldsymbol{P} & \boldsymbol{P}^T & \boldsymbol{D} \end{pmatrix} \quad \text{and} \quad \boldsymbol{A}_5 = \begin{pmatrix} \boldsymbol{D} & \boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{P}^T \\ \boldsymbol{P}^T & \boldsymbol{D} & \boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}^T & \boldsymbol{D} & \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{P}^T & \boldsymbol{D} & \boldsymbol{P} \\ \boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{P}^T & \boldsymbol{D} \end{pmatrix},$$

which are designated as the k-covers of the graphs $C_3 \circledast \boldsymbol{J}_k$ and $C_5 \circledast \boldsymbol{J}_k$.

Theorem 6. *The k-covers* $C_3 \circledast \boldsymbol{J}_k$ *and* $C_5 \circledast \boldsymbol{J}_k$ *are GSG.*

Note that the icosahedral graph is the 2-cover of $\boldsymbol{C}_3 \circledast \boldsymbol{J}_2$, and therefore it is a GSG. In Fig.1 we illustrate a couple of graphs with GS properties which are built by using the previous results.

Theorem 7. *The line graph of the complete bipartite graph, $L(K_{a,b})$ whenever $a = F_{k+1}$, $b = F_k$, are the $k+1$ and k-th Fibonacci number, is a GSG for $k \mapsto \infty$.*

An insteresting property of these GSGs is their expansibility. Recall that a d-regular graph is said to be Ramanujan if $\lambda_1(G) \leq 2\sqrt{d-1}$, where $\lambda(G)$ is the maximum of the non-trivial eigenvalues of the graph. Then, it is proved in [9] the following properties for the former graphs.

Proposition 8. *It is verified*

1. *All $C_5 \otimes \boldsymbol{J}_5$ graphs are Ramanujan for $k \leq 20$,*
2. *all k-covers $C_3 \circledast \boldsymbol{J}_k$ and $C_5 \circledast \boldsymbol{J}_k$ are Ramanujan for $k \leq 9$,*
3. *all bipartite d-regular GSG are Ramanujan for $d \leq 70$,*
4. *no d-regular GSG is Ramanujan for $d > 70$.*

Theorem 9. *A d-regular graph is GSG iff $\lambda_2 > d/\varphi^3$.*

In ref.[9] we can also found several nonexistence results and the reader is referred to this work to find them.

4. New results about golden spectral graphs

4.1. Construction of golden spectral graphs

The first new results are generalizations of some of the results previously obtained for C_5.

Proposition 10. *If G is a GSG, then $G \otimes \boldsymbol{J}_n$ is a GSG.*

Proof. Let $\text{sp}(\boldsymbol{A}) = \{\lambda_1^{m_1} \geq \lambda_2^{m_2} \geq \ldots \geq \lambda_d^{m_d}\}$ be the spectra of the graph G, then the spectra of the graph $G \otimes \boldsymbol{J}_n$ is

$$\text{sp}(G \otimes \boldsymbol{J}_n) = \{n\lambda_1^{m_1} \geq n\lambda_2^{m_2} \geq \ldots \geq n\lambda_d^{m_d}\},$$

and therefore is easy to check that

$$\omega_1(G \otimes \boldsymbol{J}_n) = \frac{n\lambda_2 - n\lambda_d}{n\lambda_1 - n\lambda_2} = \frac{\lambda_2 - \lambda_d}{\lambda_1 - \lambda_2} = \varphi.$$

□

Proposition 11. *If G is a GSG, then $G \circledast \boldsymbol{J}_n$ is a GSG.*

Proof. Let $\text{sp}(\boldsymbol{A}) = \{\lambda_1^{m_1} \geq \lambda_2^{m_2} \geq \ldots \geq \lambda_d^{m_d}\}$ be the spectra of the graph G, then the graph $G \circledast \boldsymbol{J}_n$ has spectra

$$\text{sp}(G \circledast \boldsymbol{J}_n) = \{(n\lambda_1 + n - 1)^{m_1} \geq (n\lambda_2 + n - 1)^{m_2} \geq \ldots \geq (n\lambda_d + n - 1)^{m_d}\},$$

and as a consequence it easily follows that $\omega_1(G \circledast \boldsymbol{J}_n) = \varphi$. □

The previous two results are of relevance for practical reasons. It has been observed empirically that some real-world networks are very close to GSGs. Then, by using these results we can generate algorithms for growing such graphs in a way that they keep their GS properties.
The former result can also be generalized considering the following graph H, whose adjacency is $\boldsymbol{H} = \boldsymbol{J}_{n_1} \otimes \boldsymbol{I}_{n_1} \otimes \cdots \otimes \boldsymbol{J}_{n_h} \otimes \boldsymbol{I}_{n_h}$. Now $G \circledast H = (\boldsymbol{A} + \boldsymbol{I}) \otimes \boldsymbol{H} - \boldsymbol{I}$, and the eigenvalues of this operation are $(\lambda_i - 1)\pi_h - 1$, where $\pi_h = \prod_{i=1}^{h} n_i$.

Proposition 12. *If G is a GSG, then $G \circledast H$ is a GSG.*

Proof. Following a similar argument as the preceding proposition the result holds. □

The next new result is concerning the construction of new GSGs based on the line graph of certain regular graphs.

Proposition 13. *If G is a d-regular GSG, with $\lambda_n < d - 4 < \lambda_2$, its line graph $L(G)$ is a GS graph.*

Proof. The spectrum of the complement of a regular graph can be found in [5]. That is $\lambda_i' = \lambda_i - d + 2$ for $1 \leq i \leq n$, and -2 with multiplicity $n(d-2)/2$. Therefore, if the condition $\lambda_n < d - 4 < \lambda_2$ holds, the spectra of the line graph preserve the spectral spread and the spectral gap, and this means that $L(G)$ will be a GSG (by Lemma2)

$$\lambda_1 - (d-2) > \lambda_2 - (d-2) > \ldots > -2 > \ldots > \lambda_n - (d-2).$$

□

Our next results are concerning bipartite GSGs. However we point out that up to now, no one bipartite GSG has been reported and the reader is encouraged to search for such examples.

Proposition 14. *If G_1 and G_2 are bipartite graphs, and G_2 is GSG, with spectra* $\mathrm{sp}(\boldsymbol{A}_1) = \{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d\}$ *and* $\mathrm{sp}(\boldsymbol{A}_2) = \{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_d\}$*. If $\lambda_2\mu_1 < \lambda_1\mu_2$, then $G_1 \times G_2$ is a bipartite GSG.*

Proof. The spectrum of the product of two graphs is the product of all their eigenvalues ([5]). As the spectra of a bipartite graph is symmetric, $\lambda_1 = -\lambda_d$ and $\mu_1 = -\mu_d$, and supposing $\lambda_2\mu_1 < \lambda_1\mu_2$ then the spectra of the product will be $-\lambda_1\mu_1 \leq \ldots \leq \lambda_1\mu_2 \leq \lambda_1\mu_1$, and

$$\omega_1(G_1 \times G_2) = \frac{\lambda_1\mu_2 + \lambda_1\mu_1}{\lambda_1\mu_1 - \lambda_1\mu_2} = \frac{\mu_2 - \mu_n}{\mu_1 - \mu_2} = \varphi.$$

□

Proposition 15. *If G is a bipartite GSG, then $\overline{G}$ is a GSG.*

Proof. Suppose $\mathrm{sp}(G) = \{\lambda_1^{m_1} \geq \lambda_2^{m_2} \geq \ldots \geq \lambda_d^{m_d}\}$, as it is bipartite $\lambda_1 = -\lambda_d$ and $\lambda_2 = -\lambda_{d-1}$. For Theorem 2.5 of [5] the spectra of the complement graph $\overline{G}$ is $\{(-\lambda_d - 1)^{n_1} \geq (-\lambda_{d-1} - 1)^{n_2} \geq \ldots \geq (-\lambda_1 - 1)^{n_d}\}$ and combining both results it easily follows that $\omega_1(\overline{G}) = \varphi$. □

4.2. Synchronizability in golden spectral graphs

In the previous work we reported that the GSGs display good synchronizability. A graph is found to have good synchronizability if the ratio Q is small. This ratio is defined as $Q = \theta_1/\theta_{n-1}$, where θ_i is the ith eigenvalue of the Laplacian matrix of the graph defined as $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$, where $\boldsymbol{D}$ is the diagonal degree matrix. For regular GSG it was previously found that $Q = \varphi^2 = \varphi + 1$, for $1 \leq i \leq n$. This mean, for instance, that any GSG has better synchronizability that any of the entangled networks reported by Donetti et al. [7] with the exception of the Petersen graph. Another advantage of GSGs is that they can be built analytically (see Theorems 1-3) for different number of nodes instead as the entangled networks, which are obtained by a numerical optimisation process. Here we propose a bound for the ratio of non-regular GSGs.

Recalling the Courant-Weyl inequalities (see [5]) we know that $\theta_{n-1} \leq d_1 - \lambda_2$, $d_1 - \lambda_1 \leq \theta_1$ and $d_1 - \lambda_n \leq \theta_1$. It is also well known that $\lambda_1 \leq d_1$, thus $S(G) = \lambda_1 - \lambda_n \leq d_1 - \lambda_n \leq \theta_1$. Besides $\theta_{n-1} \leq d_1 - \lambda_2 = d_1 - \lambda_1 + \lambda_1 - \lambda_2 \leq \theta_1 + gap(G)$. Combining both expressions we get

$$\frac{1}{Q} = \frac{\theta_{n-1}}{\theta_1} = 1 + \frac{\theta_{n-1} - \theta_1}{\theta_1} \leq 1 + \frac{gap(G)}{S(G)} = 1 + \frac{1}{\omega_1\omega_2}$$

As a consequence, we get the following lower bound for the synchronizability ratio of non-regular GS graphs:

$$\frac{\varphi^2}{\varphi^2+1} \leq Q. \tag{3}$$

We observe that in spite of this bound is always smaller than 1, and therefore seems to not have too much practical interest, we use it to obtain bounds for topological parameters of non-regular GSG in the next Section 4.3.

4.3. Topological parameters in golden spectral graphs

In the following we study several topological parameters of d-regular GSGs, such as the diameter, degree, mean distance,etc. and find some bounds for them, by using previous bounds obtained for the ratio Q in ref. [6]. First we consider only d-regular GSGs.

Lemma 16. *Let G be a GSG, B and C two subsets of vertices at distance $r+1$, and let $|B|$, $|C|$ be their cardinalities, then*

$$4(r-1)^2 \frac{|B||C|}{(n-|B|-|C|)\cdot(|B|+|C|)} < \varphi^2. \tag{4}$$

Despite both subsets have only one vertex at a maximum distance D, we can derive an upper bound for the diameter of a GS graph.

Proposition 17. *Let G be a regular GSG of orden n, B and C two subsets of vertices at distance $r+1$, and let $|B|$, $|C|$ be their cardinalities, then*

$$D < 2 + \frac{\sqrt{2}}{2}\varphi\sqrt{n-2}. \tag{5}$$

From a classical bound due to Mohar (ref. [12]) we derive the following upper bound for the average distance $\bar{l}$

Proposition 18. *Let G be a regular GSG of orden n and average distance $\bar{l}$, then*

$$\bar{l} < \frac{n}{n-1}\left\lceil 1+\varphi\sqrt{\frac{\alpha^2-1}{4\alpha}}\right\rceil \left(\frac{1}{2}+\left\lceil \log_\alpha \frac{n}{2}\right\rceil\right), \tag{6}$$

where $\alpha > 1$ is a parameter.

With respect to lower bounds for Q, we can use the corresponding equations of [6] to obtain two bounds relating D and the degree d

Proposition 19. *Let G be a d-regular GSG of order n and diameter D, then it holds*

$$\left(\frac{4}{\ln(n-1)}\left\lfloor\frac{D}{2}\right\rfloor - 1\right)\frac{d+1}{d} \leq \varphi^2, \tag{7}$$

$$\left(\left\lfloor\frac{2\bar{l}(n-1)-n}{2n\ln(n-1)}\right\rfloor - \frac{1}{4}\right)\frac{4(d+1)}{d} \leq \varphi^2. \tag{8}$$

From a lower bound relating the isoperimetric number we can obtain

Proposition 20. *Let G be a d-regular GSG, and let $i(G)$ be the isoperimetric number, then*

$$i(G) < 2d\sqrt{\frac{\varphi}{3\varphi+2}}. \tag{9}$$

Now consider non-regular GS graphs, we use Equation (3) to get some bounds for general parameters. For instance, from the following lower bound

$$Q \leq \Delta\Big[n\left(\bar{l}-\frac{1}{2}\right)+1-\bar{l}\Big],$$

by applying Equation (3) we get a lower bound for the mean distance.

Proposition 21. *Let G be a GSG of order n, let Δ be the maximum degree and $\bar{l}$ the average distance, then*

$$\frac{1}{n-1}\left(\frac{\varphi^2}{(\varphi^2+1)\Delta}+\frac{n}{2}-1\right) \leq \bar{l}. \tag{10}$$

Another upper bound that can be found in [6] is $Q \leq n\Delta D/2$, from which we can obtain the relation

Proposition 22. *Let G be a GSG of order n, maximum degree Δ and diameter D, then*

$$\frac{2\varphi^2}{\varphi^2+1} \leq n\Delta D. \tag{11}$$

Finally, we present a bound for the chromatic number of GSGs in terms of the adjacency eigenvalues of the graph, based on a Hoffman [10] known result.

Proposition 23. *Let G be a GSG of order n, let $\chi(G)$ be the chromatic number, then*

$$\chi(G) \geq \varphi \left(1 - \frac{\lambda_2}{\lambda_n}\right). \tag{12}$$

Proof. If G is GSG then $\lambda_2 - \lambda_n = \varphi(\lambda_1 - \lambda_2) \Rightarrow \varphi\lambda_2 + \lambda_2 - \lambda_n = \varphi\lambda_1 \Rightarrow$

$$-\frac{\varphi^2\lambda_2}{\varphi\lambda_n} + \frac{\lambda_n}{\varphi\lambda_n} = -\frac{\varphi\lambda_1}{\varphi\lambda_n} \Rightarrow \varphi\left(-\frac{\lambda_2}{\lambda_n} + 1\right) = 1 - \frac{\lambda_1}{\lambda_n} \leq \chi(G).$$

□

5. Summary

We have introduced here the concept of golden spectral graphs for a mathematical audience. These graphs are defined by considering spectral ratios which involves the spectral gap, spectral spread and the difference between the second largest and the smallest eigenvalue of the adjacency matrix. These graphs display a series of interesting properties, such as good synchronizability and the best possible robustness. Many of these graphs are Ramanujan graphs displaying the best possible expansibility and in general are the best synchronizers reported in the literature. Thus, they are interesting for several technological applications. We have shown here some new construction methods and several topological properties for these graphs, which we hope will motivate mathematicians to explore new relations for golden spectral graphs.

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