

# ON HAMILTONIAN POTENTIALS WITH CUARTIC POLYNOMIAL NORMAL VARIATIONAL EQUATIONS

PRIMITIVO B. ACOSTA-HUMÁNEZ<sup>1</sup>, DAVID BLAZQUEZ-SANZ<sup>2</sup>,  
AND CAMILO A. VARGAS CONTRERAS<sup>2</sup>

<sup>1</sup>Departament Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain.

<sup>2</sup>Escuela de Matemáticas, Universidad Sergio Arboleda, Bogotá, Colombia.

**ABSTRACT.** Here we find the complete family of two degree of freedom classical Hamiltonians with invariant plane  $\Gamma = \{q_2 = p_2 = 0\}$  whose normal variational equation around integral curves in  $\Gamma$  is a generically a Hill-Schrödinger equation with cuartic polynomial potential. In particular, these Hamiltonian form a family of non-integrable Hamiltonians through rational first integrals.

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## INTRODUCTION

The integrability of families of two degrees of freedom potentials with an invariant plane  $\Gamma = \{x_2 = y_2 = 0\}$  and normal variational equations (NVEs) of polynomial (degree two and odd degree) type along generic curves in  $\Gamma$  was studied jointly by the first and second author in [1, 2]. Related problems with this approach have been studied before by Morales and Simó (see [11]) and by Baider, Churchill and Rod (see [3]). The use of techniques of Differential Galois theory to determine the non-integrability of hamiltonian systems, appeared independently for first time in [7, 12] and [4], followed by [3], [5] and [11]. A common limitation presented in these works is that they only analyzed cases of fuchsian monodromy groups, avoiding cases of irregular singularities of linear differential equations. The case of the NVEs with irregular singularities can be approached from the Morales-Ramis framework ([8, 9], see also [6]).

In [2], in the frame of Morales-Ramis theory, analyzing the NVEs through Kovacic algorithm, were obtained the following results:

- (1) **Theorem.** *Galois group of the Hill-Schrödinger equation,*

$$\ddot{\xi} = P_n(t)\xi,$$

*with  $P_n(t) \in \mathbb{C}[t]$  non-constant polynomial, is a non-abelian connected group isomorphic either to  $SL(2, \mathbb{C})$  or to semidirect product of  $\mathbb{C}^*$  with  $\mathbb{C}$  (also known as the Borel group).*

- (2) **Non-integrability results.** *Let  $H$  be a hamiltonian system of two degrees of freedom given by  $H = T + V$ . If  $\Gamma$  is an invariant plane and the NVE around a generic integral curve in  $\Gamma$  is of Hill-Schrodinger type with degree 2 polynomial coefficient, then the potential  $V$  is written such as follows:*

$$V = \frac{\lambda_4}{(\lambda_2 + 2\lambda_3 x_1)^2} + \lambda_0 - \lambda_1 x_2^2 - \lambda_2 x_1 x_2^2 - \lambda_3 x_1^2 x_2^2 + \beta(x_1, x_2) x_2^3,$$

with  $\lambda_3 \neq 0$ ; and thus they are not integrable by terms of rational functions. If the potential  $V$  is of the following form:

$$V = \lambda_0 + Q(x_1) x_2^2 + \beta(x_1, x_2) x_2^3,$$

being  $Q(x_1)$  a non-constant polynomial, then it has NVE of Hill-Schrödinger type with non-constant polynomial coefficient along a generic integral curve in  $\Gamma = \{x_2 = y_2 = 0\}$ , and thus they are not integrable by terms of rational functions.

Along this paper we follow [2] considering the NVEs of type Hill-Schrödinger equation with quartic polynomial potential, i.e quartic polynomial normal variational equations. We analyze deeply all the possibilities for  $V$  to be this kind of NVEs.

## 1. MORALES-RAMIS THEORY

Morales-Ramis theory relates the integrability of hamiltonian systems with the integrability of linear differential equations (see [8, 9] and see also [6]). In such approach the linearization (variational equations) of hamiltonian systems along some known particular solution is studied. If the hamiltonian system is integrable, then we expect that the linearized equation has good properties in the sense of differential Galois theory (also known as Picard-Vessiot theory). To be more precise, for integrable hamiltonian systems, the Galois group of the linearized equation must be virtually abelian. This gives us the best non-integrability criterion known so far for hamiltonian systems. This approach has been extended to higher order variational equations in [10].

**1.1. Integrability of hamiltonian systems.** A symplectic manifold (real or complex),  $M_{2n}$  is a  $2n$ -dimensional manifold, provided with a non-degenerate closed 2-form  $\omega_2$ . This closed 2-form gives us a natural isomorphism between vector bundles,  $\flat: TM \rightarrow T^*M$ . Given a function  $H$  on  $M$ , there is an unique vector field  $X_H$  such that,

$$\flat(X_H) = dH$$

this is the hamiltonian vector field of  $H$ . Furthermore, it has a structure of *Poisson algebra* over the ring of differentiable functions of  $M_{2n}$  by defining:

$$\{H, F\} := X_H F.$$

We say that  $H$  and  $F$  are *in involution* if and only if  $\{H, F\} = 0$ . From our definition, it is obvious that  $F$  is a *first integral* of  $X_H$  if and only if  $H$  and  $F$  are in involution. In particular  $H$  is always a first integral of  $X_H$ . Moreover, if  $H$  and  $F$  are in involution, then their flows commute.

The equations of the flow of  $X_H$ , in a system of canonical coordinates,  $p_1, \dots, p_n, q_1, \dots, q_n$  (that is, such that  $\omega_2 = \sum_{i=1}^n p_i \wedge q_i$ ), can be written in the form

$$\dot{q} = \frac{\partial H}{\partial p} (= \{H, q\}), \quad \dot{p} = -\frac{\partial H}{\partial q} (= \{H, p\}),$$

and they are known as *Hamilton equations*.

**Theorem 1.1** (Liouville-Arnold). *Let  $X_H$  be a hamiltonian defined on a real symplectic manifold  $M_{2n}$ . Assume that there are  $n$  functionally independent first integrals  $F_1, \dots, F_n$  in involution. Let  $M_a$  be a non-singular (that is,  $dF_1, \dots, dF_n$  are independent over each point of  $M_a$ ) level manifold,*

$$M_a = \{p: F_1(p) = a_1, \dots, F_n(p) = a_n\}.$$

- (1) *If  $M_a$  is compact and connected, then it is a torus  $M_a \simeq \mathbb{R}^n / \mathbb{Z}^n$ .*
- (2) *In a neighborhood of the torus  $M_a$  there are functions  $I_1, \dots, I_n, \phi_1, \dots, \phi_n$  such that*

$$\omega_2 = \sum_{i=1}^n dI_i \wedge d\phi_i,$$

and  $\{H, I_j\} = 0$  for  $j = 1, \dots, n$ .

From now on, we will consider  $\mathbb{C}^{2n}$  as a complex symplectic manifold. Liouville-Arnold theorem gives us a notion of integrability for hamiltonian systems. A hamiltonian  $H$  in  $\mathbb{C}^{2n}$  is called *integrable in the Liouville's sense* if and only if there exists  $n$  independent first integrals of  $X_H$  in involution. We will say that  $H$  is integrable *by rational functions* if and only we can find a complete set of first integrals within the family of rational functions.

**1.2. Variational equations.** We want to relate integrability of hamiltonian systems with Picard-Vessiot theory. We deal with non-linear hamiltonian systems. But, given a hamiltonian  $H$  in  $\mathbb{C}^{2n}$  and  $\Gamma$  an integral curve of  $X_H$ , we can consider the *first variational equation* (VE), such as

$$\mathcal{L}_{X_H} \xi = 0,$$

in which the linear equation is induced over the tangent bundle ( $\xi$  represents a vector field supported on  $\Gamma$ ).

Let  $\Gamma$  be parameterized by  $\gamma: t \mapsto (x(t), y(t))$  in such way that

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

Then the VE along  $\Gamma$  is the linear system,

$$\begin{pmatrix} \dot{\xi}_i \\ \dot{\eta}_i \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 H}{\partial y_i \partial x_j}(\gamma(t)) & \frac{\partial^2 H}{\partial y_i \partial y_j}(\gamma(t)) \\ -\frac{\partial^2 H}{\partial x_i \partial x_j}(\gamma(t)) & -\frac{\partial^2 H}{\partial x_i \partial y_j}(\gamma(t)) \end{pmatrix} \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}.$$

From the definition of Lie derivative, it follows that

$$\xi_i(t) = \frac{\partial H}{\partial y_i}(\gamma(t)), \quad \eta_i(t) = -\frac{\partial H}{\partial x_i}(\gamma(t)),$$

is a solution of the VE. We can use a generalization of D'Alambert's method to reduce our VE (see [8, 9] and see also [6]), obtaining the so-called *normal variational equation* (the NVE). We can see that the NVE is a linear system of rank  $2(n-1)$ . In the case of hamiltonian systems of 2-degrees of freedom, their NVE can be seen as second order linear homogeneous differential equation.

**1.3. Non-integrability tools.** Morales–Ramis theory is conformed by several results relating the existence of first integrals of  $H$  with the Galois group of the variational equations (see for example [8], [9] and see also [6]).

Most applications of Picard-Vessiot theory to integrability analysis are studied considering meromorphic functions, but for every equation considered throughout this paper, the point at infinity, of our particular solution, plays a transcendental role: the Galois group is mainly generated by Stokes phenomenon in the irregular singularity at infinity and by the exponential torus. So that we will only work with particular solutions in the context of meromorphic functions with certain properties of regularity near to the infinity point, that is, rational functions of the *positions and momenta*. Along this paper we will use the following result:

**Theorem 1.2** ([8]). *Let  $H$  be a hamiltonian in  $\mathbb{C}^{2n}$ , and  $\gamma$  a particular solution such that the NVE has irregular singularities at points of  $\gamma$  at infinity. Then, if  $H$  is completely integrable by rational functions, then the identity component of Galois Group of the NVE is abelian.*

**Remark 1.3.** *Here, the field of coefficients of the NVE is the field of meromorphic functions on  $\gamma$ .*

## 2. METHOD TO DETERMINE FAMILIES OF HAMILTONIANS WITH SPECIFIC NVE

This method was implemented in [2] as a generalization of the method shown in [11]. This section is devoted to this method.

Let us consider a two degrees of freedom classical hamiltonian,

$$H = \frac{y_1^2 + y_2^2}{2} + V(x_1, x_2).$$

$V$  is the *potential function*, and it is assumed to be analytical in some open subset of  $\mathbb{C}^2$ . The evolution of the system is determined by Hamilton equations:

$$\dot{x}_1 = y_1, \quad \dot{x}_2 = y_2, \quad \dot{y}_1 = -\frac{\partial V}{\partial x_1}, \quad \dot{y}_2 = -\frac{\partial V}{\partial x_2}.$$

Let us assume that the plane  $\Gamma = \{x_2 = 0, y_2 = 0\}$  is an invariant manifold of the hamiltonian. We keep in mind that the family of integral curves lying on  $\Gamma$  is parameterized by the energy  $h = H|_{\Gamma}$ , but we do not need to use it explicitly. We are interested in studying the linear approximation of the system near  $\Gamma$ . Since  $\Gamma$  is an invariant manifold, we have

$$\left. \frac{\partial V}{\partial x_2} \right|_{\Gamma} = 0,$$

so that the NVE for a particular solution

$$t \mapsto \gamma(t) = (x_1(t), y_1 = \dot{x}_1(t), x_2 = 0, y_2 = 0),$$

is written,

$$\dot{\xi} = \eta, \quad \dot{\eta} = - \left[ \frac{\partial^2 V}{\partial x_2^2}(x_1(t), 0) \right] \xi.$$

Let us define,

$$\phi(x_1) = V(x_1, 0), \quad \alpha(x_1) = -\frac{\partial^2 V}{\partial x_2^2}(x_1, 0),$$

and then we write the second order Taylor series in  $x_2$  for  $V$ , obtaining the following expression for  $H$

$$(1) \quad H = \frac{y_1^2 + y_2^2}{2} + \phi(x_1) - \alpha(x_1) \frac{x_2^2}{2} + \beta(x_1, x_2) x_2^3,$$

which is the *general form of a classical analytic hamiltonian, with invariant plane*  $\Gamma$ , providing that a Taylor expansion of the potential around  $\{x_2 = 0\}$  exist. The NVE associated to any integral curve lying on  $\Gamma$  is,

$$(2) \quad \ddot{\xi} = \alpha(x_1(t))\xi.$$

**2.1. General Method.** We are interested in computing hamiltonians of the family (1), such that its NVE (2) belongs to a specific family of Linear Differential Equations. Then we can apply our results about the integrability of this LDE, and Morales–Ramis theorem to obtain information about the non-integrability of such hamiltonians.

From now on, we will write  $a(t) = \alpha(x_1(t))$ , for a generic curve  $\gamma$  lying on  $\Gamma$ , parameterized by  $t$ . Then, the NVE is written

$$(3) \quad \ddot{\xi} = a(t)\xi.$$

**Problem.** Consider a differential polynomial  $Q(\eta, \dot{\eta}, \ddot{\eta}, \dots) \in \mathbb{C}[\eta, \dot{\eta}, \ddot{\eta}, \dots]$ , being  $\eta$  a differential indeterminate ( $Q$  is polynomial in  $\eta$  and a finite number of the successive derivatives of  $\eta$ ). Compute all hamiltonians in the family (1) verifying: for all any particular solution in  $\Gamma$ , the coefficient  $a(t)$  of the corresponding NVE is a differential zero of  $Q$ , in the sense that  $Q(a, \dot{a}, \ddot{a}, \dots) = 0$ .

We should notice that, for a generic integral curve  $\gamma(t) = (x_1(t), y_1 = \dot{x}_1(t))$  lying on  $\Gamma$ , (3) depends only of the values of functions  $\alpha$ , and  $\phi$ . It depends on  $\alpha(x_1)$ , since  $a(t) = \alpha(x_1(t))$ . We observe that the curve  $\gamma(t)$  is a solution of the restricted hamiltonian,

$$(4) \quad h = \frac{y_1^2}{2} + \phi(x_1)$$

whose associated hamiltonian vector field is,

$$(5) \quad X_h = y_1 \frac{\partial}{\partial x_1} - \frac{d\phi}{dx_1} \frac{\partial}{\partial y_1},$$

thus  $x_1(t)$  is a solution of the differential equation,  $\ddot{x}_1 = -\frac{d\phi}{dx_1}$ , and then, the relation of  $x_1(t)$  is given by  $\phi$ .

Since  $\gamma(t)$  is an integral curve of  $X_h$ , for any function  $f(x_1, y_1)$  defined in  $\Gamma$  we have

$$\frac{d}{dt} \gamma^*(f) = \gamma^*(X_h f),$$

where  $\gamma^*$  denote the usual pullback of functions. Then, using  $a(t) = \gamma^*(\alpha)$ , we have for each  $k \geq 0$ ,

$$(6) \quad \frac{d^k a}{dt^k} = \gamma^*(X_h^k \alpha),$$

so that,

$$Q(a, \dot{a}, \ddot{a}, \dots) = Q(\gamma^*(\alpha), \gamma^*(X_h \alpha), \gamma^*(X_h^2 \alpha), \dots).$$

There is an integral curve of the hamiltonian passing through each point of  $\Gamma$ , so that we have proven the following.

**Proposition 2.1.** *Let  $H$  be a hamiltonian of the family (1), and  $Q(a, \dot{a}, \ddot{a}, \dots)$  a differential polynomial with constants coefficients. Then, for each integral curve lying on  $\Gamma$ , the coefficient  $a(t)$  of the NVE (3) verifies  $Q(a, \dot{a}, \ddot{a}, \dots) = 0$ , if and only if the function*

$$\hat{Q}(x_1, y_1) = Q(\alpha, X_h \alpha, X_h^2 \alpha, \dots),$$

vanishes on  $\Gamma$ .

**Remark 2.2.** *In fact, the NVE of a integral curve depends on the parameterization. Our criterion does not depend on any choice of parameterization of the integral curves. This is simple, the NVE corresponding to different parameterizations of the same integral curve are related by a translation of time  $t$ . We just observe that a polynomial  $Q(a, \dot{a}, \ddot{a}, \dots)$  with constant coefficients is invariant of the group by translations of time. So that, if the coefficient  $a(t)$  of the NVE (3) for a certain parameterization of an integral curve  $\gamma(t)$  satisfied  $\{Q = 0\}$ , then it is also satisfied for any other right parameterization of the curve.*

Next, we will see that  $\hat{Q}(x_1, y_1)$  is a polynomial in  $y_1$  and its coefficients are differential polynomials in  $\alpha, \phi$ . If we write down the expressions for successive Lie derivatives of  $\alpha$ , we obtain

$$(7) \quad X_h \alpha = y_1 \frac{d\alpha}{dx_1},$$

$$(8) \quad X_h^2 \alpha = y_1^2 \frac{d^2 \alpha}{dx_1^2} - \frac{d\phi}{dx_1} \frac{d\alpha}{dx_1}$$

$$(9) \quad X_h^3 \alpha = y_1^3 \frac{d^3 \alpha}{dx_1^3} - y_1 \left( \frac{d}{dx_1} \left( \frac{d\phi}{dx_1} \frac{d\alpha}{dx_1} \right) + 2 \frac{d\phi}{dx_1} \frac{d^2 \alpha}{dx_1^2} \right)$$

$$(10) \quad X_h^4 \alpha = y_1^4 \frac{d^4 \alpha}{dx_1^4} - y_1^2 \left( \frac{d}{dx_1} \left( \frac{d}{dx_1} \left( \frac{d\phi}{dx_1} \frac{d\alpha}{dx_1} \right) + 2 \frac{d\phi}{dx_1} \frac{d^2 \alpha}{dx_1^2} \right) + 3 \frac{d^3 \alpha}{dx_1^3} \frac{d\phi}{dx_1} \right) + \left( \frac{d}{dx_1} \left( \frac{d\phi}{dx_1} \frac{d\alpha}{dx_1} \right) + 2 \frac{d\phi}{dx_1} \frac{d^2 \alpha}{dx_1^2} \right) \frac{d\phi}{dx_1}.$$

In general form we have,

$$(11) \quad X_h^{n+1} \alpha = y_1 \frac{\partial X_h^n \alpha}{dx_1} - \frac{d\phi}{dx_1} \frac{\partial X_h^n \alpha}{\partial y_1},$$

it inductively follows that they all are polynomials in  $y_1$ , in which their coefficients are differential polynomials in  $\alpha$  and  $\phi$ . If we write it down explicitly,

$$(12) \quad X_h^n \alpha = \sum_{n \geq k \geq 0} E_{n,k}(\alpha, \phi) y_1^k$$

we can see that the coefficients  $E_{n,k}(\alpha, \phi) \in \mathbb{C} \left[ \alpha, \phi, \frac{d^r \alpha}{dx_1^r}, \frac{d^s \phi}{dx_1^s} \right]$ , satisfy the following recurrence law,

$$(13) \quad E_{n+1,k}(\alpha, \phi) = \frac{d}{dx_1} E_{n,k-1}(\alpha, \phi) - (k+1) E_{n,k+1}(\alpha, \phi) \frac{d\phi}{dx_1}$$

with initial conditions,

$$(14) \quad E_{1,1}(\alpha, \phi) = \frac{d\alpha}{dx_1}, \quad E_{1,k}(\alpha, \phi) = 0 \quad \forall k \neq 1.$$

**Remark 2.3.** *The recurrence law (13) and the initial conditions (14), determine the coefficients  $E_{n,k}(\alpha, \phi)$ . We can compute the value of some of them easily:*

- $E_{n,n}(\alpha, \phi) = \frac{d^n \alpha}{dx_1^n}$  for all  $n \geq 1$ .
- $E_{n,k}(\alpha, \phi) = 0$  if  $n - k$  is odd, or  $k < 0$ , or  $k > n$ .

### 3. MAIN RESULT: FAMILIES OF HAMILTONIANS WITH CUARTIC NVES

**Theorem 3.1.** *Let  $H = T + V$  be a classical hamiltonian with invariant plane  $\Gamma$  such that the generic NVE along integral curves in  $\Gamma$  is a Hill-Schrödinger equation with cuartic polynomial coefficient. Then, the potential  $V$  is written in the form:*

$$V = \lambda_0 + P_4(x_1)x_2^2 + \beta(x_1, x_2)x_2^3,$$

where  $\lambda_0 \in \mathbb{C}$ ,  $P_4$  is a cuartic polynomial and  $\beta(x_1, x_2)$  is analytic around  $\Gamma$ .

*Proof.* Following our general method, the family potentials of potentials satisfying the assumptions of the theorem are them given by solutions  $\alpha(x_1)$ ,  $\phi(x_1)$  of the system of differential equations

$$E_{5,5}(\alpha, \phi) = 0, \quad E_{5,3}(\alpha, \phi) = 0, \quad E_{5,1}(\alpha, \phi) = 0.$$

The first equation is just the following,

$$E_{5,5}(\alpha, \phi) = \frac{d^5 \alpha}{dx_1^5} = 0,$$

and then we know that  $\alpha$  is a cuartic polynomial in  $x_1$ .

$$\alpha = a + bx_1 + cx_1^2 + dx_1^3 + ex_1^4.$$

where  $b, c, d, e$  are complex numbers and  $e$  does not vanish. Then we substitute  $\alpha$  for his value in the equations

$$E_{5,3}(\alpha, \phi) = 0, \quad E_{5,1}(\alpha, \phi) = 0,$$

obtaining the following system of differential equations in  $\phi(x_1)$ :

$$(L) \quad \begin{aligned} & (4ex_1^3 + 3dx_1^2 + 2cx_1 + b)\phi^{iv} + (60ex_1^2 + 30dx_1 + 10c)\phi'' + \\ & + (240ex_1 + 60d)\phi'' + 240e\phi' = 0, \end{aligned}$$

$$(NL) \quad \begin{aligned} & (18d + 72ex_1)(\phi')^2 + (b + 2cx_1 + 3dx_1^2 + 4ex_1^3)(\phi'')^2 + \\ & + (14c + 42dx_1 + 84ex_1^2)\phi'\phi'' + (b + 2cx_1 + 3dx_1^2 + 4ex_1^3)\phi'\phi''' = 0. \end{aligned}$$

Equations (L) and (NL) are ordinary differential equations in  $\phi'$  with some complex parameters. We can substitute a new unknown  $y$  for  $\phi'$  in order to reduce the order by one. Secondly, by a translation of  $x_1$  by an scalar value,

$$x = x_1 - \mu,$$

we can assume that one of the coefficients of the polynomial  $\alpha(x_1)$  vanish. From now on let us write,

$$\alpha(x) = a + bx + cx^2 + ex^4,$$

and let us study the system of differential equations:

$$(L2) \quad (4ex^3 + 2cx + b)y''' + (60ex^2 + 10c)y'' + 240exy' + 240ey = 0,$$

$$(NL2) \quad 72exy^2 + (b + 2cx + 4ex^3)(y')^2 + (14c + 84ex^2)yy' + (b + 2cx + 4ex^3)yy'' = 0.$$

The first equation (L2) is a linear equation in  $y$ . In this special case we will be able to completely solve the equation (L2) and then prove that solutions of (L2) do not satisfy the non-linear equation (NL2). Then, the only solution of the system is given by the function  $y = 0$ , that correspond to  $\phi = \lambda_0$  and then the potential

$$V = \phi + \frac{\alpha(x_1)}{2}x_1^2 + \beta(x_1, x_2)x_1^3,$$

is of the form given in the statement of the theorem.

### Solution of the equation (L2)

This linear equation is solvable by elemental methods. Fortunately, its Galois group is trivial, and then we can look for a fundamental system of solutions that are rational functions over  $x, a, b, c, e$ . The main problem, is that such a system of solutions do not always specializes to a particular system of solutions when determining the values of the parameters  $a, b, c, e$ . There are some values of the parameters that correspond to degeneration of the system of solutions. The equations of this *locus of degeneration* is given by the wronskian of the fundamental system. When the wronskian vanish, the fundamental system degenerates, and then there appear a different solution. We have to consider also these restricted problems independently.

First, we find the general solution of (L2), for generic values of the parameters, depending on arbitrary constants  $K_1, K_2, K_3$ :

$$(15) \quad y = \frac{K_1N_1 + K_2N_2 + K_3N_3}{D^3} = \frac{P}{D^3},$$

where,

$$D = 4ex^3 + 2cx + b,$$

$$N_1 = x(4ec^2x^5 - 42becx^4 - (6c^3 + 48eb^2)x^3 + 9b^2cx + 6b^3)$$

$$N_2 = x(8ecx^5 - 12bce^2x^4 - (24eb^2 + 12c^3)x^3 - 12bc^2x^2 + 3b^3)$$

$$N_3 = 8c^2e^2x^6 - 84bce^2x^5 - (12c^3e + 168b^2e^2)x^4 + 21b^3ex - 3b^2c^2$$

Let us study for which parameters the above expression is not the general solution of (L2). It happens if and only if the wronskian of the fundamental system of solutions vanish. We know that the wronskian of the fundamental solutions  $N_i/D$  vanish if and only if the wronskian of the numerators  $N_i$  vanish. We compute it, obtaining  $W(N_1, N_2, N_3) = 162c^3b^7 + 1296b^6c^4x + 3888b^5c^5x^2 + (2592b^6ec^3 + 5184b^4c^6)x^3 + (2592c^7b^3 + 15552b^5c^4e)x^4 + 31104b^4c^5ex^5 + (15552b^5c^3e^2 + 20736b^3c^6e)x^6 + 62208b^4c^4e^2x^7 + 62208b^3c^5e^2x^8 + 41472b^4c^3e^3x^9 + 82944b^3c^4e^3x^{10} + 41472b^3c^3e^4x^{12}$ . The equations of the vanishing of the wronskian are given by the coefficients of this polynomial. This system is simple to solve and it has two independent solutions, that we will consider independently.

- (a)  $\{b = 0\}$
- (b)  $\{c = 0\}$

**Case A,  $b = 0$ .**

If  $b$  vanish then the considered system of equations is:

$$(L3) \quad (4ex^3 + 2cx)y''' + (60ex^2 + 10c)y'' + 240exy' + 240ey = 0,$$

$$(NL3) \quad 72exy^2 + (2cx + 4ex^3)(y')^2 + (14c + 84ex^2)yy' + (2cx + 4ex^3)yy'' = 0.$$

we obtain a new general solution for this restricted case by direct integration. The general solution is:

$$(16) \quad y = \frac{K_1 N_{3,1} + K_2 N_{3,2} + K_3 N_{3,3}}{D_3^3} = \frac{P_3}{D_3^3},$$

where,

$$\begin{aligned} D_3 &= 2ex^2 + c, \\ N_{3,1} &= 6ex^2 - c, \\ N_{3,2} &= x(-3c + 2ex^2) \\ N_{3,3} &= (c^3 + 6ec^2x^2 + 16e^3x^6)x^{-3} \end{aligned}$$

In this case, the wronskian of the numerators is,

$$W(N_{3,1}, N_{3,2}, N_{3,3}) = \frac{96ec(6ec^2x^4 + 16e^3x^6 + 5c^3)}{x^4},$$

so that this system of solutions only degenerates when  $c = 0$  which is considered as a particular case of the following.

**Case B,  $c = 0$ .**

Let us finally consider the last case  $c = 0$ . The system of equations is now as follows.

$$(L4) \quad (4ex^3 + b)y''' + 60ex^2y'' + 240exy' + 240ey = 0,$$

$$(NL4) \quad 72exy^2 + (b + 4ex^3)(y')^2 + 84ex^2yy' + (b + 4ex^3)yy'' = 0.$$

The general solution of (L4) is given by,

$$(17) \quad y = \frac{K_1 N_{4,1} + K_2 N_{4,2} + K_3 N_{4,3}}{D_4^3} = \frac{P^3}{D_4^3}$$

where

$$\begin{aligned} D_4 &= 4ex^3 + b, \\ N_{4,1} &= x(b - 8ex^3), \\ N_{4,2} &= x^2(b - 2ex^3) \\ N_{4,3} &= (b^2 - 28ebx^3 + 16e^2x^6). \end{aligned}$$

We analyze the vanishing of the wronskian,  $W(N_{4,1}, N_{4,2}, N_{4,3}) = 2b^4 + 32eb^3x^3 + 192e^2b^3x^6 + 512be^3x^9 + 512e^4x^{12}$ . Because of the coefficient in the 12th power of  $x$ , this wronskian does not vanish for any of the considered values of the parameters, and then this general solution does not degenerates.

**Common solutions with the non-linear equation**

Here we look for solutions of the linear equation that also satisfy the considered non-linear equation. We directly substitute the general solution (15) of (L2) into (NL2). Then we obtain an rational expression:

$$\frac{Q(x; b, c, e, K_1, K_2, K_3)}{D^7} = 0$$

being  $Q$  a polynomial in  $x$  depending on the parameter  $b, c, e, K_1, K_2, K_3$ . Thus, we look for the values of the parameters that force  $Q$  to vanish. If we develop  $Q$  as differential polynomial in  $P$  and  $D$  we obtain the following expression:  $Q = (72exD + (-6 + 42c + 262ex^3)DD' + 21D'^2 + D^2 - 3DD'')P^2 + (14c + 84ex)D^2 - 6DD')PP' + D^2P^2$ . Note that the polynomials  $D, D', D''$  does not depend on the parameters  $K_i$ , and  $P, P'$  are linear in such parameters. It follows that  $Q$  is polynomial in  $x$  of degree 16 whose coefficients are homogeneous polynomials of degree two in the parameters  $K_1, K_2, K_3$ .

$$Q = \sum_{i=0}^{16} C_i(K_1, K_2, K_3, b, c, e)x^i = 0,$$

where:

$$C_i = (K_1, K_2, K_3) \begin{pmatrix} \lambda_{11}^i & \lambda_{12}^i & \lambda_{13}^i \\ \lambda_{21}^i & \lambda_{22}^i & \lambda_{23}^i \\ \lambda_{31}^i & \lambda_{32}^i & \lambda_{33}^i \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}.$$

Finally, the coefficients  $\lambda_{jk}^i$  are polynomials in the parameters  $b, c, e$ . The common solution of (L2) and (NL2) correspond to values of the parameters  $b, c, e, K_1, K_2, K_3$  that are solutions of the system of 17 algebraic equations:

$$C_i(K_1, K_2, K_3, b, c, e) = 0, \quad i = 0, \dots, 16.$$

Each equation  $C_i = 0$  is the equation of a cone 3-dimensional the affine space over the field  $\mathbb{C}(b, c, e)$ ; thus the equation of a conic curve in the projective plane  $\mathbb{P}(\mathbb{C}(b, c, e))$ ; of homogeneous coordinates  $K_1, K_2, K_3$ . Two conic curves intersect in 4 points. This simplifies the computations, that have to be carried symbolically, and lead to the incompatibility of the system, provided that  $b$  and  $c$  are different from zero.

Following the same schema, we analyze the exceptional case  $b = 0$ . We put the solution (16) of (L3) into the equation (NL3). We obtain an expression,

$$\frac{Q_3}{x^7 D_3^7} = 0,$$

where,  $Q_3 = (72exD_3 + (-6 + 42c + 262ex^3)D_3D_3' + 21D_3'^2 + D_3^2 - 3D_3D_3'')P_3^2 + (14c + 84ex)D_3^2 - 6D_3D_3')P_3P_3' + D_3^2P_3^2$ . Here, we find that  $Q_3$  is a polynomial in  $x$  of degree 18,

$$Q_3 = \sum_{i=0}^{18} E_i(c, e, K_1, K_2, K_3)x^i,$$

and again the system of algebraic equations  $\{E_i = 0\}$ , is qualitatively similar to the above system  $\{C_i = 0\}$ . The same analysis is carried out in the other exceptional case  $c = 0$ . In this last case we obtain that  $Q_4 = (72exD_4 + (-6 + 262ex^3)D_4D_4' + 21D_4'^2 + D_4^2 - 3D_4D_4'')P_4^2 + 84exD_4^2 - 6D_4D_4')P_4P_4' + D_4^2P_4^2$  is of degree 18 in  $x$ , and then we have a system  $\{F_i = 0\}$  19 algebraic equations that form an incompatible system. □

#### FINAL COMMENTS AND OPEN QUESTIONS

One open problem presented in [2] is the problem of determining families of classical hamiltonians with an invariant plane and NVE of Hill-Schrödinger type with polynomial coefficient of even degree greater than two, now the case of even degree

greater than four is still open.

The problem of analyzing the monodromy of the NVE of integral curves of a two degrees of freedom hamiltonian (both, classical and general) has been studied by Baider, Churchill and Rod at the beginning of the 90's (see [3]). Their method is quite different, they have imposed the monodromy group to verify some special properties that were translated as algebraic conditions in the hamiltonian functions. Their theory were restricted to the case of fuchsian groups, which in terms of Galois theory means regular singularities, while we work in the general case. The comparison of both methods should be done.

Another problem is to apply these method to higher variational equations.

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