The Manhattan Product of Digraphs *

F. Comellas, C. Dalfó, M.A. Fiol
Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya
{comellas,cdalfo,fiol}@ma4.upc.edu

October 3, 2007

Abstract

We give a formal definition of a new product of bipartite digraphs, the Manhattan product, and we study some of its main properties. It is shown that when all the factors of the above product are (directed) cycles, then the obtained digraph is the Manhattan street network. To this respect, it is proved that many properties of such networks, such as high symmetries and the presence of Hamiltonian cycles, are shared by the Manhattan product of some digraphs.

1 Introduction

We give a formal definition of a new product of bipartite digraphs, the Manhattan product, and we prove its main properties. It is shown that when all the factors of the above product are (directed) cycles, then the resulting digraph is just the Manhattan street network M_n, which was introduced simultaneously, and in different contexts, by Morillo *et al.* [6] and Maxemchuk [5] as an unidirectional regular mesh structure resembling locally the topology of the avenues and streets of Manhattan (or *l'Eixample* in downtown Barcelona).

Recall that a digraph $G = (V, A)$ consists of a set of *vertices* V, together with a set of *arcs* A, which are ordered pairs of vertices, $A \subset V \times V = \{(u, v) : u, v \in V\}$. An arc (u, v) is usually depicted as an arrow with *tail* u (initial vertex) and *head* v (end vertex); that is, $u \to v$. The *indegree* $\delta^-(u)$ (respectively, *outdegree* $\delta^+(u)$) of a vertex u is the number of arcs with tail (respectively, head) u. Then G is δ*-regular* when $\delta^-(u) = \delta^+(u) = \delta$ for every vertex $u \in V$. Given a digraph $G = (V, A)$, its *converse* digraph $\overline{G} = (V, \overline{A})$ is obtained from G by reversing all the orientations of the arcs in A; that is, $(u, v) \in \overline{A}$ if and only if $(v, u) \in A$. The standard definitions and basic results about graphs and digraphs not defined here can be found in [1, 2].

In this paper, we first recall the definition and some of the properties of the Manhattan street network (where the Manhattan product takes its name from). Afterwards we introduce the Manhattan product of (bipartite) digraphs. It is shown that when all the factors are (directed) cycles, then the obtained digraph is just the Manhattan street network. To this respect, it is proved that many properties of such networks, such as high symmetries and the presence of Hamiltonian cycles, are shared by the Manhattan product of some digraphs.

*Research supported by the Ministerio de Educación y Ciencia, Spain, and the European Regional Development Fund under projects MTM2005-08990-C02-01 and TEC2005-03575 and by the Catalan Research Council under project 2005SGR00256.

2 Manhattan street networks

In this section, we recall the definition [3] and some basic properties [3, 4] of a class of toroidal directed networks, commonly known are Manhattan street networks.

Given n *even* positive integers $N_1, N_2, \ldots, N_n$, the *n-dimensional Manhattan street network* $M_n = M(N_1, N_2, \ldots, N_n)$ is a digraph with vertex set $V(M_n) = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \ldots \times \mathbb{Z}_{N_n}$. Thus, each of its vertices is represented by an n-vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, with $0 \le u_i \le N_i - 1$, $i = 1, 2, \ldots, n$. The arc set $A(M_n)$ is defined by the following adjacencies (here called *i-arcs*):

$$(u_1, \ldots, u_i, \ldots, u_n) \quad \to \quad (u_1, \ldots, u_i + (-1)^{\sum_{j \neq i} u_j}, \ldots, u_n) \qquad (1 \le i \le n). \qquad (1)$$

Therefore, M_n is a n-regular digraph on $N = \prod_{i=1}^{n} N_i$ vertices.

Its properties are the following:

- Stable sets: The n-dimensional Manhattan street network M_n is both a 2^n-partite and bipartite digraph.
- Vertex-symmetry: The n-dimensional Manhattan street network M_n is a vertex-symmetric digraph.
- Line digraph: For any N_1, N_2, the 2-dimensional Manhattan street network $M_2(N_1, N_2)$ is a line digraph.
- Diameter: The diameter of the n-dim Manhattan street network $M_n = M(N_1, N_2, \ldots, N_n)$, $N_i > 4$, $i = 1, 2, \ldots, n$, is
  (*a*) $D(M_n) = \frac{1}{2} \sum_{i=1}^{n} N_i + 1$, if $N_i \equiv 0 \pmod 4$ for any $1 \le i \le n$;
  (*b*) $D(M_n) = \frac{1}{2} \sum_{i=1}^{n} N_i$, otherwise.
- Hamiltonicity: The n-dimensional Manhattan street network M_n is Hamiltonian.

3 The Manhattan product and its basic properties

In this section, we present an operation on (bipartite) digraphs which, as a particular case, produces a Manhattan street network. With this aim, let $G_i = (V_i, A_i)$ be n bipartite digraphs with independent sets $V_i = V_{i0} \cup V_{i1}$, $N_i = |V_i|$, $i = 1, 2, \ldots, n$. Let π be the characteristic function of $V_{i1} \subset V_i$ for any i; that is,

$$\pi(u) = \begin{cases} 0 & \text{if } u \in V_{i0}, \\ 1 & \text{if } u \in V_{i1}. \end{cases}$$

Then, the *Manhattan product* $M_n = G_1 \# G_2 \# \cdots \# G_n$ is the digraph with vertex set $V(M_n) = V_1 \times V_2 \times \cdots \times V_n$; and each vertex $(u_1, \ldots, u_i, \ldots, u_n)$ is adjacent to vertices $(u_1, \ldots, v_i, \ldots, u_n)$, $1 \le i \le n$, when

- $v_i \in \Gamma^+(u_i)$ if $\sum_{j \neq i} \pi(u_j)$ is even,
- $v_i \in \Gamma^-(u_i)$ if $\sum_{j \neq i} \pi(u_j)$ is odd.

Fig. 1 shows an example of the Manhattan product of the circulant digraph on 6 vertices and steps 1 and 3, $Cay(\mathbb{Z}_6, \{1, 3\})$ by the symmetric complete digraph on 2 vertices, K_2^*.

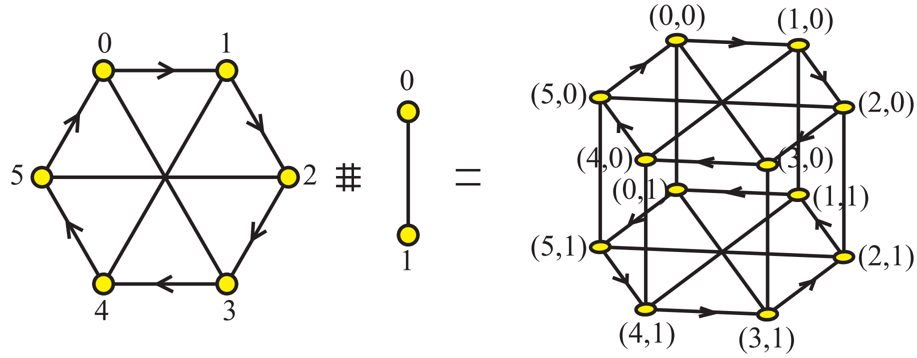

Figure 1: The Manhattan product $Cay(\mathbb{Z}_6, \{1,3\}) \# K_2^*$ (undirected lines stand for pairs of arcs in opposite directions).

Thus, if every G_i is δ_i-regular, then M_n is a δ-regular digraph, $\delta = \sum \delta_i$, on $N = \prod_{i=1}^{n} N_i$ vertices.

Some of the basic properties of the Manhattan product, which are a generalization of the properties of the Manhattan street networks given in [3], are presented in the following proposition:

Proposition 3.1. *The Manhattan product $H = G_1 \# G_2 \# \cdots \# G_n$ satisfies the following properties:*

(a) *The Manhattan product has the associative and commutative properties.*

(b) *H is a bipartite and 2^n-partite digraph.*

(c) *For any $n-k$ fixed vertices $x_i \in V_i$, $i = k+1, k+2, \ldots, n$, the subdigraph of H induced by the vertices $(u_1, u_2, \ldots, u_k, x_{k+1}, \ldots, x_n)$ is either the Manhattan product $H_k = G_1 \# G_2 \# \cdots \# G_k$ or its converse $\overline{H}_k$, depending on if $\alpha := \sum_{i=k+1}^{n} \pi(x_i)$ is even or odd, respectively.*

(d) *There exists an homomorphism from H to the symmetric digraph of the hypercube Q_n^*.*

(e) *If each G_i, $i = 1, 2, \ldots, n$, is isomorphic to its converse, then H also is.*

Proof. We only prove the last property because the others can be proved similarly as those of the Manhattan street network in [3]. As the Manhattan product is associative, we only need to prove it for the case $H = G_1 \# G_2$. By hypothesis, $G_i \cong \overline{G}_i$, then there exist isomorphisms ψ_i, such that $\Gamma^{\pm}_{G_i}(\psi_i(u_i)) = \psi_i(\Gamma^{\mp}_{G_i}(u_i))$, for all $u_i \in V_i$. As ψ_i is a mapping between stable sets, the parity π in $\overline{G}_i$ can be defined in such a way that $\pi(u_i)$ is even if and only if $\pi(\psi_i(u_i))$ is also even. Then, the mapping Ψ defined in H as

$$\Psi(u_1, u_2) := (\psi_1(u_1), \psi_2(u_2))$$

is the automorphism from H to its converse $\overline{H}$. Indeed, assuming that, for instance, $\pi(u_1), \pi(u_2)$ are even, we have

$$\begin{aligned}
\Psi(\Gamma^+_H(u_1,u_2)) &= \Psi(\Gamma^+_{G_1}(u_1),u_2) \cup \Psi(u_1,\Gamma^+_{G_2}(u_2)) \\
&= (\psi_1(\Gamma^+_{G_1}(u_1)),\psi_2(u_2)) \cup (\psi_1(u_1),\psi_2(\Gamma^+_{G_2}(u_2))) \\
&= (\Gamma^-_{G_1}(\psi_1(u_1)),\psi_2(u_2)) \cup (\psi_1(u_1),\Gamma^-_{G_2}(\psi_2(u_2))) \\
&= \Gamma^-_H(\psi_1(u_1),\psi_2(u_2)) \\
&= \Gamma^-_H(\Psi(u_1,u_2)).
\end{aligned}$$

The other case depending on the parities of $\pi(u_1), \pi(u_2)$ can be proved similarly. □

As an example of a Manhattan product satisfying the property 3.1(e), see again Fig. 1.

4 Manhattan product and Manhattan street network

In this section we shown the relationship between the digraphs obtained by the Manhattan product and the Manhattan street networks.

Lemma 4.1. *The Manhattan product of directed cycles with an even number N_i of vertices is a Manhattan street network. More precisely, if $G_i = C_{N_i}$, then*

$$C_{N_1} \# C_{N_2} \# \cdots \# C_{N_n} = M(N_1, N_2, \ldots, N_n).$$

Proof. Each cycle C_{N_i} has set of vertices $V_i = \mathbb{Z}_{N_i}$, and adjacencies $\Gamma^+(u_i) = \{u_i + 1 \pmod{N_i}\}$ and $\Gamma^-(u_i) = \{u_i - 1 \pmod{N_i}\}$, such that V_{i0} and V_{i1} are the sets of even and odd vertices, respectively. Thus, the set of vertices in the Manhattan product of directed cycles is $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \ldots \times \mathbb{Z}_{N_n}$ and each vertex $(u_1, \ldots, u_i, \ldots, u_n)$ is adjacent to the vertices $(u_1, \ldots, v_i, \ldots, u_n)$, $1 \le i \le n$, when

- $v_i = u_i + 1$ if and only if $\sum_{j \ne i} \pi(u_j)$ is even and, hence, $\sum_{j \ne i} u_j$ is also even,
- $v_i = u_i - 1$ if and only if $\sum_{j \ne i} \pi(u_j)$ is odd and, hence, $\sum_{j \ne i} u_j$ is also odd,

which corresponds to the definition of the Manhattan street network. □

Another expected result of the Manhattan product is the following:

Lemma 4.2. *The Manhattan product of two Manhattan street networks is a Manhattan network. More precisely, if $M^1 = M(N^1_1, N^1_2, \ldots, N^1_{n_1})$ and $M^2 = M(N^2_1, N^2_2, \ldots, N^2_{n_2})$, then*

$$M^1 \# M^2 = M,$$

where $M = M(N^1_1, \ldots, N^1_{n_1}, N^2_1, \ldots, N^2_{n_2})$.

Proof. Both M^1 and M^2 are bipartite digraphs with vertex sets $V^\alpha = \mathbb{Z}_{N^\alpha_1} \times \mathbb{Z}_{N^\alpha_2} \times \cdots \times \mathbb{Z}_{N^\alpha_{n_\alpha}}$, $\alpha = 1, 2$; whereas $M^1 \# M^2$ has vertex set $V = V^1 \times V^2$. Let $V(M)$ be the vertex set of M. Then, we claim that the natural mapping $\Psi : V \to V(M)$, defined by $\Psi(\boldsymbol{u}^1, \boldsymbol{u}^2) = (u^1_1, \ldots, u^1_{n_1}, u^2_1, \ldots, u^2_{n_2})$ is an isomorphism between the corresponding digraphs. In proving this, let V^α_0 and V^α_1 be the stable sets of M^α constituted, respectively, by the vertices $\boldsymbol{u}^\alpha = (u^\alpha_1, \ldots, u^\alpha_{n_\alpha})$ whose sum of components $\sum_{k=1}^{n_\alpha} u^\alpha_k$ is even or odd. With this convention, each vertex $(\boldsymbol{u}^1, \boldsymbol{u}^2)$ of the Manhattan product $M^1 \# M^2$ is adjacent to the vertices $(\boldsymbol{v}^1, \boldsymbol{u}^2)$ and $(\boldsymbol{u}^1, \boldsymbol{v}^2)$ where, for the first ones,

- $\boldsymbol{v}^1 \in \Gamma^+(\boldsymbol{u}^1)$ (in M^1) if $\pi(\boldsymbol{u}^2)$, and hence $\sum_{k=1}^{n_2} u^2_k$, is even;

- $\boldsymbol{v}^1 \in \Gamma^-(\boldsymbol{u}^1)$ (in M^1) if $\pi(\boldsymbol{u}^2)$, and hence $\sum_{k=1}^{n_2} u_k^2$, is odd.

In the first case,

$$
\begin{aligned}
(\boldsymbol{v}^1, \boldsymbol{u}^2) &\xrightarrow{\Psi} (u_1^1, \ldots, u_i^1 + (-1)^{\sum_{j\neq i} u_j^1}, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2) \\
&= (u_1^1, \ldots, u_i^1 + (-1)^{\sum_{j\neq i} u_j^1 + \sum_{k=1}^{n_2} u_k^2}, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2) \qquad (1 \leq i \leq n_1).
\end{aligned}
$$

Analogously, in the second case,

$$
\begin{aligned}
(\boldsymbol{v}^1, \boldsymbol{u}^2) &\xrightarrow{\Psi} (u_1^1, \ldots, u_i^1 - (-1)^{\sum_{j\neq i} u_j^1}, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2) \\
&= (u_1^1, \ldots, u_i^1 + (-1)^{\sum_{j\neq i} u_j^1 + \sum_{k=1}^{n_2} u_k^2}, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2) \qquad (1 \leq i \leq n_1).
\end{aligned}
$$

Altogether, we obtain the vertices adjacent to $\Psi(\boldsymbol{u}^1, \boldsymbol{u}^2) = (u_1^1, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2)$ in M (through all the i-arcs, $1 \leq i \leq n_1$). The adjacencies through the other i-arcs, $n_1 + 1 \leq i \leq n_1 + n_2$ come from the vertices $(\boldsymbol{u}^1, \boldsymbol{v}^2)$. $\square$

5 Symmetries

In this section we study the symmetries in the digraphs obtained by the Manhattan product.

Lemma 5.1. *Let G_i be vertex-symmetric digraphs, which are isomorphic to their converses, $1, 2, \ldots, n$. Then, the Manhattan product $H = G_1 \# G_2 \# \cdots \# G_n$ is vertex-symmetric.*

Proof. As before, let $G_i = (V_i, A_i)$ be digraphs with $V_i = V_{i0} \cup V_{i1}$, $i = 1, 2, \ldots, n$.

First, we show that there exists an automorphism Φ in H, which transforms a vertex $(u_1, u_2, \ldots, u_n)$ into a vertex $(v_1, v_2, \ldots, v_n)$, such that $u_i, v_i \in V_{ij_i}$, for each $i \in \{1, 2, \ldots, n\}$ and some $j_i \in \{0, 1\}$ (that is, both components u_i, v_i are in the same stable set). By hypothesis, there exist automorphisms ϕ_i in G_i, $\Gamma_{G_i}^+(\phi_i(w_i)) = \phi_i(\Gamma_{G_i}^+(w_i))$, for every $w_i \in V_i$, such that $\phi_i(u_i) = v_i$. Then, we define

$$\Phi(w_1, w_2, \ldots, w_n) := (\phi_1(w_1), \phi_2(w_2), \ldots, \phi_n(w_n)).$$

Then, assuming that $\sum_{j\neq i} \pi(w_j)$ is even and, hence, $\sum_{j\neq i} \pi(\phi_j(w_j))$ is also even, we have

$$
\begin{aligned}
\Phi(\Gamma_H^+(w_1, \ldots, w_i, \ldots, w_n)) &= \Phi(w_1, \ldots, \Gamma_{G_i}^+(w_i), \ldots, w_n) \\
&= (\phi_1(w_1), \ldots, \phi_i(\Gamma_{G_i}^+(w_i)), \ldots, \phi_n(w_n)) \\
&= (\phi_1(w_1), \ldots, \Gamma_{G_i}^+(\phi_i(w_i)), \ldots, \phi_n(w_n)) \\
&= \Gamma_H^+(\phi_1(w_1), \ldots, \phi_i(w_i), \ldots, \phi_n(w_n)) \\
&= \Gamma_H^+(\Phi(w_1, \ldots, w_i, \ldots, w_n)),
\end{aligned}
$$

which proves that Φ is an automorphism. The proof is similar for $\sum_{j\neq i} \pi(w_j)$ odd, by using $\Gamma_{G_i}^-(\phi_i(w_i)) = \phi_i(\Gamma_{G_i}^-(w_i))$.

Moreover, we need an automorphism Ψ, which transforms a vertex $(u_1, \ldots, u_i, \ldots, u_n)$ into a vertex $(v_1, \ldots, v_i, \ldots, v_n)$, such that, for $k \neq i$, $u_k, v_k \in V_{kj_k}$ as before, while u_i and v_i belong to different stable sets, for example, $u_i \in V_{i0}$ and $v_i \in V_{i1}$. In this case, the automorphism Ψ is built up in the following way. As each G_i is isomorphic to its converse, there exist automorphisms ψ_k, with $k \neq i$, from G_k to $\overline{G}_k$, $\Gamma_{G_k}^+(\psi_k(w_k)) = \psi_k(\Gamma_{G_k}^-(w_k))$,

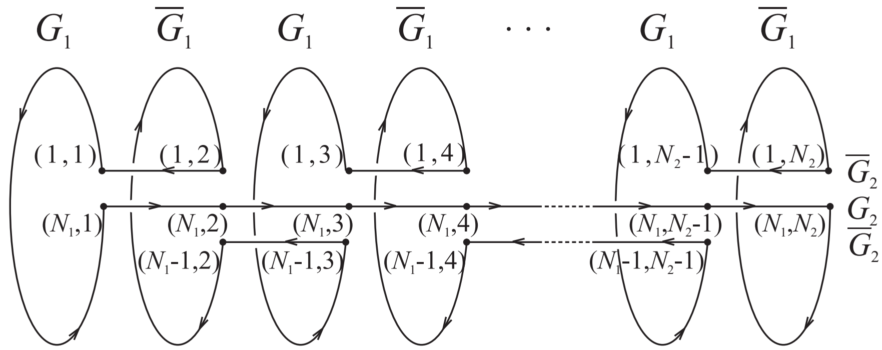

Figure 2: A Hamiltonian cycle in the Manhattan product $G_1 \# G_2$.

for every $w_k \in V_k$, such that $\psi_k(u_k) = v_k$; and $\psi_i = \phi_i$ (as in the first case). Then, we define Ψ as

$$\Psi(w_1, \ldots, w_i, \ldots, w_n) := (\psi_1(w_1), \ldots, \psi_i(w_i), \ldots, \psi_n(w_n)).$$

Let us now assume that $k = 1 \neq i$ and that $\sum_{j\neq 1} \pi(w_j)$ is even, so that, $\pi(\phi_i(w_i)) + \sum_{j\neq 1,i} \pi(\psi_j(w_j))$ is odd. Then, we have

$$\begin{aligned}
\Psi(\Gamma^+_H(w_1, \ldots, w_i, \ldots, w_n)) &= \Psi(\Gamma^+_{G_1}(w_1), \ldots, w_i, \ldots, w_n) \\
&= (\psi_1(\Gamma^+_{G_1}(w_1)), \ldots, \phi_i(w_i), \ldots, \psi_n(w_n)) \\
&= (\Gamma^-_{G_1}(\psi_1(w_1)), \ldots, \phi_i(w_i), \ldots, \psi_n(w_n)) \\
&= \Gamma^+_H(\psi_1(w_1), \ldots, \phi_i(w_i), \ldots, \psi_n(w_n)) \\
&= \Gamma^+_H(\Psi(w_1, \ldots, w_i, \ldots, w_n)).
\end{aligned}$$

Thus, Ψ is an automorphism. For the case $\sum_{j\neq 1} \pi(w_j)$ odd, the proof is similar, using $\Gamma^-_{G_k}(\psi_k(w_k)) = \psi_k(\Gamma^+_{G_k}(w_k))$. On the other hand, the case $k = i$ is as before, because assuming that $\sum_{j\neq i} \pi(w_j)$ is even, hence $\sum_{j\neq i} \pi(\psi_j(w_j))$ is also even. This completes the proof. □

6 Hamiltonian Cycles

Next we give a result on the Hamiltonicity of the Manhattan product of two Hamiltonian digraphs, as a generalization of a theorem in [3] about the hamiltonicity of the Manhattan street network.

Theorem 6.1. *If G_1 and G_2 have both a Hamiltonian path, then their Manhattan product $H = G_1 \# G_2$ is Hamiltonian.*

Proof. We use the same idea as in the proof of Theorem 5.1 in [3], which allows to construct a Hamiltonian cycle in H, from the Hamiltonian paths in G_1 and G_2, say $1 \to 2 \to \cdots \to N_1$ and $1' \to 2' \to \cdots \to N_2$ respectively. With this aim, we appropriately joint N_2 Hamiltonian paths (some of them without an arc) of N_2 subdigraphs isomorphic to G_1 or $\overline{G}_1$ (see Prop. 3.1(c)). Such paths are joined by using three copies of the Hamiltonian path (two of them with alternative arc removed) of subdigraphs isomorphic to G_2 or $\overline{G}_2$. See the selfexplanatory Fig. 2. □

References

[1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications,* Springer Monographs in Mathematics, Springer, London, 2003.

[2] G. Chartrand and L. Lesniak, *Graphs & Digraphs,*, Chapman and Hall, London (1996), third edition.

[3] F. Comellas, C. Dalfó and M.A. Fiol, The multidimensional Manhattan networks, *Siam J. Discrete Math.*, submitted.

[4] M.A. Fiol, J.L.A Yebra and I. Alegre, Line digraph iterations in the (d,k) digraph problem, *IEEE Trans. Comput.* **C-33** (1984), no. 5, 400–403.

[5] N.F. Maxemchuk, Routing in the Manhattan Street Network, *IEEE Trans. Commun.* **35** (1987), no. 5, 503–512.

[6] P. Morillo, M.A. Fiol, and J. Fàbrega, The diameter of directed graphs associated to plane tessellations, *Ars Comb.* **20A** (1985), 17–27.