# STABILIZED CONTINUOUS AND DISCONTINUOUS GALERKIN TECHNIQUES FOR DARCY FLOW*

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**Abstract.** We design stabilized methods based on the variational multiscale decomposition of Darcy's problem. A model for the subscales is designed by using a heuristic Fourier analysis. This model involves a characteristic length scale, that can go from the element size to the diameter of the domain, leading to stabilized methods with different stability and convergence properties. These stabilized methods mimic the different possible functional settings of the continuous problem. The optimal method depends on the velocity and pressure approximation order. They also involve a subgrid projector that can be either the identity (when applied to finite element residuals) or can have an image orthogonal to the finite element space. In particular, we have designed a new stabilized method that allows the use of piecewise constant pressures. We consider a general setting in which velocity and pressure can be approximated by either continuous or discontinuous approximations. All these methods have been analyzed, proving stability and convergence results. In some cases, duality arguments have been used to obtain error bounds in the $L^2$-norm.

**Key words.** Darcy's problem, stabilized finite element methods, characteristic length scale, orthogonal subgrid scales

**AMS subject classifications.** 65N30, 35Q30

## 1. Introduction.

Darcy's problem governs the flow of an incompressible fluid through a porous medium. It is composed by the Darcy law that relates the fluid velocity (the flux) and the pressure gradient and the mass conservation equation. In flow in porous media, a proper functional setting for this problem is to consider the flux in $H(\mathrm{div},\Omega)$ and the pressure in $L^2(\Omega)$. This yields a saddle-point problem that is well posed due to inf-sup conditions known to hold at the continuous level, and that allow one to obtain stability estimates for the pressure and the velocity divergence.

The Galerkin approximation of this indefinite system is a difficult task, because the continuous inf-sup conditions are not naturally inherited by most finite element (FE) velocity-pressure spaces. We can avoid these problems by invoking the Darcy law in the mass conservation equation, getting a pressure Poisson problem; this is an elliptic problem that can be easily approximated by the Galerkin technique and Lagrangian elements. The fluxes can be obtained as a postprocess by using a $L^2$-projection. This approach is computationally appealing because pressure and velocity computations are decoupled and the implementation is easy. Unfortunately, this approach has two drawbacks: the loss of accuracy for the velocity and the very weak enforcement of the mass conservation equation. Improved post-processing techniques that reduce these problems can be found e.g. in [15, 17]. This approach has been restricted to continuous ($H^1$-conforming) pressure FE spaces. However, the continuous pressure admits discontinuities, e.g. in regions with jumps of the physical properties (conductivity), and this approach leads to poor accuracy in the vicinity of these regions.

The indefinite problem can be approximated by the Galerkin technique and mixed FE formulations (see [5]) that satisfy the inf-sup conditions required for the well-posedness of the discrete problem. As an example, the combination of the Raviart-Thomas FE velocity space introduced in [27] with piecewise constant or linear pressures leads to stable approxi-

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mations. The Raviart-Thomas FE space is $H(\mathrm{div}, \Omega)$-conforming; it is composed by vector functions with continuous normal traces and discontinuous tangential traces on the element boundaries, even though discontinuous Galerkin Raviart-Thomas FE methods have recently been proposed in [8]. The element unknowns are the normal fluxes on the faces, but all components are needed inside every element domain. This makes the implementation involved, specially for three dimensional problems. On the other hand, this FE space experiments a loss of accuracy in some meshes (see [2]). Finally, when dealing with a coupled Stokes-Darcy problem it is hard to find mixed FE methods that are stable for both the Stokes and the Darcy problems (see [1, 23]). The FE spaces that satisfy these conditions are expensive and restricted to particular typologies of meshes that complicate their use in real applications. For the same reasons, they are not appealing when solving the Biot system that couples in a particular way the elastic problem and the Darcy problem (possibly coupled with the Navier-Stokes equations too).

A third alternative is to resort to stabilization techniques that perturb the indefinite problem in such a way that the FE approximation can violate the inf-sup condition in the functional setting of the continuous problem. Stabilization techniques for the Darcy problem have been designed in [25]. Therein, the stabilized problem mimics the mixed Laplacian functional setting (the pressure belongs to $H^1(\Omega)$ and the velocity belongs to $L^2(\Omega)$) and leads to the same order of convergence that is attained when using the pressure Poisson problem plus postprocessing. This method has been extended to discontinuous FE spaces for velocities and pressures in [6, 21]. The stabilization term is the inner product of the residual times the adjoint of the Darcy differential operator applied to the test function. Correa and Loula have considered an alternative stabilized formulation in [16] that gives very strong stability bounds; both velocity and pressure are in $H^1(\Omega)$, even though the authors use the continuous embedding of $H(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$ in $H^1(\Omega)$, which is false for domains with re-entrant corners (see e.g. [19]). As a consequence, no convergence is attained for the natural norm, and only $L^2$-norms of the errors can be bounded using elliptic regularity properties that are not true in general; the error estimates do not apply for non-convex domains.

In this work, we motivate stabilized methods based on the variational multiscale (VMS) decomposition of the Darcy problem which is in fact an adjoint formulation (see [20, 26]). A matrix of algorithmic stabilization parameters appears, which we design using a heuristic Fourier analysis. The definition of this matrix involves a characteristic length scale. The choice of this characteristic length, which can be either the element size or the diameter of the domain, leads to stabilized methods with different stability and convergence properties. In this frame, we get numerical methods that mimic the typical setting in Darcy's flow (the velocity belongs to $H(\mathrm{div}, \Omega)$ and the pressure to $L^2(\Omega)$) as well as others that mimic the mixed Laplacian formulation. Intermediate settings with unclear continuous counterpart but interesting convergence properties are also designed. Roughly speaking, we can increase the velocity stability reducing pressure stability and vice-versa, and analogously for the convergence rate. The optimal method depends on the velocity and pressure approximation order.

The methods motivated by VMS also involve a subgrid projection of the residual of the finite element solution. If the subgrid projection is considered the identity (the method called ASGS in this article) we recover, up to the definition of the stabilization parameters, the methods discussed in [21, 26, 25]. We will also consider the case in which the subgrid projection is orthogonal to the finite element space (the method termed OSS below), as suggested in [9]. We thus motivate in a unified way a wide set of stabilized methods that can keep symmetry and mimic the different functional settings of the continuous problem (as well as other methods). In particular, we suggest a new stabilized method that allows the use of piecewise constant pressure—as far as we know, the first of this kind.

We have considered a general setting in which velocity and pressure can be approximated by using either continuous or discontinuous approximations. All these methods have been analyzed, proving stability and convergence results. In some cases, Aubin-Nitsche-type duality arguments have been used to obtain error bounds in the $L^2$-norm. We have previously suggested a unified stabilization of the coupled Stokes-Darcy problem and performed the numerical analysis in [4] using these ideas.

Let us give the outline of the paper. In Section 2 we introduce the continuous problem and analyze its stability. Section 3 introduces a (non-conforming) discontinuous Galerkin (dG) approximation of the problem. We motivate the stabilization methods in the VMS framework and suggest and expression for the stabilization parameters and subgrid projector in Section 4. Section 5 is devoted to the stability and convergence analysis of these stabilized FE approximations. Improved error estimates obtained by duality arguments are presented in Section 6. We draw some recommendations about the method to use in Section 7, depending on the order of approximation of velocities and pressures. Numerical tests that show experimental convergence rates can be found in Section 8. We close the paper with some conclusions.

## 2. Continuous problem.

**2.1. Problem statement.** Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polyhedral domain (with Lipschitz boundary) where we consider the Darcy problem, which consists in finding a velocity $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d$ and a pressure $p : \Omega \longrightarrow \mathbb{R}$ such that

$$\sigma \boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{2.1a}$$

$$\nabla \cdot \boldsymbol{u} = g, \tag{2.1b}$$

where $\boldsymbol{f}$ and $g$ are given functions and the physical parameter $\sigma$ is the inverse of the permeability. As boundary conditions we will consider $\boldsymbol{n} \cdot \boldsymbol{u} = \psi$ on $\partial\Omega$, $\boldsymbol{n}$ being the unit exterior normal. The body force $\boldsymbol{f}$ is usually zero for flow in porous media. However, we will keep $\boldsymbol{f}$ because a non-zero $\boldsymbol{f}$ is needed for some interesting applications governed by system (2.1), like in magnetohydrodynamics, where the current density is governed by Ohm's law and the conservation of charge.

Let us introduce some standard notation. The space of functions whose $p$ power ($1 \leq p < \infty$) is integrable in a domain $\omega$ is denoted by $L^p(\omega)$, $L^\infty(\omega)$ being the space of bounded functions in $\omega$ (in the Lebesgue sense). The space of functions whose distributional derivatives of order up to $m \geq 0$ (integer) belong to $L^2(\omega)$ is denoted by $H^m(\omega)$. The space $H_0^1(\omega)$ consists of functions in $H^1(\omega)$ vanishing on $\partial\omega$. The topological dual of $H_0^1(\omega)$ is denoted by $H^{-1}(\omega)$. The space of vector-valued functions with components in $L^2(\omega)$ is denoted with $L^2(\omega)^d$, and analogously for the rest of scalar spaces. $H(\mathrm{div}, \omega)$ is the space of functions in $L^2(\omega)^d$ with their divergence in $L^2(\omega)$. $H_0(\mathrm{div}, \omega)$ is the space of vector fields in $H(\mathrm{div}, \omega)$ with zero normal trace on $\partial\omega$. We also recall that the space of traces of $H^1(\omega)$ on a line (surface for three dimensions) $\beta \subset \omega$ is denoted by $H^{1/2}(\beta)$. The topological dual of $H^{1/2}(\beta)$ is the space of fluxes denoted by $H^{-1/2}(\beta)$.

The Darcy problem can be thought in two different ways:

1. The *typical* setting for flow in porous media:

$$\begin{aligned} &\boldsymbol{u} \in H(\mathrm{div}, \Omega), \quad p \in L^2(\Omega)/\mathbb{R}, \\ &\boldsymbol{f} \in H(\mathrm{div}, \Omega)', \quad g \in L^2(\Omega), \quad \psi \in L^2(\partial\Omega) \end{aligned} \tag{2.2}$$

   with the essential boundary condition $\boldsymbol{n} \cdot \boldsymbol{u} = \psi$.

2. A mixed formulation of the Poisson problem. In this case, the functional setting is:

$$\begin{aligned} &\boldsymbol{u} \in L^2(\Omega)^d, \quad p \in H^1(\Omega)/\mathbb{R}, \\ &\boldsymbol{f} \in L^2(\Omega)^d, \quad g \in H^{-1}(\Omega), \quad \psi \in H^{-1/2}(\partial\Omega). \end{aligned} \tag{2.3}$$

Note that for an arbitrary function $\boldsymbol{v} \in L^2(\Omega)^d$, the normal trace of $\boldsymbol{v}$ is not defined and cannot be enforced. The boundary condition $\boldsymbol{n} \cdot \boldsymbol{u} = \psi$ (which is essential in the previous setting) is natural and holds in $H^{-1/2}(\partial\Omega)$. In this case, (essential) pressure boundary conditions can be imposed too, since the pressure trace belongs to $H^{1/2}(\partial\Omega)$.

In fact, whichever the situation is, it will be determined by the data. In the next subsection we will obtain an inf-sup condition that can be trivially translated into velocity-pressure stability if the data are regular enough. For the sake of clarity we have considered $\sigma$ to be a positive constant, but all the results obtained in this work apply for the general case in which $\sigma \in L^\infty(\Omega)$ and $\sigma_+ \geq \sigma(\boldsymbol{x}) \geq \sigma_- > 0$ for all $\boldsymbol{x} \in \Omega$ (up to sets of zero measure), where $\sigma_+$ and $\sigma_-$ are constants.

Let us denote by $\langle f_1, f_2 \rangle$ the integral of two (generalized) functions $f_1$ and $f_2$ (either scalar or vector-valued) in $\Omega$. The regularity of both is such that the integral is well defined. For example, if $f_1 \in H_0^1(\Omega)$ we may take $f_2 \in H^{-1}(\Omega)$. When both $f_1, f_2 \in L^2(\Omega)$ we will write their $L^2(\Omega)$ inner product as $\langle f_1, f_2 \rangle \equiv (f_1, f_2)$. The associated norm will be denoted by $\|f_1\|_{L^2(\Omega)} \equiv \|f_1\|$.

Either in the situation (2.2) or in (2.3) the variational formulation of the problem consists in finding a velocity-pressure pair $[\boldsymbol{u}, p]$, with $\boldsymbol{n} \cdot \boldsymbol{u} = \psi$ on $\partial\Omega$, such that

$$B_c([\boldsymbol{u}, p], [\boldsymbol{v}, q]) = L_c([\boldsymbol{v}, q]), \tag{2.4}$$

for all the $[\boldsymbol{v}, q]$ in the test space, where the bilinear form $B_c$ and the linear form $L_c$ are defined by

$$B_c([\boldsymbol{u}, p], [\boldsymbol{v}, q]) = \sigma(\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u}), \tag{2.5a}$$

$$L_c([\boldsymbol{v}, q]) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle + \langle g, q \rangle. \tag{2.5b}$$

The correct functional setting of the problem is a consequence of the inf-sup condition stated in the next subsection.

**2.2. *A priori* stability bounds.** A key ingredient in the following discussion is the introduction of a characteristic length scale of the problem, that we denote by $L_0$, which may be taken as the diameter of the computational domain $\Omega$. Whereas for the Stokes problem its introduction is unnecessary, it will play a key role in the Darcy problem. The ultimate reason to explain this fact is that in the Stokes case the seminorm $\|\nabla \boldsymbol{u}\|$ controls the whole norm in $H_0^1(\Omega)^d$ because of the Poincaré-Friedrichs inequality, and thus a stability estimate in this seminorm suffices; an analogous situation occurs for the elastic problem and Korn's inequality (see [7]). However, for the Darcy problem we need to control both $\boldsymbol{u}$ and $\nabla \cdot \boldsymbol{u}$ to obtain stability in $H(\mathrm{div}, \Omega)$, and the only way to incorporate both norms in a dimensionally correct one is through the introduction of a length scale. Thus, we introduce the following norm:

$$\|\boldsymbol{v}\|_{H(\mathrm{div},\Omega)} = \|\boldsymbol{v}\| + L_0 \|\nabla \cdot \boldsymbol{v}\|.$$

While this discussion might seem unnecessary to obtain theoretical stability estimates (and thus to determine the functional framework of the problem), it will lead to very important consequences in the discrete finite element problem.

The correct functional setting of the problem (2.4)-(2.5) is a consequence of the inf-sup condition

$$\inf_{q\in L^2(\Omega)} \sup_{\boldsymbol{v}\in H(\mathrm{div},\Omega)} \frac{(q,\nabla\cdot\boldsymbol{v})}{\|q\|(\|\boldsymbol{v}\|+L_0\|\nabla\cdot\boldsymbol{v}\|)} \tag{2.6}$$

which is true due to the surjectivity of the divergence operator from $H(\mathrm{div},\Omega)$ onto $L^2(\Omega)$ (see e.g. [18]).

Let now $V$ (the velocity space) be the closure of $C^\infty(\Omega)^d$ with respect to the norm $\sqrt{\sigma}\|\boldsymbol{v}\|+\sqrt{\sigma}L_0\|\nabla\cdot\boldsymbol{v}\|$ and $Q$ the closure of $C^\infty(\Omega)/\mathbb{R}$ with respect to $(\sqrt{\sigma}L_0)^{-1}\|q\|$. The pair $V\times Q$ reduces to $H(\mathrm{div},\Omega)\times L^2(\Omega)/\mathbb{R}$. On this space we define

$$|\!|\!|[\boldsymbol{v},q]|\!|\!|_c^2 := \sigma\|\boldsymbol{v}\|^2+\sigma L_0^2\|\nabla\cdot\boldsymbol{v}\|^2+\frac{1}{\sigma L_0^2}\|q\|^2. \tag{2.7}$$

We will denote by $V_\psi$ the subspace of $V$ of functions $\boldsymbol{v}\in V$ such that $\boldsymbol{n}\cdot\boldsymbol{v}=\psi$, and $V_0$ the subspace of functions such that $\boldsymbol{n}\cdot\boldsymbol{v}=0$. For the sake of simplicity, $\psi=0$ is considered in the following theorem, although non-homogeneous conditions will be taken into account at the discrete level.

In what follows, $C$ denotes a positive constant, in our case *independent of $\sigma$ and $L_0$*. When dealing with the finite element problem, $C$ will be independent also of the mesh size $h$. The value of $C$ may be different at different occurrences. We will use the notation $A\gtrsim B$ and $A\lesssim B$ to indicate that $A\geq CB$ and $A\leq CB$, respectively, where $A$ and $B$ are expressions depending on functions that in the discrete case may depend on $h$ as well. Analogously, $A\eqsim B$ will mean that $B\lesssim A\lesssim B$.

The following theorem is a simplified version of the corresponding one in [4].

THEOREM 2.1 (Stability of the continuous problem). *For all $[\boldsymbol{u},p]\in V_0\times Q$ there exists $[\boldsymbol{v},q]\in V_0\times Q$ for which*

$$B_c([\boldsymbol{u},p],[\boldsymbol{v},q])\geq C|\!|\!|[\boldsymbol{u},p]|\!|\!|_c\,|\!|\!|[\boldsymbol{v},q]|\!|\!|_c,$$

*where the bilinear form $B_c$ is given in (2.5a) and the norm $|\!|\!|\cdot|\!|\!|_c$ in (2.7).*

**Proof**. Taking $[\boldsymbol{v}_1,q_1]=[\boldsymbol{u},p]$ we get:

$$B_c([\boldsymbol{u},p],[\boldsymbol{v}_1,q_1])=\sigma\|\boldsymbol{u}\|^2. \tag{2.8}$$

The inf-sup condition (2.6) states that

$$\forall p\in L^2(\Omega)\ \exists\,\boldsymbol{v}_p\in H_0(\mathrm{div},\Omega)\ |\ -(p,\nabla\cdot\boldsymbol{v}_p)\gtrsim\|p\|\left(\frac{1}{L_0}\|\boldsymbol{v}_p\|+\|\nabla\cdot\boldsymbol{v}_p\|\right).$$

We can choose $\boldsymbol{v}_p$ such that

$$\|\boldsymbol{v}_p\|+L_0\|\nabla\cdot\boldsymbol{v}_p\|=\frac{1}{\sigma L_0}\|p\|,$$

which is a dimensionally consistent norm. Taking $[\boldsymbol{v}_2,q_2]=[\boldsymbol{v}_p,0]$ we have:

$$B_c([\boldsymbol{u},p],[\boldsymbol{v}_2,q_2])\gtrsim-\sqrt{\sigma}\|\boldsymbol{u}\|_{H(\mathrm{div},\Omega)}+\frac{1}{\sigma L_0^2}\|p\|^2.$$

Since $\boldsymbol{u}\in V_0$, we have that $\nabla\cdot\boldsymbol{u}\in L^2(\Omega)$. For $[\boldsymbol{v}_3,q_3]=[\boldsymbol{0},\sigma L_0^2\nabla\cdot\boldsymbol{u}]$ we get:

$$B_c([\boldsymbol{u},p],[\boldsymbol{v}_3,q_3])=\sigma L_0^2\|\nabla\cdot\boldsymbol{u}\|^2. \tag{2.9}$$

Let $[\boldsymbol{v}, q] = \sum_{i=1}^{3} \alpha_i [\boldsymbol{v}_i, q_i] \in V_0 \times Q$, $\alpha_i \in \mathbb{R}$. The coefficients $\alpha_i$ can be chosen so that

$$B_c([\boldsymbol{u}, p], [\boldsymbol{v}, q]) \gtrsim |\!|\!| [\boldsymbol{u}, p] |\!|\!|_c^2.$$

It is easily checked that $|\!|\!|[\boldsymbol{v}, q]|\!|\!|_c \lesssim |\!|\!|[\boldsymbol{u}, p]|\!|\!|_c$ for any combination of coefficients $\alpha_i \in \mathbb{R}$. This proves the theorem. □

REMARK 2.1. *The inf-sup condition of Theorem 2.1 leads to stability bounds for velocity and pressure provided the data are regular; that is to say, $L_c$ is continuous with respect to $|\!|\!|\cdot|\!|\!|_c$. This continuity is true for $\boldsymbol{f} \in H(\mathrm{div}, \Omega)'$ and $g \in L^2(\Omega)$.*

REMARK 2.2. *If there is more regularity of the data, that is, if $\boldsymbol{f} \in L^2(\Omega)^d$ and $g \in L^2(\Omega)$, the pressure belongs to $H^1(\Omega)$ and we can pose the problem in a different functional setting. Let now the pressure space be the closure of $C^\infty(\Omega)/\mathbb{R}$ with respect to $(\sigma L_0^2)^{-1/2}\|q\| + \sigma^{-1/2}\|\nabla q\|$, that reduces to $H^1(\Omega)$. We consider the following weak formulation: find $[\boldsymbol{u}, p] \in H(\mathrm{div}, \Omega) \times H^1(\Omega)$ (trial space) such that*

$$B_c([\boldsymbol{u}, p], [\boldsymbol{v}, q]) = L_c([\boldsymbol{v}, q]), \qquad \forall [\boldsymbol{v}, q] \in L^2(\Omega)^d \times L^2(\Omega).$$

*with $\boldsymbol{n} \cdot \boldsymbol{u} = \psi$ on $\partial\Omega$. Note that the trial and test spaces are different. Control over $\frac{1}{\sigma}\|\nabla p\|^2$ can be obtained by taking as test function in (2.5a) $[\boldsymbol{v}_4, q_4] = [\nabla p, 0] \in L^2(\Omega)^d \times L^2(\Omega)$. Now, taking a linear combination of this test function and the test functions in the proof of Theorem 2.1, $[\boldsymbol{v}, q] = \sum_{i=1}^{4} \alpha_i [\boldsymbol{v}_i, q_i] \in L^2(\Omega)^d \times L^2(\Omega)$, and picking appropriate coefficients $\alpha_i \in \mathbb{R}$, we get stability over $|\!|\!|[\boldsymbol{u}, p]|\!|\!|_c + \frac{1}{\sqrt{\sigma}}\|\nabla p\|$. This is the functional setting in which stability of the continuous problem has been proved in [4].*

**3. Non-conforming finite element approximation.** Let us introduce some notation. The FE partition will be denoted by $\mathcal{T}_h = \{K\}$, and summation over all the elements will be indicated by $\sum_K$. For conciseness, $\mathcal{T}_h = \{K\}$ will be assumed quasi-uniform, being $h$ the mesh size. The broken integral $\sum_K \int_K$ will be denoted by $\int_{\mathcal{T}_h}$. The collection of all *edges* (faces, for $d = 3$) will be written as $\mathcal{E}_h = \{E\}$ and summation over all these edges will be indicated as $\sum_E$. The set of internal and boundary edges will be denoted by $\mathcal{E}_h^0 = \{E_0\}$ and $\mathcal{E}_h^\partial = \{E_\partial\}$ respectively. The broken integral $\sum_E \int_E$ will be written as $\int_{\mathcal{E}_h}$, using $\int_{\mathcal{E}_h^0}$ and $\int_{\mathcal{E}_h^\partial}$ when the edges are interior or on the boundary, respectively.

Suppose now that elements $K_1$ and $K_2$ share an edge $E$, and let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be the normals to $E$ exterior to $K_1$ and $K_2$, respectively. For a scalar function $f$, possibly discontinuous across $E$, we define its jump and average as

$$[\![f]\!] := \boldsymbol{n}_1 f|_{\partial K_1 \cap E} + \boldsymbol{n}_2 f|_{\partial K_2 \cap E},$$
$$\{f\} := \frac{1}{2}(f|_{\partial K_1 \cap E} + f|_{\partial K_2 \cap E}),$$

whereas for vectorial quantities we will use

$$[\![\boldsymbol{v}]\!] := \boldsymbol{n}_1 \cdot \boldsymbol{v}|_{\partial K_1 \cap E} + \boldsymbol{n}_2 \cdot \boldsymbol{v}|_{\partial K_2 \cap E},$$
$$\{\boldsymbol{v}\} := \frac{1}{2}(\boldsymbol{v}|_{\partial K_1 \cap E} + \boldsymbol{v}|_{\partial K_2 \cap E}).$$

Let us consider piecewise discontinuous FE spaces for the velocity and the pressure, given respectively by

$$V_h := \{\boldsymbol{v} \in (L^2(\Omega))^d |\; \boldsymbol{v}|_K \in R_k(K)^d \; \forall K \in \mathcal{T}_h\},$$
$$Q_h := \{q \in L^2(\Omega)/\mathbb{R} |\; q|_K \in R_l(K) \; \forall K \in \mathcal{T}_h\},$$

where $R_m$ consists of polynomials in $x_1, ..., x_d$ of degree less than or equal to $m$ when $K$ is a simplex and of degree less than or equal to $m$ in each coordinate when $K$ is a quadrilateral (hexahedron, when $d = 3$). Thus, $k$ and $l$ are the order of approximation of velocity and pressure, respectively. This is a non-conforming approximation of problem (2.4). The notion of *non-conforming approximation* depends on the way the continuous problem is posed. In particular, a discontinuous approximation of the velocity is not conforming for the first functional setting introduced above (because $V_h \not\subset H(\mathrm{div}, \Omega)$) whereas it is conforming in the mixed Laplacian setting. Similarly, if instead of using (2.4) the problem is posed using hybrid methods in which the continuity of the (*a priori* discontinuous) solution is enforced via Lagrange multipliers, a discontinuous approximation is conforming. In what follows, the concept of conforming (and subsequently non-conforming) approximation is considered with respect to the velocity-pressure space $H(\mathrm{div}, \Omega) \times L^2(\Omega)$. Likewise, we will use the term *discontinuous Galerkin* (dG) referring to the discontinuous functions in the interpolation spaces, even if the discrete formulations we will analyze are not of Galerkin type.

With the aim of obtaining a well-defined weak formulation of the continuous problem (2.1) for dG approximations, let us test (2.1) against functions in $V_h \times Q_h$.[1] Taking the FE test functions $[\boldsymbol{v}_h^K, q_h^K]$ with support in an element $K$ and integrating some terms by parts, we obtain

$$\int_K \sigma \boldsymbol{u} \cdot \boldsymbol{v}_h^K \mathrm{d}\Omega - \int_K p \nabla \cdot \boldsymbol{v}_h^K \mathrm{d}\Omega + \int_{\partial K} p\, \boldsymbol{n} \cdot \boldsymbol{v}_h^K \mathrm{d}\Gamma - \int_K \boldsymbol{u} \cdot \nabla q_h^K \mathrm{d}\Omega + \int_{\partial K} q_h^K \boldsymbol{n} \cdot \boldsymbol{u} \mathrm{d}\Gamma = \int_K \boldsymbol{f} \cdot \boldsymbol{v}_h^K \mathrm{d}\Omega + \int_K g q_h^K \mathrm{d}\Omega. \tag{3.1}$$

The discontinuous FE space $V_h \times Q_h$ is spanned by discontinuous functions with support in a single element, so that for any $[\boldsymbol{v}_h, q_h] \in V_h \times Q_h$, $[\boldsymbol{v}_h, q_h] = \sum_K [\boldsymbol{v}_h^K, q_h^K]$. Adding up (3.1) for all $K \in \mathcal{T}_h$, using formula

$$\begin{aligned} \sum_K \int_{\partial K} \phi\, \boldsymbol{n} \cdot \boldsymbol{w} \mathrm{d}\Gamma &= \int_{\mathcal{E}_h^0} [\![\phi]\!] \cdot \{\boldsymbol{w}\} \mathrm{d}\Gamma + \int_{\mathcal{E}_h} [\![\boldsymbol{w}]\!] \{\phi\} \mathrm{d}\Gamma \\ &= \int_{\mathcal{E}_h} [\![\phi]\!] \cdot \{\boldsymbol{w}\} \mathrm{d}\Gamma + \int_{\mathcal{E}_h^0} [\![\boldsymbol{w}]\!] \{\phi\} \mathrm{d}\Gamma. \end{aligned}$$

and invoking the continuity of velocities and fluxes

$$[\![\boldsymbol{u}]\!] = 0, \qquad [\![p]\!] = 0$$

for every internal edge $E_0$ in $\mathcal{E}_h^0$ and the boundary condition $[\![\boldsymbol{u}]\!] = \psi$ for every boundary edge $E_\partial$ in $\mathcal{E}_h^\partial$, we get a variational problem that, after replacing the continuous unknowns by their discrete counterparts, reads

$$\begin{aligned} &\int_{\mathcal{T}_h} \sigma \boldsymbol{u}_h \cdot \boldsymbol{v}_h \mathrm{d}\Omega - \int_{\mathcal{T}_h} p_h \nabla \cdot \boldsymbol{v}_h \mathrm{d}\Omega + \int_{\mathcal{E}_h} [\![\boldsymbol{v}_h]\!] \{p_h\} \mathrm{d}\Gamma = \int_{\mathcal{T}_h} \boldsymbol{f} \cdot \boldsymbol{v}_h \mathrm{d}\Omega, \\ &- \int_{\mathcal{T}_h} \boldsymbol{u}_h \cdot \nabla q_h \mathrm{d}\Omega + \int_{\mathcal{E}_h^0} [\![q_h]\!] \cdot \{\boldsymbol{u}_h\} \mathrm{d}\Gamma = \int_{\mathcal{T}_h} g q_h \mathrm{d}\Omega - \int_{\mathcal{E}_h^0} \psi q_h \mathrm{d}\Gamma. \end{aligned}$$

In this discrete problem the continuity constraints and the boundary condition over the normal velocity have been enforced in a weak way. Re-integrating by parts the pressure gradient and/or the divergence of the velocity, and using the previous identities (no jumps cancel for the discontinuous FE approximations), we get the equivalent formulations:

[1] We cannot use (2.4) since $V_h \times Q_h \not\subset V \times Q$ in general.

1. Gradient form:

$$\int_{\mathcal{T}_h} \sigma \boldsymbol{u}_h \cdot \boldsymbol{v}_h \mathrm{d}\Omega + \int_{\mathcal{T}_h} \nabla p_h \cdot \boldsymbol{v}_h \mathrm{d}\Omega - \int_{\mathcal{E}_h^0} [\![ p_h ]\!] \cdot \{\boldsymbol{v}_h\} \mathrm{d}\Gamma = \int_{\mathcal{T}_h} \boldsymbol{f} \cdot \boldsymbol{v}_h \mathrm{d}\Omega,$$
$$- \int_{\mathcal{T}_h} \boldsymbol{u}_h \cdot \nabla q_h \mathrm{d}\Omega + \int_{\mathcal{E}_h^0} [\![ q_h ]\!] \cdot \{\boldsymbol{u}_h\} \mathrm{d}\Gamma = \int_{\mathcal{T}_h} g q_h \mathrm{d}\Omega - \int_{\mathcal{E}_h^0} \psi q_h \mathrm{d}\Gamma$$

2. Divergence form:

$$\int_{\mathcal{T}_h} \sigma \boldsymbol{u}_h \cdot \boldsymbol{v}_h \mathrm{d}\Omega - \int_{\mathcal{T}_h} p_h \nabla \cdot \boldsymbol{v}_h \mathrm{d}\Omega + \int_{\mathcal{E}_h} [\![ \boldsymbol{v}_h ]\!] \{p_h\} \mathrm{d}\Gamma = \int_{\mathcal{T}_h} \boldsymbol{f} \cdot \boldsymbol{v}_h \mathrm{d}\Omega,$$
$$\int_{\mathcal{T}_h} \nabla \cdot \boldsymbol{u}_h q_h \mathrm{d}\Omega - \int_{\mathcal{E}_h} [\![ \boldsymbol{u}_h ]\!] \{q_h\} \mathrm{d}\Gamma = \int_{\mathcal{T}_h} g q_h \mathrm{d}\Omega - \int_{\mathcal{E}_h^0} \psi q_h \mathrm{d}\Gamma. \tag{3.2}$$

All these formulations are *equivalent*.

Consistently with the notation introduced above, the symbol $\langle f_1, f_2 \rangle_D$ will be used to denote the integral of the product of functions $f_1$ and $f_2$ over $D$, with $D = K$ (an element), $D = \partial K$ (an element boundary) or $D = E$ (an edge). Likewise, $\|f_1\|_D^2 := \langle f_1, f_1 \rangle_D$. With all this notation, let us write the problem in a compact manner, e.g. using the divergence form (3.2). It consists in finding $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ such that

$$B_d([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) = L_d([\boldsymbol{v}_h, q_h]) \qquad \forall [\boldsymbol{v}_h, q_h] \in V_h \times Q_h,$$

where

$$\begin{aligned} B_d([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) =& \sigma\left(\boldsymbol{u}_h, \boldsymbol{v}_h\right) - \sum_K \langle p_h, \nabla \cdot \boldsymbol{v}_h \rangle_K + \sum_K \langle \nabla \cdot \boldsymbol{u}_h, q_h \rangle_K \\ &+ \sum_E \langle \{p_h\}, [\![ \boldsymbol{v}_h ]\!] \rangle_E + \sum_E \langle \{q_h\}, [\![ \boldsymbol{u}_h ]\!] \rangle_E, \end{aligned} \tag{3.3a}$$

$$L_d([\boldsymbol{v}_h, q_h]) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle + \langle g, q_h \rangle - \sum_{E_\partial} \langle \psi, q_h \rangle_{E_\partial}. \tag{3.3b}$$

We have ended up with a FE formulation that allows us to use piecewise discontinuous functions; the continuity of normal velocities and pressures has already been enforced in a weak way, as well as the normal velocity boundary condition. Unfortunately, this formulation is not stable and the weak enforcement of normal velocity boundary conditions is too weak. In the next section we motivate *stabilizing* terms that lead to a well-posed discrete problem with a weak (but effective) enforcement of the normal trace of the velocity on the boundary.

**4. A stabilized finite element method.** In this section we introduce some stabilization techniques for the FE approximation of the Darcy problem. These stabilization techniques are motivated by the variational multiscale (VMS) framework introduced in [20]. The use of the VMS approach for the Darcy problem can also be found in [26]. Our approach is different to the one in these references; we motivate a different set of stabilization parameters and stabilization terms that open a new discussion, namely, *the choice of the characteristic length*. Different expressions for the length scales that appear in our stabilization parameters lead to a set of methods with different stability and convergence properties. We motivate methods that mimic both variational frameworks in Section 2 and some intermediate situations, whereas the approaches in [26, 24] can only mimic the mixed Laplancian setting. Furthermore, we consider two different choices of the so-called subgrid projection that are well-settled for the Stokes problem (see e.g. [20, 10]).

We target a unified method that will accommodate continuous and discontinuous approximations. Therefore, the FE spaces for both velocity and pressure, denoted by $V_h$ and $Q_h$, respectively, are free to be either continuous (conforming) or discontinuous. In all cases, the stabilization methods can be stated as follows: find $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ such that

$$B_s\left([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]\right) = L_s\left([\boldsymbol{v}_h, q_h]\right) \qquad \forall [\boldsymbol{v}_h, q_h] \in V_h \times Q_h. \tag{4.1}$$

**4.1. Variational multiscale formulation.** Let us start with a brief motivation of our stabilization techniques in the VMS framework, that consists in splitting the continuous solution $[\boldsymbol{u}, p]$ of (2.4)-(2.5) into its FE component $[\boldsymbol{u}_h, p_h]$ and the subgrid scale $[\boldsymbol{u}', p']$. In order to have a unique decomposition, we consider a subgrid space such that $V \times Q = V_h \times Q_h \oplus V' \times Q'$, so that, for the moment, we consider $V_h \times Q_h \subset V \times Q$. Invoking this decomposition in the continuous problem for both the solution and test functions, we get the two-scale system:

$$\begin{aligned} B_c\left([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]\right) + B_c\left([\boldsymbol{u}', p'], [\boldsymbol{v}_h, q_h]\right) &= L_c([\boldsymbol{v}_h, q_h]), \\ B_c\left([\boldsymbol{u}_h, p_h], [\boldsymbol{v}', q']\right) + B_c\left([\boldsymbol{u}', p'], [\boldsymbol{v}', q']\right) &= L_c([\boldsymbol{v}', q']), \end{aligned}$$

for all $[\boldsymbol{v}_h, q_h] \in V_h \times Q_h$ and $[\boldsymbol{v}', q'] \in V' \times Q'$. This is an infinite-dimensional problem equivalent to (2.4)-(2.5). Further approximations must be considered in order to get a discrete problem (see [10, 3] for a very detailed exposition). After integration-by-parts of some terms, and formally assuming that the subgrid component can be localized inside every finite element, we get:

$$B_c\left([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]\right) + \langle [\boldsymbol{u}', p'], \mathcal{L}^*[\boldsymbol{v}_h, q_h] \rangle = L([\boldsymbol{v}_h, q_h]), \tag{4.2a}$$

$$\mathcal{P}'\left(\mathcal{L}[\boldsymbol{u}', p']\right) = \mathcal{P}'\left([\boldsymbol{f}, g] - \mathcal{L}[\boldsymbol{u}_h, p_h]\right), \tag{4.2b}$$

where the operator $\mathcal{P}'$ is the broken $L^2$-projection onto $V'$ (see Subsection 4.2) and $\mathcal{L}^*$ is the adjoint of the Darcy operator $\mathcal{L}$, defined by $\mathcal{L}[\boldsymbol{u}, p] = [\sigma \boldsymbol{u} + \nabla p, \nabla \cdot \boldsymbol{u}]$. The second term in (4.2a) is the *stabilization term*, whereas the second equation is the (still infinite-dimensional) *subgrid* equation. The next step consists in replacing the differential operator $\mathcal{L}$ by an algebraic one. Inside every element, this operator is approximated by a *matrix of stabilization parameters* $\boldsymbol{\tau}^{-1}$, and the subgrid projection $\mathcal{P}'$ by an appropriate approximation $\mathcal{P}'_h := [\mathcal{P}'_{h,u}, \mathcal{P}'_{h,p}]$. Then, (4.2b) can be approximated by

$$\boldsymbol{\tau}^{-1}[\boldsymbol{u}', p'] = \mathcal{P}'_h\left([\boldsymbol{f}, g] - \mathcal{L}[\boldsymbol{u}_h, p_h]\right),$$

from where the subscale component has a closed form in terms of the FE component. Let us assume the stabilization matrix to be a diagonal matrix $\boldsymbol{\tau} = \mathrm{diag}(\tau_u, \tau_p)$. In this case, we have

$$\begin{aligned} \boldsymbol{u}' &= \tau_u \mathcal{P}'_{h,u}(\boldsymbol{f} - \sigma \boldsymbol{u}_h - \nabla p_h), \\ p' &= \tau_p \mathcal{P}'_{h,q}(g - \nabla \cdot \boldsymbol{u}_h). \end{aligned}$$

Using these expressions for the subscales in the FE problem (4.2a), we get the stabilized versions of $B_c$ and $L_c$:

$$\begin{aligned} B_{sc}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) =& B_c([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\ &+ \tau_p \sum_K \left\langle \mathcal{P}'_{h,p}(\nabla \cdot \boldsymbol{u}_h), \nabla \cdot \boldsymbol{v}_h \right\rangle_K \\ &+ \tau_u \sum_K \left\langle \mathcal{P}'_{h,u}(\sigma \boldsymbol{u}_h + \nabla p_h), -\sigma \boldsymbol{v}_h + \nabla q_h \right\rangle_K, \end{aligned} \tag{4.3a}$$

$$\begin{aligned} L_{sc}([\boldsymbol{v}_h, q_h]) =& L_c([\boldsymbol{v}_h, q_h]) \\ &+ \tau_p \sum_K \left\langle \mathcal{P}'_{h,p}(g), \nabla \cdot \boldsymbol{v}_h \right\rangle_K \\ &+ \tau_u \sum_K \left\langle \mathcal{P}'_{h,u}(\boldsymbol{f}), -\sigma \boldsymbol{v}_h + \nabla q_h \right\rangle_K. \end{aligned} \tag{4.3b}$$

As we shall see, for appropriate choices of the subgrid projectors, the stabilization terms allow us to get control over $\sum_K \tau_p \|\nabla \cdot \boldsymbol{u}_h\|_K^2$ and $\sum_K \tau_u \|\nabla p_h\|_K^2$. Using continuous FE spaces for both velocity and pressure this control is effective; the broken norms are identical to $\tau_p \|\nabla \cdot \boldsymbol{u}_h\|^2$ and $\tau_u \|\nabla p_h\|^2$, respectively.

When considering dG formulations, and therefore the possibility to use non-conforming approximations, $B_c$ and $L_c$ have to be replaced by $B_d$ and $L_d$ defined in (3.3a) and (3.3b), respectively. However, the introduction of the edge stabilization terms in $B_d$ and $L_d$, and the stabilization terms motivated by the VMS approach in $B_{sc}$ and $L_{sc}$ defined in (4.3a) and (4.3b) are not enough because they only give control in broken norms of the velocity divergence and the pressure gradient. A dimensionally correct norm that gives all the control needed for discontinuous velocities is

$$\sum_K \tau_p \|\nabla \cdot \boldsymbol{u}_h\|_K^2 + \sum_E \frac{\tau_p}{h} \|[\![\boldsymbol{u}_h]\!]\|_E^2,$$

and analogously for the pressure

$$\sum_K \tau_u \|\nabla p_h\|_K^2 + \sum_E \frac{\tau_u}{h} \|[\![p_h]\!]\|_E^2.$$

In order to get stability in these norms, to account for non-conforming approximations and, at the same time, to incorporate non-homogeneous velocity boundary conditions $\boldsymbol{n} \cdot \boldsymbol{u} = \psi$ on $\partial\Omega$, we modify $B_{sc}$ to $B_s$ and $L_{sc}$ to $L_s$, defined respectively as

$$\begin{aligned} B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) =& B_d([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\ &+ \tau_p \sum_K \left\langle \mathcal{P}'_{h,p}(\nabla \cdot \boldsymbol{u}_h), \nabla \cdot \boldsymbol{v}_h \right\rangle_K \\ &+ \tau_u \sum_K \left\langle \mathcal{P}'_{h,u}(\sigma \boldsymbol{u}_h + \nabla p_h), -\sigma \boldsymbol{v}_h + \nabla q_h \right\rangle_K \\ &+ \frac{\tau_p}{h} \sum_E \langle [\![\boldsymbol{u}_h]\!], [\![\boldsymbol{v}_h]\!] \rangle_E \\ &+ \frac{\tau_u}{h} \sum_E \langle [\![p_h]\!], [\![q_h]\!] \rangle_{E_0}, \end{aligned} \tag{4.4a}$$

$$
\begin{aligned}
L_s([\boldsymbol{v}_h, q_h]) =& L_d([\boldsymbol{v}_h, q_h]) \\
&+ \tau_p \sum_K \left\langle \mathcal{P}'_{h,p}(g), \nabla \cdot \boldsymbol{v}_h \right\rangle_K \\
&+ \tau_u \sum_K \left\langle \mathcal{P}'_{h,u}(\boldsymbol{f}), -\sigma \boldsymbol{v}_h + \nabla q_h \right\rangle_K \\
&+ \frac{\tau_p}{h} \sum_E \langle \psi, [\![\boldsymbol{v}_h]\!] \rangle_{E_\partial}.
\end{aligned} \tag{4.4b}
$$

Here and below we have considered $\tau_u$ and $\tau_p$ constant for all the elements, in accordance with the assumption of quasi-uniformity of the family of finite element meshes.

It is easy to see that the last two terms in (4.4a) provide the desired control over the jumps. Furthermore, these terms are consistent, in the sense that they vanish when $[\boldsymbol{u}_h, p_h]$ is replaced by $[\boldsymbol{u}, p]$ (for sufficiently smooth $p$). Let us point out that the velocity boundary condition has already been enforced in a weak sense, *à la* Nitsche, with a penalty coefficient $\frac{\tau_p}{h}$ (see e.g. [28]). We refer to [14] for a different motivation of stabilizing jump terms based on the VMS decomposition.

We have ended up with a stabilized discrete problem for continuous and discontinuous FE approximations. The definition of $\boldsymbol{\tau}$ is an essential ingredient of any stabilization technique, and in particular of this one. We motivate an expression for these parameters in the next subsection.

REMARK 4.1. *For the Darcy problem, the pressure subscale cannot be neglected, since the Galerkin terms do not control the velocity in $H(\mathrm{div}, \Omega)$. At the continuous level, this stability comes from the surjectivity of the divergence operator from $H(\mathrm{div}, \Omega)$ onto $L^2(\Omega)$, which can be understood as an inf-sup condition. Therefore, both velocity and pressure stability rely on inf-sup conditions. The Stokes problem is very different, since only the pressure stability requires an inf-sup condition; the pressure subscale can be neglected because the $H^1(\Omega)$ velocity stability comes from Galerkin terms.*

**4.2. The length scale and $\boldsymbol{\tau}$.** In order to get an effective choice of $\boldsymbol{\tau}$, we apply the approach in [12] to the Darcy problem. Let us consider the one-dimensional case for simplicity: find $u$ and $p$ such that

$$
\begin{aligned}
\sigma u + p_{,x} &= f, \\
u_{,x} &= g,
\end{aligned}
$$

where the subscript $(\cdot)_{,x}$ denotes the spatial derivative. Let $U = [u, p]$ be the unknown of the problem and $F = [f, g]$ the force vector, and let $M$ be a positive definite matrix that defines a pointwise product in the space of admissible force vectors. Up to factors, the only diagonal matrix that defines a dimensionally correct inner product (all terms with the same dimensions) is:

$$
M = \left[ \begin{array}{cc} \frac{1}{\sigma} & 0 \\ 0 & \sigma \ell^2 \end{array} \right],
$$

where $\ell$ *has dimensions of length*. This matrix defines the pointwise norm $|F|_M^2 = F \cdot MF$. We will also make use of the norm $\|F\|_{K,M}^2 = \int_K |F|_M^2 \mathrm{d}\Omega$ restricted to an element $K$.

Since $U'$ is the part of the solution that cannot be captured by the FE space, we assume that its Fourier transform is dominated by wave numbers of order $h^{-1}\tilde{k}$, where $\tilde{k}$ is an order $\mathcal{O}(1)$ dimensionless quantity. Therefore, the Fourier transform of $\mathcal{P}'(\mathcal{L}U')$ inside an element

$K$ (neglecting boundary values) can be approximated by $\mathcal{S}(\tilde{k})\widehat{U}'$, where

$$\mathcal{S}(\tilde{k}) = \left[ \begin{array}{cc} \sigma & \frac{\mathrm{i}\tilde{\mathrm{k}}}{h} \\ \frac{\mathrm{i}\tilde{\mathrm{k}}}{h} & 0 \end{array} \right],$$

with $\mathrm{i} = \sqrt{-1}$. Using Plancherel's formula we easily get

$$\begin{aligned} \|\mathcal{P}'(\mathcal{L}U')\|^2{}_{K,M} &\approx \int |\mathcal{S}(\tilde{k})\widehat{U}'|^2_{K,M} \mathrm{d}\tilde{k} \\ &\leq \|\mathcal{S}(\tilde{k}_0)\|^2{}_{K,M} \|\widehat{U}'\|^2{}_{K,M^{-1}} \\ &\approx \|\mathcal{S}(\tilde{k}_0)\|^2{}_{K,M} \|U'\|^2{}_{K,M^{-1}}, \end{aligned}$$

where $\tilde{k}_0$ is a mean wave number whose existence is established by the mean value theorem and the symbol $\approx$ has been used because boundary terms have been disregarded.

We want our choice of $\boldsymbol{\tau}$ to be real, *diagonal* and spectrally similar to $\mathcal{S}(\tilde{k}_0)$. Let $\boldsymbol{\tau} = \mathrm{diag}(\tau_u, \tau_p)$. We require that

$$\mathrm{spec}(\bar{S}(\tilde{k}_0)^t M S(\tilde{k}_0)) \approx \mathrm{spec}((\boldsymbol{\tau}^{-1})^t M \boldsymbol{\tau}^{-1}),$$

where the spectrum is computed with respect to matrix $M^{-1}$. The two eigenvalues $\lambda_i$ (for $i = 1, 2$) of $\bar{S}(\tilde{k}_0)^t M S(\tilde{k}_0)$ that satisfy

$$\det(\bar{S}(\tilde{k}_0)^t M S(\tilde{k}_0) - \lambda_i M^{-1}) = 0$$

are

$$\lambda_1 = \frac{1}{2}\left(1 + \frac{2\tilde{k}^2\ell^2}{h^2} + \sqrt{1 + \frac{4\tilde{k}^2\ell^2}{h^2}}\right), \quad \lambda_2 = \frac{1}{2}\left(1 + \frac{2\tilde{k}^2\ell^2}{h^2} - \sqrt{1 + \frac{4\tilde{k}^2\ell^2}{h^2}}\right).$$

Both eigenvalues are strictly positive. Similarly, we get the eigenvalues of $(\boldsymbol{\tau}^{-1})^t M \boldsymbol{\tau}^{-1}$:

$$\lambda_1' = \frac{\tau_u^{-2}}{\sigma^2}, \quad \lambda_2' = \tau_p^{-2}\sigma^2\ell^4. \tag{4.5}$$

Therefore, we take the stabilization parameters as

$$\tau_u = \frac{1}{\sigma\sqrt{\lambda_1}}, \quad \tau_p = \frac{\sigma\ell^2}{\sqrt{\lambda_2}}.$$

The expression of $\boldsymbol{\tau}$ will depend on the length $\ell$. We have considered four different choices of $\ell$ that lead to numerical methods with interesting properties:

1. Method A: $\ell = h$, the element size. In this case, the scaling is mesh-dependent, and gives

$$\tau_u \sim \frac{1}{\sigma}, \quad \tau_p \sim \sigma h^2.$$

2. Method B: $\ell = L_0$, where $L_0$ is a characteristic length of the problem under consideration. This implies an *a priori* scaling of the continuous problem that leads to

$$\tau_u \sim \frac{h}{\sigma L_0}, \quad \tau_p \sim \sigma L_0 h.$$

3. Method C: $\ell = L_0^2/h$, again a mesh-dependent scaling. In this case, we get

$$\tau_u \sim \frac{h^2}{\sigma {L_0}^2}, \quad \tau_p \sim \sigma L_0^2.$$

4. Method D: Another choice that leads to a method with interesting properties is to take a different value of $\ell$ in $\tau_u$ and $\tau_p$. In particular, we can consider $\ell = L_0$ in $\tau_u$ and $\ell = L_0^2 h$ in $\tau_p$ , getting

$$\tau_u \sim \frac{h}{\sigma L_0}, \quad \tau_p \sim \sigma L_0^2.$$

The reasons for this choice will be clear later.

Let us write the stabilization parameters in a unified way that includes all these cases:

$$\tau_u = \frac{h^2}{\sigma \ell_u^2}, \quad \tau_p = \sigma \ell_p^2, \tag{4.6}$$

where $\ell_u$ and $\ell_p$ are parameters with dimension of length that allow us to write the expression of $\tau_u$ and $\tau_p$ of the previous four methods if we define them as

- Method A: $\ell_u = c_u h$ and $\ell_p = c_p h$.
- Method B: $\ell_u = c_u L_0^{1/2} h^{1/2}$ and $\ell_p = c_p L_0^{1/2} h^{1/2}$.
- Method C: $\ell_u = c_u L_0$ and $\ell_p = c_p L_0$.
- Method D: $\ell_u = c_u h$ and $\ell_p = c_u L_0$.

In these expressions, $c_u$ and $c_p$ are algorithmic dimensionless constants.

**4.3. The subgrid projection.** Two choices of the approximated subgrid projection $\mathcal{P}_h'$ will be considered (see [22] for a discussion about another subgrid projection based on the $H^1$-inner product). The first and simplest is to take $\mathcal{P}_h'$ as the identity operator when acting on the FE residual (see [20]). Assuming this, we end up with a stabilized method that we call *algebraic subgrid scale* (ASGS) method. Invoking the closed form of the subgrid scale in terms of the FE component, we get the following stabilized forms $B_s$ and $L_s$:

$$\begin{aligned} B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) =& B_d([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\ &+ \tau_p \sum_K \langle \nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h \rangle_K \\ &+ \tau_u \sum_K \langle \sigma \boldsymbol{u}_h + \nabla p_h, -\sigma \boldsymbol{v}_h + \nabla q_h \rangle_K \\ &+ \frac{\tau_p}{h} \sum_E \langle [\![\boldsymbol{u}_h]\!], [\![\boldsymbol{v}_h]\!] \rangle_E \\ &+ \frac{\tau_u}{h} \sum_E \langle [\![p_h]\!], [\![q_h]\!] \rangle_{E_0}, \end{aligned} \tag{4.7a}$$

$$\begin{aligned} L_s([\boldsymbol{v}_h, q_h]) =& L_d([\boldsymbol{v}_h, q_h]) \\ &+ \tau_p \sum_K \langle g, \nabla \cdot \boldsymbol{v}_h \rangle_K \\ &+ \tau_u \sum_K \langle \boldsymbol{f}, -\sigma \boldsymbol{v}_h + \nabla q_h \rangle_K \\ &+ \frac{\tau_p}{h} \sum_{E_\partial} \langle \psi, [\![\boldsymbol{v}_h]\!] \rangle_{E_\partial}, \end{aligned} \tag{4.7b}$$

To define the second subgrid projector, let us introduce some additional ingredients. Given a function $g$ such that $g|_K \in L^2(K)$ for any element $K \in \mathcal{T}_h$, the *broken* $L^2$-projection over a Hilbert space $X$, denoted by $\Pi_X(g)$, is defined as the solution of:

$$(\Pi_X(g), v) = \sum_K (g, v)_K\,, \qquad \forall v \in X.$$

We also define $\Pi_X^\perp(g) = g - \Pi_X(g) \in L^2(\Omega)$. Using this notation, we define the orthogonal projection $\mathcal{P}'_h([x, y]) := [\Pi_{V_h}^\perp(x), \Pi_{Q_h}^\perp(y)]$. This method is called as *orthogonal subgrid scales* method (see e.g. [10]). This choice is in concordance with the VMS decomposition, because the subgrid velocity component belongs to a subgrid space $V'$ that satisfies $V' \cap V_h = \{\mathbf{0}\}$. Let us note that the ASGS method does not necessarily satisfy this property for the Darcy problem. Again, writing the problem in terms of the FE component only, $B_s$ and $L_s$ for the OSS formulation read as follows:

$$\begin{aligned}
B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) =& B_d([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\
&+ \tau_p \sum_K \left\langle \Pi_{Q_h}^\perp(\nabla \cdot \boldsymbol{u}_h), \nabla \cdot \boldsymbol{v}_h \right\rangle_K \\
&+ \tau_u \sum_K \left\langle \Pi_{V_h}^\perp(\nabla p_h), \nabla q_h \right\rangle_K \\
&+ \frac{\tau_p}{h} \sum_E \langle [\![\boldsymbol{u}_h]\!], [\![\boldsymbol{v}_h]\!] \rangle_E \\
&+ \frac{\tau_u}{h} \sum_E \langle [\![p_h]\!], [\![q_h]\!] \rangle_E,
\end{aligned} \tag{4.8a}$$

$$L_s([\boldsymbol{v}_h, q_h]) = L_d([\boldsymbol{v}_h, q_h]) + \frac{\tau_p}{h} \sum_{E_\partial} \langle \psi, [\![\boldsymbol{v}_h]\!] \rangle_{E_\partial}. \tag{4.8b}$$

The set of stabilization parameters designed in the previous section can be applied to both the ASGS and the OSS methods. Therefore, we have ended up with a number of methods, depending on the choice of the lengths $\ell_u$ and $\ell_p$ and the subgrid projection. In the next section we analyze the stability and convergence properties in all these cases. Finally, let us remark that in case of using continuous FE approximations, we recover a stabilized conforming formulation with Nitsche's enforcement of the normal trace of the velocity on the boundary.

REMARK 4.2. *Whereas the ASGS is a consistent algorithm, the OSS method (4.8a)-(4.8b) introduces a consistency error that does not spoil the accuracy of the discrete solution. In any case, consistency can be recovered replacing (4.8b) by*

$$\begin{aligned}
L_s([\boldsymbol{v}_h, q_h]) =& L_d([\boldsymbol{v}_h, q_h]) \\
&+ \tau_p \sum_K \left\langle \Pi_{Q_h}^\perp(g), \nabla \cdot \boldsymbol{v}_h \right\rangle_K \\
&+ \tau_u \sum_K \left\langle \Pi_{V_h}^\perp(\boldsymbol{f}), \nabla q_h \right\rangle_K \\
&+ \frac{\tau_p}{h} \sum_{E_\partial} \langle \psi, [\![\boldsymbol{v}_h]\!] \rangle_{E_\partial}.
\end{aligned}$$

*In the next section, we analyze the non-consistent version of the OSS method; the following results apply to the consistent formulation simply considering the consistency error equal to zero.*

REMARK 4.3. *Given* $\boldsymbol{v}_h \in V_h$*, if* $\sigma\boldsymbol{v}_h \notin V_h$*,* $\Pi_{V_h}(\sigma\boldsymbol{v}_h) \neq \mathbf{0}$*. However, using the non-consistent approach, we can still neglect this term without spoiling the accuracy.*

REMARK 4.4. *Control over* $\sum_K \|\nabla\cdot\boldsymbol{u}_h\|_K$ *and* $\sum_K \|\nabla p_h\|_K$ *is obtained from the Galerkin terms when* $\nabla\cdot V_h \subset Q_h$ *and* $\nabla Q_h \subset V_h$*, respectively (abusing of notation). This is true for some dG velocity-pressure pairs. In those cases, the element interior stability terms vanish for the OSS method, leaving only the inherent Galerkin stability. For the ASGS method, these terms are still there, even though they are not needed. The OSS formulation introduces less dissipation to the system than the ASGS method; we refer to [9] for a discussion about this topic in another setting, when using conforming approximations.*

**5. Analysis of stabilized formulations for discontinuous approximations.** Let us introduce the mesh dependent norms

$$\begin{aligned}
|||[\boldsymbol{v}_h, q_h]|||_h^2 &= \sigma\|\boldsymbol{v}_h\|^2 + \sigma\ell_p^2 \sum_K \|\nabla\cdot\boldsymbol{v}_h\|_K^2 + \frac{\sigma\ell_p^2}{h}\sum_E \|[\![\boldsymbol{v}_h]\!]\|_E^2 \\
&\quad + \frac{h^2}{\sigma\ell_u^2}\sum_K \|\nabla q_h\|_K^2 + \frac{h}{\sigma\ell_u^2}\sum_E \|[\![q_h]\!]\|_E^2, \\
|||[\boldsymbol{v}_h, q_h]|||^2 &= |||[\boldsymbol{v}_h, q_h]|||_h^2 + \frac{1}{\sigma L_0^2}\|q_h\|^2.
\end{aligned} \tag{5.1}$$

These are the norms in which the numerical analysis will be performed for both the ASGS and the OSS methods.

We define the interpolation error function

$$E_I(h)^2 = \sigma\ell_p^2(h^{-2}\varepsilon_0^2(\boldsymbol{u}) + \varepsilon_1^2(\boldsymbol{u})) + \sigma\varepsilon_0^2(\boldsymbol{u}) + \frac{h^2}{\sigma\ell_u^2}(h^{-2}\varepsilon_0^2(p) + \varepsilon_1^2(p)). \tag{5.2}$$

where, given a function $g$, $\varepsilon_i(g) = \|g - \tilde{g}_h\|_{H^i(\Omega)}$ and $\tilde{g}_h$ is an optimal FE interpolant of $g$. It will be proved that this is precisely the error function in the previous norm of the formulations introduced.

For the OSS method, we have to introduce the consistency error function

$$E_C(h)^2 = \sigma\ell_p^2\|\Pi_{Q_h}^{\perp}(\nabla\cdot\boldsymbol{u})\|^2 + \frac{h^2}{\sigma\ell_u^2}\|\Pi_{V_h}^{\perp}(\nabla p)\|^2.$$

Let us recall that we will consider for the sake of conciseness quasi-uniform FE partitions (for the analysis of a stabilized formulation in the more general non-degenerate case, see [11]). Therefore, we assume that there is a constant $C_{\rm inv}$, independent of the mesh size $h$ (the maximum of all the element diameters), such that

$$\|\nabla v_h\|_K \leq C_{\rm inv} h^{-1}\|v_h\|_K, \quad \|\Delta v_h\|_K \leq C_{\rm inv} h^{-1}\|\nabla v_h\|_K,$$

for all FE functions $v_h$ defined on $K \in \mathcal{T}_h$. This inequality can be used for scalars, vectors or tensors. Similarly, the trace inequality

$$\|v\|_{\partial K}^2 \leq C_{\rm tr}\left(h^{-1}\|v\|_K^2 + h\|\nabla v\|_K^2\right) \tag{5.3}$$

is assumed to hold for functions $v \in H^1(K)$, $K \in \mathcal{T}_h$. If $\psi_h$ is a piecewise (continuous or discontinuous) polynomial, the last term in the previous inequality can be dropped using an inverse inequality, getting $\|\psi_h\|_{\partial K}^2 \lesssim h^{-1}\|\psi_h\|_K^2$.

Using (5.3), for a given function $g$ we have that:

$$\sum_E \|[\![g - \tilde{g}_h]\!]\|_E^2 \lesssim \left(h^{-1}\varepsilon_0^2(g) + h\varepsilon_1^2(g)\right) \lesssim h^{2j-1}\|g\|^2_{H^j(\Omega)},\ j = 1,2. \tag{5.4}$$

Analogously, for a continuous function $g$ it holds:

$$\sum_E \|g - \tilde{g}_h\|_E^2 \lesssim \left(h^{-1}\varepsilon_0^2(g) + h\varepsilon_1^2(g)\right).$$

**5.1. Analysis of the OSS method.** In order to prove stability and convergence of the OSS method (4.1)-(4.8), we need the following preliminary result:

LEMMA 5.1 (Equivalence of norms). *Let $[\tilde{\boldsymbol{u}}_h, \tilde{p}_h]$ be an optimal interpolator of $[\boldsymbol{u}, p]$, the solution of the continuous problem (2.4)-(2.5). Let $[\boldsymbol{u}_h, p_h]$ be the solution of the OSS stabilized FE problem (4.1)-(4.8). Then, assuming that $k \geq 1$, the following inequalities are true*

$$\begin{aligned}
|\!|\!|[\boldsymbol{u}_h, p_h]|\!|\!| &\lesssim |\!|\!|[\boldsymbol{u}_h, p_h]|\!|\!|_h, \\
|\!|\!|[\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \tilde{p}_h - p_h]|\!|\!| &\lesssim |\!|\!|[\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \tilde{p}_h - p_h]|\!|\!|_h + E_I(h) + E_C(h).
\end{aligned}$$

**Proof**. From the inf-sup condition in the continuous case, for all $p \in L^2(\Omega)$ there exists a $\boldsymbol{v}_p \in H_0^1(\Omega)^d$ such that:

$$(p, \nabla\cdot\boldsymbol{v}_p) \gtrsim \frac{1}{\sqrt{\sigma}L_0}\|p\|\left(\sqrt{\sigma}\|\boldsymbol{v}_p\| + \sqrt{\sigma}L_0\|\nabla\boldsymbol{v}_p\|\right),$$

with $\|\boldsymbol{v}_p\|_1 = \frac{1}{\sigma L_0}\|p\|$, where we consider a dimensionally consistent norm $\|\boldsymbol{v}\|_1 := \|\boldsymbol{v}\| + L_0\|\nabla\boldsymbol{v}\|$. Then, for $p_h$ there exists $\boldsymbol{v}_p$ for which

$$\begin{aligned}
\frac{1}{\sigma L_0^2}\|p_h\|^2 &\lesssim (p_h, \nabla\cdot\boldsymbol{v}_p) \\
&= (p_h, \nabla\cdot\tilde{\boldsymbol{v}}_{p,h}) - \sum_K \langle\nabla p_h, \boldsymbol{v}_p - \tilde{\boldsymbol{v}}_{p,h}\rangle_K \\
&\quad + \sum_{E_0}\langle[\![p_h]\!], \{\boldsymbol{v}_p - \tilde{\boldsymbol{v}}_{p,h}\}\rangle_{E_0} + \sum_{E_0}\langle\{p_h\}, [\![\boldsymbol{v}_p - \tilde{\boldsymbol{v}}_{p,h}]\!]\rangle_{E_0},
\end{aligned} \tag{5.5}$$

where $\tilde{\boldsymbol{v}}_{p,h}$ is the Scott-Zhang interpolation[2] of $\boldsymbol{v}_p$ onto $V_h \cap H_0^1(\Omega)$. Therefore, $\tilde{\boldsymbol{v}}_{p,h} \in \mathcal{C}^0(\Omega)$, and $k \geq 1$ is required (where $k$ is the order of the velocity FE space). In any case, $k \geq 1$ is needed for proving convergence. We note that $[\![\boldsymbol{v}_p]\!] = 0$ and $[\![\tilde{\boldsymbol{v}}_{p,h}]\!] = 0$ on the set of edges $\mathcal{E}_h$. Using the interpolation property $\|\boldsymbol{v}_p - \tilde{\boldsymbol{v}}_{p,h}\| \lesssim \frac{h}{L_0}\|\boldsymbol{v}_p\|_1$ and the fact that $h \lesssim \ell_u \lesssim L_0$, we get for the second term in the right-hand side of (5.5):

$$-\sum_K \langle\nabla p_h, \boldsymbol{v}_p - \tilde{\boldsymbol{v}}_{p,h}\rangle_K \lesssim \sum_K \frac{h}{\sqrt{\sigma}\ell_u}\|\nabla p_h\|_K \frac{1}{\sqrt{\sigma}L_0}\|p_h\|.$$

[2] We explicitly consider this interpolation since the Scott-Zhang operator preserves homogeneous boundary conditions and it is a projection (see e.g. [18]). It allows us to use integration by parts without the introduction of terms on $\partial\Omega$.

Using the trace inequality (5.3) and the $H^1$-continuity of the Scott-Zhang projector, we obtain for the edge terms:

$$\sum_{E_0} \langle [\![ p_h ]\!], \{\boldsymbol{v}_p - \tilde{\boldsymbol{v}}_{p,h}\}\rangle_{E_0} \lesssim \sum_{E_0} \frac{h^{1/2}}{\sqrt{\sigma}\ell_u} \| [\![ p_h ]\!] \|_{E_0} \frac{1}{\sqrt{\sigma} L_0} \|p_h\|,$$
$$\sum_{E_0} \langle \{p_h\}, [\![ \boldsymbol{v}_p - \tilde{\boldsymbol{v}}_{p,h} ]\!]\rangle_{E_0} = 0.$$

Finally, testing (4.8a) with $[\boldsymbol{v}_h, q_h] = [\tilde{\boldsymbol{v}}_{p,h}, 0]$ and using the fact that $h \lesssim \ell_p \lesssim L_0$ and $\|v_h\| \leq \|v_h\|_1$, we get:

$$\begin{aligned}(p_h, \nabla \cdot \tilde{\boldsymbol{v}}_{p,h}) &= \sigma\left(\boldsymbol{u}_h, \tilde{\boldsymbol{v}}_{p,h}\right) + \sigma \ell_p^2 \left(\Pi_{Q_h}^{\perp}(\nabla \cdot \boldsymbol{u}_h), \nabla \cdot \tilde{\boldsymbol{v}}_{p,h}\right) \\ &\lesssim \left(\sqrt{\sigma}\|\boldsymbol{u}_h\| + \sqrt{\sigma}\ell_p \|\Pi_{Q_h}^{\perp}(\nabla \cdot \boldsymbol{u}_h)\|\right) \frac{1}{\sqrt{\sigma} L_0} \|p_h\|.\end{aligned}$$

With these ingredients, we prove the first part of the lemma. For the second part, the only difference is the control over the last term. Taking $\boldsymbol{v}_p$ such that $\|\boldsymbol{v}_p\|_1 = \frac{1}{\sigma L_0}\|\tilde{p}_h - p_h\|$, we proceed as above, the only difference being the treatment of the last term:

$$\begin{aligned}(\tilde{p}_h - p_h, \nabla \cdot \tilde{\boldsymbol{v}}_{p,h}) =& \sigma\left(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \tilde{\boldsymbol{v}}_{p,h}\right) + \sigma \ell_p^2 \left(\Pi_{Q_h}^{\perp}(\nabla \cdot (\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h)), \nabla \cdot \tilde{\boldsymbol{v}}_{p,h}\right) \\ &+ \sigma\left(\boldsymbol{u} - \tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_{p,h}\right) + \sigma \ell_p^2 \left(\Pi_{Q_h}^{\perp}(\nabla \cdot (\boldsymbol{u} - \tilde{\boldsymbol{u}}_h)), \nabla \cdot \tilde{\boldsymbol{v}}_{p,h}\right) \\ &- \sigma \ell_p^2 \left(\Pi_{Q_h}^{\perp}(\nabla \cdot \boldsymbol{u}), \nabla \cdot \tilde{\boldsymbol{v}}_{p,h}\right) - (p - \tilde{p}_h, \nabla \cdot \tilde{\boldsymbol{v}}_{p,h}) \\ \lesssim& \left( |\!|\!| [\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \tilde{p}_h - p_h] |\!|\!|_h + E_I(h) + E_C(h)\right) \frac{1}{\sqrt{\sigma} L_0} \|\tilde{p}_h - p_h\|.\end{aligned}$$

This proves the lemma. □

In the next theorem, we prove the stability properties of the OSS method in the working norms defined above. The OSS technique leads to a stabilized method that satisfies a discrete inf-sup condition and gives control over the velocity and pressure approximations in appropriate norms. The proof is constructive in the sense that we build a test function that implies the inf-sup condition.

THEOREM 5.2 (Stability). *Let $[\boldsymbol{u}_h, p_h]$ be the solution of the OSS stabilized FE problem (4.1)-(4.8) with a choice of the length scales that satisfies $\ell_p \lesssim \ell_u$. Then, the bilinear form $B_s$ satisfies a discrete inf-sup condition*

$$\inf_{[\boldsymbol{u}_h, p_h] \in V_h \times Q_h} \sup_{[\boldsymbol{v}_h, q_h] \in V_h \times Q_h} \frac{B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h])}{|\!|\!| [\boldsymbol{u}_h, p_h] |\!|\!|_h |\!|\!| [\boldsymbol{v}_h, q_h] |\!|\!|_h} \geq \beta.$$

*In particular, for $k \geq 1$*

$$B_s([\boldsymbol{u}_h, p_h], \Lambda([\boldsymbol{u}_h, p_h])) \gtrsim |\!|\!| [\boldsymbol{u}_h, p_h] |\!|\!|^2,$$

*with*

$$\Lambda([\boldsymbol{u}_h, p_h]) = \left[\boldsymbol{u}_h + \alpha \frac{h^2}{\sigma \ell_u^2} \Pi_{V_h}(\nabla p_h), p_h + \beta \sigma \ell_p^2 \Pi_{Q_h}(\nabla \cdot \boldsymbol{u}_h)\right],$$

*for $\alpha,\ \beta$ small enough constants that depend on $C_{\text{inv}}$ and $C_{\text{tr}}$.*

**Proof**. Stability is proved in three steps. First, taking $\boldsymbol{v}_h = \boldsymbol{u}_h$ and $q_h = p_h$ we obtain

$$\begin{aligned}B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{u}_h, p_h]) =& \sigma \|\boldsymbol{u}_h\|^2 + \sigma \ell_p^2 \|\Pi_{Q_h}^{\perp}(\nabla \cdot \boldsymbol{u}_h)\|^2 + \frac{h^2}{\sigma \ell_u^2} \|\Pi_{V_h}^{\perp}(\nabla p_h)\|^2 \\ &+ \frac{\sigma \ell_p^2}{h} \sum_{E} \| [\![ \boldsymbol{u}_h ]\!] \|_E^2 + \frac{h}{\sigma \ell_u^2} \sum_{E_0} \| [\![ p_h ]\!] \|_{E_0}^2 =: |\!|\!| [\boldsymbol{u}_h, p_h] |\!|\!|_*^2.\end{aligned}$$

Now, taking $[\boldsymbol{v}_h, q_h] = [\mathbf{0}, \sigma\ell_p^2\Pi_{Q_h}(\nabla\cdot\boldsymbol{u}_h)]$ we get

$$\begin{aligned}
B_s([\boldsymbol{u}_h, p_h], [0, \ell_p^2\Pi_{Q_h}(\nabla\cdot\boldsymbol{u}_h)]) \geq & \sigma\ell_p^2\|\Pi_{Q_h}(\nabla\cdot\boldsymbol{u}_h)\|^2 \\
& - c\frac{h}{\sqrt{\sigma}\ell_u}\|\Pi_{V_h}^{\perp}(\nabla p_h)\|\sqrt{\sigma}\ell_p\|\Pi_{Q_h}(\nabla\cdot\boldsymbol{u}_h)\| \\
& - c\frac{h^{1/2}}{\sqrt{\sigma}\ell_u}\sum_{E_0}\|[\![p_h]\!]\|_{E_0}\sqrt{\sigma}\ell_p\|\Pi_{Q_h}(\nabla\cdot\boldsymbol{u}_h)\| \\
& - c\frac{\sqrt{\sigma}\ell_p}{h^{1/2}}\sum_{E}\|[\![\boldsymbol{u}_h]\!]\|_{E}\sqrt{\sigma}\ell_p\|\Pi_{Q_h}(\nabla\cdot\boldsymbol{u}_h)\| \\
\geq & \frac{\sigma\ell_p^2}{2}\|\Pi_{Q_h}(\nabla\cdot\boldsymbol{u}_h)\|^2 - \frac{1}{4\alpha}|\!|\!|[\boldsymbol{u}_h, p_h]|\!|\!|_*^2,
\end{aligned}$$

for an appropriate constant $\alpha$, where we have used the assumption $\ell_p \lesssim \ell_u$. Now, let us consider the gradient form of the stabilized momentum equation, which is obtained by using

$$-\sum_K\langle p_h, \nabla\cdot\boldsymbol{v}_h\rangle_K + \sum_E\langle\{p_h\}, [\![\boldsymbol{v}_h]\!]\rangle_E = \sum_K\langle\nabla p_h, \boldsymbol{v}_h\rangle_K - \sum_{E_0}\langle[\![p_h]\!], \{\boldsymbol{v}_h\}\rangle_{E_0},$$

and take $[\boldsymbol{v}_h, q_h] = [\sigma\frac{h^2}{\sigma\ell_u^2}\Pi_{V_h}(\nabla p_h), 0]$. After some manipulation we get

$$\begin{aligned}
B_s([\boldsymbol{u}_h, p_h], [\frac{h^2}{\ell_u^2}\Pi_{V_h}(\nabla p_h), 0]) \geq & \frac{h^2}{\ell_u^2}\|\Pi_{V_h}(\nabla p_h)\|^2 \\
& - c\sqrt{\sigma}\|\boldsymbol{u}_h\|\frac{h}{\sqrt{\sigma}\ell_u}\|\Pi_{V_h}(\nabla p_h)\| \\
& - c\ell_p\|\Pi_{Q_h}^{\perp}(\nabla\cdot\boldsymbol{u}_h)\|\frac{h}{\ell_u}\|\Pi_{V_h}(\nabla p_h)\| \\
& - c\frac{h^{1/2}}{\sqrt{\sigma}\ell_u}\sum_{E_0}\|[\![p_h]\!]\|_{E_0}\frac{h}{\sqrt{\sigma}\ell_u}\|\Pi_{V_h}(\nabla p_h)\| \\
& - c\sqrt{\sigma}h^{-1/2}\sum_{E}\ell_p|[\![\boldsymbol{u}_h]\!]\|_{E}\frac{h}{\sqrt{\sigma}\ell_u}\|\Pi_{V_h}(\nabla p_h)\| \\
\geq & \frac{h^2}{2\ell_u^2}\|\Pi_{V_h}(\nabla p_h)\|^2 - \frac{1}{4\beta}|\!|\!|[\boldsymbol{u}_h, p_h]|\!|\!|_*^2
\end{aligned}$$

for an appropriate constant $\beta$, where we have used the fact that $h \lesssim \ell_u$. Combining all these results we get

$$B_s([\boldsymbol{u}_h, p_h], \Lambda([\boldsymbol{u}_h, p_h])) \geq 2|\!|\!|[\boldsymbol{u}_h, p_h]|\!|\!|_h^2. \tag{5.6}$$

In order to prove the theorem, we need the continuity of $\Lambda$, that is to say, $|\!|\!|\Lambda([\boldsymbol{u}_h, p_h])|\!|\!| \lesssim |\!|\!|[\boldsymbol{u}_h, p_h]|\!|\!|$. It is easily seen that

$$
\begin{aligned}
\|\Lambda([\boldsymbol{u}_h,p_h])\|^2 \lesssim\ & |\!|\!|[\boldsymbol{u}_h,p_h]|\!|\!|^2 + \frac{h^4}{\sigma\ell_u^4}\|\Pi_{V_h}(\nabla p_h)\|^2 \\
& + \frac{h^4\ell_p^2}{\sigma\ell_u^4}\sum_K \|\nabla\cdot\Pi_{V_h}(\nabla p_h)\|_K^2 + \frac{h^3\ell_p^2}{\sigma\ell_u^4}\sum_E \|[\![\Pi_{V_h}(\nabla p_h)]\!]\|_E^2 \\
& + \frac{\sigma\ell_p^4}{L_0^2}\|\Pi_{Q_h}(\nabla\cdot\boldsymbol{u}_h)\|^2 + \frac{\sigma\ell_p^4 h^2}{\ell_u^2}\sum_K \|\nabla\cdot\Pi_{Q_h}(\nabla\cdot\boldsymbol{u}_h)\|_K^2 \\
& + \frac{\sigma\ell_p^4 h}{\ell_u^2}\sum_E \|[\![\Pi_{Q_h}(\nabla\cdot\boldsymbol{u}_h)]\!]\|_E^2 \\
\lesssim\ & |\!|\!|[\boldsymbol{u}_h,p_h]|\!|\!|^2 + \frac{1}{\sigma L_0^2}\|q_h\|^2,
\end{aligned}
$$

where we have used inverse inequalities, trace inequalities, and the relations

$$
h \lesssim \ell_p \lesssim \ell_u \lesssim L_0.
$$

Analogously, we get $|\!|\!|\Lambda([\boldsymbol{u}_h,p_h])|\!|\!|_h \lesssim |\!|\!|[\boldsymbol{u}_h,p_h]|\!|\!|_h$. All these results are not only true for $[\boldsymbol{u}_h,p_h]$ but for any FE function in $V_h\times Q_h$. From (5.6) and using the continuity of $\Lambda(\cdot)$ for the norm $|\!|\!|\cdot|\!|\!|_h$ we get the inf-sup condition. Using the previous lemma and (5.6) we prove the second part of the theorem.□

From this theorem we conclude that the OSS technique leads to a stable method in the working norms (5.1). In order to prove the accuracy of the algorithm, we split the numerical error into two contributions, the interpolation and the consistency error. Let us start bounding the former:

LEMMA 5.3 (Interpolation error). *Let $[\boldsymbol{u},p]$ be the solution of the continuous problem (2.4)-(2.5) and $[\tilde{\boldsymbol{u}}_h,\tilde{p}_h]$ an optimal interpolator in $V_h\times Q_h$. We also assume that the length scales in the stabilization parameters satisfy $\ell_u\lesssim\ell_p$. Then, the following interpolation error estimate holds:*

$$
B_s([\boldsymbol{u}-\tilde{\boldsymbol{u}}_h,p-\tilde{p}_h],[\boldsymbol{v}_h,q_h]) \le E_I(h)|\!|\!|[\boldsymbol{v}_h,q_h]|\!|\!|_h
$$

**Proof**. The symmetric terms can be easily bounded by using the Cauchy-Schwarz inequality. The rest of the terms can be bounded as follows:

$$
\begin{aligned}
&\sum_K\langle\nabla\cdot(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h),q_h\rangle_K - \sum_E\langle[\![\boldsymbol{u}-\tilde{\boldsymbol{u}}_h]\!],\{q_h\}\rangle_E \\
&\quad = -\sum_K\langle\boldsymbol{u}-\tilde{\boldsymbol{u}}_h,\nabla q_h\rangle_K + \sum_{E_0}\langle\{\boldsymbol{u}-\tilde{\boldsymbol{u}}_h\},[\![q_h]\!]\rangle_{E_0} \\
&\quad \lesssim \frac{\sqrt{\sigma}\ell_u}{h}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h\|\left(\frac{h}{\sqrt{\sigma}\ell_u}\sum_K\|\nabla q_h\|_K + \frac{h^{1/2}}{\sqrt{\sigma}\ell_u}\sum_E\|[\![q_h]\!]\|_E\right) \\
&\quad \lesssim \frac{\sqrt{\sigma}\ell_p}{h}\varepsilon_0(\boldsymbol{u})|\!|\!|[\boldsymbol{v}_h,q_h]|\!|\!|,
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_K \langle p-\tilde{p}_h, \nabla\cdot \boldsymbol{v}_h\rangle_K + \sum_E \langle \{p-\tilde{p}_h\}, [\![\boldsymbol{v}_h]\!]\rangle_E \\
&\quad\lesssim \frac{1}{\sqrt{\sigma}\ell_p}\sum_K \|p-\tilde{p}_h\|_K \left(\sqrt{\sigma}\ell_p\|\nabla\cdot\boldsymbol{v}_h\| + \sum_E \frac{\sqrt{\sigma}\ell_p}{h^{1/2}}\|[\![\boldsymbol{v}_h]\!]\|_E\right) \\
&\quad\lesssim \frac{1}{\sqrt{\sigma}\ell_u}\varepsilon_0(p)|\!|\!|[\boldsymbol{v}_h, q_h]|\!|\!|_h.
\end{aligned}
$$

Using the definition of $E_I(h)$ (5.2) we finish the proof of the lemma. □

With regard to the consistency error, we have obtained the following bound:

LEMMA 5.4 (Consistency error). *The following inequality holds:*

$$
B_s([\boldsymbol{u}-\boldsymbol{u}_h, p-p_h],[\boldsymbol{v}_h,q_h]) \leq E_C(h)|\!|\!|[\boldsymbol{v}_h,q_h]|\!|\!|_h \qquad \forall [\boldsymbol{v}_h,q_h]\in V_h\times Q_h.
$$

**Proof**. The consistency error is

$$
\begin{aligned}
&B_s([\boldsymbol{u}-\boldsymbol{u}_h, p-p_h],[\boldsymbol{v}_h,q_h]) \\
&= \sigma\ell_p^2\sum_K \left(\Pi_{Q_h}^{\perp}(\nabla\cdot\boldsymbol{u}),\nabla\cdot\boldsymbol{v}_h\right)_K + \frac{h^2}{\sigma\ell_u^2}\sum_K\left(\Pi_{V_h}^{\perp}(\nabla p),\nabla q_h\right)_K \\
&\leq \sqrt{\sigma}\ell_p\|\Pi_{Q_h}^{\perp}(\nabla\cdot\boldsymbol{u})\|\sqrt{\sigma}\ell_p\sum_K\|\nabla\cdot\boldsymbol{v}_h\|_K + \frac{h}{\sqrt{\sigma}\ell_u}\|\Pi_{V_h}^{\perp}(\nabla p)\|\frac{h}{\sqrt{\sigma}\ell_u}\sum_K\|\nabla q_h\|_K,
\end{aligned}
$$

from where the result easily follows. □

Using the stability properties in Theorem 5.2 and the bounds for the interpolation and consistency error in Lemmata 5.3-5.4, we can prove the following convergence result:

THEOREM 5.5 (Convergence). *Let $[\boldsymbol{u},p]$ be the solution of the continuous problem (2.4)-(2.5) and let $[\boldsymbol{u}_h,p_h]$ be the solution of the OSS stabilized FE problem (4.1)-(4.8). We also assume that the length scales in the stabilization parameters satisfy $\ell_u \eqsim \ell_p$ and $k\geq 1$. Then, the following error estimate holds:*

$$
|\!|\!|[\boldsymbol{u}-\boldsymbol{u}_h, p-p_h]|\!|\!| \lesssim (E_I(h)+E_C(h)).
$$

**Proof**. Let $[\tilde{\boldsymbol{u}}_h,\tilde{p}_h]$ be an optimal interpolator of $[\boldsymbol{u},p]$ in $V_h\times Q_h$. From the previous results it follows that

$$
\begin{aligned}
&|\!|\!|[\tilde{\boldsymbol{u}}_h-\boldsymbol{u}_h,\tilde{p}_h-p_h]|\!|\!|_h|\!|\!|[\boldsymbol{v}_h,q_h]|\!|\!|_h \lesssim B_s([\tilde{\boldsymbol{u}}_h-\boldsymbol{u},\tilde{p}_h-p_h],[\boldsymbol{v}_h,q_h]) \\
&\lesssim B_s([\tilde{\boldsymbol{u}}_h-\boldsymbol{u},\tilde{p}_h-p_h],[\boldsymbol{v}_h,q_h]) + B_s([\boldsymbol{u}-\boldsymbol{u}_h,p-p_h],[\boldsymbol{v}_h,q_h]) \\
&\lesssim (E_I(h)+E_C(h))\,|\!|\!|[\boldsymbol{v}_h,q_h]|\!|\!|_h,
\end{aligned}
$$

where $[\boldsymbol{v}_h,q_h]$ is chosen so that Theorem 5.2 holds. We conclude the proof using the second result in Lemma 5.1, the triangle inequality and the fact that $|\!|\!|[\boldsymbol{u}-\tilde{\boldsymbol{u}}_h,p-\tilde{p}_h|\!|\!| \lesssim E_I(h)$. □

REMARK 5.1. *For the OSS stabilization technique, $\ell_p\lesssim\ell_u$ is needed for stability and $\ell_u\lesssim\ell_p$ for convergence, so that we require $\ell_p\eqsim\ell_u$. Therefore, the choice of the stabilization parameters in Method D with the OSS stabilized system (4.1)-(4.8) is out of this analysis.*

**5.2. Analysis of the ASGS method.** The stability and convergence analysis for the ASGS method is similar to the one for the OSS formulation, but not identical. The main difference, as we will show below, is the different nature of the stability in every case. As in the previous section, let us start with the relation between the two working norms for the FE solution and interpolation error.

LEMMA 5.6 (Equivalence of norms). *Let $[\tilde{\boldsymbol{u}}_h, \tilde{p}_h]$ be an optimal interpolator of $[\boldsymbol{u}, p]$, the solution of the continuous problem (2.4)-(2.5). Let $[\boldsymbol{u}_h, p_h]$ be the solution of the ASGS stabilized FE problem (4.1)-(4.7). Then, assuming that $k \geq 1$, the following inequalities are true*

$$
\begin{aligned}
\|[\boldsymbol{u}_h, p_h]\| &\lesssim \|[\boldsymbol{u}_h, p_h]\|_h, \\
\|[\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \tilde{p}_h - p_h]\| &\lesssim \|[\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \tilde{p}_h - p_h]\|_h + E_I(h).
\end{aligned}
$$

**Proof**. The proof only differs from the one for the OSS method in obtaining bounds for the following terms:

$$
\begin{aligned}
&(p_h, \nabla \cdot \tilde{\boldsymbol{v}}_{p,h}) \\
&= \sigma\,(\boldsymbol{u}_h, \tilde{\boldsymbol{v}}_{p,h}) + \sigma \ell_p^2 \sum_K \langle \nabla \cdot \boldsymbol{u}_h, \nabla \cdot \tilde{\boldsymbol{v}}_{p,h} \rangle_K + \frac{h^2}{\sigma \ell_u^2} \sum_K \langle \sigma \boldsymbol{u}_h + \nabla p_h, -\sigma \tilde{\boldsymbol{v}}_{p,h} \rangle_K \\
&\lesssim \|[\boldsymbol{u}_h, p_h]\|_h \frac{1}{\sigma L_0} \|p_h\|,
\end{aligned}
$$

where we have used that $h \lesssim \ell_u$, and

$$
\begin{aligned}
(\tilde{p}_h - p_h, \nabla \cdot \tilde{\boldsymbol{v}}_{p,h}) \; &= \sigma\,(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \tilde{\boldsymbol{v}}_{p,h}) + \sigma \ell_p^2 \sum_K \langle \nabla \cdot (\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h), \nabla \cdot \tilde{\boldsymbol{v}}_{p,h} \rangle_K \\
&+ \frac{h^2}{\sigma \ell_u^2} \sum_K \langle \sigma(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h) + \nabla(\tilde{p}_h - p_h), -\sigma \tilde{\boldsymbol{v}}_{p,h} \rangle_K \\
&+ \sigma\,(\boldsymbol{u} - \tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_{p,h}) + \sigma \ell_p^2 \sum_K \langle \nabla \cdot (\boldsymbol{u} - \tilde{\boldsymbol{u}}_h), \nabla \cdot \tilde{\boldsymbol{v}}_{p,h} \rangle_K \\
&+ \frac{h^2}{\sigma \ell_u^2} \sum_K \langle \sigma(\boldsymbol{u} - \tilde{\boldsymbol{u}}_h) + \nabla(p - \tilde{p}_h), -\sigma \tilde{\boldsymbol{v}}_{p,h} \rangle_K \\
&\lesssim (\|[\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h, \tilde{p}_h - p_h]\|_h + E_I(h)) \frac{1}{\sigma L_0} \|\tilde{p}_h - p_h\|,
\end{aligned}
$$

from where the second part of the Theorem follows. □

In the next theorem, we prove the coercivity of $B_s$ for the ASGS stabilization.

THEOREM 5.7 (Stability). *Let $[\boldsymbol{u}_h, p_h]$ be the solution of the ASGS stabilized FE problem (4.1)-(4.7) with a choice of the length scales that satisfies $\ell_p \lesssim \ell_u$. Let us also assume that the algorithmic constant in the definition of $\ell_u$ is $c_u > 1$ and that $k \geq 1$. Then, the bilinear form $B_s$ satisfies the coercivity property*

$$
B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{u}_h, p_h]) \gtrsim \|[\boldsymbol{u}_h, p_h]\|^2.
$$

**Proof**. For the ASGS method, stability is simply proved taking $[\boldsymbol{v}_h, q_h] = [\boldsymbol{u}_h, p_h]$:

$$B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{u}_h, p_h]) = \left(1 - \frac{1}{c_u^2}\right) \sigma \|\boldsymbol{u}_h\|^2 + \sigma \ell_p^2 \sum_K \|\nabla \cdot \boldsymbol{u}_h\|_K^2 + \frac{h^2}{\sigma \ell_u^2} \sum_K \|\nabla p_h\|_K^2 + \frac{\sigma \ell_p^2}{h} \|[\![\boldsymbol{u}_h]\!]\|_{\mathcal{E}_h}^2 + \frac{h}{\sigma \ell_u^2} \|[\![p_h]\!]\|_{\mathcal{E}_h^0}^2.$$

The first term in the right-hand side of this equality is positive under the assumption that $c_u > 1$, that implies $h < \ell_u$. □

The previous theorem proves that the ASGS technique leads to a positive definite bilinear form, whereas the OSS technique leads to a bilinear form that satisfies a discrete inf-sup condition (see [13]), that is to say, $B_s$ is an indefinite bilinear form, as its continuous counterpart $B_c$. This is an essential difference between both stabilization techniques that makes the analysis of the OSS method slightly more involved. However, the lack of coercivity for the OSS approach is not a drawback at all; the stabilized problem in this case only introduces what is not controlled by the Galerkin terms and inherits the stability mechanism of the continuous problem. More precisely, this fact means that the OSS method introduces less numerical dissipation than the ASGS formulation, as it has been shown for some numerical tests in [9].

Another difference between the ASGS and the OSS methods is the fact that the first one is consistent whereas the second one can introduce a consistency error (see Remark 4.2). Therefore, the convergence analysis of the former is more straightforward because it only involves an interpolation error, for which we have the following bound:

LEMMA 5.8 (Interpolation error). *Let $[\boldsymbol{u}, p]$ be the solution of the continuous problem (2.4)-(2.5) and $[\tilde{\boldsymbol{u}}_h, \tilde{p}_h]$ an optimal interpolator in $V_h \times Q_h$. We also assume that the length scales in the stabilization parameters satisfy $\ell_u \lesssim \ell_p$. Then, the following interpolation error estimate holds:*

$$B_s([\boldsymbol{u} - \tilde{\boldsymbol{u}}_h, p - \tilde{p}_h], [\boldsymbol{v}_h, q_h]) \leq E_I(h) |\!|\!|[\boldsymbol{v}_h, q_h]|\!|\!|_h.$$

**Proof**. All the terms can be easily bounded by using the Cauchy-Schwarz inequality and the bounds proved in Lemma 5.3 for the OSS method.□

The convergence result for this algorithm is stated in the following theorem:

THEOREM 5.9 (Convergence). *Let $[\boldsymbol{u}, p]$ be the solution of the continuous problem (2.4)-(2.5) and $[\tilde{\boldsymbol{u}}_h, \tilde{p}_h]$ an optimal interpolator in $V_h \times Q_h$. Let $[\boldsymbol{u}_h, p_h]$ be the solution of the ASGS stabilized FE problem (4.1)-(4.7). We also assume that the length scales in the stabilization parameters satisfy $\ell_u \lesssim \ell_p$ and $k \geq 1$. Then, the following error estimate holds:*

$$|\!|\!|[\boldsymbol{u} - \boldsymbol{u}_h, p - p_h]|\!|\!| \lesssim E_I(h).$$

The proof is very similar to the one for Theorem 5.5 and has been omitted.

REMARK 5.2. *For the ASGS method , the assumption $\ell_p \lesssim \ell_u$ is not needed. Therefore, the previous result applies for Method D introduced earlier. Let us remark that $\ell_u \lesssim \ell_p$ is still needed for convergence. It does not allow us to take $\ell_u = c_u L_0$ and $\ell_p = c_p h$.*

In any case, both the ASGS and the OSS algorithms lead to the same orders of convergence. Another important aspect of this analysis is the effect of the stabilization parameters in the stability and convergence results. We will discuss this effect in Section 7.

**6. Duality arguments and improved convergence estimates.** In the previous section *a priori* error estimates have been obtained for both the ASGS and the OSS methods. For conforming FE approximations of the velocity, sharper error estimates in $L^2(\Omega)$ for

$$\boldsymbol{e}_u = \boldsymbol{u} - \boldsymbol{u}_h, \qquad e_p = p - p_h$$

have been obtained by the authors in [4] by using Aubin-Nitsche-type duality arguments. These results are obtained assuming that the adjoint problem

$$\begin{aligned} \sigma \boldsymbol{w} - \nabla \xi &= \sigma \boldsymbol{f} && \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{w} &= \frac{1}{\sigma L_0^2} g && \text{in } \Omega, \\ \boldsymbol{n} \cdot \boldsymbol{w} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

satisfies the elliptic regularity assumptions

$$\|\xi\|_2 \lesssim \frac{1}{L_0^2}\|g\| + \sigma \|\nabla \cdot \boldsymbol{f}\| \qquad \text{if } \boldsymbol{f} \in H(\mathrm{div}, \Omega), \tag{6.1}$$

$$\|\boldsymbol{w}\|_1 \lesssim \frac{1}{\sigma L_0^2}\|g\| \qquad \text{if } \boldsymbol{f} = \boldsymbol{0}, \tag{6.2}$$

together with the obvious general stability estimate

$$\|\boldsymbol{w}\| \leq \|\boldsymbol{f}\| \quad \text{if } g = 0. \tag{6.3}$$

It is known that (6.1)-(6.2) hold if $\Omega$ is convex and polyhedral or with twice differentiable boundary. The improved error estimate for the pressure is obtained in [4] taking $\boldsymbol{f} = \boldsymbol{0}$ and $g = e_p$. Therefore, since $e_p \in L^2(\Omega)$, the regularity assumptions (6.1)-(6.2) can be used. For the sharper velocity estimates we should take $\boldsymbol{f} = \boldsymbol{e}_u$ and $g = 0$. Since $\nabla \cdot \boldsymbol{e}_u$ does not belong to $L^2(\Omega)$ for velocity approximations that are not conforming in $H(\mathrm{div}, \Omega)$, (6.1) is meaningless and the classical Aubin-Nitsche-type duality arguments do not apply.

The error estimates obtained in Theorems 5.5-5.9 can be written as

$$\begin{aligned} &\sigma \|\boldsymbol{e}_u\|^2 + \sigma \ell_p^2 \sum_K \|\nabla \cdot \boldsymbol{e}_u\|_K^2 + \frac{\sigma \ell_p^2}{h} \sum_E \|[\![\boldsymbol{e}_u]\!]\|_E^2 \\ &+ \frac{1}{\sigma L_0^2}\|e_p\|^2 + \frac{h^2}{\sigma \ell_u^2} \sum_K \|\nabla e_p\|_K^2 + \frac{h}{\sigma \ell_u^2} \sum_E \|[\![e_p]\!]\|_E^2 \\ &\quad \lesssim \sigma \ell_p^2 h^{2k} \|\boldsymbol{u}\|_{k+1}^2 + \sigma h^{2k+2} \|\boldsymbol{u}\|_{k+1}^2 + \frac{1}{\sigma \ell_u^2} h^{2l+2} \|p\|_{l+1}^2. \end{aligned} \tag{6.4}$$

Using duality arguments for the OSS method, we get improved error estimates for the pressure in the next theorem.

THEOREM 6.1. *Assume the same conditions as in Theorem 5.5 and, moreover, assume (6.1)-(6.2) to hold. Furthermore, for $\ell_u = h$ and piecewise constant pressures ($l = 0$) we also require the constant $c_u$ in Section 4.2 to be large enough. Under these assumptions, there holds*

$$\|e_p\|^2 \lesssim \sigma^2 \ell_p^4 \|\nabla \cdot \boldsymbol{e}_u\|^2 + h^2 \sum_K \|\nabla e_p\|_K^2. \tag{6.5}$$

*When $V_h \subset \mathcal{C}^0(\Omega)$, we also have:*

$$\|\boldsymbol{e}_u\|^2 \lesssim \Big(h^2 + \frac{\ell_p^4}{L_0^2} + h^2\frac{\ell_p^4}{\ell_u^4}\Big)\|\nabla\cdot\boldsymbol{e}_u\|^2 + \frac{1}{\sigma^2}\Big(\frac{h^4}{\ell_u^4} + \frac{h^2}{L_0^2}\Big)\sum_K \|\nabla e_p\|_K^2, \qquad (6.6)$$

**Proof**. We have assumed that the order of the piecewise polynomial functions that span $V_h$ are of order greater or equal to one ($k \geq 1$), that is to say, piecewise constant velocity approximations cannot be used. Thanks to that, we can pick an optimal FE interpolant $\tilde{\boldsymbol{w}}_h$ of $\boldsymbol{w}$ such that $\tilde{\boldsymbol{w}}_h \in V_h \cap H^1(\Omega)^d$. Therefore, all the terms involving jumps of $\boldsymbol{w}$ and $\tilde{\boldsymbol{w}}_h$ cancel. At this point, the proof of the improved error estimate over the pressure follows the one for conforming FE approximations for the velocity, that can be found in [4].□

Let us use the same duality arguments for the ASGS method.

THEOREM 6.2. *Assume the same conditions as in Theorem 5.9 and, moreover, assume (6.1)-(6.2) to hold. Furthermore, for $\ell_u = h$ and piecewise constant pressures ($l = 0$) we also require the constant $c_u$ in Section 4.2 to be large enough. For $l > 1$, we simply require $c_u > 1$. Under these assumptions, (6.5) holds. When $V_h \subset \mathcal{C}^0(\Omega)$, (6.6) is also true.*

**Proof**. Again, we note that $\boldsymbol{w}$ can be approximated by a $\mathcal{C}^0$ FE interpolant that belongs to $V_h$. Therefore, the proof in [4] for continuous FE velocity spaces can be extended to dG approximations. □

**7. The right choice of $\ell_u$ and $\ell_p$.** In the previous section we have proved the error estimate (6.4) with respect to what could be called the energy norm of the stabilized methods. An improved bound (6.5) for $\|e_p\|$ has been obtained using duality arguments. This estimate is always true for methods B and C; when piecewise constant pressures are used together with methods A and D this result only holds for $c_u$ large enough. The sharper bound for $\|\boldsymbol{e}_u\|$ in (6.6) is only true for conforming approximations; it does not apply for dG velocity approximations. We have collected all these results in Table 7.1, where the convergence rate of the different error quantities is indicated for all the methods introduced above, in terms of $k$ and $l$. We have also marked the results that are not always true, and in which cases these bounds are false.

All these rates of convergence allow us to draw some recommendations about the method to use, depending on the order of the velocity-pressure approximation, that is to say, the pair $(k, l)$:

- $k < l$: This situation has limited interest since it is not used in flow in porous media applications and because of the fact that the velocity field cannot be approximated by piecewise constant velocities in our analysis. In any case, Method A should be the one to take in this case. This method becomes optimal for $k = l - 1$ with $l > 1$ since $k > 0$ has to be assumed. On the other hand, this is the natural method for the mixed Laplacian formulation.
- $k = l$: For equal velocity-pressure approximations Method B is the most accurate one. Furthermore, it is optimal for conforming FE approximations. When using Method D, the choice of $k = l$ is the best one. Anyways, this method is far from being optimal and is always worse than Method A. The nice property of Method D is the fact that it exhibits the same stability as the continuous problem for $\boldsymbol{f} \in L^2(\Omega)^d$ (see Remark 2.2).
- $k > l$: Method C is the one that performs best when using this fairly used choice. In fact, the method is optimal when $k = l + 1$ for any interpolation pair. It is important to remark that Method C is the only one that allows us to take $l = 0$ . As far as we know, this is the first stabilized formulation of the Darcy problem that allows to

| **Method** $(\ell_p, \ell_u) \eqsim$ | A $(h, h)$ | B $(L_0^{1/2} h^{1/2}, L_0^{1/2} h^{1/2})$ | C $(L_0, L_0)$ | D $(L_0, h)$ |
|---|---|---|---|---|
| $\|\boldsymbol{e}_u\|$ | $h^{k+1} + h^l$ | $h^{k+1/2} + h^{l+1/2}$ | $h^k + h^{l+1}$ | $h^k + h^l$ (†) |
| $\|\boldsymbol{e}_u\|$ (duality) | $h^{k+1} + h^l$ | $h^{k+1} + h^{l+1}$ ($\star$) | $h^k + h^{l+1}$ | $h^k + h^l$ (†) |
| $\|e_p\|$ | $h^{k+1} + h^l$ | $h^{k+1/2} + h^{l+1/2}$ | $h^k + h^{l+1}$ | $h^k + h^l$ (†) |
| $\|e_p\|$ (duality) | $h^{k+2} + h^{l+1}$ (‡) | $h^{k+1} + h^{l+1}$ | $h^k + h^{l+1}$ | $h^k + h^l$ (†) |
| $\|\nabla \cdot \boldsymbol{e}_u\|$ | $h^k + h^{l-1}$ | $h^k + h^l$ | $h^k + h^{l+1}$ | $h^k + h^l$ (†) |
| $\|\nabla e_p\|$ | $h^{k+1} + h^l$ | $h^k + h^l$ | $h^{k-1} + h^l$ | $h^k + h^l$ (†) |
| Optimal $(k, l)$ | $k + 1 = l$ | $k = l$ | $k = l + 1$ | $k = l$ |

TABLE 7.1

*Convergence rates according to the choice of the length scale in the stabilization parameters. When using piecewise constant pressures, the results marked with (‡) are only true for large enough $c_u$. The results marked with ($\star$) are only true for $V_h \subset \mathcal{C}^0(\Omega)$. The results marked with (†) only apply to the ASGS formulation.*

use piecewise constant pressures. Furthermore, this method has been proved to be optimal for the Stokes-Darcy problem in [4].

**8. Numerical testing.** In this section we carry out some numerical experiments in order to check the theoretical convergence rates proved in Sections 5 and 6. We have considered both the ASGS and the OSS techniques with all the possible choices of the stabilization parameters that have been analyzed previously. Let us denote the spaces of discontinuous piecewise linear functions as $P1^d$, continuous piecewise linear functions as $P1^c$ and piecewise constant (obviously discontinuous) functions as $P0^d$. This notation is used for both the velocity and the pressure interpolation. With regard to the FE approximations, we have considered four velocity-pressure pairs: $P1^c/P0^d$, $P1^c/P1^d$, $P1^d/P0^d$ and $P1^d/P1^d$. Numerical experiments for the $P1^c/P1^c$ pair have not been included for the sake of conciseness, but they can be found in [4] in the frame of the Stokes-Darcy system.

All test problems are defined in the domain $\Omega \equiv (0,1) \times (0,1)$. We have considered structured and regular meshes. The family of FE partitions used in the convergence analysis consist of 3200, 7200 and 12800 linear triangular elements.

The definition of the stabilization parameters in (4.6) include the algorithmic constants $c_u$ and $c_p$ and a characteristic length $L_0$. Let us consider $c_u = \gamma c_p$. We have used $c_p = 2$ and $L_0 = 0.1 \sqrt[d]{\mathrm{meas}(\Omega)}$ in all cases. Based on numerical experimentation, we have taken $\gamma = 1$ for methods A and B and $\gamma = 0.1$ for methods C and D.

In order to evaluate the error introduced by the numerical approximations, we have solved a test problems with analytical solution:

$$\boldsymbol{u} = \left(-2\pi \cos(2\pi x)\sin(2\pi y), -2\pi \sin(2\pi x)\cos(2\pi y)\right), \quad p = \sin(2\pi x)\sin(2\pi y),$$

that can be obtained with the appropriate choice of $\boldsymbol{f}$, $g$ and boundary conditions. This test has been used in [24]. The analytical solution is obtained for $\boldsymbol{f} = \boldsymbol{0}$. Let us remark that, due to the regularity of the solution, only the normal component of the velocity can be enforced on the boundary.

With all the experimental convergence rates obtained, we want to support the recommendations of the previous sections:

- $k < l$: The lower order pair that could be used is the $P1^d/P2^d$ (or its continuous counterpart); since this FE space is of limited interest, we do not consider this case in the numerical experiments.

| **Method** $\ell_p, \ell_u =$ | A $h, h$ | B $L_0^{1/2}h^{1/2}, L_0^{1/2}h^{1/2}$ | C $L_0, L_0$ | D $L_0, h$ |
|---|---|---|---|---|
| $\Vert \boldsymbol{e}_u \Vert$ | -0.09 (-) | 0.74 (1) | 1.84 (1) | -0.03 (-) |
| $\Vert e_p \Vert$ | 0.01 (1) | 0.94 (1) | 1.88 (1) | -0.01 (-) |
| $\Vert \nabla \cdot \boldsymbol{e}_u \Vert$ | -0.38 (-) | 0.48 (-) | 1.54 (1) | -0.03 (-) |
| $\Vert \nabla e_p \Vert$ | -0.98 (-) | -0.03 (-) | 0.54 (-) | -0.99 (-) |

TABLE 8.1

*Experimental convergence rates for the ASGS method according to the choice of the length scale in the stabilization parameters. The $P1^c/P0^d$ pair.*

| **Method** $\ell_p, \ell_u =$ | A $h, h$ | B $L_0^{1/2}h^{1/2}, L_0^{1/2}h^{1/2}$ | C $L_0, L_0$ | D $L_0, h$ |
|---|---|---|---|---|
| $\Vert \boldsymbol{e}_u \Vert$ | -0.03 (-) | 0.80 (0.5) | 1.86 (1) | 0.07 (-) |
| $\Vert e_p \Vert$ | -0.02 (1) | 0.84 (1) | 1.83 (1) | 0.01 (-) |
| $\Vert \nabla \cdot \boldsymbol{e}_u \Vert$ | -0.37 (-) | 0.48 (-) | 1.06 (1) | -0.12 (-) |
| $\Vert \nabla e_p \Vert$ | -0.99 (-) | -0.14 (-) | 0.83 (-) | -0.98 (-) |

TABLE 8.2

*Experimental convergence rates for the ASGS method according to the choice of the length scale in the stabilization parameters. The $P1^d/P0^d$ pair.*

- $k = l$: The numerical orders of convergence obtained for the $P1^c/P1^d$ case are collected in Table 8.3 for the ASGS method and in Table 8.6 for the OSS method. The theoretical order of convergence is indicated in parenthesis and (-) is used when no convergence is expected. It becomes clear from these results that Method B is the optimal one. Anyway, all the methods exhibit super-convergence. The results for the $P1^d/P1^d$ case are shown in Tables 8.4 and 8.8 for the ASGS and the OSS methods, respectively. Again, the superiority of Method B is clear; Method C still keeps super-convergence. Methods A and D have lost this superconvergence for the ASGS formulation but Method A keeps it for the OSS approach.
- $k = l-1$: The results for the $P1^c/P0^d$ interpolation are included in Table 8.1 for the ASGS method and in Table 8.5 for the OSS formulation. As expected, when using piecewise constant pressures, Methods A and D fail to converge. The superiority of Method C is even clearer than expected thanks to super-convergence. Method B only converges in $L^2$-norms, and always exhibits lower orders of convergence than Method C. For $P1^d/P0^d$, with discontinuous velocities, the orders of convergence can be found in Table 8.2 for ASGS method and Table 8.7 for the OSS approach. Again, Method C is clearly the method to use.

These results are a numerical evidence of the recommendations stated in the previous section.

**9. Conclusions.** In this article we have motivated a set of stabilized methods for the numerical approximation of the Darcy problem in mixed form. Two of these methods are particularly interesting in flow in porous media applications. One is optimal for equal order velocity-pressure approximation (called Method B) whereas the other one is particularly well suited when the order of the velocity FE space is one order higher than the pressure one. This method is denoted Method C and, as far as we know, is the first stabilized method that allows piecewise constant pressures.

Both continuous and discontinuous approximations have been considered and the stabil-

| **Method** $\ell_p, \ell_u =$ | A $h, h$ | B $L_0^{1/2}h^{1/2}, L_0^{1/2}h^{1/2}$ | C $L_0, L_0$ | D $L_0, h$ |
|---|---|---|---|---|
| $\|\boldsymbol{e}_u\|$ | 1.50 (1) | 1.86 (2) | 1.89 (1) | 1.69 (1) |
| $\|e_p\|$ | 2.05 (2) | 2.39 (2) | 1.67 (1) | 2.07 (1) |
| $\|\nabla \cdot \boldsymbol{e}_u\|$ | 1.32 (-) | 1.47 (1) | 1.53 (1) | 1.76 (1) |
| $\|\nabla e_p\|$ | 1.04 (1) | 0.99 (1) | 0.01 (-) | 1.04 (1) |

TABLE 8.3

*Experimental convergence rates for the ASGS method according to the choice of the length scale in the stabilization parameters. The $P1^c/P1^d$ pair.*

| **Method** $\ell_p, \ell_u =$ | A $h, h$ | B $L_0^{1/2}h^{1/2}, L_0^{1/2}h^{1/2}$ | C $L_0, L_0$ | D $L_0, h$ |
|---|---|---|---|---|
| $\|\boldsymbol{e}_u\|$ | 1.00 (1) | 1.94 (1.5) | 1.98 (1) | 1.00 (1) |
| $\|e_p\|$ | 1.99 (2) | 2.31 (2) | 1.59 (1) | 1.98 (1) |
| $\|\nabla \cdot \boldsymbol{e}_u\|$ | 0.58 (-) | 1.01 (1) | 1.21 (1) | 1.04 (1) |
| $\|\nabla e_p\|$ | 1.05 (1) | 0.98 (1) | 0.06 (-) | 1.06 (1) |

TABLE 8.4

*Experimental convergence rates for the ASGS method according to the choice of the length scale in the stabilization parameters. The $P1^d/P1^d$ pair.*

ity and convergence analyses have been performed in a general setting that include all the stabilized methods that have been designed. We have also used duality arguments to obtain improved error estimates in $L^2$-norms.

The theoretical analysis has allowed us to draw recommendations about the method to be used, depending on the order of approximation of velocities and pressures. These recommendations have been proved to be accurate using numerical experimentation.

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| **Method** $\ell_p, \ell_u =$ | A $h, h$ | B $L_0^{1/2}h^{1/2}, L_0^{1/2}h^{1/2}$ | C $L_0, L_0$ |
|---|---|---|---|
| $\|\boldsymbol{e}_u\|$ | -0.09 (-) | 0.75 (1) | 1.84 (1) |
| $\|e_p\|$ | 0.01 (1) | 0.95 (1) | 1.89 (1) |
| $\|\nabla \cdot \boldsymbol{e}_u\|$ | -0.38 (-) | 0.49 (-) | 1.54 (1) |
| $\|\nabla e_p\|$ | -0.98 (-) | -0.03 (-) | 0.54 (-) |

TABLE 8.5

*Experimental convergence rates for the OSS method according to the choice of the length scale in the stabilization parameters. The $P1^c/P0^d$ pair.*

| **Method** $\ell_p, \ell_u =$ | A $h, h$ | B $L_0^{1/2}h^{1/2}, L_0^{1/2}h^{1/2}$ | C $L_0, L_0$ |
|---|---|---|---|
| $\|\boldsymbol{e}_u\|$ | 1.78 (1) | 1.91 (2) | 1.77 (1) |
| $\|e_p\|$ | 1.96 (2) | 2.34 (2) | 1.69 (1) |
| $\|\nabla \cdot \boldsymbol{e}_u\|$ | 0.65 (-) | 1.44 (1) | 1.51 (1) |
| $\|\nabla e_p\|$ | 1.12 (1) | 0.99 (1) | 0.03 (-) |

TABLE 8.6

*Experimental convergence rates for the OSS method according to the choice of the length scale in the stabilization parameters. The $P1^c/P1^d$ pair.*


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| **Method** $\ell_p, \ell_u =$ | A $h, h$ | B $L_0^{1/2}h^{1/2}, L_0^{1/2}h^{1/2}$ | C $L_0, L_0$ |
|---|---|---|---|
| $\Vert \boldsymbol{e}_u \Vert$ | -0.02 (-) | 0.87 (1) | 1.89 (1) |
| $\Vert e_p \Vert$ | -0.04 (1) | 0.78 (1) | 1.85 (1) |
| $\Vert \nabla \cdot \boldsymbol{e}_u \Vert$ | -0.90 (-) | -0.01 (-) | 0.91 (1) |
| $\Vert \nabla e_p \Vert$ | -1.02 (-) | -0.20 (-) | 0.86 (-) |

TABLE 8.7

*Experimental convergence rates for the OSS method according to the choice of the length scale in the stabilization parameters. The $P1^d/P0^d$ pair.*

| **Method** $\ell_p, \ell_u =$ | A $h, h$ | B $L_0^{1/2}h^{1/2}, L_0^{1/2}h^{1/2}$ | C $L_0, L_0$ |
|---|---|---|---|
| $\Vert \boldsymbol{e}_u \Vert$ | 1.79 (1) | 2.00 (2) | 1.99 (1) |
| $\Vert e_p \Vert$ | 2.19 (2) | 2.33 (2) | 1.47 (1) |
| $\Vert \nabla \cdot \boldsymbol{e}_u \Vert$ | 0.09 (-) | 1.07 (1) | 1.06 (1) |
| $\Vert \nabla e_p \Vert$ | 1.62 (1) | 1.02 (1) | 0.03 (-) |

TABLE 8.8

*Experimental convergence rates for the OSS method according to the choice of the length scale in the stabilization parameters. The $P1^d/P1^d$ pair.*


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