

Defensive alliances in regular graphs and circulant graphs

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September 27, 2008

Abstract

In this paper we study defensive alliances in some regular graphs. We determine which subgraphs could a critical defensive alliance of a graph G induce, if G is 6-regular and the cardinality of the alliance is at most 8.

In particular, we study the case of circulant graphs, i.e. Cayley graphs on a cyclic group. The critical defensive alliances of a circulant graph of degree at most 6 are completely determined. For the general case, we give tight lower and upper bounds on the alliance number of a circulant graph with d generators.

Keywords: Alliance, induced subgraph.

1 Introduction

An alliance in a graph is a kind of *community*, in the sense that nodes in the alliance either protect each other from attacks of other nodes, in the case of defensive alliances, or are able to collaborate to attack other nodes, in the case of offensive alliances.

Alliances, which were introduced in [13], can be defined as follows. A *defensive alliance* is a set of vertices satisfying that each vertex has at least as many neighbors in the alliance (including itself) than neighbors not belonging to the alliance. A defensive alliance is *strong* if each vertex has more neighbors in the alliance than outside, and it is *critical* if it doesn't include other defensive alliances. An *offensive alliance* [6] is a set of vertices satisfying that each vertex in its boundary has at least as many neighbors in the alliance than neighbors not belonging to the alliance (including itself). Strong and critical offensive alliances are defined similarly to the strong and critical defensive ones.

An alliance is called *global* if it is also a dominating set. Global defensive alliance and global offensive alliances were first studied in [11] and [22], respectively.

Though the concept of alliance is relatively new, it is related with some other well known concepts and problems. Moreover, it has given rise to new concepts and problems that are worth to mention. In the context of complex networks, the definition of web community, as in [9], coincides with the definition of offensive alliance. Some works relate alliances with community detection and partitioning [9, 12]. Other related concepts are modules [17] and, in the context of distributed computing, coalition and monopolies [10, 16, 20]. From an algorithmic point of view, the clustering coefficient is defined in terms of small alliances in [4], and a study of algorithms

for global alliances is given in [27]. Some of the works related with alliances in the context of graph theory are [8, 25], where the concept of k -alliance is defined and studied, and [21, 24], in which the authors focus on the spectral properties of alliances. The questions about complexity and alliances are studied in [7].

In this paper we study defensive alliances in regular graphs. In a d -regular graph, a defensive alliance is a set of vertices that induces a subgraph with minimum degree at least $\lfloor \frac{d}{2} \rfloor$ and maximum degree at most d . We are interested in the following problem: which graphs can a critical defensive alliance induce?

The answer is known for degree $d \leq 5$. For 6-regular graphs, it turns out to be a difficult question. We study alliances in graphs of degree 6, and of given cardinality $k \leq 8$. Even in these restricted cases, there is not an easy description of such alliances. Because of the complexity of the problem, we also restrict the question to a family of very symmetric graphs, the well known circulant graphs.

Circulant digraphs were first defined as graphs whose adjacency matrix is a circulant matrix [5]. Circulant digraphs are, in fact, Cayley graphs on the cyclic group \mathbb{Z}_n . If the set of generators is closed under inversion, then the digraph is symmetric and it can be seen as a graph.

The regularity and the underlying algebraic structure of Cayley graphs and, particularly, circulant (di)graphs make them good candidates for interconnecting nodes of a network [14]. A problem that has been widely studied is the isomorphism of circulant graphs. The Ádám conjecture, proposed in 1967 in [1], gave rise to a large amount of literature. It is worth to cite the work [2] which gave a good view of the state of the question, about ten years ago. The problem has been recently closed in [19]. Other studied problems on circulant graphs are, among others, the automorphism groups of circulant graphs [18], the spanning trees [3], the arboricity [26], and extremal problems [15].

The paper is organized as follows. Basic definitions and properties are given in Section 2. Section 3 deals with alliances in regular graphs of small degrees. We give some results about alliances in circulant (undirected) graphs in Section 4. We finish with some conclusions and open problems.

2 Definition and basic properties

First, we introduce some notation and basic definitions. Given a graph $G = (V, E)$ we denote by n and m its order and size, respectively. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) := \{u \in V : u \sim v\}$, and the *closed neighborhood* of v is the set $N[v] := N(v) \cup \{v\}$. The degree of v is $d(v) := |N(v)|$. We denote by δ_G the minimum degree of G .

Given a non-empty set of vertices S , the neighborhood of v in S is $N_S(v) := \{u \in S : u \sim v\} = N(v) \cap S$. Denoting by \overline{S} the complement in V of S , we have $N(v) = N_S(v) \cup N_{\overline{S}}(v)$. We denote by $\langle S \rangle$ the subgraph of G induced by S .

2.1 Alliances

The following definitions are taken from [13].

Definition 2.1 (Defensive alliance) *A non-empty set $S \subseteq V$ is a defensive alliance of G if, for every $v \in S$,*

$$|N_S[v]| \geq |N_{\overline{S}}(v)|. \quad (1)$$

We say that the alliance is strong if, for every $v \in S$, the inequality is strict.

The inequality (1) is called the (defensive) boundary condition.

Definition 2.2 (Offensive alliance) A non-empty set $S \subseteq V$ is an offensive alliance of G if, for every $v \in \partial(S)$,

$$|N_S(v)| \geq |N_{\overline{S}}[v]|. \quad (2)$$

We say that the alliance is strong if, for every $v \in \partial(S)$, the inequality is strict.

The inequality (2) is called the (offensive) boundary condition.

An alliance (of any type) is said to be *global* if it is also a dominating set of the graph. (Recall that S is a *dominating set* if every vertex of G is in S or has a neighbor in S , that is, $N[S] = V$.) An alliance (of any type) is said to be *critical* if none of its proper subsets is an alliance (of the same type). A *dual (or powerful) alliance* is a set that is both a defensive and an offensive alliance.

In the remaining of the paper we will focus on defensive alliances. Notice that, the whole graph G is a defensive alliance in G . Moreover, if S is a critical (strong) defensive alliance in G , then $\langle S \rangle$ is connected.

2.2 Alliance numbers

From the definition of alliance, some problems naturally arise. The first studied problem is to find the minimum cardinality of a defensive alliance of given a graph G . The problem we are interested in is which subsets of V , or the induced subgraphs of G , are critical defensive alliances and, among them, which are the minimal ones.

For a graph G , we can consider the following classes.

- $\mathcal{A}(G)$, the class of critical defensive alliances.
- $\hat{\mathcal{A}}(G)$, the class of critical strong defensive alliances.

Associated with this classes, the following invariants are defined.

- The defensive alliance number of G , $a(G) := \min\{|S| : S \in \mathcal{A}(G)\}$.

The upper defensive alliance number of G , $A(G) := \max\{|S| : S \in \mathcal{A}(G)\}$.

- The strong defensive alliance number of G , $\hat{a}(G) := \min\{|S| : S \in \hat{\mathcal{A}}(G)\}$.

The upper strong defensive alliance number of G , $\hat{A}(G) := \max\{|S| : S \in \hat{\mathcal{A}}(G)\}$.

For the defensive alliance number of a graph, or alliance number from now, it is easy to find tight lower bounds in terms of the minimum degree of the graph, as well as tight upper bounds in terms of the order:

$$\left\lfloor \frac{\delta_G}{2} \right\rfloor + 1 \leq a(G) \leq \left\lceil \frac{n}{2} \right\rceil, \quad (3)$$

$$\left\lceil \frac{\delta_G}{2} \right\rceil + 1 \leq \hat{a}(G) \leq \left\lceil \frac{n}{2} \right\rceil + 1. \quad (4)$$

The alliance number of a graph G is also related with its girth $g(G)$, i.e., the length of the shortest cycle of the graph (if any): If $\delta_G \geq 4$ then

$$g(G) \leq a(G).$$

The classes of critical offensive alliances and critical strong offensive alliances, with their corresponding alliance numbers can be analogously defined. Also, we can define the classes and alliance numbers for global alliances of any type.

It is worth mentioning that the decision problems associated to the different variation of alliances are all NP-complete (see [7] and the references therein). Therefore, it makes sense to study both the properties of the different types of alliance numbers and the alliance number of restricted classes of graphs.

3 Defensive alliances in regular graphs

The alliance numbers of regular graphs are known only for small degrees [13, 23].

We denote by $g(G)$ the girth of G and by $lc(G)$ the maximum length of an induced cycle in G . If G is d -regular, then it is known that:

- $d = 1 \Rightarrow a(G) = A(G) = 1, \hat{a}(G) = \hat{A}(G) = 2;$
- $d = 2 \Rightarrow a(G) = A(G) = \hat{a}(G) = \hat{A}(G) = 2;$
- $d = 3 \Rightarrow a(G) = A(G) = 2, \hat{a}(G) = g(G), \text{ and } \hat{A}(G) = lc(G);$
- $d = 4 \Rightarrow a(G) = \hat{a}(G) = g(G), A(G) = \hat{A}(G) = lc(G); \text{ and}$
- $d = 5 \Rightarrow a(G) = g(G), A(G) = lc(G).$

If $G = (V, E)$ is a graph, we say that a vertex $v \in S \subset V$ is defended in S if and only if it satisfies the boundary condition with respect to S . Similarly, if v satisfies the strong boundary condition with respect to S we say that v is strongly defended in S . Let $G = (V, E)$ a graph, and $v \in S \subset V$. The following properties are direct consequences of the definition of alliance and strong alliance.

Property 3.1 *If $d(v) = 2k$, v is defended in S if and only if $d_S(v) \geq k$. Moreover, the strong boundary condition is equivalent to the boundary condition, i.e., v is defended in S if and only if it is strongly defended in S .*

Property 3.2 *If $d(v) = 2k + 1$, v is defended in S if and only if $d_S(v) \geq k$; v is strongly defended in S if and only if $d_S(v) \geq k + 1$.*

Property 3.3 *If G is d -regular, then S is an alliance in G if and only if S induces a subgraph of minimum degree $\delta_S \geq \lfloor \frac{d}{2} \rfloor$; S is a strong alliance in G if and only if it induces a subgraph of minimum degree $\delta_S \geq \lceil \frac{d}{2} \rceil$.*

In fact, the known results for regular graphs of degree $d \leq 5$ allow us to completely characterize critical alliances for these graphs:

- If G is 1-regular, the critical alliances are exactly the singletons.
- The strong critical alliances in a 1-regular or 2-regular graph and the critical alliances in a 2-regular or 3-regular graph are exactly the edges.
- The strong critical alliances in a 3-regular or 4-regular graph and the critical alliances in a 4-regular or 5-regular graph are exactly the induced cycles.

Given a d -regular graph, G , we are concerned with two basic problems: determine $a(G)$, $\hat{a}(G)$, $A(G)$, and $\hat{A}(G)$, and characterize critical alliances in G , i.e., if S is a critical alliance in G , which graphs could $\langle S \rangle$ be isomorphic to?

Unfortunately, there is no simple characterization of the alliances, respectively strong alliances, of d -regular graphs if $d > 5$, respectively $d > 4$. So, we will concentrate on alliances of given cardinality. For that purpose, we give the following definition.

Definition 3.4 (Induced alliances set) *The (k, d) -induced alliances set is the set of graphs H of order k , minimum degree $\delta_H \geq \lfloor \frac{d}{2} \rfloor$, and maximum degree $\Delta_H \leq d$, with no proper subgraph of minimum degree greater than $\lfloor \frac{d}{2} \rfloor$. We denote this set by $\mathcal{S}_{(k,d)}$.*

Similarly, the (k, d) -induced strong alliances set is the set of graphs H of order k , minimum degree $\delta_H \geq \lceil \frac{d}{2} \rceil$, and maximum degree $\Delta_H \leq d$, with no proper subgraph of minimum degree greater than $\lceil \frac{d}{2} \rceil$. We denote this set by $\hat{\mathcal{S}}_{(k,d)}$.

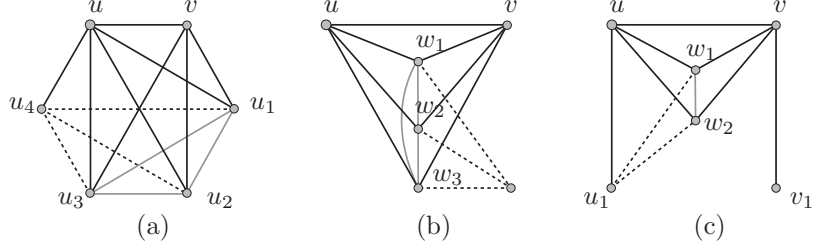


Figure 1: Proof of Lemma 3.7.

For instance, $\mathcal{S}_{(2,2)} = \mathcal{S}_{(2,3)} = \{K_2\}$, and $\mathcal{S}_{(k,2)} = \mathcal{S}_{(k,3)} = \emptyset$, if $k \geq 3$; $\mathcal{S}_{(5,4)} = \{C_5\}$, and $\mathcal{S}_{(k,4)} = \mathcal{S}_{(k,5)} = \{C_k\}$, if $k \geq 6$.

The following result is a consequence of the definitions of defensive alliance and (k, d) -induced alliances set, or (k, d) -ias for short.

Proposition 3.5 *If G is d -regular, then S is a critical alliance of G of cardinality k , if and only if $\langle S \rangle \in \mathcal{S}_{(k,d)}$.*

Proof. It follows straightforward from Property 3.3. ■

Notice that Proposition 3.5 says that alliances in regular graphs are defined by induced subgraphs of given minimum degree. The family of graphs that can be induced by a critical alliance can be described by its degree sequence.

Definition 3.6 (Admittable sequence) *A sequence $\mathbf{s} = (d_1, d_2, \dots, d_k)$ is a (k, d) -admittable sequence, or an admittable sequence, if there is a graph $G_{\mathbf{s}}$ in $\mathcal{S}_{(k,d)}$ with degree sequence \mathbf{s} .*

3.1 Defensive alliances in 6-regular graphs

In this section we pay attention to 6-regular graphs. Our study is based on determining all $(k, 6)$ -admittable sequences and then describing the corresponding $(k, 6)$ -induced alliance sets.

- If $|S| = 4$ then $\langle S \rangle = K_4$ and its associated degree sequence is $(3, 3, 3, 3)$. That is, $\mathcal{S}_{(4,6)} = \{K_4\}$.
- If $|S| = 5$ then $\langle S \rangle = W_4$ and its associated degree sequence is $(4, 3, 3, 3, 3)$. That is, $\mathcal{S}_{(5,6)} = \{W_4\}$.

Lemma 3.7 *If G is 6-regular and contains a critical alliance S of cardinality 6, then the associated degree sequence of $\langle S \rangle$ is one of the following:*

$$(3, 3, 3, 3, 3, 3), (4, 4, 3, 3, 3, 3), \text{ or } (5, 3, 3, 3, 3, 3)$$

Any other degree sequence with minimum degree 3 gives graphs containing K_4 or W_4 .

Proof. Notice that if S is a critical alliance of G of cardinality 6 then any vertex v in $\langle S \rangle$ satisfies $3 \leq d_S(v) \leq 5$. Moreover, there must be at least one vertex of degree 3 in $\langle S \rangle$.

First, we prove that if S is an alliance of cardinality 6 then $\langle S \rangle$ cannot have two vertices, u and v , with $d_S(u) = 5$ and $d_S(v) \geq 4$. For that purpose, assume $d(u) = 5$ and $d(v) \geq 4$. We can assume, w.l.o.g., that $N(u) = \{v, u_1, u_2, u_3, u_4\}$ and $\{u, u_1, u_2, u_3\} \subseteq N(v)$ (see Figure 1 (a)).

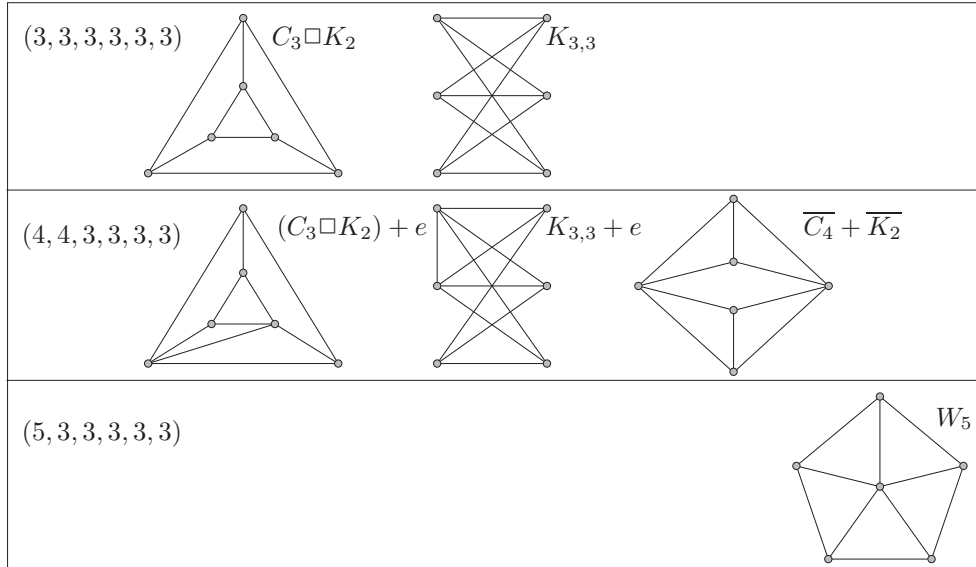


Figure 2: The $(6, 6)$ -induced alliances set, with their associated degree sequence.

Now, there is no edge between the vertices u_1, u_2 , and u_3 , that is, none of the grey edges in Figure 1 (a) is in $\langle S \rangle$, otherwise there is an induced K_4 . But every vertex has degree at least 3. So these three vertices must be all adjacent to u_4 , that is, the dotted edges in Figure 1 (a) must be in $\langle S \rangle$, and then there is at least one induced W_4 .

Notice that W_5 does not contain K_4 , neither W_4 as a subgraph. Its degree sequence is $(5, 3, 3, 3, 3, 3)$. So, this is the only admissible sequence with one vertex of degree 5.

Let us consider now degree sequences with only vertices of degree 3 and 4, i.e., the sequences $(3, 3, 3, 3, 3, 3)$, $(4, 4, 3, 3, 3, 3)$, $(4, 4, 4, 4, 3, 3)$, and $(4, 4, 4, 4, 4, 4)$. The graph $K_{3,3}$ has degree sequence $(3, 3, 3, 3, 3, 3)$ and contains no K_4 nor W_5 . The graph $K_{3,3} + e$ has degree sequence $(4, 4, 3, 3, 3, 3)$ and contains no K_4 nor W_5 . Thus, both sequences $(3, 3, 3, 3, 3, 3)$ and $(4, 4, 3, 3, 3, 3)$ are admissible.

We only need to show that any graph H of order 6 with at least four vertices of degree 4 contains either K_4 or W_4 . The graph H must contain two adjacent vertices of degree 4, say u and v . There are two possibilities: u and v have three common neighbors, w_1, w_2 and w_3 (see Figure 1 (b)), or u and v share only two neighbors, w_1 and w_2 (see Figure 1 (c)).

In the first case, there is no edge between the vertices w_1, w_2 and w_3 , that is, none of the grey edges in Figure 1 (b) is in $\langle S \rangle$, otherwise there is an induced K_4 . But, then, none of them can have degree 4, a contradiction. In the second case, assume that u_1 is adjacent to u but not to v , and v_1 adjacent to v but not to u . Now, w_1 and w_2 cannot be adjacent, that is, the grey edge in Figure 1 (c) cannot be in $\langle S \rangle$, otherwise, there is a K_4 , induced by $\{u, v, w_1, w_2\}$. Since there are at least four vertices of degree 4, at least one of the vertices u_1 or v_1 , say u_1 , is adjacent to w_1 and w_2 , that is, the dotted edges in Figure 1 (c) must be in $\langle S \rangle$. Then, $\{u, v, u_1, w_1, w_2\}$ induce a subgraph isomorphic to W_4 .

This completes the proof. ■

Notice that, by using Definition 3.6, this lemma can be reformulated as: the only $(6, 6)$ -admissible sequences are $(3, 3, 3, 3, 3, 3)$, $(4, 4, 3, 3, 3, 3)$ and $(5, 3, 3, 3, 3, 3)$.

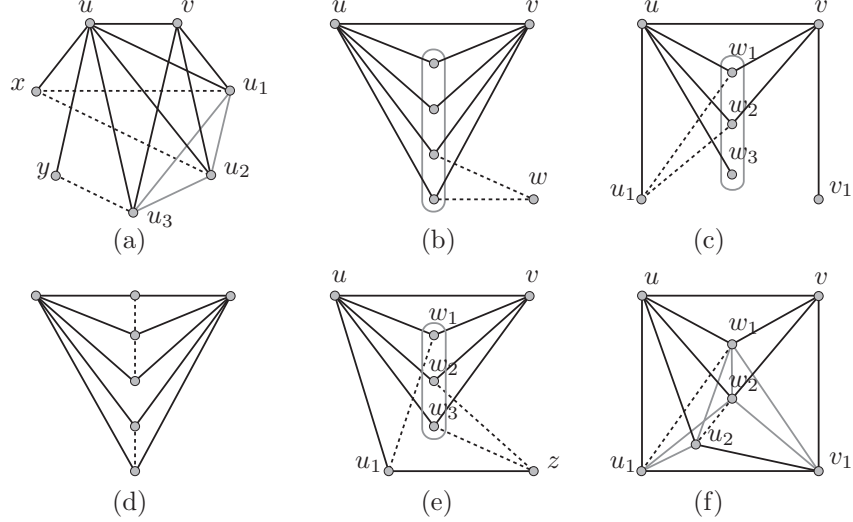


Figure 3: Proof of Lemma 3.9.

Proposition 3.8 *The (6, 6)-ias are:*

$$\mathcal{S}_{(6,6)} = \{C_3 \square K_2, K_{3,3}, (C_3 \square K_2) + e, K_{3,3} + e, \overline{C_4} + \overline{K_2}, W_5\}$$

This set contains exactly the six graphs in Figure 2.

Proof. Let H be a graph in $\mathcal{S}_{(6,6)}$. Its degree sequence is one of the sequences in Lemma 3.7, i.e., $(3, 3, 3, 3, 3, 3)$, $(4, 4, 3, 3, 3, 3)$, and $(5, 3, 3, 3, 3, 3)$.

If the degree sequence of H is $(5, 3, 3, 3, 3, 3)$, then $H \cong W_5$. If the degree sequence of H is $(3, 3, 3, 3, 3, 3)$, then we consider two cases: if H is triangle free, then $H \cong K_{3,3}$; otherwise H contains a triangle and then $H \cong C_3 \square K_2$.

Finally, if the degree sequence of H is $(4, 4, 3, 3, 3, 3)$, and $d(u) = d(v) = 4$, we consider the following two cases: if $u \sim v$ then the graph $H - e$, with $e = \{u, v\}$, has degree sequence $(3, 3, 3, 3, 3, 3)$ and this implies that either $H \cong K_{3,3} + e$, or $H \cong (C_3 \square K_2) + e$; otherwise $u \not\sim v$, then $H - \{u, v\} \cong \overline{C_4}$ and thus, $H \cong \overline{C_4} + \overline{K_2}$. ■

Lemma 3.9 *If G is 6-regular and contains a critical alliance S of cardinality 7, then the associated degree sequence of $\langle S \rangle$ is one of the following:*

$$(4, 3, 3, 3, 3, 3, 3), (4, 4, 4, 3, 3, 3, 3), (5, 4, 3, 3, 3, 3, 3), \text{ or } (6, 3, 3, 3, 3, 3, 3)$$

Any other degree sequence with minimum degree 3 gives graphs containing K_4 , W_4 , or some graph in $\mathcal{S}_{(6,6)}$.

Proof. Notice that if S is a critical alliance of G of cardinality 7 then any vertex v in $\langle S \rangle$ satisfies $3 \leq d_S(v) \leq 6$. Moreover, in $\langle S \rangle$ there must be at least two vertices of degree 3. Assume that there are only two vertices of degree 3, u and v . Then, if $u \sim v$, there is a vertex w adjacent only to vertices of degree greater than 3. By removing w , we obtain an induced subgraph of $\langle S \rangle$ with minimum degree at least 3. On the other hand, if u and v are not adjacent, then u is adjacent only to vertices of degree greater than 3. By removing u , we obtain an induced

subgraph of $\langle S \rangle$ with minimum degree at least 3. But this is a contradiction, because S is a critical alliance. Thus, $\langle S \rangle$ has at least three vertices of degree 3.

To find all the $(7, 6)$ -admittable sequences, we first prove that $\langle S \rangle$ cannot have two vertices, u and v , with $d_S(u) = 6$ and $d_S(v) \geq 4$. Assume that $d(u) = 6$ and $d(v) \geq 4$, and let u_1, u_2 , and u_3 be three common neighbors of u and v (see Figure 3 (a)). There are two more vertices x, y in $\langle S \rangle$, which must be adjacent to (at least) u . Since there is no induced K_4 , u_1, u_2 , and u_3 are independent. That is, none of the grey edges in Figure 3 (a) is in $\langle S \rangle$. Since the minimum degree of $\langle S \rangle$ is 3, each of the vertices u_1, u_2 , and u_3 has to be adjacent to one of the vertices x, y . We can assume, w.l.o.g., that $u_1 \sim x, u_2 \sim x$, and $u_3 \sim y$. That is, the dotted edges in Figure 3 (a) are in $\langle S \rangle$. Then, $\{u, u_1, v, u_2, x\}$ induce W_4 , a contradiction.

Since W_6 is clearly in $\mathcal{S}_{(6,7)}$, we have that the only admitted degree sequence for $\langle S \rangle$ with maximum degree 6 is $(6, 3, 3, 3, 3, 3)$.

Now we prove that, if $3 \leq d_S(v) \leq 5$, at most one vertex can have degree 5. Moreover, if there is one vertex u with degree 5, only one vertex v has degree 4.

- If there are two adjacent vertices of degree 5, say u and v , with four common neighbors, then there is one vertex, w , not adjacent to u nor v (see Figure 3 (b)). In this case, w has to be adjacent to at least 3 of the common neighbors of u and v , and then there is an induced $K_{3,3}$.
- If there are two adjacent vertices of degree 5, say u and v , with three common neighbors (which have to be pairwise independent) then there is one vertex, u_1 adjacent to u but not to v , and a vertex v_1 adjacent to v but not to u (see Figure 3 (c)). Since the minimum degree is 3, we can assume, w.l.o.g., that u_1 is adjacent to two of the common neighbors of u and v . That is, the dotted edges in Figure 3 (c) have to be in $\langle S \rangle$. But then, there is an induced W_4 , a contradiction.
- If there are two non adjacent vertices of degree 5, then there are at least three edges between their five common neighbors (see Figure 3 (d)). Two of these three edges must be incident and thus, $\langle S \rangle$ contains a W_4 .

Assume now that there is exactly one vertex u of degree 5. Assume also that there is more than one vertex of degree 4. Then, one of them is adjacent to u , say v .

- If u and v share three neighbors, we have: w_1, w_2 and w_3 the common neighbors of u and v , one vertex u_1 adjacent to u and not to v , and one vertex z not adjacent to u neither to v (see Figure 3 (e)). To avoid the existence of induced $K_{3,3}$, z can only be adjacent to two of the common neighbors. So, z is adjacent to u_1, w_2 and w_3 . Since w_1, w_2 and w_3 have to be independent, and the minimum degree in $\langle S \rangle$ is 3, w_1 is adjacent to u_1 . But now, we cannot add more edges, without introducing one of the forbidden induced subgraphs. So there are no more vertices of degree 4, a contradiction.
- If u and v share only two neighbors, we have: two vertices u_1 and u_2 , adjacent to u and not to v , one vertex v_1 , adjacent to v and not to u , and w_1 and w_2 the common neighbors to u and v (see Figure 3 (f)). Now, u_1 cannot be adjacent to both w_1 and w_2 , and the same is true for u_2 . We can assume, w.l.o.g., that $u_1 \sim w_1$ and $u_2 \sim w_2$. That is, the dotted edges in Figure 3 (f) are in $\langle S \rangle$. Since the minimum degree is 3, both u_1 and u_2 have to be adjacent to v_1 . We also have that v_1 cannot be adjacent to w_1 neither w_2 , because this would induce a W_4 . That is, none of the grey edges in Figure 3 (f) is in $\langle S \rangle$. This implies that the maximum degree of w_1, w_2 and v_1 is 3. Now, the only way to obtain two vertices of degree 4 is adding an edge between u_1 and u_2 . But then, there is an induced W_5 .

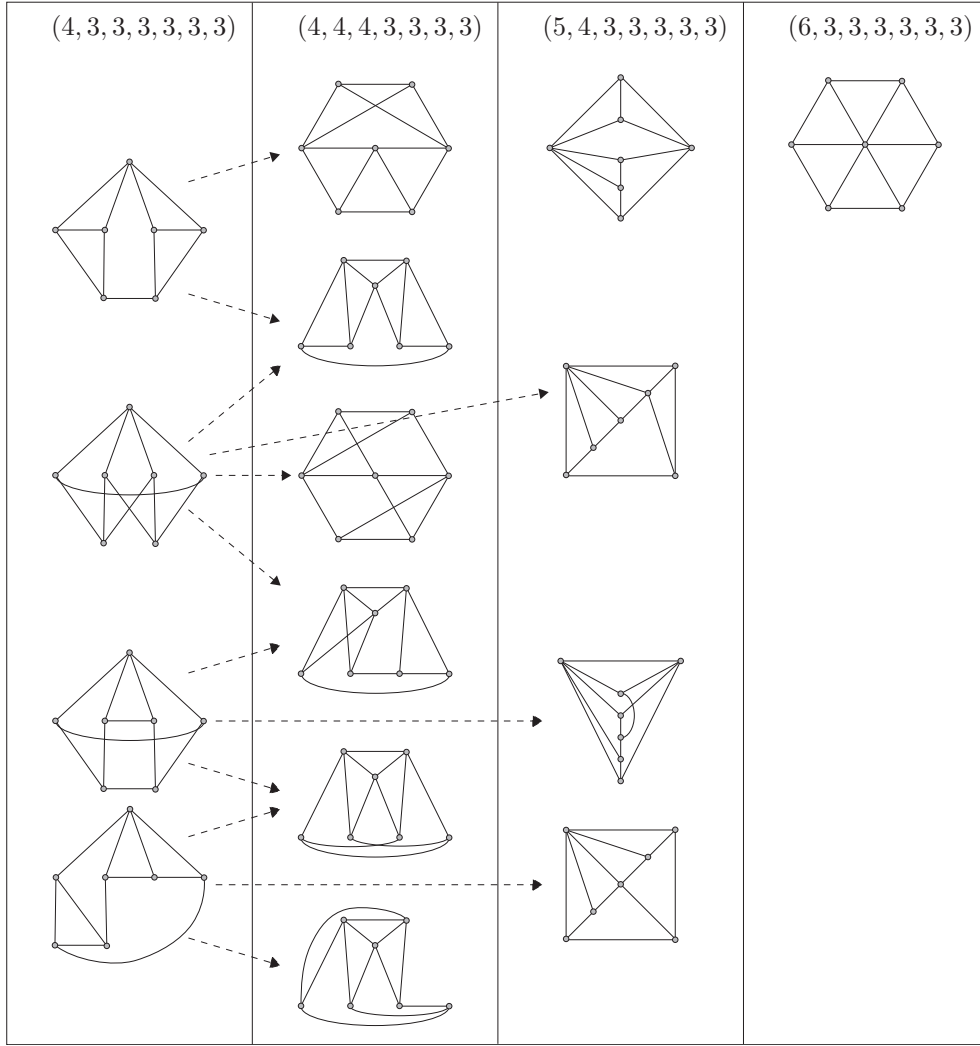


Figure 4: The $(7, 6)$ -induced alliances set, with their associated degree sequence. The arrows indicate the subgraph relation.

The graph G_1 obtained from $P_5 + \overline{K_2}$ by removing two edges linking the same vertex of $\overline{K_2}$ with any two internal vertices of P_5 has degree sequence $(5, 4, 3, 3, 3, 3, 3)$. Moreover, G_1 does not contain K_4 , nor W_4 , nor a graph of $\mathcal{S}_{(6,6)}$, as induced subgraphs.

Finally, if $3 \leq d_S(v) \leq 4$, since we have seen that there are at least three vertices of degree 3, the degree sequence is either $(4, 4, 4, 3, 3, 3, 3)$ or $(4, 3, 3, 3, 3, 3, 3)$. The graph G_2 obtained identifying an arbitrary pair of adjacent vertices of the cube Q_3 in a single vertex v , has degree sequence $(4, 3, 3, 3, 3, 3, 3)$. The graph G_3 obtained by adding one edge to G_2 , between two of the vertices adjacent to v , has degree sequence $(4, 4, 4, 3, 3, 3, 3)$. None of these graphs contains K_4 , nor W_4 , nor a graph of $\mathcal{S}_{(6,6)}$. ■

Proposition 3.10 *The set $\mathcal{S}_{(7,6)}$ of the $(7, 6)$ -ias contains exactly the 15 graphs in Figure 4.*

The proof of this proposition, which is omitted, is similar to but longer than that of Proposition 3.8. For every admissible sequence \mathbf{s} in Lemma 3.9 we can constructively find every graph in $\mathcal{S}_{(7,6)}$ with degree sequence \mathbf{s} . The obtained graphs are exactly the 15 graphs in Figure 4.

Corollary 3.11 *Let $G = (V, E)$ be a 6-regular graph.*

- $a(G) = 4 \Leftrightarrow K_4$ is an induced subgraph of G ;
- $a(G) = 5 \Leftrightarrow W_4$ is an induced subgraph of G and K_4 is not; and
- $a(G) = 6 \Leftrightarrow$ some graph in $\mathcal{S}_{(6,6)}$ is an induced subgraph of G , and neither K_4 nor W_4 are.
- $a(G) = 7 \Leftrightarrow$ some graph in $\mathcal{S}_{(7,6)}$ is an induced subgraph of G , and neither K_4 nor W_4 , nor any of the graphs in $\mathcal{S}_{(6,6)}$ are.

The number of $(m, 6)$ -admittable sequences and, consequently, the number of graphs in $\mathcal{S}_{(m,6)}$, increases with the cardinality, m . A similar but longer reasoning gives the set of degree sequences associated to $(8, 6)$ -induced alliances. In this case, the number of graphs is significantly larger. However, an exhaustive search allows us to give the following two claims.

Claim 3.12 *The $(8, 6)$ -admittable sequences are*

$$(6, 4, 3, 3, 3, 3, 3, 3), (5, 5, 4, 4, 3, 3, 3, 3), (5, 5, 3, 3, 3, 3, 3, 3), (5, 4, 4, 4, 4, 3, 3, 3), \\ (5, 4, 4, 3, 3, 3, 3, 3), (4, 4, 4, 4, 3, 3, 3, 3), (4, 4, 3, 3, 3, 3, 3, 3), \text{ and } (3, 3, 3, 3, 3, 3, 3, 3).$$

Claim 3.13 *The set $\mathcal{S}_{(8,6)}$ of the $(8, 6)$ -ias contains exactly the 65 graphs in Figure 5.*

3.2 Defensive alliances in 7-regular graphs

We can easily extend the results in the previous section to 7-regular graphs.

Indeed, we only need to notice that, if $m \leq 7$, then the $(m, 7)$ -admittable sequences coincide with the $(m, 6)$ -admittable sequences, and the $(m, 7)$ -ias are the same as the $(m, 6)$ -ias. Moreover, the set of $(8, 7)$ -admittable sequences is exactly the set of $(8, 6)$ -admittable sequences, adding the sequence $(7, 3, 3, 3, 3, 3, 3, 3)$. This implies that the set of $(8, 7)$ -ias, $\mathcal{S}_{(8,7)}$, contains exactly the graphs in $\mathcal{S}_{(8,6)}$, plus W_7 , which corresponds to the degree sequence $(7, 3, 3, 3, 3, 3, 3, 3)$. To summarize, we have

$$\mathcal{S}_{(6,7)} = \mathcal{S}_{(6,6)}, \quad \mathcal{S}_{(7,7)} = \mathcal{S}_{(7,6)}, \quad \mathcal{S}_{(8,7)} = \mathcal{S}_{(8,6)} \cup \{W_7\}.$$

(See Figures 2 and 4.)

3.3 Strong defensive alliances in regular graphs

Defensive alliances and strong defensive alliances coincide if G is d -regular, with d even. For d odd, a defensive alliance is a set of vertices that induces a subgraph with minimum degree at least $\frac{d-1}{2}$ and maximum degree at most d , while a strong defensive alliance is a set of vertices that induces a subgraph with minimum degree at least $\frac{d+1}{2}$ and maximum degree at most d . (See Property 3.3.)

5-regular graphs. We have that $\hat{\mathcal{S}}_{(m,5)}$, defined in Definition 3.4, is the set of graphs of minimum degree at least 3 and maximum degree at most 5. Thus, if $m \leq 6$, $\hat{\mathcal{S}}_{(m,5)} = \mathcal{S}_{(m,6)}$. For $m = 7, 8$, we have to remove from $\mathcal{S}_{(m,6)}$ the graphs with maximum degree 6. To be precise, $\hat{\mathcal{S}}_{(7,5)} = \mathcal{S}_{(7,6)} \setminus \{W_6\}$ and $\hat{\mathcal{S}}_{(8,5)}$ contains the 59 graphs in $\mathcal{S}_{(8,6)}$ (see Figure 5) corresponding to the degree sequences

$$(5, 5, 4, 4, 3, 3, 3, 3), (5, 5, 3, 3, 3, 3, 3, 3), (5, 4, 4, 4, 4, 3, 3, 3), (5, 4, 4, 3, 3, 3, 3, 3), \\ (4, 4, 4, 4, 3, 3, 3, 3), (4, 4, 3, 3, 3, 3, 3, 3), \text{ and } (3, 3, 3, 3, 3, 3, 3, 3).$$

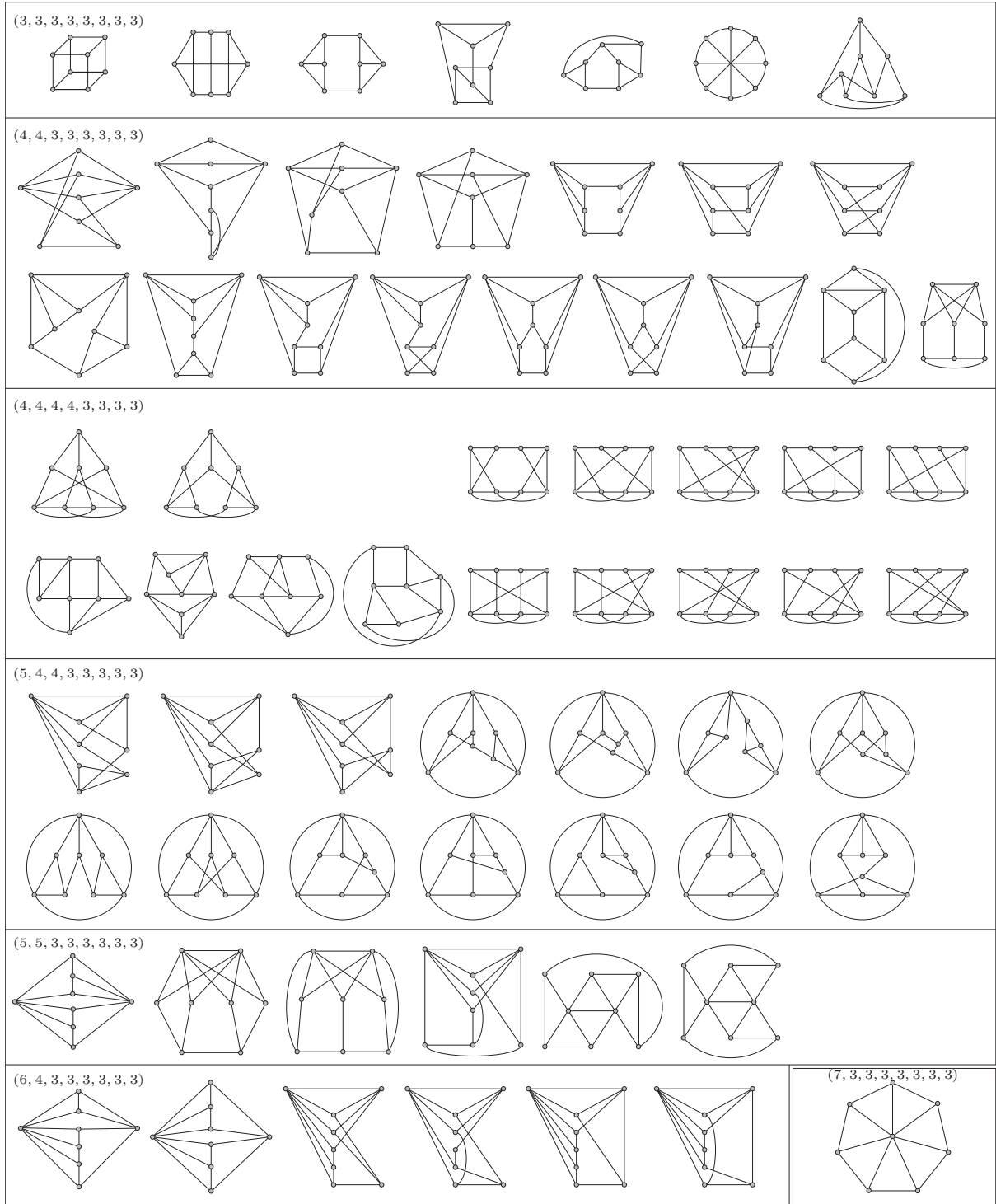


Figure 5: The $(8, 6)$ - and the $(8, 7)$ -induced alliances set, with their associated degree sequence. The first 65 graphs are the graphs in $\mathcal{S}_{(8,6)}$, which are also in $\mathcal{S}_{(8,7)}$. The set $\mathcal{S}_{(8,7)} \setminus \mathcal{S}_{(8,6)}$ contains only the graph W_7 .

6-regular graphs. We have that $\hat{\mathcal{S}}_{(m,6)} = \mathcal{S}_{(m,6)}$.

7-regular graphs. A graph is in $\hat{\mathcal{S}}_{(m,7)}$ if it has minimum degree 4 and maximum degree at most 7, and it contains no subgraph isomorphic to a graph in $\hat{\mathcal{S}}_{(m',7)}$, for any $4 \leq m' < m$.

In this case, we cannot derive any result about the $(m, 7)$ -induced strong alliances set from the $(m, 7)$ -induced alliances set.

4 Defensive alliances in circulant graphs

We have seen that the study of defensive alliances in regular graphs becomes more and more complex as the degree increases. Therefore, it makes sense to restrict the study of alliances to more symmetric graphs. In this section we begin our study of defensive alliances on a family of highly symmetric graphs, the well known (undirected) circulant graphs. Circulant graphs are Cayley graphs on the cyclic group \mathbb{Z}_n . Since we are studying undirected graphs, the set of generators must be closed under additive inversion. An undirected circulant graph can be defined as follows.

Definition 4.1 (Circulant graph of order n with generators c_1, c_2, \dots, c_d) *The circulant graph of order n with generators c_1, c_2, \dots, c_d is the graph $G = C_n(c_1, c_2, \dots, c_d)$ with vertex set \mathbb{Z}_n and adjacencies defined by:*

$$v \sim v \pm c_i$$

for every $v \in \mathbb{Z}_n$ and $i = 1, \dots, d$.

Remark. The usual notation for a symmetric or undirected circulant graph with generators c_1, c_2, \dots, c_d is $C_n(\pm c_1, \pm c_2, \dots, \pm c_d)$. Since we are only dealing with undirected graphs, we use the simpler notation $C_n(c_1, c_2, \dots, c_d)$, assuming that both c_i and $-c_i$ are in the set of generators.

According to this notation, a permutation of the set of generators, which gives an isomorphic circulant graph, is given by a permutation of the set $\{c_1, c_2, \dots, c_d\}$, but also by the change of the signs of an arbitrary subset of $\{c_1, c_2, \dots, c_d\}$.

4.1 Properties of circulant graphs

We first recall some well known properties of circulant graphs.

- The circulant graph $C_n(c_1, c_2, \dots, c_d)$ is connected if and only if $\gcd(c_1, c_2, \dots, c_d, n) = 1$.
If $\gcd(c_1, \dots, c_d, n) = m$ then $C_n(c_1, c_2, \dots, c_d)$ is isomorphic to m copies of the connected circulant graph $C_{\frac{n}{m}}(\frac{c_1}{m}, \frac{c_2}{m}, \dots, \frac{c_d}{m})$.
Thus, we can restrict our study to the case of connected circulant graphs and, therefore, we always assume the connectedness condition $\gcd(c_1, \dots, c_d, n) = 1$.
- If $\frac{n}{2} \notin \{c_1, \dots, c_d\}$, then $C_n(c_1, \dots, c_d)$ is $2d$ -regular.
If $\frac{n}{2} \in \{c_1, c_2, \dots, c_d\}$, then $C_n(c_1, \dots, c_d)$ is $(2d - 1)$ -regular.
- If $\lambda \in \mathbb{Z}_n^*$ then $C_n(c_1, c_2, \dots, c_d) \cong C_n(\lambda \cdot c_1, \lambda \cdot c_2, \dots, \lambda \cdot c_d)$. This kind of isomorphism is called Ádám-isomorphism [1]. In particular, if one of the generators of a circulant graph, say c_1 , is invertible, then we can always assume that it is equal to 1. Indeed,

$$C_n(c_1, c_2, \dots, c_d) \cong C_n(1, c'_2, \dots, c'_d),$$

where $c'_i = c_1^{-1} \cdot c_i$, for $i = 2, \dots, c_d$.

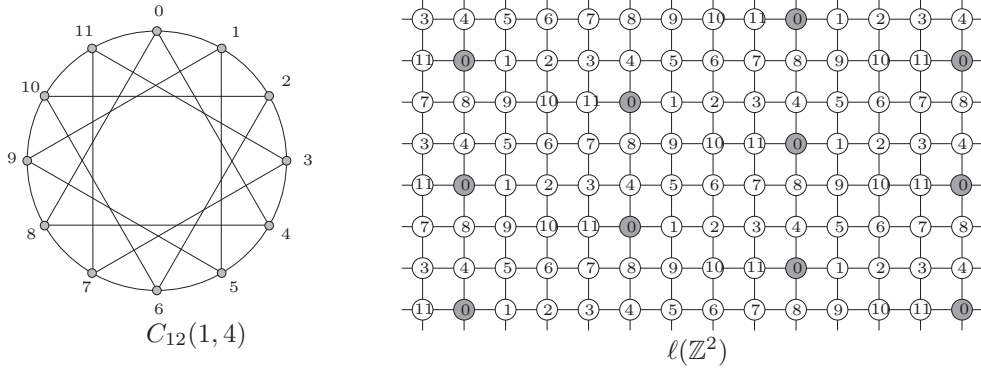


Figure 6: The circulant graph $C_{12}(1,4)$ with its lattice representation. The vertices of the sublattice $\ell^{-1}(0)$ are filled in grey.

- Every circulant graph is vertex-symmetric. For every $v \in \mathbb{Z}_n$, the mapping $f_v : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ defined by $f_v(u) = u + v$ is an automorphism of $C_n(c_1, c_2, \dots, c_d)$, which applies the edge $\{u, u + c\}$ to the edge $\{u + v, u + v + c\}$, for any $u \in \mathbb{Z}_n$ and c in the set of generators.
- A circulant graph $C_n(c_1, c_2, \dots, c_d)$ contains triangles if, for some i, j and k pairwise distinct, $c_i \pm c_j \pm c_k = 0$, or $2c_i \pm c_j = 0$, or $3c_i = 0$, with the additions and products modulo n .

Lattice representation. Let $C_n(c_1, \dots, c_d)$ a circulant graph, and consider the infinite integer lattice \mathbb{Z}^d , with the usual adjacencies $(x_1, \dots, x_i, \dots, x_d) \sim (x_1, \dots, x_i \pm 1, \dots, x_d)$, for $i = 1, \dots, d$.

The vertices of this lattice can be labeled in \mathbb{Z}_n by

$$\ell(x_1, \dots, x_d) = x_1 \cdot c_1 + x_2 \cdot c_2 + \dots + x_d \cdot c_d \pmod{n}.$$

Notice that, for every $v \in \mathbb{Z}_n$, the set $\ell^{-1}(v)$ is an infinite set of vertices of \mathbb{Z}^d . It can be easily seen that, since the map ℓ is linear, $\ell^{-1}(v)$ is a sublattice of \mathbb{Z}^d . If $v, w \in \mathbb{Z}_n$, the lattices $\ell^{-1}(v)$ and $\ell^{-1}(w)$ are isomorphic. This surjective map provides a useful geometric representation of circulant graphs which, in fact, can be easily generalized to Cayley graphs on Abelian groups. See Figure 6 for an example of a circulant graph and its lattice representation.

4.2 Alliances in circulant graphs

Using the geometric representation of a circulant graph $G = C_n(c_1, \dots, c_d)$, we can give a first bound on its alliance number.

Proposition 4.2 *Let $G = C_n(c_1, \dots, c_d)$, a circulant graph with d generators.*

1. If $\delta_G = 2d$, i.e., $\frac{n}{2} \notin \{c_1, \dots, c_d\}$, then the alliance number of G satisfies $d+1 \leq a(G) \leq 2^d$.
2. If $\delta_G = 2d - 1$, i.e., $\frac{n}{2} \in \{c_1, \dots, c_d\}$, then the alliance number of G satisfies $d \leq a(G) \leq 2^{d-1}$.

Proof. Assume that $\frac{n}{2} \notin \{c_1, \dots, c_d\}$. Since $\delta_G = 2d$, we know that $a(G) \geq d + 1$.

To show that $a(G) \leq 2^d$, we show that the set $S = \ell(\{0, 1\}^d)$ is a defensive alliance of G of cardinality $|S| \leq 2^d$. Indeed, every vertex in S has at least d neighbors in S . Moreover, S

contains 2^d d -tuples. However, since the labeling ℓ is not injective, some of the d -tuples might be assigned by ℓ to the same vertex. Thus, $|S| \leq 2^d$.

In the second case, that is, if $\frac{n}{2} \notin \{c_1, \dots, c_d\}$, we can assume, w.l.o.g., that $\frac{n}{2} = c_d$. Then, the set $S = \ell(\{0, 1\}^{d-1} \times \{0\})$, i.e., the d -tuples with coordinates in $\{0, 1\}$ with 0 in the last one, is a defensive alliance of G of cardinality $|S| \leq 2^{d-1}$. As in the previous case, every vertex in S has at least $d - 1$ neighbors in S , and S contains 2^{d-1} d -tuples. Again, some of the d -tuples might be assigned by ℓ to the same vertex. Thus, $|S| \leq 2^{d-1}$. ■

For $d = 3$ and $\frac{n}{2} \notin \{a, b, c\}$, we have that $G = C_n(a, b, c)$ is 6-regular and its alliance number satisfies

$$4 \leq a(G) \leq 8.$$

In Section 3.1 we have characterized all the alliances of cardinality at most 8 of 6-regular graphs. In what follows, we study the alliances in circulant graphs, up to 3 generators.

4.2.1 Alliances in circulant graphs of small degrees

For the sake of completeness we give a short study of circulant graphs with 1 and 2 generators.

- $G = C_n(c_1) \cong C_n$ is the n -cycle. In this case, the alliance numbers are $a(G) = \hat{a}(G) = 2$. The critical alliances are the edges.
- $G = C_n(c_1, c_2)$ is a 3-regular graph if and only if $n = 2m$ and $c_2 = m$. In this case, $G \cong C_{2m}(1, m)$ or $G \cong C_{2m}(2, m)$. The alliance numbers are $a(G) = 2$ and $\hat{a}(G) = g(G)$. G contains triangles (and thus, $\hat{a}(G) = 3$) if and only if $n = 4$, which implies $G \cong K_4$, and if $n = 6$ and $G \cong C_6(2, 3)$. In the remaining cases, $\hat{a}(G) = g(G) = 4$.

The critical alliances are the edges, and the strong critical alliances are the induced cycles.

- $G = C_n(c_1, c_2)$, with $c_2 \neq \frac{n}{2}$, is a 4-regular graph. Therefore $a(G) = \hat{a}(G) = g(G)$. In this case, G contains triangles if and only if $c_2 = 2c_1$, that is, $G \cong C_n(1, 2)$, or $n = 3m$ and $c_2 = \pm m$, that is, $G \cong C_{3m}(1, m)$ or $G \cong C_{3m}(3, m)$.

If $G \cong C_n(1, 2)$, $G \cong C_{3m}(1, m)$, or $G \cong C_{3m}(3, m)$, then $a(G) = \hat{a}(G) = 3$. Otherwise, $a(G) = \hat{a}(G) = 4$.

Circulant graphs of degree 5. Let $G = C_{2m}(a, b, m)$ be a circulant graph of degree 5. Notice that, since $\delta = 5$, $2m \geq 6$. But, if $2m = 6$ then Equations 3 and 4 imply that $a(K_6) = 3$ and $\hat{a}(K_6) = 4$. Thus, we can assume that $2m \geq 8$.

The known bounds give $a(G) = g(G) \leq 4$.

- G contains triangles (and thus, $g(G) = 3$) if and only if $a + b = m$, or $m = 2m'$ is even and $b = \pm m'$, or $m = 3m'$ and $b = \pm 2m'$. The connectedness condition, $\gcd(a, b, m, 2m) = 1$, implies that these three cases correspond to $G \cong C_{2m}(1, 2, m)$, $G \cong C_{4m'}(1, m', 2m')$ and $G \cong C_{6m'}(1, 2m', 3m')$, respectively.
- Otherwise, $g(G) = 4$.

4.2.2 Alliances in circulant graphs of degree 6

Let us concentrate now on circulant graphs with 3 generators, $C_n(a, b, c)$, with $\frac{n}{2} \notin \{a, b, c\}$. Notice that, since $\delta = 6$, $n \geq 7$. But, if $n = 7$ then $C_n(a, b, c) = K_7$ and Equations 3 and 4 imply that $a(K_7) = \hat{a}(K_7) = 4$. Thus, we can assume that $n \geq 8$.

Recall that $4 \leq a(G) \leq 8$ (see Proposition 4.2), and also that G contains triangles if and only if, up to a permutation of the generators,

$$a + b + c = 0 \pmod{n}, 2a + b = 0 \pmod{n}, \text{ or } 3c = 0 \pmod{n}.$$

In Section 3.1 we found that $\mathcal{S}_{(4,6)} = \{K_4\}$, $\mathcal{S}_{(5,6)} = \{W_4\}$, $\mathcal{S}_{(6,6)} = \{C_3 \square K_2, K_{3,3}, (C_3 \square K_2) + e, K_{3,3} + e, \overline{C_4} + \overline{K_2}, W_5\}$, and $\mathcal{S}_{(7,6)}$ contains exactly the 15 graphs in Figure 4. To apply these results to 6-regular circulant graphs, it is worth noticing that every graph in $\mathcal{S}_{(6,7)}$ contains triangles, while the only triangle free graph in $\mathcal{S}_{(6,6)}$ is $K_{3,3}$.

In the remaining of the section, we give a classification of 6-regular circulant graphs according to their alliance number, showing which is a minimal alliance in these graphs. In summary, we show that, either $G = C_n(a, b, c)$ contains triangles and then $a(G)$ ranges from 4 to 7, or G is triangle free. In this later case, either G contains $K_{3,3}$ and $a(G) = 6$, or $a(G) = 8$ and the minimal alliance is the cube Q_3 . Before the complete classification theorem we prove some technical lemmas.

In all of the following lemmas we use the vertex-symmetry of circulant graphs, and also that a circulant graph is isomorphic to any circulant graph obtained by a permutation of its generators.

Lemma 4.3 *If $G = C_n(a, b, c)$ is 6-regular then $a(G) = 4$ if and only if $G \cong C_n(1, 2, 3)$.*

Proof. Assume that $G = C_n(a, b, c)$ contains a subgraph isomorphic to K_4 . Because of the vertex-symmetry of G , we can fix 0 to be any of the vertices of this induced K_4 . Now, the set of vertices that induce K_4 is, up to a permutation of generators, either $\{0, a, b, -a\}$ or $\{0, a, b, c\}$.

- If $\langle \{0, a, b, -a\} \rangle \cong K_4$, then all the integers $\pm 2a, \pm(b-a), \pm(b+a)$ are generators of G .
Let us first study which of the generators equals $2a$. The case $2a = a$ is impossible. The case $2a = -b$ is, up to a permutation of the generators, the same as $2a = b$. The case $2a = -c$ is, up to a permutation of the generators, the same as $2a = c$. Thus, $2a \in \{-a, b, c\}$.
 - $2a = -a$ gives $3a = 0$, that is $n = 3m$ and $a = m$. Now, $b-a$ can only be equal to $-b$ or c . If $b-a = -b$ then $n = 6m'$, $a = 2m'$, $b = m'$, which implies $a+b = 3m' = \pm c$. But then, the connectedness condition is not fulfilled. If, otherwise, $b-a = c$, then $b+a = -b$, which implies $2b = -a = 2a$. In both cases we get a contradiction.
 - $2a = b$ gives $b-a = a$ and then $a+b = 3a = \pm c$, which implies $G \cong C_n(a, 2a, 3a) \cong C_n(1, 2, 3)$.
 - $2a = c$ implies that $b-a = -b$, that is $2b = a$, and thus, $a+b = 3b$. This implies $b, 2b, 3b$, and $4b$ are all generators of G and this is a contradiction if $n \geq 7$.
- If $\langle \{0, a, b, c\} \rangle \cong K_4$, then all the integers $\pm(a-b), \pm(b-c), \pm(c-a)$ are generators of G .
Let us first study which of the generators equals $c-a$. The cases $c-a = c$ and $c-a = -a$ are both impossible. Thus, $c-a \in \{a, \pm b, c\}$.
 - $c-a = a$ gives $c = 2a$. Now, $c-b$ can only be equal to $-a$ or $-c$. If $c-b = -a$ we get $3a = b$, which implies $G \cong C_n(a, 2a, 3a) \cong C_n(1, 2, 3)$. The case $c-b = -c$ implies $2c = 4a = b$ and then $b-a = 3a$. So $a, 2a, 3a$, and $4a$ are all generators of G and this is a contradiction if $n \geq 7$.
 - $c-a = b$ gives $a+b = c$ and $b-c = a$. Now, $a-b$ can only be equal to $-a$ or b . Both cases imply $G \cong C_n(1, 2, 3)$.

- $c - a = -b$ implies that $a - b = c$ and $c - b$ can only be equal to b or $-c$. As in the previous case, both cases imply $G \cong C_n(1, 2, 3)$.
- $c - a = -c$ gives $2c = a$. Now, $c - b$ can only be equal to $-a$ or b . If $c - b = -a$ then $3c = b$, which implies $G \cong (2c, 3c, c) \cong C_n(1, 2, 3)$. If, otherwise, $c - b = b$, we get $2b = c$ and thus $a = 4b$. This implies $b, 2b, 3b$, and $4b$ are all generators of G and this is a contradiction if $n \geq 7$.

All possible cases give $G \cong C_n(1, 2, 3)$. ■

Lemma 4.4 *If $G = C_n(a, b, c)$ is 6-regular then $a(G) = 5$ if and only if $G \cong C_{3m}(a, m - a, m)$ or $G \cong C_n(1, 2, 4)$.*

Proof. Assume that $G = C_n(a, b, c)$ contains an induced subgraph isomorphic to W_4 , and G does not contain any subgraph isomorphic to K_4 .

By symmetry, we can assume w.l.o.g. that, either $\langle\{0, \pm a, \pm b\}\rangle \cong W_4$ or $\langle\{0, \pm a, b, c\}\rangle \cong W_4$. That is, either $\langle\{\pm a, \pm b\}\rangle \cong C_4$ or $\langle\{\pm a, b, c\}\rangle \cong C_4$.

For each case there are two possibilities:

- $\langle\{0, \pm a, \pm b\}\rangle \cong W_4$ and $(a, b, -a, -b, a)$ is an induced 4-cycle.

This implies $\pm(b - a)$ and $\pm(b + a)$ are generators of G . It is easy to see that, in this case, $b - a \in \{a, -b, c\}$ and in all the three cases, $G \cong C_n(1, 2, 3)$, which satisfies $a(G) = 4$.

- $\langle\{0, \pm a, \pm b\}\rangle \cong W_4$ and $(a, b, -b, -a, a)$ is an induced 4-cycle.

This implies $\pm(b - a)$, $\pm 2a$, and $\pm 2b$ are generators of G . It is easy to see that, in this case, $2a \in \{-a, \pm b, c\}$. By easy computations, we obtain:

- $2a = -a$ implies $n = 9m$ and $a = 3m, b = m, c = 2m$, which gives that either $G \cong C_9(1, 2, 3)$, which satisfies $a(G) = 4$, or G disconnected.
- $2a = b$ implies $G \cong C_n(1, 2, 4)$.
- $2a = -b$ implies $G \cong C_n(1, 2, 3)$, which satisfies $a(G) = 4$.
- $2a = c$ is only possible if $b - a = -b$, and this implies $G \cong C_n(1, 2, 4)$.

Summarizing, $G \cong C_n(1, 2, 4)$.

- $\langle\{0, \pm a, b, c\}\rangle \cong W_4$ and $(a, b, -a, c, a)$ is an induced 4-cycle.

This implies $\pm(b - a)$, $\pm(b + a)$, $\pm(c - a)$, and $\pm(c + a)$ are generators of G . It is easy to see that, in this case, $b - a \in \{a, -b, \pm c\}$. Reasoning and computing as in the previous case we obtain: (1) $b - a = a$ implies $2a = b$ and $b + a = 3a$; (2) $b - a = -b$ implies $2b = a$ and $b + a = 3b$; (3) $b - a = c$ gives that $b + a$ can only be equal to $-b$ and, thus $3b = c$; and (4) $b - a = -c$ gives that $b + a$ can only be equal to $-b$ and, thus $3b = -c$. Any of this four cases implies $G \cong C_n(1, 2, 3)$, which satisfies $a(G) = 4$.

- $\langle\{0, \pm a, b, c\}\rangle \cong W_4$ and $(a, b, c, -a, a)$ is an induced 4-cycle.

This implies $\pm(b - a)$, $\pm 2a$, and $\pm(c + a)$ are generators of G . It is easy to see that, in this case, $2a \in \{-a, \pm b, \pm c\}$. In this case we obtain: (1) $2a = -a$ and $b - c = \pm a$ implies $G \cong C_{3m}(a, m - a, m)$; (2) $2a = -a$ and $b - c = -b$ implies $G \cong C_n(1, 2, 4)$; (3) $2a = b$ and $b - c = -b$ implies $G \cong C_n(1, 2, 4)$; and (4) $2a = -c$ and $b - c = c$ implies $G \cong C_n(1, 2, 4)$. All the remaining cases either give impossible values for the generators or imply $G \cong C_n(1, 2, 3)$, which satisfies $a(G) = 4$.

This completes the proof. ■

The two following lemmas characterize the critical defensive alliances for a circulant graph G of degree 6 not containing K_4 , neither W_4 . These graphs satisfy $6 \leq a(G) \leq 8$. Lemma 4.5 deals with graphs containing triangles, and Lemma 4.6 deals with triangle-free graphs.

Lemma 4.5 *If $G = C_n(a, b, c)$ contains a triangle and $a(G) > 5$, then one of the following conditions hold.*

1. $3c = 0 \pmod{n}$. Then $n = 3m$ and $G \cong C_{3m}(a', b', m)$, with $a' + b' \neq m$, but $G \not\cong C_{12}(1, 2, 4)$. In this case, $C_3 \square K_2$ is an induced subgraph of G . Thus, $a(G) = 6$.
2. $2a + b = 0 \pmod{n}$. Then $G = C_n(a, -2a, c)$, but $G \not\cong C_n(1, 2, 3)$ and $G \not\cong C_n(1, 2, 4)$. In this case, $C_3 \square K_2$ is an induced subgraph of G . Thus, $a(G) = 6$.
3. $a + b + c = 0 \pmod{n}$. Then, $G \not\cong C_n(1, 2, 3)$ and there are two possibilities:
 - $G \cong C_{2m}(a, m - a, m - 2a)$ and it has an induced $\overline{C_4} + \overline{K_2}$. Thus, $a(G) = 6$.
 - Otherwise, none of the graphs in $\mathcal{S}_{(6,6)}$ is an induced subgraph of G , but G contains W_6 . Thus, $a(G) = 7$.

Proof. The three cases in the statement of this lemma correspond to the three cases for which G contains triangles.

Case 1: $3c = 0$ It is clear that $n = 3m$ and $G \cong C_{3m}(a', b', m)$. Moreover, if $a' + b' = m$, we have $G \cong C_{3m}(a', m - a', m)$. Lemma 4.4 says that both $C_{3m}(a', m - a', m)$ and $C_{12}(1, 2, 4)$ contain W_4 . On the other hand, $\langle \{0, m, -m, a', m + a', -m + a'\} \rangle \cong C_3 \square K_2$, if $a' + b' \neq m$.

Case 2: $2a + b = 0$ It is clear that $G = C_n(a, -2a, c)$. Since $a(G) > 5$, we know that $G \not\cong C_n(1, 2, 3)$ and $G \not\cong C_n(1, 2, 4)$. One can also see that, if $n = 3m$, $C_{3m}(a, 2a, c)$ cannot be isomorphic to $C_{3m}(a', m - a', m)$. Moreover, $\langle \{0, a, 2a, c, a + c, 2a + c\} \rangle \cong C_3 \square K_2$.

Case 3: $a + b + c = 0$ The condition $a(G) > 5$ implies that $G \not\cong C_n(1, 2, 3)$. We have that, if $S = \{0, \pm a, \pm b, \pm c\}$ then $\langle S \rangle \cong W_6$ and thus, $6 \leq a(G) \leq 7$. The adjacencies in $\langle S \rangle$ are given by $d_S(0) = 6$ and $(a, -b, c, -a, b, -c, a)$ is a 6-cycle.

Now, $a(G) = 6$ if one of the graphs in

$$\mathcal{S}_{(6,6)} = \{C_3 \square K_2, K_{3,3}, (C_3 \square K_2) + e, K_{3,3} + e, \overline{C_4} + \overline{K_2}, W_5\}$$

is an induced subgraph of G (see Proposition 3.8). First, we prove that none of the graphs W_5 , $K_{3,3}$, $K_{3,3} + e$, $C_3 \square K_2$, or $(C_3 \square K_2) + e$, can be an induced subgraph of G .

- If there is an induced subgraph isomorphic to W_5 , by symmetry we can assume that its vertex of 5 degree is 0. Now, up to a permutation of the generators, we can assume that $W_5 \cong \langle \{0, a, -b, c, -a, b\} \rangle$, which clearly implies that a and b are adjacent. But then, $K_4 \cong \langle \{0, a, b, -c\} \rangle$ is an induced subgraph of G , a contradiction.
- If either $K_{3,3}$ or $K_{3,3} + e$ is isomorphic to an induced subgraph of G , we can assume, w.l.o.g., that this subgraph is induced by a set $\{0, a, b, c, x, y\}$, where x and y are at distance 2 of 0 and are adjacent, both, to a, b and c . If $x \asymp y$, then the induced subgraph is $K_{3,3}$, and if $x \sim y$, then the induced subgraph is $K_{3,3} + e$.

Easy computations give that, up to symmetries and permutations of the generators, either $x = a - b \asymp y = a - c$ or $x = a - b \sim y = 2a$. There are three possible values for $a - b$:

- $a - b = c - a$ implies $3a = 0$, which corresponds to case 1,
 - $a - b = c - b$ implies $a = c$, which is impossible, and
 - $a - b = 2c$ implies $2a = c$ and $3a = -b$, which corresponds to $G \cong C_n(1, 2, 3)$.
- If $C_3 \square K_2$ is isomorphic to an induced subgraph of G . We can assume, w.l.o.g., that the disjoint triangles in this graph are $T_1 = \{0, a, -c\}$ and $T_2 = \{2a, x, y\}$, with either $x \sim 0$ and $y \sim -c$, or $x \sim -c$ and $y \sim 0$. One can see that, up to symmetries, $x = 3a$, and $y = 2a - c$ or $y = 2a - b$. A careful analysis shows that neither case is possible.
 - If $(C_3 \square K_2) + e$ is isomorphic to an induced subgraph of G , we can assume, w.l.o.g., that this subgraph is induced by a set $\{0, a, -b, c, -a, y\}$, with y a common neighbor of a, c and $-a$. One can see that this implies $y \in \{2c, c - a\} \cap \{2a, a - c\} \cap \{c - a, -2a, b - a\}$, which is impossible.

Let us assume now that G contains an induced subgraph isomorphic to $\overline{C_4} + \overline{K_2}$. Let $\{0, x, y, z, t, w\}$ be its set of vertices and $x \sim y, z \sim t$ the neighbors of 0 in this induced subgraph. We can assume, w.l.o.g., that $\{x, y, z, t\} = \{a, \pm b, -c\}$ or $\{x, y, z, t\} = \{\pm a, \pm b\}$. In the first case, we get $w = a - b = b - c$ and $3b = 0$, which corresponds to case 1. Otherwise, if $w = a - b = b - a$, then $2(b - a) = 0$, which implies $n = 2m, b = m + a$ and $c = -m - 2a$. Thus, $G \cong C_{2m}(a, m - a, m - 2a)$.

This completes the proof. ■

Lemma 4.6 *Let $G = C_n(a, b, c)$ be 6-regular and triangle-free. $K_{3,3}$ is an induced subgraph of G if and only if $G \cong C_n(1, 3, 5)$. In this case, $a(G) = 6$. Otherwise, the minimal alliance of G is the cube Q_3 and, thus, $a(G) = 8$.*

Proof. First, remember that the only triangle-free graph in $\mathcal{S}_{(4,6)} \cup \mathcal{S}_{(5,6)} \cup \mathcal{S}_{(6,6)} \cup \mathcal{S}_{(7,6)}$ is $K_{3,3}$. Moreover, if $G = C_n(a, b, c)$ is 6-regular, by Proposition 4.2, the set of vertices $\{0, a, b, c, a + b, a + c, b + c, a + b + c\}$ induce an alliance of cardinality at most 8 that, if G is triangle-free, is isomorphic to the cube Q_3 . To prove the Lemma, we only need to show that G , triangle-free, contains an induced $K_{3,3}$ if and only if $G \cong C_n(1, 3, 5)$. Notice that, in $C_n(1, 3, 5)$, the set $\{0, 1, 2, 3, 4, 5\}$ induce a subgraph isomorphic to $K_{3,3}$.

Let us assume that G has an induced $K_{3,3}$. We can assume, w.l.o.g., that $K_{3,3}$ is induced by $\{0, a, b, c, x, y\}$, with x and y common neighbors of a, b and c , at distance 2 from 0, or $\{0, a, b, -a, x, y\}$, with x and y common neighbors of a, b and $-a$, at distance 2 from 0. But easy computations show that only the first possibility can hold, provided x and y are conveniently chosen. Again by symmetry, $x \in \{2a, a + b, a - b\}$. For every possible value of x , we have to see which values are possible for y and how they determine the graph G .

- If $x = 2a$, since b and c are adjacent to x , we have that $2a - b$ and $2a - c$ are both in $\{\pm a, \pm b, \pm c\}$. In this case, $2a - b$ can only be equal to $-a$, and $\pm c$. We can see that $2a - b = -a$ implies $3a = b$ and $2a - b = -c$, which gives $G \cong C_n(1, 3, 5)$. If $2a - b = -c$, then $2a - c$ can only be equal to a , which implies $b = -3a, c = -5a$, and again, $G \cong C_n(1, 3, 5)$. On the other hand, for $2a - b = c$, and thus $2a - c = b$, we have to distinguish according to the values of y . It can be seen that $y \in \{a \pm b, a \pm c\}$. Some of these cases are impossible; the possible ones give again, $G \cong C_n(1, 3, 5)$.
- If $x = a + b$, then we need only to consider the cases $y \in \{a - b, a \pm c\}$. One can see that the only valid possibilities give $G \cong C_n(1, 3, 5)$.
- Finally, if $x = a - b$, then we only need to consider the case $y = a - c$. This case can be reduced by symmetry to the previous ones.

Thus, we have show that the only triangle-free circulant graph of degree 6 with an induced $K_{3,3}$ is, up to isomorphism, $G \cong C_n(1, 3, 5)$. This completes the proof. ■

Theorem 4.7 *The alliance number of $G = C_n(a, b, c)$, circulant graph of degree 6 and order $n \geq 8$ is*

- $a(G) = 4 \Leftrightarrow G \cong C_n(1, 2, 3)$ or $G \cong C_n(1, 2, 4)$.
- $a(G) = 5 \Leftrightarrow G \cong C_{3m}(a, m - a, m)$
- $a(G) = 6 \Leftrightarrow G$ contains triangles and it is isomorphic to one of the graphs $C_{3m}(a, b, m)$, with $a + b \neq m$, $C_{2m}(m - 2c, m + c, c)$, $C_{4m}(m, b, -m - b)$, $C_n(1, 3, 4)$, $C_n(a, -2a, c)$, but G is not isomorphic to $C_n(1, 2, 3)$ nor $C_n(1, 2, 4)$; or G is triangle-free and it is isomorphic to $C_n(1, 3, 5)$ or $C_n(1, 5, 7)$.
- $a(G) = 7 \Leftrightarrow G \cong C_n(a, b, -(a + b))$, but G is not isomorphic to none of the graphs $C_{2m}(m - 2c, m + c, c)$, $C_{4m}(m, b, -m - b)$, $C_n(1, 3, 4)$.
- $a(G) = 8 \Leftrightarrow G$ is triangle-free and it is not isomorphic to $C_n(1, 3, 5)$ nor $C_n(1, 5, 7)$.

Proof. It follows straightforward from the previous lemmas 4.3, 4.4, 4.5, and 4.6. ■

5 Conclusions and open problems

We have studied defensive alliances of cardinality $k \leq 8$ in regular graphs of degree 6. We have also restricted the problem to circulant graphs.

Open problems. We let some open problems about defensive alliances in regular graphs in general and circulant graphs in particular.

- Can we describe, in some constructive way, the graphs in $\mathcal{S}_{(m,6)}$?
- We think that the number of graphs in $\mathcal{S}_{(m,6)}$ exponentially increases with m .
- We have shown that, if $G = C_n(a, b, c)$ is 6-regular, then $4 \leq a(G) \leq 8$. Moreover, in the family of circulant graphs of degree 6, there are graphs with alliance number ranging from 4 to 8. Is this the same for circulant graphs of higher degree?
- On the other hand, assuming that the bounds in Proposition 4.2 are tight, an interesting problem is: Find a lower bound for n , the order of a circulant graph $G = C_n(c_1, \dots, c_d)$, such that $a(G) = 2^d$ (or $a(G) = 2^{d-1}$, if $\frac{n}{2} \in \{c_1, \dots, c_d\}$).

Acknowledgments. Research supported by: CONACyT-México under project 5737 and Intercambio Académico, Coordinación de la Investigación Científica UNAM, under project "Construcción de Gráficas: Jaulas y Alianzas en redes"; and the Education and Science Ministry, Spain, and the European Regional Development Fund under projects MTM2005-08990-C02-01 and TEC2005-03575 and by the Catalan Research Council under project 2005SGR00256.

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