MODEL CATEGORIES AND CUBICAL DESCENT

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ABSTRACT. We prove that the subcategory of fibrant objects of a simplicial model category is a cohomological descent category, in the sense of Guillén and Navarro, if and only if an acyclicity criterion holds.

1. INTRODUCTION

Let k be a field of characteristic zero, $Sm(k)$ the category of smooth schemes over k and $Sch(k)$ the category of separated and finite type schemes over k. Guillén and Navarro have proved [6] an extension result for cohomological functors defined on $Sm(k)$ to cohomological functors defined on $Sch(k)$. Classical cohomogical functors take values in the category of graded abelian groups or in the category of abelian chain complexes. In order to apply their main result to non-abelian situations, such as the rational homotopy type of $\mathbb{C}$-schemes or to motives of singular varieties, they introduced the notion of (cohomological) descent category, a higher category variation of Verdier triangulated categories, as a good class of categories in which cohomology theories take values.

A *descent category* [6] is, essentially, a triple $(\mathcal{D}, E, \mathbf{s})$ given by a cartesian category $\mathcal{D}$ with initial object 0, a saturated class of morphisms E of $\mathcal{D}$, called *weak equivalences*, and for every cubical type $\square$ (see section 4.1) a functor

$$\mathbf{s}_\square : (\square, \mathcal{D}) \to \mathcal{D},$$

called *simple*, natural in $\square$ in a precise sense, which satisfies the following properties:

1.*Multiplicativity.* The simple of an object X considered as a diagram is isomorphic to X, and for every pair $(\mathbf{X}, \mathbf{Y})$ of $\square$-diagrams there is an isomorphism $\mathbf{s}_\square(\mathbf{X} \times \mathbf{Y}) \to \mathbf{s}_\square\mathbf{X} \times \mathbf{s}_\square\mathbf{Y}$.

2.*Factorisation.* For every $\square \times \square'$-diagram $\mathbf{X} = (\mathbf{X}_{\alpha\beta})$ there is an isomorphism $\mu : \mathbf{s}_{\alpha\beta}\mathbf{X}_{\alpha\beta} \to \mathbf{s}_\alpha\mathbf{s}_\beta\mathbf{X}_{\alpha\beta}$.

3.*Exactness.* Let $f : \mathbf{X} \to \mathbf{Y}$ be a morphism of $\square$-diagrams. If it is a pointwise weak equivalence, then the morphism $\mathbf{s}_\square f : \mathbf{s}_\square\mathbf{X} \to \mathbf{s}_\square\mathbf{Y}$ is a weak equivalence.

4.*Acyclicity criterion.* A morphism $f : X_0 \to X_1$ is a weak equivalence if and only if the simple of the $\square_1$-diagram

$$X_0 \xrightarrow{f} X_1 \leftarrow 0$$

is acyclic, that is, the initial object of $\mathcal{D}$ is weakly equivalent to it.

In fact, an extended acyclicity criterion must hold:

4'.*Extended acyclicity criterion.* For every augmented diagram $\mathbf{X}^+$ of type $\square_n^+$, the morphism $\mathbf{X}_0 \to \mathbf{s}_{\square_n}\mathbf{X}$ is a weak equivalence if and only if the morphism $0 \to \mathbf{s}_{\square^+}\mathbf{X}^+$ is a weak equivalence.

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One problem that appears is to have sufficiently many examples of descent categories. Model categories, with the homotopy limit functor as simple functor, are natural candidates to be descent categories, as the homotopy limit has the factorisation and exactness properties for fibrant objects. We prove that the subcategory of fibrant objects of a simplicial model category is a cohomological descent category if and only if the acyclicity criterion holds. In particular, in a simplicial model category the extended acyclicity criterion is equivalent to the acyclicity criterion.

All stable model categories satisfy acyclicity criterion. Thus we have as a corollary that the category of fibrant spectra is a cohomological descent category. This result has applications in algebraic K-theory [12].

We remark that the category of topological spaces and homotopy weak equivalences does not verify the acyclicity criterion. We have to take homology isomorphisms. We prove that the category of CW-complexes with h_*-isomorphisms is a homological descent category, where h is a homology theory.

We restrict to simplicial model categories, where an homotopy limit functor is defined. This should not be an important restriction, as it is known that a cofibrantly generated, proper model category whith a realization axiom is Quillen equivalent to a simplicial model category [14].

2. Recollections and notations

2.1. Model categories were introduced by Quillen [13]. We use the definition adopted by Hirschhorn [7, 7.1], which has stronger conditions than Quillen's (all small limits and colimits are required to exist, and also functorial factorisations in axiom five). Modern references of model categories use this definition, but we observe that in our case the stronger conditions are not necessary.

Given $\mathcal{M}$ a simplicial model category, tensoring an object X by a simplicial set K is denoted by $X \otimes K$, and $F(K, X)$ denotes the cotensorisation (usually denoted by X^K).

By adjointness properties [4, II 2.1 and 2.2], $F(K, -)$ preserves limits and $F(-, X)$ converts colimits to limits. We also have $F(K \times L, X) \cong F(K, F(L, X))$ [7, 9.1.11].

2.2. A model category is *pointed* if the initial and final objects coincide. In this case the initial and final object is denoted by $*$. Let $*$ also denote the one point simplicial set. There are isomorphisms $F(*, X) = X$ and $F(K, *) = *$.

An object X in a pointed model category is *acyclic* if the natural morphism $* \to X$, or,equivalently, if the morphism $X \to *$ is a weak equivalence.

2.3. Recall that X is *fibrant* if $X \to *$ is a fibration. Fibrant objects are closed by cotensoring and by taking finite products.

2.4. The homotopy category $Ho(\mathcal{C})$ of a model category $\mathcal{C}$ is obtained by inverting the weak equivalences. Weak equivalences are *saturated*: a morphism $f : X \to Y$ is a weak equivalence if, and only if, it is an isomorphism in the homotopy category [7, 8.3.10].

2.5. If $\mathcal{C}$ is a small category, $B\mathcal{C}$ denotes the classifying space or nerve of $\mathcal{C}$. If α is an object of $\mathcal{C}$, $\mathcal{C} \downarrow \alpha$ denotes the category of objects of $\mathcal{C}$ over α. We use the notations and definitions of [7] concerning classifying spaces, overcategories, homotopy limits and homotopy cofinal functors.

3. Homotopy limits

In this section we recall briefly the definition of the homotopy limit of a diagram in a simplicial model category and its main properties.

3.1. Given $\mathcal{C}$ a small category and $\mathcal{M}$ a category, a *$\mathcal{C}$-codiagram of $\mathcal{M}$*, or a *codiagram of type* $\mathcal{C}$ is a functor $\mathcal{C} \to \mathcal{M}$ and $(\mathcal{C}, \mathcal{M})$ denotes the category of codiagrams of type $\mathcal{C}$.

If X is an object of $\mathcal{M}$, $\mathcal{C} \times X$ denotes the constant codiagram of type $\mathcal{C}$ (with all morphisms equal to the identity of X).

We use the prefix co- here to be consistent with the terminology used in [6], where a diagram of type $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to \mathcal{M}$.

Definition 3.1. Let $\mathbf{X}$ be $\mathcal{C}$-codiagram of a simplicial model category $\mathcal{M}$. The *homotopy limit* of $\mathbf{X}$, $\operatorname{holim} \mathbf{X}$, is the equaliser of the morphisms

$$\prod_{\alpha \in Ob(\mathcal{C})} F(B(\mathcal{C}\downarrow\alpha), \mathbf{X}_\alpha) \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{(\sigma:\alpha\to\alpha')\in\mathcal{C}} F(B(\mathcal{C}\downarrow\alpha), \mathbf{X}_{\alpha'})$$

where the projection of ϕ in the factor $\sigma : \alpha \to \alpha'$ is the composition of the natural projection from the product with the morphism

$$\sigma_*^{1_{B(\mathcal{C}\downarrow\alpha)}} : F(B(\mathcal{C}\downarrow\alpha), \mathbf{X}_\alpha) \to F(B(\mathcal{C}\downarrow\alpha), \mathbf{X}_{\alpha'})$$

and the projection of ψ to the factor $\sigma : \alpha \to \alpha'$ is the composition of the natural projection from the product with the morphism

$$F(B(\sigma_*), 1_{\mathbf{X}_{\alpha'}}) : F(B(\mathcal{C}\downarrow\alpha'), \mathbf{X}_{\alpha'}) \to F(B(\mathcal{C}\downarrow\alpha), \mathbf{X}_{\alpha'}),$$

where $\sigma_* : (\mathcal{C}\downarrow\alpha) \to (\mathcal{C}\downarrow\alpha')$.

Example 3.2. Given an object X of a simplicial model category $\mathcal{M}$, $\operatorname{holim} \mathcal{C} \times X = F(B\mathcal{C}, X)$, as is easily seen from [2, XI, 2.3].

3.2. We recall the basic properties of homotopy limits in a simplicial model category $\mathcal{M}$. Observe that homotopy invariance and cofinality require pointwise fibrant diagrams. We could drop the fibrant hypothesis doing a functorial fibrant replacement in the definition of the homotopy limit. In that case functorial factorizations in the definition of model category are necessary.

3.2.1. The homotopy limit is an end. The homotopy limit of a $\mathcal{C}$-codiagram is the end of the functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{M}$, $(\alpha, \alpha') \mapsto F(B(\mathcal{C}\downarrow\alpha), \mathbf{X}_{\alpha'})$. We can write

$$\operatorname{holim} \mathbf{X} = \int_\alpha F(B(\mathcal{C}\downarrow\alpha), \mathbf{X}_\alpha)$$

with the notation of [10] (cf. [7, 18.3.2 and 18.3.6]). Therefore end properties as Fubini theorem hold [10].

3.2.2. The homotopy limit is functorial with respect to both variables:

a) If $\mathbf{X}$ and $\mathbf{Y}$ are $\mathcal{C}$-codiagrams, a morphism $f : \mathbf{X} \to \mathbf{Y}$ induces a morphism

$$\operatorname{holim} f : \operatorname{holim} \mathbf{X} \to \operatorname{holim} \mathbf{Y}.$$

b) If $F : \mathcal{C} \to \mathcal{D}$ is a functor between small categories and $\mathbf{X}$ a $\mathcal{D}$-codiagram, a $\mathcal{C}$-codiagram $F^*\mathbf{X}$ is induced. Then there is a natural morphism

$$\operatorname*{holim}_{\mathcal{D}} \mathbf{X} \to \operatorname*{holim}_{\mathcal{C}} F^*\mathbf{X}$$

induced by the morphisms $F_* : B(\mathcal{C}\downarrow\alpha) \to B(\mathcal{D}\downarrow F\alpha)$ [7, 19.1.8].

3.2.3. The homotopy limit is particularly well behaved with respect to pointwise fibrant diagrams:

i) If $\mathbf{X}$ and $\mathbf{Y}$ are pointwise fibrant and f is a pointwise fibration, then $\operatorname{holim} f$ is a fibration [7, 18.5.1]. In particular, the homotopy limit of a pointwise fibrant diagram is fibrant.
ii) *Homotopy invariance* [7, 18.5.3]. If $\mathbf{X}$ and $\mathbf{Y}$ are pointwise fibrant and f is a pointwise weak equivalence, then $\operatorname{holim} f$ is a weak equivalence of fibrant objects.
iii) *Cofinality theorem* [7, 19.6.7b]. If the functor $F : \mathcal{C} \to \mathcal{D}$ is homotopy left cofinal and $\mathbf{X}$ is pointwise fibrant, the natural morphism $\operatorname{holim}_{\mathcal{D}} \mathbf{X} \to \operatorname{holim}_{\mathcal{C}} F^*\mathbf{X}$ is a weak equivalence.

3.2.4. The homotopy limit considered as a functor from the category of $\mathcal{C}$-codiagrams to $\mathcal{M}$ preserves limits. This property is deduced as in [16, lemma 5.11], where is stated for spectra.

3.3. **Homotopy fibre and acyclicity criterion.** We define the *homotopy fibre* of a morphism $f : X \to Y$ in a pointed simplicial model category $\mathcal{M}$ as $\operatorname{hofib} f = \operatorname{holim}(X \to Y \leftarrow *)$. The *fibre* of f is $\lim(X \to Y \leftarrow *)$.

Definition 3.3 (Acyclicity criterion)**.** A simplicial model category $\mathcal{M}$ satisfyies the *acyclicity criterion* if a morphism $f : X \to Y$ with X and Y fibrant is a weak equivalence if and only if the homotopy fibre $\operatorname{hofib} f$ is acyclic.

4. Simple functor of a cubical codiagram

In this section we define the simple functor of a codiagram by the homotopy limit. This functor is defined over the category $Codiag_{\Pi}\mathcal{M}$ of cubical codiagrams with variable type in the category Π. We begin by recalling the definition of the category Π of cubical types [6, 1.1.1]. After defining the category of codiagrams we define the simple functor and then we explain how to extend the definition to augmented cubical codiagrams.

4.1. **Cubical types.** Associate to a non-empty set S the set of non-empty subsets, ordered by inclusion: that defines the category $\Box_S$. The category $\Box_{\{0,1,\ldots,n\}}$ is denoted by $\Box_n$. An element α of $\Box_n$ is identified with a non zero element of $\{0,1\}^{n+1}$

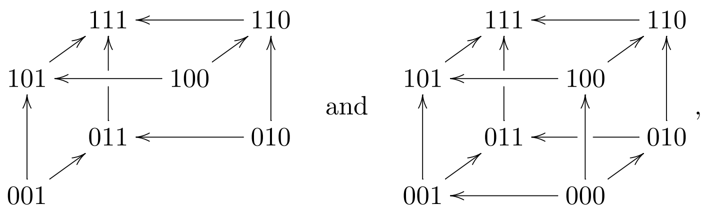
Given S and T two finite sets, any injective map $u : S \to T$ defines a functor $\Box_u : \Box_S \to \Box_T$.

Associate to a family $S = (S_i)_{i\in I}$, with I a finite set, the cartesian product $\Pi_{i\in I}\Box_{S_i}$ with the product order. Write $\Box_S = \Pi_{i\in I}\Box_{S_i}$.

The objects of the category Π of *cubical types* are the families $(S_i)_{i\in I}$ of non-empty finite sets, with I a finite set. Given $S = (S_i)_{i\in I}$ and $T = (T_j)_{j\in J}$, a morphism $u : S \to T$ of Π is an injective map $u : \Pi_i S_i \to \Pi_j T_j$ such that, for every $\alpha = (\alpha_i) \in \Box_S$, exists $\beta = (\beta_j) \in \Box_T$ such that $u(\Pi\alpha_i) = \Pi\beta_j$.

Now we turn to augmented cubical types. Associate to a finite set S, possibly empty, the set $\Box_S^+$ of subsets of S, ordered by inclusion. Observe that $\Box_n^+$ is a cube of dimension $(n+1)$.

For example, the categories $\Box_2$ and $\Box_2^+$ are represented by



where identities and morphisms composition of two are not represented.

4.2. **The category** $Codiag_{\Pi}\mathcal{M}$**.** Given $\delta : \Box \to \Box'$ a morphism of Π, there is an induced inverse image functor

$$\delta^* : (\Box', \mathcal{M}) \to (\Box, \mathcal{M})$$

defined by $F \mapsto \delta^*(F) := F \circ \delta$.

The category $Codiag_{\Pi}\mathcal{M}$ of *cubical codiagrams* is defined as follows. An object is a pair $(X, \Box)$, where $\Box \in Ob\Pi$ and $X : \Box \to \mathcal{M}$ is a $\Box$-codiagram. A morphism $(\mathbf{Y}, \Box') \to (\mathbf{X}, \Box)$ is a pair (a, δ) where $\delta : \Box \to \Box'$ is a morphism of Π and $a : \delta^*\mathbf{Y} \to \mathbf{X}$ is a natural transformation of functors of $(\Box, \mathcal{M})$.

The category $Codiag_{\Pi}\mathcal{D}$ of codiagrams is analogous to the category $Diag_{\Pi}\mathcal{D}$ of diagrams [6, 1.2.1].

4.3. **Simple functor of a codiagram.** Let $\mathcal{M}_f$ be the category of fibrant objects of a simplicial model category. Let $\Box$ be an object of Π. The *simple functor*

$$\mathbf{s} : Codiag_{\Pi}\mathcal{M}_f \to \mathcal{M}_f$$

associates to a $\Box$-codiagram $\mathbf{X}$ the object defined by the homotopy limit $\mathbf{s}_{\Box}(\mathbf{X}) = \operatorname{holim} \mathbf{X}$.

If we have $(\mathbf{Y}, \Box') \to (\mathbf{X}, \Box)$ a morphism in $Codiag_{\Pi}\mathcal{M}_f$ given by $\delta : \Box \to \Box'$ and $a : \delta^*\mathbf{Y} \to \mathbf{X}$, the functorial properties of the homotopy limit (see section 3.2.3) allow us to define the composition

$$\mathbf{s}_{\Box'}\mathbf{Y} = \operatorname*{holim}_{\Box'} \mathbf{Y} \to \operatorname*{holim}_{\Box} \delta^*\mathbf{Y} \to \operatorname*{holim}_{\Box} \mathbf{X} = \mathbf{s}_{\Box}\mathbf{X},$$

which gives the covariance of the functor.

A codiagram $\mathbf{X}$ is *acyclic* if the object $\mathbf{sX}$ is acyclic.

5. Simple of an augmented cubical codiagram

In this section we explain first how the simple functor is extended to augmented cubical codiagrams. Then we prove that the acyclicity criterion is equivalent to an extended acyclicity criterion in a simplicial model category. The proof generalizes a property of cubes of spectra, following [17, 1.1]. For a similar result for the case of topological spaces see [5, 1.1].

First observe that the homotopy limit of an augmented cubical codiagram $\mathbf{X}^+$ is $\mathbf{X}_0$, as the codiagram has an inital object. Thus the simple functor of an augmented cubical codiagram is not the homotopy limit.

The functor $\mathbf{s}$ is extended to augmented cubical codiagrams by using the cone construction [6, 1.4.3]. It can be calculated as follows. Given a $\Box_n^+$-codiagram $\mathbf{X}^+$, view it as a morphism of two $\Box_{n-1}^+$-codiagrams, $f : \mathbf{X}_0^+ \to \mathbf{X}_1^+$. The simple object associated to $\mathbf{X}^+$ is obtained as the simple of the $\Box_{n-1}^+$-codiagram which in each degree α has the homotopy fibre of f_α. This construction does not depend on the order chosen [6, 1.4.3].

In particular if $f : X \to Y$ is a $\Box_0^+$-codiagram, its simple is the homotopy fibre of f.

The acyclic augmented cubes are also called *homotopy cartesian* cubes.

Augmented cubical types allow induction, as $\square_n^+ = \square_1^+ \times \square_{n-1}$. Non-augmented cubical types do not have this property. The lemmas below give homotopy left cofinal functors which allow induction in some cases.

Lemma 5.1. *The functor $f : \square_n \to \square_{n-1}^+$ defined by $(i,j) \mapsto (j)$ is homotopy left cofinal.*

Proof. Given $\alpha = (1,j) \in \square_n$, the category $(f \downarrow f(\alpha))$ is isomorphic to the category $(\square_n \downarrow \alpha)$ and thus contractible. □

Lemma 5.2. *The functor $f : \square_1 \times \square_{n-1} \to \square_n$ defined by $f((0,1),k) = (0,k)$, $f((1,1),k) = (1,k)$ and $f((1,0),k) = (1,0,\dots,0)$ is homotopy left cofinal.*

Proof. It is easy to see that the required categories are contractible: the category $(f \downarrow (1,0,\dots,0))$ is isomorphic to $\square_{n-1}$, the category $(f \downarrow (0,k))$ is isomorphic to $(\square_{n-1} \downarrow (k))$ and the category $(f \downarrow (1,k))$ is isomorphic to $(\square_n \downarrow (1,k))$. □

Lemma 5.3. *Let $\mathcal{M}$ be a pointed simplicial model category. Given a functor $\mathbf{E} : \square_n \to \mathcal{M}$ such that $\mathbf{E}_{1,k}$ is acyclic for every $k \in \square_{n-1}$, then there is a weak equivalence* $\operatorname{holim}_{\square_n} \mathbf{E} \simeq \operatorname{holim}_{\square_{n-1}} \mathbf{E}_{0,-}$.

Proof. Let f be the functor of lemma 5.2. The result is deduced by cofinality, Fubini and homotopy invariance ($\mathbf{E}_{1,k}$ is acyclic):

$$\operatorname*{holim}_{\square_n} \mathbf{E} \xrightarrow{\simeq} \operatorname*{holim}_{\square_1 \times \square_{n-1}} f^*\mathbf{E} \cong \operatorname*{holim}_{j \in \square_1} \operatorname*{holim}_{k \in \square_{n-1}} \mathbf{E}_{f(j,k)} \simeq$$
$$\simeq \operatorname{holim}(* \to * \leftarrow \operatorname*{holim}_{k \in \square_{n-1}} \mathbf{E}_{0,k}) \cong \operatorname*{holim}_{\square_{n-1}} \mathbf{E}_{0,-}.$$

□

Proposition 5.4. *Let $\mathcal{M}_f$ be the category of fibrant objects of a pointed simplicial model category where acyclicity criterion holds. There is a weak equivalence*

$$\mathbf{s}_{\square^+}\mathbf{X}^+ \simeq \operatorname{hofib}(\mathbf{X}_0 \to \operatorname*{holim}_{\square_n} \mathbf{X}).$$

Proof. We use induction on n. The case $n = 0$ holds by definition.

Define $\mathbf{E}^+ : \square_n^+ \to \mathcal{M}_f$ by $\mathbf{E}_{0,\mathbf{j}} = \operatorname{hofib}(\mathbf{X}_{0,\mathbf{j}} \to \mathbf{X}_{1,\mathbf{j}})$, and $\mathbf{E}_{1,\mathbf{j}} = \operatorname{hofib}(\mathbf{X}_{1,\mathbf{j}} \to \mathbf{X}_{1,\mathbf{j}})$. Thus, $\mathbf{s}_{\square_n^+}\mathbf{X}^+ = \mathbf{s}_{\square_{n-1}^+}\mathbf{E}_{0,-}^+$.

By induction and lemma 5.3 above $\mathbf{s}_{\square_{n-1}^+}\mathbf{E}_{0,-}^+ \simeq \operatorname{hofib}(\mathbf{E}_0 \to \operatorname{holim}_{\square_{n-1}} \mathbf{E}_{0,-}) \simeq \operatorname{hofib}(\mathbf{E}_0 \to \operatorname{holim}_{\square_n} \mathbf{E})$.

To finish it is enough to see that $\operatorname{hofib}(\mathbf{X}_0 \to \operatorname{holim}_{\square_n} \mathbf{X})$ is weakly equivalent to $\operatorname{hofib}(\mathbf{E}_0 \to \operatorname{holim}_{\square_n} \mathbf{E})$.

To this end, define $\mathbf{Y} : \square_n^+ \to \mathcal{M}_f$ by $\mathbf{Y}_{0,\mathbf{j}} = \mathbf{Y}_{1,\mathbf{j}} = \mathbf{X}_{1,\mathbf{j}}$, where $\mathbf{j} \in \square_{n-1}^+$. Consider the functor $f : \square_n \to 1 \times \square_{n-1}^+$ defined by $(i,\mathbf{j}) \mapsto (1,\mathbf{j})$. By lemma 5.1 above this functor is homotopy left cofinal. As $\square_{n-1}^+$ has an initial object and by cofinality theorem we have:

$$\mathbf{X}_{1,0,\dots,0} \xrightarrow{\simeq} \operatorname*{holim}_{\square_{n-1}^+} \mathbf{X}_{1,-}^+ \xrightarrow{\simeq} \operatorname*{holim}_{\square_n} f^*(\mathbf{X}_{1,-}^+) = \operatorname*{holim}_{\square_n} \mathbf{Y}.$$

Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbf{X_0} & \longrightarrow & \mathbf{Y_0} = \mathbf{X}_{1,0,\dots,0} \\ {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow{\scriptstyle\simeq} \\ \operatorname{holim}_{\square_n} \mathbf{X} & \longrightarrow & \operatorname{holim}_{\square_n} \mathbf{Y} \end{array}.$$

If we calculate its simple by rows we obtain $\operatorname{hofib}(\mathbf{E_0} \to \operatorname{holim}_{\square_n} \mathbf{E})$, by definition of $\mathbf{E}$. If we calculate its simple by columns we obtain the homotopy fiber of the first column, $\operatorname{hofib}(\mathbf{X_0} \to \operatorname{holim}_{\square_n} \mathbf{X})$, as the homotopy fiber of the second column is trivial by the acyclicity criterion. □

Corollary 5.5. *Let $\mathcal{M}_f$ be the category of fibrant objects of a simplicial model category where acyclicity criterion holds. An augmented codiagram $\mathbf{X}^+ : \square_n^+ \to \mathcal{M}_f$ is acyclic if, and only if, the canonical morphism $\mathbf{X}_0 \to \operatorname{holim}_{\square_n} \mathbf{X}$ is a weak equivalence.*

Proof. This follows from the above proposition and the acyclicity criterion. □

6. Main result

A *cohomological descent category* [6, 1.5.3, 1.7.1] is given by $(\mathcal{D}, E, \mathbf{s}, \mu, \lambda)$ satisfying the eight properties $(\mathrm{CD1})^{op}$ to $(\mathrm{CD8})^{op}$ below, which are stated there for $\mathcal{D} = \mathcal{M}_f$ and E the class of weak equivalences.

Let $\mathcal{M}$ be a pointed simplicial model category and $\mathcal{M}_f$ the subcategory of fibrant objects. Let $\mathbf{s}$ be the simple functor defined by the homotopy limit.

By the general properties of ends [10], given $\square, \square' \in \Pi$, we have a natural transformation of functors

$$\mu_{\square,\square'} : \mathbf{s}_\square \circ \mathbf{s}_{\square'} \to \mathbf{s}_{\square\times\square'}$$

such that $\mu_{\square,\square'}(\mathbf{X}) : \mathbf{s}_\square \circ \mathbf{s}_{\square'}(\mathbf{X}) \to \mathbf{s}_{\square\times\square'}(\mathbf{X})$ is an isomorphism for every $\square \times \square'$-codiagram $\mathbf{X}$. This isomorphism is called Fubini isomorphism.

The morphism $\lambda_{\square_S}(X) : X = F(*, X) \to F(B(\square_S), X)$ is the one induced by the simplicial set morphism $B(\square_S) \to *$.

Theorem 6.1. *The category $\mathcal{M}_f$ of fibrant objects of a pointed simplicial model category where the acyclicity criterion holds, with the class E of weak equivalences, the simple functor $\mathbf{s}$ and μ and λ defined above, is a cohomological descent category.*

Proof. See $(\mathrm{CD1})^{op}$ to $(\mathrm{CD8})^{op}$ below. □

$(\mathrm{CD1})^{op}$. *$\mathcal{M}_f$ is a cartesian category with initial object.*

Proof. Recall that a category is cartesian if it has all finite products, and $\mathcal{M}_f$ has them. □

$(\mathrm{CD2})^{op}$. *The class of weak equivalences is a saturated class of morphisms, stable by products: if $f : X \to X'$ and $g : Y \to Y'$ are weak equivalences, then $f \times g : X \times Y \to X' \times Y'$ is a weak equivalence.*

Proof. In every model category, the class of weak equivalences is saturated [7, 8.3.10].

The stability by products is seen obtaining $X \times Y$ as a homotopy limit of a discrete diagram and using $(\mathrm{CD5})^{op}$. □

If $\delta : \square \to \square'$ is a morphism of Π, there is a *direct image* functor

$$\delta_* : (\square, \mathcal{M}) \to (\square', \mathcal{M})$$

such that if $\mathbf{X}$ is a $\square$-codiagram of $\mathcal{M}$ then $\delta_*\mathbf{X}$ is the $\square'$-codiagram defined by

$$(\delta_*\mathbf{X})_\beta = \begin{cases} \mathbf{X}_\alpha & \text{if } \beta = \delta(\alpha), \alpha \in \square \\ * & \text{if } \beta \in \square' \setminus \delta(\square) \end{cases}$$

with the evident morphisms. This definition is dual to the one for diagrams [6, 1.2.2].

(CD3)^{op}. *$\mathbf{s}: Codiag_{\Pi}\mathcal{M}_f \to \mathcal{M}_f$ is a covariant functor such that if $\delta : \Box \to \Box'$ is a morphism of Π and $\mathbf{X}$ is a $\Box$-codiagram of $\mathcal{M}_f$, the morphism $\mathbf{s}_{\Box'}\delta_*\mathbf{X} \to \mathbf{s}_{\Box}\mathbf{X}$ is a weak equivalence.*

Proof. The functor $\mathbf{s}$ has been defined in section 4.3.

We may assume that $\Box = \Box_S = \Box_{S_1} \times \cdots \times \Box_{S_s}$ and $\Box' = \Box_T = \Box_{T_1} \times \cdots \times \Box_{T_t}$ with $t \geq s$, and that the morphism δ is induced by inclusions $S_i \subset T_i$ and constants $\gamma_{s+1} \in T_{s+1}, \ldots, \gamma_t \in T_t$.

Thus δ embeds $\Box_S$ as a full subcategory of $\Box_T$. Moreover, there are no arrows from a vertex of $\Box_T \setminus \delta(\Box_S)$ to a vertex of $\delta(\Box_S)$. These properties and the definition of δ_* gives us an isomorphism $\operatorname{holim}_{\Box_T}(\delta_*\mathbf{X}) \cong \operatorname{holim}_{\Box_S}\mathbf{X}$.

□

(CD4)^{op}. *For every object $\Box$ of Π, the functor $\mathbf{s}_{\Box} : (\Box, \mathcal{M}_f) \to \mathcal{M}_f$ is op-monoidal and quasistrict.*

Proof. The Künneth morphism $\sigma = \sigma_{\Box}(\mathbf{X}, \mathbf{Y}) : \mathbf{s}_{\Box}(\mathbf{X} \times \mathbf{Y}) \to \mathbf{s}_{\Box}\mathbf{X} \times \mathbf{s}_{\Box}\mathbf{Y}$ is an isomorphism because holim preserves limits:

$$\mathbf{s}_{\Box}(\mathbf{X} \times \mathbf{Y}) = \operatorname{holim}(\mathbf{X} \times \mathbf{Y}) \cong \operatorname{holim}(\mathbf{X}) \times \operatorname{holim}(\mathbf{Y}) = \mathbf{s}_{\Box}(\mathbf{X}) \times \mathbf{s}_{\Box}(\mathbf{Y})$$

Observe that σ is natural in $(\mathbf{X}, \mathbf{Y})$.

The unit morphism $\sigma^1_{\Box} : \mathbf{s}_{\Box}(1 \times \Box) \to 1$ is clearly an isomorphism: the realization of a constant codiagram in the initial object 1 is $F(K, 1) = 1$ (see example 3.2).

Finally, it is clear that σ and σ^1 verify the associativity and unit restrictions and that $\mathbf{s}_{\Box}$ is an op-monoidal functor. □

(CD5)^{op}. *If $f : \mathbf{X} \to \mathbf{Y}$ is a morphism of $\Box$-codiagrams of $\mathcal{M}_f$ such that for every $\alpha \in \Box$, f_α is a weak equivalence, then $\mathbf{s}_{\Box}f : \mathbf{s}_{\Box}\mathbf{X} \to \mathbf{s}_{\Box}\mathbf{Y}$ is a weak equivalence of fibrant objects.*

Proof. This is exactly the homotopy invariance property of the homotopy limit (3.2.3). □

We introduce now the category $Coreal_{\Pi}\mathcal{M}$ of corealisations, which is analogous to the category $Real_{\Pi}\mathcal{M}$ of realisations [6, 1.2.1]

Given $\delta : \Box \to \Box'$ a morphism of Π, there is an induced direct image functor

$$\delta_* : ((\Box, \mathcal{M}), \mathcal{M}) \to ((\Box', \mathcal{M}), \mathcal{M})$$

defined by $f \mapsto \delta_*(f) := f \circ \delta^*$.

The category $Coreal_{\Pi}\mathcal{M}$ of corealisations of cubical codiagrams is defined as follows. An object is a functor $s_{\Box} \in ((\Box, \mathcal{M}), \mathcal{M})$. A morphism from $s_{\Box'} \in ((\Box', \mathcal{M}), \mathcal{M})$ to $s_{\Box} \in ((\Box, \mathcal{M}), \mathcal{M})$ is a morphism $\delta : \Box \to \Box'$ of Π and a natural transformation of functors $s_{\Box'} \to \delta_* s_{\Box}$ of $((\Box', \mathcal{M}), \mathcal{M})$.

Remark 6.2. The category $Coreal_{\Pi}\mathcal{M}$ has a structure of monoidal category: Given $\Box, \Box' \in \Pi$, $\mathbf{s}_{\Box} \in ((\Box, \mathcal{M}), \mathcal{M})$, $\mathbf{s}_{\Box'} \in ((\Box', \mathcal{M}, \mathcal{M}))$, the composition

$$\mathbf{s}_{\Box} \circ \mathbf{s}_{\Box'} : ((\Box \times \Box'), \mathcal{M}) \to \mathcal{M}$$

is defined by

$$\mathbf{s}_{\Box} \circ \mathbf{s}_{\Box'}(\mathbf{X}) = \mathbf{s}_{\Box}(\alpha \mapsto \mathbf{s}_{\Box'}(\beta \mapsto \mathbf{X}_{\alpha\beta})).$$

The unit object is the evaluation functor $Av : (\Box_0, \mathcal{M}) \to \mathcal{M}$.

(CD6)^{op}. *$(\mathbf{s}, \mu, \lambda_0) : \Pi^{op} \to Coreal_{\Pi}\mathcal{M}_f$, $\Box \mapsto (\mathbf{s}_{\Box} : (\Box, \mathcal{M}_f) \to \mathcal{M}_f)$, is a strict monoidal functor.*

Proof. A (strict) monoidal functor $(\mathbf{s}, \mu, \lambda_0) : (\Pi, \times, \Box_0) \to (Codiag_{\Pi}\mathcal{M}_f, \circ, Av)$ is given by

(i) a functor $\mathbf{s} : \Pi \to Coreal_\Pi \mathcal{M}_f$,
(ii) for every pair $(\Box, \Box')$ of $\Pi \times \Pi$, a (iso)morphism of $Coreal_\Pi \mathcal{M}_f$ (i.e. a natural transformation of functors)

$$\mu_{\Box,\Box'} : \mathbf{s}_\Box \circ \mathbf{s}_{\Box'} \to \mathbf{s}_{\Box\times\Box'}$$

natural in $(\Box, \Box')$, and
(iii) a (iso)morphism of $Coreal_\Pi \mathcal{M}_f$

$$\lambda_0 : Av \to \mathbf{s}_{\Box_0}$$

compatible with the associativity and unit restrictions.

If we consider $X \in \mathcal{M}_f$ as a $\Box_0$-codiagram, $\mathbf{s}_{\Box_0} X = F(*, X) \cong X$ and $Av(X) = X$. It is clear that we have a natural transformation of functors $\lambda_0 : Av \to \mathbf{s}_{\Box_0}$ such that $\lambda_0(X)$ is an isomorphism.

It is easy to see that μ and λ_0 satisfy the associativity and unit restrictions, so we are done. □

Given S a non-empty finite set, $\mathbf{s}_{\Box_S}(\Box_S \times X) = F(B(\Box_S), X) = F(\Delta^S, X)$ (see example 3.2). For $S = \prod_i S_i$, if we set $\Delta^S = \prod_i \Delta^{S_i}$, the equality also holds.

We denote by $i_\Box$ the functor $X \mapsto \Box \times X$.

(CD7)^{op}. *The morphism λ is a monoidal natural transformation from the functor $G : \Box \mapsto id_{\mathcal{M}_f}$ to the monoidal functor $H : \Box \mapsto \mathbf{s}_\Box \circ i_\Box$, which coincides with λ_0 over $\Box_0$*

Proof. For every $\Box \in \Pi$ and $X \in \mathcal{M}_f$, the morphism $\lambda_\Box(X)$ is natural in $\Box$ and X, and therefore defines a natural transformation from $G : \Pi^{op} \to (\mathcal{M}_f, \mathcal{M}_f)$, $\Box \mapsto id_{\mathcal{M}_f}$, to $H : \Pi^{op} \to (\mathcal{M}_f, \mathcal{M}_f)$, $\Box \mapsto \mathbf{s}_\Box \circ i_\Box$.

The naturality of λ in $\Box$ is clear: for every morphism $\Box_S \to \Box_T$ of Π, the diagram

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ \downarrow{\scriptstyle \lambda_{\Box_S}(X)} & & \downarrow{\scriptstyle \lambda_{\Box_T}(X)} \\ \mathbf{s}_{\Box_T}(i_{\Box_T} X) = F(\Delta^T, X) & \longrightarrow & \mathbf{s}_{\Box_S}(i_{\Box_S} X) = F(\Delta^S, X) \end{array}$$

commutes.

It is also clear that λ is natural in X: if $X \to Y$ is a morphism of $\mathcal{M}_f$,

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle \lambda_\Box(X)} & & \downarrow{\scriptstyle \lambda_\Box(Y)} \\ F(\Delta, X) & \longrightarrow & F(\Delta, Y) \end{array}$$

commutes.

We have to see that the natural transformation λ is monoidal. The functors G and H are between monoidal categories: $(\Pi^{op}, \times, \Box_0)$ and $((\mathcal{M}_f, \mathcal{M}_f), \circ, id_{\mathcal{M}_f})$.

We observe that, given $X \in \mathcal{M}_f$,

$$H(\Box) \circ H(\Box')(X) = (\mathbf{s}_\Box \circ i_\Box) \circ (\mathbf{s}_{\Box'} \circ i_{\Box'})(X) = \mathbf{s}_\Box(i_\Box[\mathbf{s}_{\Box'}(i_{\Box'} X)]) =$$
$$= F(\Delta, F(\Delta', X)) \cong F((\Delta \times \Delta'), X),$$

and that

$$H(\Box \times \Box')(X) = \mathbf{s}_{\Box\times\Box'} \circ i_{\Box\times\Box'}(X) = \mathbf{s}_{\Box\times\Box'}(i_{\Box\times\Box'} X) = F((\Delta \times \Delta'), X)$$

where $\Delta = B\Box$ and $\Delta' = B\Box'$. This defines the morphism in $(\mathcal{M}_f, \mathcal{M}_f)$

$$H_2(\Box, \Box') : \mathbf{s}_{\Box\times\Box'} \circ i_{\Box\times\Box'} \to (\mathbf{s}_\Box \circ i_\Box) \circ (\mathbf{s}_{\Box'} \circ i_{\Box'}).$$

For all $X \in \mathcal{M}_f$ the isomorphism $\mathbf{s}_{\Box_0}(i_{\Box_0}X) = F(*, X) \to X$ defines a morphism in $(\mathcal{M}_f, \mathcal{M}_f)$

$$H_0 : id_{\mathcal{M}_f} \to \mathbf{s}_{\Box_0} \circ i_{\Box_0}.$$

We have that (H, H_2, H_0) is an monoidal functor.

Now we see that the natural transformation λ is monoidal and that it coincides with λ_0 over $\Box_0$. □

(CD8)^{op}. *Suppose that the acyclicity criterion holds in $\mathcal{M}_f$. For every codiagram $\mathbf{X} : \Box_S \to \mathcal{M}_f$, where S is a finite non-empty set and every augmentation $\varepsilon : \mathbf{X}_0 \to \mathbf{X}$, the morphism $\lambda_\varepsilon := \mathbf{s}_\Box(\varepsilon) \circ \lambda_\Box(\mathbf{X}_0) : \mathbf{X}_0 \to \mathbf{s}_\Box\mathbf{X}$ is a weak equivalence if and only if the canonical morphism $0 \to \mathbf{s}_{\Box^+}\mathbf{X}^+$ is a weak equivalence.*

Proof. This result is exactly corollary 5.5. □

Everything can be dualized, and we obtain the following dual statemnt of the main theorem:

Theorem 6.3 (Dual statement)**.** *The category $\mathcal{M}_c$ of cofibrant objects of a pointed simplicial model category where the dual acyclicity criterion holds, with the class E of weak equivalences, the simple functor defined by the homotopy colimit and μ and λ defined dually as those above, is a homological descent category.*

7. Examples

Our initial target was to prove that the category of fibrant spectra, in the sense of Bousfield and Friedlander [1], is a cohomological descent category with the homotopy limit. The result applies to any of the model categories of spectra available, including symmetric spectra [9] and orthogonal spectra [11]. The acyclicity criterion follows from the fact that model categories of spectra are stable.

7.1. **Stable model categories.** Recall that the homotopy category of a pointed model category supports a suspension functor Σ with a right adjoint loop functor Ω. A *stable model category* is a pointed model category where the functors Ω and Σ in the homotopy category are inverse equivalences.

The homotopy category of a stable model category is triangulated [8, 7.1.6]. The cofibre and fibre sequences of [8, 6.2.6] (see also [13, 1.3]) coincide up to sign [8, 7.1.11] and define the distinguished triangles.

Proposition 7.1 (Acyclicity criterion for stable simplicial model categories)**.** *Let $\mathcal{M}$ be a stable simplicial model category. A morphism $f : X \to Y$ with X and Y fibrant is a weak equivalence if and only if the homotopy fibre* hofib f *is acyclic.*

Proof. By saturation and passing to the homotopy category, the result follows from the following property of triangulated categories: If

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X$$

is a distinguished triangle in a triangulated category, then g is an isomorphism if and only if $X \cong 0$. □

The dual acyclicity criterion also holds, replacing the homotopy fibre by the homotopy cofibre. Thus stable simplicial model categories are homological and cohomological descent categories with the homotopy colimit and the homotopy limit respectively.

Example 7.2. See [15] for a list of examples of interesting stable simplicial model categories, to which our main result applies, including modules over ring spectra, presheaves of spectra, and spectra categories related to equivariant stable homotopy theory and motivic stable homotopy of schemes.

7.2. **Simplicial sets and topological spaces.** We have worked with pointed model categories for simplicity, but everything can be done in unpointed model categories. In that case the extra hypotesis of fibrant initial object is needed.

Topological spaces with homology isomorphisms is the second example of homological descent category of [6]. Usual weak homotopy equivalences can not be used because the acyclicity criterion does not hold if spaces are not arc-wise connected.

Here we recover this example and, furthermore, we obtain the same result for simplicial spaces and CW-complexes with h_*-equivalences respect to a homology theory h.

We only have the homological case because there is not an acyclicity criterion with the homotopy fibre for topological or simplicial spaces.

Proposition 7.3. *Let h be a homology theory on the category of simplicial sets. The category of simplicial sets with the homotopy colimit and the h_*-equivalences is a homological descent category.*

Proof. Bousfield localization respect to a homology theory h_* gives a model category structure on simplicial sets where weak equivalences are those morphisms which induce isomorphism in homology and cofibrations are the usual ones.

Moreover, with its usual simplicial structure it is a simplicial model category [4, X.3]. As the simplicial structure does not change, the homotopy colimit in this simplicial structure is the ususal. As cofibrations remain unchanged, all simplicial sets are h_*-cofibrant.

The result follows from the dual statement of the main theorem, observing that The mapping cone sequence gives the aciclicity criterion for any h_*. □

The same result is true for CW-complexes, as Bousfield localization can be done in topological spaces and the simplcial structure part of the proof of [4, X.3] only relies in the Mayer-Vietoris sequence.

In the case of singular homology we do not need to take CW-complexes because the homotopy colimit has the homotopy invariance property with respect to homology equivalences without the assumption of pointwise cofibrant diagrams [3, 5.16].

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