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Mengoli's mathematical ideas in Leibniz's excerpts

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Dedicated to the memory of Jacqueline

In the seventeenth century many changes occurred in the practice of mathematics. An essential change was the establishment of a symbolic language, so that the new language of symbols and techniques could be used to obtain new results. Pietro Mengoli (1626/7–86), a pupil of Cavalieri, considered the use of symbolic language and algebraic procedures essential for solving all kinds of problems. Following the algebraic research of Viète, Mengoli constructed a geometry of species, *Geometriae Speciosae Elementa* (1659), which allowed him to use algebra in geometry in complementary ways to solve quadrature problems, and later to compute the quadrature of the circle in his *Circolo* (1672). In a letter to Oldenburg as early as 1673, Gottfried Wilhelm Leibniz (1646–1716) expressed an interest in Mengoli's works, and again later in 1676, when he wrote some excerpts from Mengoli's *Circolo*. The aim of this paper is to show how in these excerpts Leibniz dealt with Mengoli's ideas as well as to provide new insights into Leibniz's mathematical interpretations and comments.

Introduction

In the seventeenth century a considerable number of changes occurred in the practice of mathematics. An essential change was the establishment of a symbolic language as a formal language in mathematics. The new language of symbols and techniques could be used in operations and geometrical constructions to obtain new mathematical results. Therefore, the use of symbolic language and algebraic procedures for solving geometrical problems was extremely significant for the transformation of seventeenth-century mathematics.

The publication in 1591 of *In artem analyticen isagoge* by François Viète (1540–1603) constituted an important step forward in the development of this symbolic language. Viète used symbols not only to represent unknown quantities but also to represent known ones. In this way he was able to investigate equations in a completely general form, though still rhetorical.[1] In addition, Viète introduced a new algebra with his 'specious logistic'; that is, calculations with 'species', in contrast to the 'numerous logistic'; that is, calculations with numbers, which was already used in the Renaissance algebras. The 'species' used in Viète's algebra consisted of all kinds of magnitudes, numerical magnitudes, such as natural and rational numbers, and also geometrical magnitudes such as lengths, areas, volumes or angles. Viète showed the usefulness of algebraic

[1] If we observe Viète's equation, we can appreciate the rhetorical form. We provide one example to show how Viète writes an equation: 'B in $A - A$quad. Aequatur Zquad', which in modern notation would be written $Bx - x^2 = Z^2$.

procedures for analysing and solving problems in arithmetic, geometry and trigonometry (Viète 1983; Giusti 1992; Bos 2001; Stedall 2011).[2]

As Viète’s work came to prominence at the beginning of the seventeenth century, other authors, such as Pietro Mengoli (1626/7–86) began to consider the use of algebraic procedures for solving all kinds of problems.[2] Mengoli’s name appears in the register of the University of Bologna in the period 1648–86. He studied with Bonaventura Cavalieri (1598–1647) and ultimately succeeded him in the Chair of Mechanics. He graduated in Philosophy in 1650 and three years later in Canon and Civil Law. In an initial period he wrote three works of pure mathematics: *Novae Quadraturae Arithmeticae seu de Additione Fractionum* (1650), *Via Regia ad Mathematicas per Arithmeticam, Algebram Speciosam et Planimetriam ornate Maiestati Serenisimae D. Christinae Reginae Suecorum* (1655), *Geometriae Speciosae Elementa* (1659, hereafter *Geometria*), and later, *Circolo* (1672). He took holy orders in 1660 and until his death was prior of the church of Santa Maria Maddalena in Bologna.[4]

We divide his scientific life into two phases; up to the year 1660 and from 1676 on, when, in addition to diversifying his field of research, quotations from his work diminished in scientific circles and Mengoli became increasingly isolated from his contemporaries.[5]

From 1673, Mengoli’s first work, *Novae Quadraturae Arithmeticae* (1650), was mentioned in many letters from European scientists and gave rise to a debate between Gottfried Wilhelm Leibniz (1646–1716) and Henry Oldenburg (1615–77) on the series studied by Mengoli. Indeed, he added infinite series in his *Novae quadraturae* and proved the divergence of the harmonic series (Giusti 1991). In Oldenburg’s correspondence, we find a first letter dated 26 February 1673 in which Leibniz expressed an interest in this work by Mengoli, during his stay in Paris (Oldenburg 1986). Oldenburg replied (6 March 1673) forwarding a letter to him by John Collins (1624–83), where this author explained that

[2] Viete’s algebra was spread by some authors, thereby facilitating the algebraicization of mathematics. One example is Thomas Harriot (*c*. 1560–1621) in his mathematical writings. On Harriot’s algebraic works we can see Stedall’s seminal researches in Stedall (2003, 2007). Another example is the encyclopaedic work by Pierre Hérigone (1580–1643), *Cursus mathematicus*, Paris (1634, 1637, 1642), which consists of six volumes, one of which is on algebra. On Hérigone’s work we can see a comparative analysis between Viète’s and Hérigone’s algebra in Massa-Esteve (2008), the treatment of Euclid’s *Elements* in Hérigone’s work in Massa-Esteve (2010) and the influence of Viète’s work on Hérigone’s work and from that to Mengoli’s work in Massa-Esteve (2012).

[3] Pierre de Fermat (1607–65) was among the mathematicians who used algebraic analysis to solve geometric problems. He did not publish any of his work during his lifetime, although it circulated in the form of letters and manuscripts and was referred to in other publications. On Fermat’s works see Fermat (1891–1922, 65–71 and 286–292) and Mahoney (1973, 229–232). The most prominent figure in this process of algebraicization was René Descartes (1596–1650), who published *La géométrie* in 1637. There are many excellent useful studies on Descartes, including Giusti (1987, 409–432), Mancosu (1996, 62–84), and Bos (1981 and 2001).

[4] For more biographical information on Mengoli, see Natucci 1970–91; Massa-Esteve 1998, 2006b; and Baroncini and Cavazza 1986.

[5] Although he published nothing between 1660 and 1670, the latter year saw the appearance of three works: *Refrattioni e parallase solare* (1670), *Speculationi di musica* (1670), and *Circolo* (1672). These reflected Mengoli’s new aim of pursuing research not on pure but on mixed mathematics, such as astronomy, chronology, and music. Furthermore, his research was clearly in defence of the Catholic faith. Mengoli went on writing in this line, publishing *Anno* (1675) and *Mese* (1681) on the subject of cosmology and Biblical chronology, and *Arithmetica rationalis* (1674) and *Arithmetica realis* (1675) on logic and metaphysics.

Mengoli had found the sum of infinite series with the reciprocal figurate numbers and the proof of the impossibility of the sum of the harmonic series. However, Mengoli was unable to add the reciprocal square numbers (Basel problem). Leibniz replied (14 May 1673) by saying that he did not think Mengoli would have added infinite series, but rather a finite sum. He also claimed: 'Yet if Mengoli has done the same thing I shall not be astonished because different people commonly fall into agreement'.[6]

In addition, it has been known since the 1920s that in April 1676 Leibniz had the opportunity to study Mengoli's *Circolo* (1672), and that he had made extensive excerpts from this work (Hofmann 1974). Thus, we focus on the relationship between Mengoli's arithmetic-algebraic method of quadratures and Leibniz's arithmetical quadrature (Leibniz 1993; Knobloch 2002). In *Geometria* (1659), Mengoli showed the properties of geometric figures through algebraic expressions and calculated their quadratures. Thus, he proved that the areas between 0 and 1 of all geometric figures determined by $y = Kx^m(1-x)^n$, with adequate coefficients, have value 1, when m and n are natural, which in modern notation we would write as follows (Massa-Esteve 1998, 2006a):

$$(m+n+1)\cdot\binom{m+n}{n}\cdot\int_0^1 x^m\cdot(1-x)^n\mathrm{d}x = 1$$

And later in *Circolo* (1672) he derived as a corollary:

$$\int_0^1 x^m(1-x)^n\mathrm{d}x = \frac{1}{(m+n+1)\cdot\binom{m+n}{n}}$$

Furthermore, in this work Mengoli interpolated geometric figures, and proved the values of these quadratures when the exponents are rationales with denominator 2, expressed in modern notation as:

$$\int_0^1 x^{m/2}(1-x)^{n/2}\mathrm{d}x = \frac{1}{\left((m+n)/2+1\right)\cdot\binom{(m+n)/2}{n/2}}$$

In particular, he made a computation of the quadrature of the circle with an approximation of the number π up to eleven decimal places (Massa-Esteve 1998; Massa-Esteve and Delshams 2009).

Four years later, in 1676, Leibniz wrote down some excerpts from Mengoli's *Circolo*, after reading the main ideas on Mengoli's quadrature. Recently in 2015, Probst described the contents of these excerpts, although he did not address the matter in

[6]'Si tamen idem et Mengolus praestitit, non miror; saepe enim concurrere solent diverse' (Oldenburg 1986, vol IX, 648). According to Probst (2015), Leibniz probably did not get access to the *Novae quadraturae* of Mengoli during his stay in Paris. However when he visited London a second time in October 1676 he made excerpts from the correspondence between James Gregory and John Collins. In the sections copied by Leibniz, there is also a passage on Mengoli's proof of the divergence of the harmonic series, characterized by Leibniz in a marginal note as 'ingeniose' (A III, 1 N 88_2 p 486f).

relation to Mengoli's ideas in *Circolo* (1672). Therefore, in this paper our aim is to provide an analysis of the main mathematical ideas in Mengoli's quadrature, which Leibniz referred to in his excerpts, in order to discuss Leibniz's mathematical interpretations and comments. These analyses show the relevance of Mengoli's algebraic procedure and the originality of his quadrature approach, which may have inspired Leibniz in some aspects of his own arithmetical quadrature or of his further researches in 1679.

Main mathematical ideas in Mengoli's quadrature

The mathematical ideas in quadratures are found in both of Mengoli's works: *Geometria* (1659) and Circolo (1672). Mengoli's *Geometria* is a 472-page text on pure mathematics with six *Elementa* whose title: 'Elements of Specious Geometry' already indicates the singular use of symbolic language in this work, and particularly in geometry.[7] Mengoli unintentionally created a new field, a 'specious geometry' modelled on Viète's 'specious algebra' since he worked with 'specious' language, which means that symbols are used to represent not just numbers but also values of any abstract magnitudes.

Thus, Mengoli used this idea to compute quadratures of geometric figures through their algebraic expressions. Mengoli's aim was the computation of the quadrature of the semicircle of diameter 1, which corresponds to the integral $\int_0^1 \sqrt{x(1-x)}\mathrm{d}x$, as we can read at the beginning of the *Circolo*: 'Since I was young I have been researching the problem of the quadrature of the circle, the foremost problem in the Geometry...'.[8] In fact, according to Mengoli the results of the *Circolo* date back to 1660.[9] Instead of computing only this quadrature, Mengoli created a new arithmetic-algebraic method for the computation of countless quadratures. From our previous researches (Massa-Esteve 2006a, 2006b; Massa-Esteve and Delshams 2009), it is evident that there are three main ideas that are essential for Mengoli's determination of quadratures: the use of algebraic expressions for working with geometric figures; the role of triangular tables of geometric figures, and the identification

[7]The *Geometriae Speciosae Elementa* (1659) consists of an introduction entitled *Lectori elementario*, which provides an overview of the six individually titled chapters or *Elementa* that follow. In the first *Elementum, De potestatibus, à radice binomia, et residua* (1659, 1–19), Mengoli shows the first 10 powers of a binomial given with letters for both addition and subtraction, and explains that it is possible to extend his result to higher powers. The second, *De innumerabilibus numerosis progressionibus* (1659, 20–94), contains calculations of numerous summations of powers and products of powers in Mengoli's own notation, as well as demonstrations of some identities. In the third, *De quasi proportionibus* (1659, 95–147), he defines the ratios 'quasi zero', 'quasi infinity', 'quasi equality' and 'quasi a number'. With these definitions, he constructs a theory of quasi proportions on the basis of the theory of proportions found in the fifth book of Euclid's *Elements*. The fourth *Elementum, De rationibus logarithmicis* (1659, 148–200), provides a complete theory of logarithmical proportions. He constructed a theory of proportions between the ratios in the same manner as Euclid with the magnitudes in the fifth book of *Elements*. From this new theory in the fifth *Elementum, De propriis rationum logarithmis* (1659, 201–347) he found a method of calculation of the logarithm of a ratio and deduced many useful properties of the ratios and their powers. Finally, in the sixth *Elementum, De innumerabilibus quadraturis* (1659, 348–392) he calculates the quadrature of figures determined by the ordinates now represented by $y = K \cdot x^m \cdot (t-x)^n$. A detailed analysis of this work can be found in Massa-Esteve (1998).

[8]'Cercai, fino da giovinetto, il Problema Della quadratura del Circolo, il più desiderato di tutti nella Geometria' (Mengoli 1672, 1).

[9]In the opening pages of the *Circolo*, Mengoli explains that he had found this result, the quadrature of the circle, in 1660, but had not published it because, according to him, he only wanted to publish the mathematics he needed to explain natural events (Mengoli 1672, 1).

of the triangular tables of geometric figures with the triangular tables of values of quadratures of these geometric figures.

Mengoli's first idea: geometric figures expressed by algebraic expressions

The relationship between the ordinates and the abscissas of a geometric figure or of the curve that describes it, as currently understood, was not yet established in the mathematics of the seventeenth century. Some authors made several attempts to introduce algebra into geometry in order to construct algebraic curves using the new algebraic language (Bos 1981). Others defined and used curves through their properties, while still others defined geometric figures or the curves that determined these figures through the description of their construction point-to-point or through their movement (Bos 2001). However, these authors did not identify the algebraic expressions with the geometric figures using a coordinate system, and even less without using the drawing when working on it. As shown below, Mengoli presented a procedure that was original and innovative for its time, by using algebra in geometry in a singular way.

In the first definitions of *Elementum sextum* in *Geometria*, Mengoli stated his own system of co-ordinates, abscissa, and ordinate. He proposed a line segment, which he named 'Rationalis', and put it in a straight line that he called 'Tota' and represented with the letter *t* (sometimes with the letter *u*, if its value is 1). He defined a base as a straight-line segment of length *t* or 1 and he used the word abscissa[10] represented by the letter '*a*', for our *x*, but within this finite base in a horizontal axis. The remainder was represented by the letter '$r = t - a$' or '$1 - a$', depending on whether the base was a given value *t* or the unit *u*.

With regard to the word ordinate,[11] he first defined the ordinates of known figures, such as the square (or rectangle) and the triangle, from his construction on every point of the base. For instance, he defined the ordinates of a square as follows:

> 10. Over a base is described a square, and I assume that from any of the points of the base a straight line will be drawn to the opposite side, maintaining itself parallel at all times to the sides of the square; this will be called *ordinate in [a] square*.[12]

Mengoli did not define the ordinates through their constructions in the case of mixed-line figures (determined by a straight part and a curved part), but he explained that the ordinates were equal to the abscissas or power of the abscissas and named

[10]The word abscissa had appeared in 1646 in Fermat's works (1891–1922, 195), in Torricelli's work (1919, III, 366), in Cavalieri's work in 1647 (1966, 858–859), and in Degli Angeli (1659, 175–179). Other authors also used the word 'diameter' with the same meaning.

[11]Mengoli used the word 'ordinata' instead of the word 'applicata', which was commonly in use at that time. Descartes defined the ordinates as 'celles qui s'appliquent par ordre' (Descartes 1954, 67). Smith in this note explains: 'The equivalent of "ordinate application" was used in the fifteenth-century by translating Apollonius'. The note also cites that the *Mathematical dictionnary* of Hutton (1796) gave 'applicata' as the word corresponding to the ordinate, and explained that the expression 'ordinata applicata' was also used. In fact, Fermat and Cavalieri used 'applicata'. Mengoli in *Circolo* (1672) named them 'ordinatamente applicate' (Mengoli 1672, 5).

[12]'10. Super basi describatur quadratum: & ab uno quolibet puncto in basi sumpto, recta ducatur, usque ad oppositum latus, reliquis lateribus quadrati parallela: quae dicetur, Ordinata in quadrato.' (Mengoli 1659, 368)

them ‘ordinate in form’. So, for the ordinates corresponding to a parabolic figure he claimed: ‘any one ordinate is abscissa square’.[13] Later, in some proofs of the properties of the figures, he used the equality between the ordinates and the powers of abscissas expressed by means of proportions such as, $1: y = (1: x)^n$.

He did not use a vertical axis, and always drew the ordinates as lines perpendicular to the base. In fact, in *Geometria* he made only three figures and in *Circolo* he did not include any drawing.

He described the geometric figures that he wanted to square as ‘extended by their ordinates’. He denoted these geometric figures (which he referred to as ‘forms’)[14] by means of an algebraic expression written, in Mengoli’s notation, as $FO.a^m r^n.$, where ‘*FO.*’ denotes the form, a expresses the abscissa (x) and r the remainder ($1 - x$). He called this expression ‘Form of all products of m abscissae and n remainders’. Mengoli again began with known figures such as the square (*FO.u*) and the triangle (*FO.a*) and then progressed to any mixed-line figures.

> 23. And generalising, if over the base a figure is constructed, not extended more than by ordinates within the square, in which any ordinate is considered as some element of the proportional table [$a^m.r^n$ with *a abscissa* and *r remainder*]. [This figure] is called the ‘Form of all of possible proportional’ and an appropriate character will represent it. For instance, ‘Form of all third abscissae’, $FO.a^3$, ‘Form of all products of the second abscissae and the remainders’, *biprimae*, $FO.a^2.r$, ‘Form of all products of the abscissa and second remainders’, *unisecundae*, $FO.a.r^2$, ‘Form of all third remainders’, $FO.r^3$ and so on.[15]

Nevertheless, Mengoli had to ensure that each of the defined algebraic expressions, which were new algebraic objects, could be associated with a definite geometric figure. He enunciated the proposition as a problem, and proved how algebraic expressions could be used to construct the ordinate in a geometric figure at any given point. Mengoli here drew one horizontal axis AR and a perpendicular line (not in the middle) with the letter B over the base and the letter C at the top of the perpendicular line (see Figure 1). Next we present the transcription of Mengoli’s proof.

Problem I. Proposition 3. Find the ordinate of a proposed [geometric] figure at a given point and from a given base.[16] Hypoth

> That is, given $FO.10a^2r^3$, over a given base AR, in which is given a point B. It is necessary to find the ordinate of [the point] B.[17]

[13]‘Unaquaelibet ordinata, est abscissa secunda’ (Mengoli 1659, 369).

[14]The word forma dates from the previous century and was identified by measuring the intensity of a given quality. (Clagett 1968, 91–92; Crombie 1980, 82–95).

[15]‘23. Et generaliter, si super basi concipiatur figura, extensa non nisi per ordinatas in quadrato: & in qua, unaquaelibet ordinata, est assumpta quaedam in tabula proportionalium: dicetur, Forma omnes tales proportionales aptoque significabitur charactere. vt Forma omnes abscissae tertiae, *FO.a3*: Forma omnes biprimae, *FO.a2r*: Forma omnes unisecundae, *FO.ar2*: Forma omnes residuae tertiae, *FO.r3.* & sic deinceps’ (Mengoli 1659, 369).

[16]‘Formae propositae, in data basi, per datum punctum, ordinatam invenire’ (Mengoli 1659, 377).

[17]‘Esto proposita *FO.10 a2r3*, super data basi AR, in qua datum punctum B. Oportet per B ordinatam invenire’ (Mengoli 1659, 377).

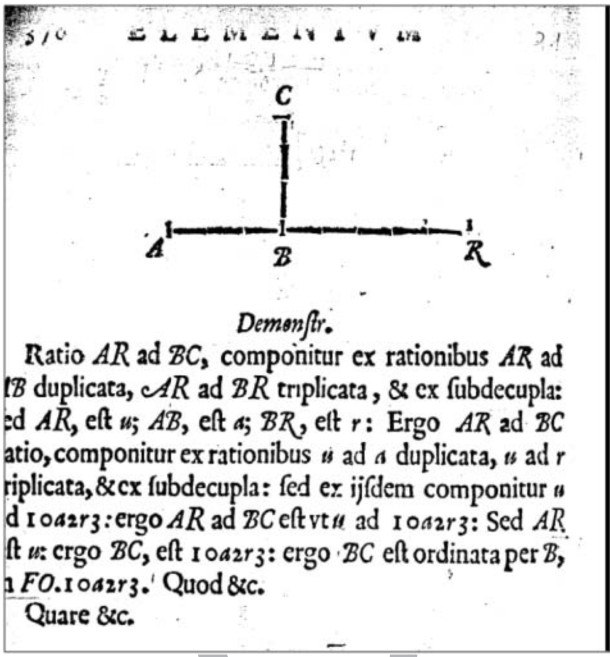


Figure 1. Illustration of Problem 3 (Mengoli 1659, 376)

Construction.[18]

Given AR, and given AB, BR, the recta BC will be found, to which AR is a ratio composed of given ratios AR to AB squared, AR to BR cubed, and of the ratio one tenth: and BC will be put perpendicular to AR. I affirm that BC is the ordinate of B, in *FO*.10 a^2r^3.[19]

Demonstration

The ratio AR to BC will be composed of ratios AR to AB squared, AR to BR cubed, and of one tenth; but AR is *u*; AB, is *a*; BR, is *r*. Thus, the ratio AR to BC will be composed of ratios '*u* to *a*', squared, '*u* to *r*', cubed, and of one tenth. But *u* to 10 a^2r^3 will be composed of these: then AR to BC is like *u* to 10 a^2r^3. But AR is *u*, so BC is 10 a^2r^3: then BC is the ordinate of [the point] B, in *FO*. 10 a^2r^3.[20]

Note here that Mengoli not only worked with proportions of segments but also identified the length of segments with the letters *u*, *a*, *r*. He equated the product of

[18]Throughout the book, Mengoli presented Theorems and Problems. In this case, he wrote the word Construction, as Euclid, before the demonstration and explained the construction used in it, as Euclid did in Mengoli's source Hérigone (Massa-Esteve 2010).

[19]'Data AR, datisque AB, BR, inveniatur recta BC, ad quàm AR, rationem habet compositam ex datis rationibus, AR ad AB duplicata, AR ad BR triplicata, & ex ratione subdecupla:& collocetur BC perpendiculariter ad AR. Dico BC, esse ordinatam per B, in *FO.10 a2r3*' (Mengoli 1659, 377).

[20]'Ratio AR ad BC, componitur ex rationibus AR ad AB duplicata, AR ad BR triplicata, & ex subdecupla: sed AR, est *u*; AB est *a*; BR est *r*: Ergo AR ad BC ratio, componitur ex rationibus *u* ad *a* duplicata, *u* ad *r* triplicata, & ex subdecupla: sed ex ijsdem componitur *u* ad 10 *a2r3*: ergo AR ad BC est ut *u* ad *10 a2r3*: sed AR est *u*: ergo BC est *10 a2r3*: ergo BC est ordinata per B, in *FO. 10 a2r3*.Quod&c' (Mengoli 1659, 378).

segments with the composition of ratios because he knew the Euclidean theory of proportions very well and took it as a link. However, unlike Descartes, Mengoli did not define previously an algebra of segments; rather he demonstrated, for a given measure u, the unity, how to construct the ordinate from the algebraic form corresponding to a curve using the composition of ratios defined at the beginning of his *Geometria*. His approach here was deeply original because he used these new symbols, which he had associated with geometric figures, for algebraic calculations.

In this way, Mengoli's first idea can be described as the establishment of an isomorphic relationship between algebraic objects and geometric figures, thereby allowing him to deal with these geometric figures by means of their algebraic expressions.

Mengoli's second idea: infinite triangular tables of geometric figures

After defining the given geometric figures: square, rectangle and any mixed-line figures, and assigning algebraic expressions to them, Mengoli proceeded to display these algebraic expressions representing geometric figures in an infinite triangular table (*Tabula Formosa*), a table of 'forms', inspired by the combinatorial triangle (also known as Pascal's triangle),[21] without making a graphical representation. In fact, the triangular table of geometric figures could be extended indefinitely according to the binomial development. The expression placed in the vertex represented a square of side 1. The two algebraic expressions of the first row represented two triangles. The first '*FO.a*' is determined by the diagonal of the first quadrant $y = x$, the axis of the abscissas and the straight line $x = 1$, and the second triangle '*FO.r*' is determined by the straight line $y = 1 - x$ traced from the point (1, 0) to the point (0, 1) and the axis of the abscissas. The three algebraic expressions of the second row are determined by the ordinates of a parabola, the axis of the abscissas and the straight line $x = 1$, respectively. The first figure, '$FO.a^2$', is determined by the ordinates $y = x^2$, the second, '*FO.ar*', is determined by the ordinates $y = x.(1 - x)$, and the third, '$FO.r^2$', is determined by the ordinates $y = (1 - x)^2$ and in the same way in the other rows. See Mengoli's table of forms (Figure 2) and my sketches of these geometric figures arranged as a triangular table (Figure 3).

As for the graphical representation of these geometric figures, it should be emphasized that there are only three drawings of geometric figures in the *Geometria*, and in his later work, *Circolo*, in which he made a computation of the quadrature of the circle, he did not include any drawings. Mengoli did not draw these geometric figures, but made it clear that the drawings could be deduced from their definitions and their positions in the triangular table. In fact, he considered three groups of geometric figures in the triangular table: the first, in the outside left diagonal of the *Tabula Formosa*, $FO.a^m$, geometric figures determined by increasing ordinates; the second, in the opposite diagonal of the table, $FO.r^n$, geometric figures determined by decreasing ordinates, and the third, in the middle of the table, $FO.a^m r^n$, geometric figures determined by ordinates that are first increasing, and after decreasing. He proved that the latter geometric figures achieved the maxima value in an abscissa that divides the base in the same ratio as the ratio between the exponents (Massa-Esteve 2016).

[21] The combinatorial triangle is known in history as Pascal's triangle, because Blaise Pascal (1623–62) explained and proved the properties in a very clear style (Bosmans 1924, 25–36; Pascal 1954, 91–107; Edwards 2002, 57–86). Mengoli may not have known about Pascal's treatise since it was published in 1665, but he knew its source very well, which was Hérigone's work. On Mengoli's triangular tables see Massa (1997).

$FO.\,u.$

$FO.\,a.$ $FO.\,r.$

$FO.\,a^2.$ $FO.\,ar.$ $FO.\,r^2.$

$FO.\,a^3.$ $FO.\,a^2r.$ $FO.\,ar^2.$ $FO.\,r^3.$

Figure 2. *Tabula Formosa* (Mengoli 1659, 366)

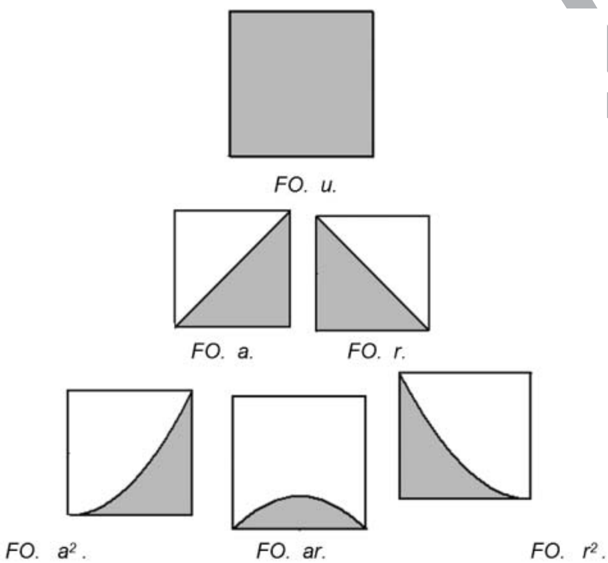


Figure 3. Our sketches of Mengoli's geometric figures (Massa-Esteve 1998)

He proves the characteristics for each group of geometric figures for only one specific entry, although he took this demonstration as true for all the entries due to the symmetry of the triangular tables, and the regularity of their rows. Thus, the role of the triangular tables for classifying, and for establishing the generality of the results for each group of geometric figures becomes essential.

Mengoli's third idea: the identification of the infinite triangular table of geometric figures with the infinite triangular table of values of their quadratures

In *Geometria*, Mengoli introduced his method based on the construction of triangular tables and used the theory of quasi-proportions to compute the quadratures of these geometric figures $FO.a^m r^n$ (Massa 1997). Indeed, he proved that all quadratures of geometric figures with the appropriate coefficients have value 1.

Later, in *Circolo* he displayed in an infinite triangular table the numerical values of their quadratures, where the notation with halves in the symbols is introduced (see Figure 4), which is nothing other than the harmonic triangle (also known as Leibniz's triangle).[22] Mengoli then identified these numbers, the inverse of the coefficients of

[22]The harmonic triangle, also called Leibniz's triangle, is formed by the reciprocal of the elements (binomial coefficients) of the binomial triangle times their own numbers. Its definition is related to the successive differences of the harmonic sequence. See Edwards 2002, 106.

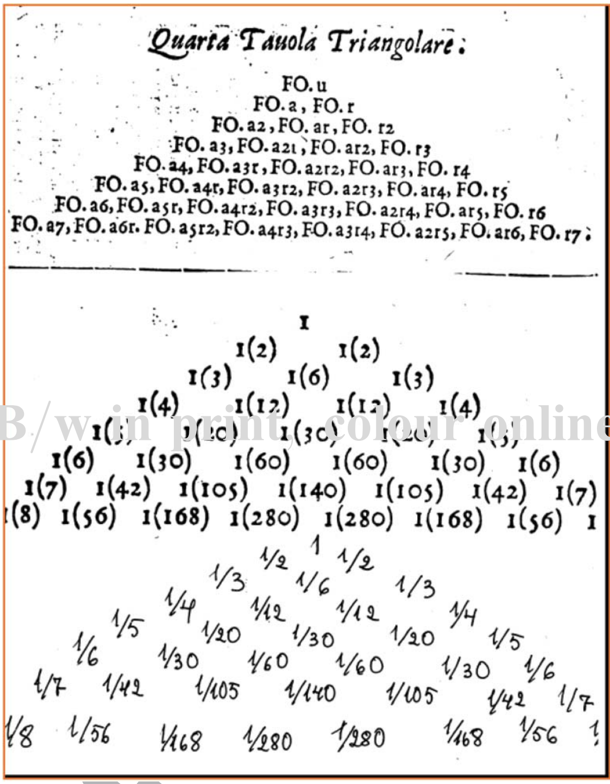


Figure 4. (a) Mengoli's *Tabula Formosa*; (b) Mengoli's harmonic triangle (Mengoli 1672, 4 and 7), and (c) author's transcription with fractions

algebraic expressions in geometric figures with the values of their quadratures of the *Tabula Formosa* by homology. Homologous terms for Mengoli are terms situated in the same place in each of the tables, which also preserve their proportions. Thus, the square of the vertex is homologous to the unity, and the geometric figures in the first row are homologous to $\frac{1}{2}$ and so on (Figure 4).

Mengoli explained this relation of the homologous terms rhetorically:

> 15. Which [figures] I have proved to be proportional to the quantities arranged in the third triangular table; and the square of the Rational $FO.u$, is homologous to the unity; and the triangles $FO.a$, and $FO.r$, are homologous to the half; and the [parabolas] FO $FO.ar$, $FO.r_2$, are homologous to a third, a sixth and a third of the unity; and $FO.a_3$, $FO.a_2.r$, $FO.a.r_2$, $FO.r_3$, are homologous to a fourth, a twelfth, a twelfth and a fourth of the unity, and of the square itself; and thus all the other forms in order, as als the sixth element, as also in the sixth element one may deduce by corollary from Proposition 10.[23]

[23] '15. Le quali tutte hò dimostrato, che sonoproportionali, come le quantità disposte nella terza tavola triangulare; ed è il quadrato Della Rationale FO. u, homologa all'unità; e i triangoli FO. a, e FO. r, homologhi allà metà; e le FO. a2, FO. ar, FO. r2, homologhe alle parti quarta, duodecima, duodecima, e quarta, dell'unità, e dello stesso quadrato; e cosí tutte le altre forme per ordine: como ivi en el sesto elemento si può dedurrè per corollario dalla prop. 10' (Mengoli 1672, 6).

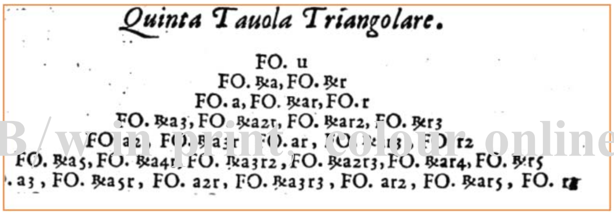


Figure 5. Mengoli's interpolated *Tabula Formosa* (Mengoli 1672, 5)

Mengoli subsequently applied his method to compute infinitely many values of geometric figures through the interpolation of both tables for half-integer values of the exponents. First, he displayed the interpolated geometric figures $\int_0^1 x^{m/2}(1-x)^{n/2}\mathrm{d}x$ for natural numbers m and n, in an infinite interpolated triangular table, which I will call interpolated *Tabula Formosa* (Figure 5).

He then obtained an infinite interpolated triangular table of values of their quadratures. With the help of the properties of combinatorial triangle, Mengoli was now able to fill the interpolated combinatorial triangle (Figure 6), except for an unknown number 'a' (Massa-Esteve and Delshams 2009).

The number 'a' is closely related to the quadrature of the circle ($1/2a = \pi/8$), thereby constructing the interpolated harmonic triangle (Figure 7). In fact, he obtained successive approximations of the number 'a' in order to approximate the number π up to eleven decimal places (Massa-Esteve 1998).

The proportion between the geometric figures in the interpolated *Tabula Formosa* and the homologous values of their areas in the interpolated harmonic triangle is preserved thanks to their construction. Mengoli makes clear that this relation referred to by him as 'the co-ordination of the two tables', is maintained, since the first tables (geometric figures and values of areas) are reproduced inside the interpolated tables. Later, these ideas were read by Leibniz and he himself wrote some excerpts from Mengoli's work as we describe below.

Mengoli's ideas in Leibniz's excerpts (1676) from *Circolo* (1672)

Leibniz acquired new mathematical knowledge during his stay in Paris in the years 1672–76 and wrote some excerpts from Mengoli's *Circolo*. According to Probst (2015), the first part of the excerpts from *Circolo*, declared as missing in the *Catalogue critique* of Leibniz's manuscripts (Rivaud 1914–24), is probably at least partly identical to the manuscript LH 35 XII 1 fol 9–10, entitled *Arithmetica infinitorum et interpolationum figuris applicata et summa harmonicorum sub finem adiecta*, printed in A VII, 3 No 57_ 2 (Leibniz 2003), and which formerly had been located together with the excerpts by Leibniz. Indeed, in a note written by Leibniz we can read 'I put this sheet to the Excerpta ex Mengoli Circolo'.[24] There, Leibniz essentially discusses the first triangular tables used by Mengoli in his *Circolo* and tries to find a method for the computation of areas using the harmonic series.

Later Leibniz made other excerpts from *Circolo* entitled by the editors: *Aus und zu Mengolis Circolo*. These manuscripts consist of two different parts: the first part, the construction of the interpolated triangular tables, and the second, the computation of

[24] 'Hoc schediasma collocavi apud Excerpta ex Mengoli Circulo' (Leibniz A VII, 3 No 57_1). This was later removed to another place within the collection of Leibniz's manuscripts (at an unknown date before the end of the nineteenth century).

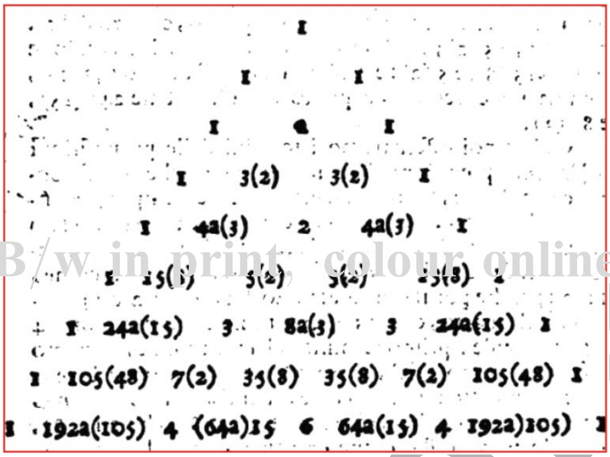


Figure 6. Mengoli's interpolated combinatorial triangle (Mengoli 1672, 13)

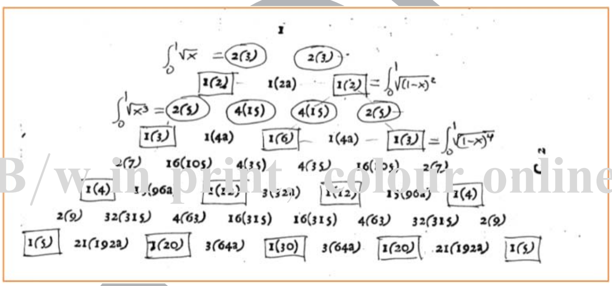
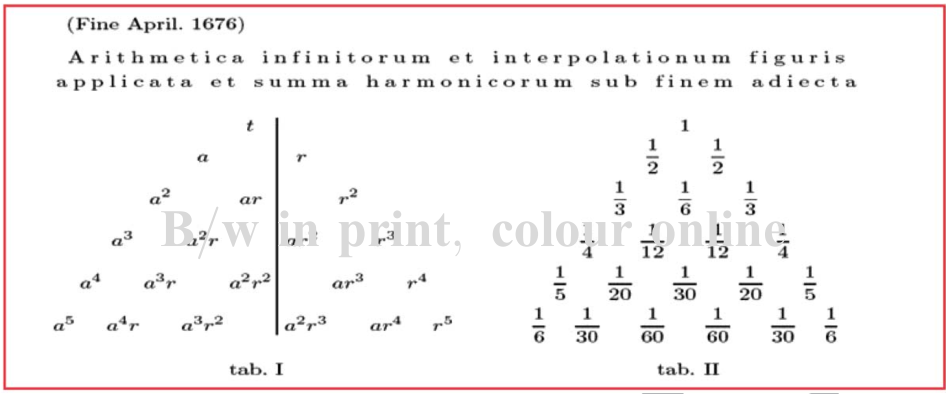


Figure 7. Mengoli's interpolated harmonic triangle with our annotations (Mengoli 1672, 19)

the number π. The first part is printed in A VII 6 N. 13_1, 113–120 (Leibniz 2012), which corresponds to Mengoli's *Circolo*, from page 1 to 20. The second part is entitled by Leibniz: *Pars 2 Excerptorum ex Circulo Mengoli, et Ad Eum Annotatorum*, printed in A VII 6 N. 13_ 2, 120–131 (Leibniz 2012), which corresponds to Mengoli's *Circolo* from page 26. In this paper we focus only on the first part of these excerpts, that is, Leibniz's interpretation of the interpolated geometric figures, and their quadratures through the interpolated triangular tables. Thus, the computation procedure of the number π that Leibniz dealt with in the second part of these excerpts has not been compared with the corresponding calculations in Mengoli's *Circolo* yet.

Leibniz's interpretation of Mengoli's ideas in *Arithmetica infinitorum et interpolationum figuris applicata* (Fine April 1676)

In the first part, these excerpts consist of some triangular tables and their explanations (Leibniz 2003). In fact, the *Tabula Formosa* with geometrical figures in Mengoli's *Circolo* was reproduced in Leibniz's excerpts with a triangular table, where the terms are the ordinates of these geometric figures. Mengoli's harmonic triangle with the values of quadratures of geometric figures in Mengoli's *Circolo* was reproduced in Leibniz's excerpts with a harmonic triangle, but they were written in a different notation. Mengoli wrote the denominator in brackets like 1(2), while in contrast Leibniz wrote the fraction $\frac{1}{2}$ (see Figure 4 and Figure 8).


(Fine April. 1676)

Arithmetica infinitorum et interpolationum figuris applicata et summa harmonicorum sub finem adiecta

$$\begin{array}{ccccccccccc} & & & & & t & & & & & \\ & & & & a & & r & & & & \\ & & & a^2 & & ar & & r^2 & & & \\ & & a^3 & & a^2r & & ar^2 & & r^3 & & \\ & a^4 & & a^3r & & a^2r^2 & & ar^3 & & r^4 & \\ a^5 & & a^4r & & a^3r^2 & & a^2r^3 & & ar^4 & & r^5 \end{array}$$

tab. I

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & \frac{1}{2} & & \frac{1}{2} & & & & \\ & & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & & \\ & & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & & \\ & \frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} & \\ \frac{1}{6} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{60} & & \frac{1}{30} & & \frac{1}{6} \end{array}$$

tab. II

Figure 8. Leibniz's table of ordinates of geometric figures (tab. I) and Leibniz's harmonic triangle (tab. II) (Leibniz 1676, AVII 3, No 57_2, 2003, 736)

In addition, there is a sharp contrast between Mengoli's rhetorical justification of identification of terms in *Tabula Formosa* and Harmonic Triangle by homology and Leibniz's interpretation of this identification. Indeed, Leibniz justified the identification of table of ordinates of geometric figures and their quadratures by designing a geometrical figure and explaining all the segments of the drawing (Figure 9).

For the coordinates, Leibniz took abscissas in a vertical axis and ordinates in a horizontal axis, in contrast with Mengoli's system of coordinates defined in *Geometria*.[25] In the drawing of the geometrical figure, Leibniz identified the segments as the tota (AR), the abscissa (AB), the remainder (BR) and the ordinate (BC) (Figure 9):

> Let trilateral space ARDCA be closed by two straight lines AR, RD and a curve ACD. The line AR tota is t. Abscissa AB is a. Residua BR is r. The ordinate BC is y.[26]

Furthermore, Leibniz identified the geometric figures: the square (ARDE), the triangle (ARD) and the parabolic curve (ARDGA) in the same figure, changing the value of the ordinate BC and assuming the value of tota equal to 1 (Figure 9).

> If we put [iam] that $y = t$, it is as if by geometric figure ARDCA would be the square ARDE. If y were equal to a, or well, to r, or if the ordinate BF were equal to AB, the geometric figure ARDCA would be the triangle a semi-squareARD. If $y = a^2/t$ or well put $t = 1$ if $y = a^2 =$ BG, the geometric figure ARDCA would be the geometric figure ARDGA trilateral space parabolic and also for other figures, as if y were equal a^3, or well a^4.[27]

[25]Leibniz may not have read Mengoli's system of coordinates, because in *Circolo* Mengoli did not include any drawing of geometric figures.
[26]Sit trilineum *ARDCA* duabus rectis *AR.RD* et curva *ACD*. Inclusum. Recta *AR* tota sit *t*.Abscissa AB sit a. Residua BR sit r. Ordinata BC sit *y* (Leibniz 1676[illegible]2, 2003, 737).
[27]Si ponamus iam esse y Π t. ut si pro figura ARDCA erit quadratum *ARDE*. Si y sit Π a vel r, seu si ordinata BF sit Π AB. pro figura ARDCA, erit triangulum a semiquadratum ARD. Sin y Πa^2/t vel posita tΠ 1, si y Πa^2 Π BG, pro figura ARDCA, erit ARDGA trilineum parabolicum, et ita de caeteris altioribus, ut si y sit a^3, vel a^4 (Leibniz 1676, AVII 3, 57_2, 2003, 737).

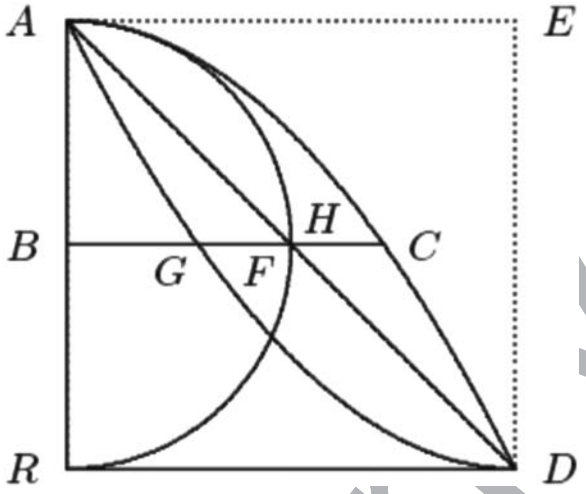


Figure 9. Leibniz's geometrical figure (Leibniz 1676, AVII 3 No 57_2, 2003, 737)

On the other hand, in his *Circolo* Mengoli did not identify these geometric figures through a drawing. Indeed, Mengoli did not include any drawing of geometric figures and always worked with algebraic expressions of these geometric figures.

Leibniz then identified the terms in the triangular Table I, which are ordinates of geometric figures, with the terms of the triangular Table II (Figure 8), which are values of areas. And for this identification with the values of areas he used the sum of all ordinates of geometric figures. Leibniz explained this idea rhetorically:

> Further, since it has all the (geometric) figures whose ordinates are shown in Tab. 1, it will yield the quadratures (and they are obviously all of the kind of Parabolic, for rational whole numbers); therefore, the complete area of the (geometric) figures or indeed the sums of all [*omnium*][28] ordinates from A to R that will be expressed in the Tab. 2. Of course, all of *a*, or indeed the area ARD is $\frac{1}{2}$ of the square ARDE. All of a^2, or indeed the area ARDGA is 1/3 of the same (square); and so on.[29]

Mengoli did not identify the terms of the triangular table with ordinates, perhaps to avoid the idea of summing all ordinates, influenced by Cavalieri's controversial ideas on indivisibles. Leibniz, on the other hand, identified the terms of the triangular table with ordinates, drawing a figure for identifying the ordinate of each geometric figure and making the summation of ordinates for squaring geometrical figures without any problem. Besides, Leibniz did not state that

[28] Obviously the expression was originated with Cavalieri.

[29] Porro quoniam harum figurarum omnium quarum ordinatae tab. I exhibentur, datae sunt quadraturae (sunt enim omnes ex genere paraboloeidum, quippe rationales integrae) ideo area figurarum completarum seu summas omnium ordinatarum ab A ad R. expressimus tab. II. Nempe omnes a. seu area ARD est $\frac{1}{2}$ quadrati ARDE. Omnes a^2, seu area ARDGA, est 1/3 eiusdem; et ita de caeteris (Leibniz 1676, AVII 3, 57_2, 2003, 738).

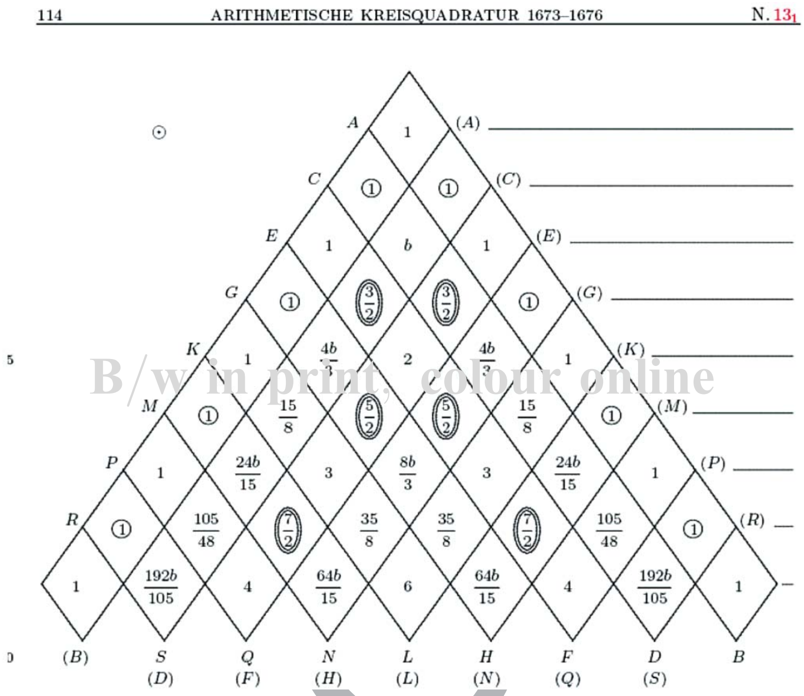


Figure 10. Leibniz's interpolated combinatorial triangle (Leibniz 1676, AVII 6 No 13_1, 2012, 114)

it was his own interpretation, because in Mengoli's *Circolo* there were no drawings of the geometrical figures, only algebraic expressions of geometric figures and the identification of triangular tables.

Leibniz's interpretation of Mengoli's ideas in *Aus und zu Mengolis Circolo* (1676)

The first part of *Aus und Zu Mengolis Circolo* consists of a large sheet with three triangular tables: an interpolated combinatorial triangle, an interpolated harmonic triangle and an interpolated table of ordinates of geometric figures, containing explanations of their constructions (Leibniz 2012). In the first paragraphs, Leibniz explained the construction of the interpolated combinatorial triangle (Figure 10) and taking AB, the unities: 1, 1, 1,...; EF, the arithmetic numbers: 1, 2, 3,...; KL, the triangular numbers 1, 3, 6,...; PQ, the pyramidal numbers, 1, 4, ... and so forth, he calculated the interpolated lines CD, GH, MN, RS, and so on. In fact, these explanations for computing the interpolated combinatorial triangle in Figure 10 are easier and more understandable than Mengoli's explanations in *Circolo* for computing a similar table (Massa-Esteve and Delshams 2009).

The construction of Leibniz's interpolated harmonic triangle (Figure 11) from the interpolated combinatorial triangle (Figure 10) was explained by some operations of multiplication and the inversion of the results, as in Mengoli's *Circolo*, without any additional comment:

> The series of the table will be multiplied in the horizontal, the first or well the vertex 1, by 1, or well by 2/2; the second or well 1, 1, by 3/2; the third, or well, 1, b, 1,

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			☽					1								
							$\frac{2}{3}$		$\frac{2}{3}$							
						$\frac{1}{2}$		$\frac{1}{2b}$		$\frac{1}{2}$						
					$\frac{2}{5}$		$\frac{4}{15}$		$\frac{4}{15}$		$\frac{2}{5}$					
				$\frac{1}{3}$		$\frac{1}{4}$		$\frac{1}{6}$		$\frac{1}{4}$		$\frac{1}{3}$				
			$\frac{2}{7}$		$\frac{16}{105}$		$\frac{4}{35}$		$\frac{4}{35}$		$\frac{16}{105}$		$\frac{2}{7}$			
		$\frac{1}{4}$		$\frac{15}{96b}$		$\frac{1}{12}$		$\frac{3}{32b}$		$\frac{1}{12}$		$\frac{15}{96b}$		$\frac{1}{4}$		
	$\frac{2}{9}$		$\frac{32}{315}$		$\frac{4}{63}$		$\frac{16}{315}$		$\frac{16}{315}$		$\frac{4}{63}$		$\frac{32}{315}$		$\frac{2}{9}$	
$\frac{1}{5}$		$\frac{21}{192b}$		$\frac{1}{20}$		$\frac{3}{64b}$		$\frac{1}{30}$		$\frac{3}{64b}$		$\frac{1}{20}$		$\frac{21}{192[b]}$		$\frac{1}{5}$

Figure 11. Leibniz's interpolated harmonic triangle (Leibniz, 1676, A VII 6 No 13_1, 2012, 115)

by 4/2 or well 2; the fourth 1, 3/2, 3/2, 1, by 5/2, and so on. All the terms of products that will be inverted, the unities in the numerator and the product in the denominator, make up the table☽.[30]

Nevertheless, if we compare Leibniz's interpolated harmonic triangle (Figure 11) with Mengoli's interpolated harmonic triangle (Figure 7), two small differences can be observed: Mengoli gave the letter a, while Leibniz put the letter b for finding the number π, and as in the former triangular tables, Leibniz wrote fractions, and Mengoli put the denominator in brackets.

However, the most remarkable differences are in Leibniz's construction of the triangular table of interpolated geometric figures. Thus, although Leibniz wrote the ordinates of these interpolated geometric figures (Figure 12), he did not write the expressions of geometric figures (Forms) as Mengoli did in his *Circolo* (Figure 5). Furthermore, Leibniz did not refer to Mengoli's way of expressing his Forms.

As in the other excerpts, Leibniz drew a geometric figure (Figure 13) for establishing the identification between the interpolated table of ordinates of geometric figures (Figure 12), and the values of their quadratures in the interpolated harmonic triangle (Figure 11).

Leibniz explained rhetorically the interpolated tables of ordinates, identifying each ordinate of the corresponding geometric figure. He explained this identification in his additional geometric figure, taking again the vertical axis as the x-axis, and the horizontal axis the y-axis. Leibniz identified the abscissa (AF), the remainder (FC) and the tota (AC). In addition, he expressed the ordinates of the circle and of the parabola (FG) (Figure 13):

[30]Series Tabulae ⊙ horizontales multiplicentur, prima seu vertex 1. per 1. seu 2/2. secunda seu 1. 1. Per 3/2. tertia seu 1. b. 1. Per 4/2 seu 2. quarta 1. 3/2. 3/2. 1. per 5/2. Et ita porro. Termini singuli producti invertantur, unitato numeratore in nominatorem et contra, fiet Tabula ☽ (Leibniz 1676, AVII 6 No 13_1, 2012, 118).

♀	1							
$\sqrt{a}$	$\sqrt{r}$							
a	$\sqrt{ar}$	r						
$a\sqrt{a}$	$a\sqrt{r}$	$r\sqrt{a}$	$r\sqrt{r}$					
a^2	$a\sqrt{ar}$	ar	$r\sqrt{ar}$	r^2				
$a^2\sqrt{a}$	$a^2\sqrt{r}$	$ar\sqrt{a}$	$ar\sqrt{r}$	$r^2\sqrt{a}$	$r^2\sqrt{r}$			
a^3	$a^2\sqrt{ar}$	a^2r	$ar\sqrt{ar}$	r^2a	$r^2\sqrt{ar}$	r^3		
$a^3\sqrt{a}$	$a^3\sqrt{r}$	$a^2r\sqrt{a}$	$a^2r\sqrt{r}$	$ar^2\sqrt{a}$	$r^2a\sqrt{r}$	$r^3\sqrt{a}$	$r^3\sqrt{r}$	
a^4	$a^3\sqrt{ar}$	a^3r	$a^2r\sqrt{ar}$	a^2r^2	$r^2a\sqrt{ar}$	r^3a	$r^3\sqrt{ar}$	r^4

Figure 12. Leibniz's interpolated table of ordinates of geometric figures (Leibniz 1676, AVII 6 No 13_1, 2012, 115)

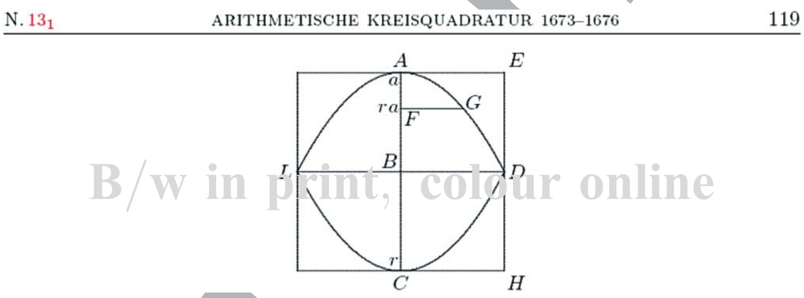

Figure 13. Leibniz's geometric figure (Leibniz, 1676, AVII 6 No 13_1, 2012, 119)

> The table ♀ can be explained from the additional geometric figure. AF $=$ *a* abscissa, FC $= r$ remainder, AC tota. The ordinates FG will be the other terms of the table ♀ and if AGD were the quadrant of the circle, FG would be $\sqrt{ar}$, and if parabola were with axis AB, [the ordinate] FG would be, $\sqrt{a}$ or well $\sqrt{r}$.[31]

Leibniz justified the values of quadratures of the interpolated geometric figures by the drawing of geometric figure (Figure 13), specifying that by approximating the quadrature of the circle, he needed to find the letter *b* and to put the value of tota equal to 1:

> Also terms of series explained the areas of geometric figures AGDCLA, put AC square of tota AC, 1.[32]

Leibniz made similar interpretations for the interpolated triangular tables as in the former triangular tables. However, we wish to point out that, from page 21 to 25 in *Circolo*, Mengoli justified the quadratures of interpolated geometrical figures

[31]Tabula ♀ ex figura adjecta explicatur AF Π a. abscissa, FC Π r residua, AC tota. FG ordinata erit aliquis terminus tabulae, ♀. ut si AGD sit quadrans circuli FG erit $\sqrt{ar}$, si parabola sit cujus axis AB, erit FG, $\sqrt{a}$ vel $\sqrt{r}$ (Leibniz 1676, AVII 6 No 13_1, 2012, 119).

[32]Termini autem seriei explicant aream figurae AGDCLA, posito AC, quadrato totius AC, 1 (Leibniz 1676, AVII 6 No 13_1, 2012, 119).

through a classification of kinds of quadratures and also computed some particular examples of each kind (Massa-Esteve and Delshams 2009).

Some final remarks

Mengoli used ‘specious’ language both as a means of expression and as an analytic tool. Mengoli dealt with species, forms, triangular tables and quasi ratios using his specious language. We claim that the triangular tables of quadratures that Mengoli constructed indefinitely as visual structures were true algebraic tools. Through these tables, he classified the geometric figures into three groups, and proved the properties of each group by using the symmetry of the triangular tables and the regularity of their rows (Massa-Esteve 2016). However, we also argue that the most innovative aspect of Mengoli’s algebraic procedure was his use of letters to deal directly with the algebraic expression of the geometric figure. This identification and the triangular tables allowed him to work with geometric figures and to calculate their quadrature via their algebraic expressions, while at the same time deriving known and unknown values for the areas of a large class of geometric figures.

As for Leibniz’s interpretation, this author learned of Mengoli’s new mathematical ideas only a few years after 1672, when they were published. In his excerpts of 1676 from Mengoli’s *Circolo*, Leibniz constructed the triangular table of ordinates, whereas Mengoli had constructed the triangular table of geometrical figures through algebraic expressions, and in this way Leibniz made his own interpretation. In addition, Leibniz drew a geometrical figure to justify rhetorically the quadratures of the figures through the segments, although he did not mention that this geometrical figure and this interpretation were not present in Mengoli’s *Circolo*. In fact, Mengoli justified the identification of the triangular tables rhetorically without the help of some figure. He worked only with the algebraic expressions, so the relation between the *Tabula Formosa* and the harmonic triangle preserves the proportion between the terms situated in the same place in each of the tables referred to by Mengoli as ‘homologous’ terms. This interpretation is also absent in Leibniz’s excerpts on interpolated figures. Perhaps Leibniz thought that his own interpretation of Mengoli’s work by means of a drawing was easier and more understandable than Mengoli’s weak rhetorical explanation by homology.

A further difference is the use of the sums of all ordinates for the identification of the values of quadratures in Leibniz’s excerpts. Mengoli did not make this sum in his method of quadratures, and his aim was not to make it. Mengoli could not pursue this approach because he was the pupil of Cavalieri, and was very familiar with the controversy of the indivisibles method, that is, the idea that the sum of all ordinates of one dimension could not give one figure of two dimensions (Massa-Esteve 2015). Leibniz was convinced that he could add the ordinates, and furthermore he attempted ‘to set the *methodus indivisibilium* on solid grounds’ (Rabouin 2015). However, the option of avoiding any mention of Mengoli’s algebraic way concealed the originality of Mengoli’s work.

Nevertheless, it should be pointed out that in 1679 Leibniz again took Mengoli’s tables in order to arrive at new results, and in this case the treatment is different and involved infinite series. We therefore surmise that Mengoli’s mathematical ideas based on algebraic procedures may have influenced Leibniz’s later developments regarding quadratures and the use of triangular tables as a tool for the computation of areas.

Indeed, as Probst (2015) also has mentioned, Mengoli was quoted in Leibniz's works as one of the scholars with whom Leibniz would like to collaborate. In one of his projects for an international science organization, *Consultatio de naturae cognitione* (1679), Leibniz mentions Mengoli among the scholars whose cooperation he desires (A IV, 3 No 133 p 868). In fact, not all authors of the seventeenth century understood Mengoli's new ideas. It is possible that his complex and confusing writing style, as well as the complicated nature of his notation, made his works too hard to read. However, it is equally possible that his original introduction of algebra into geometry failed to accord with the prevailing mathematical practice of the seventeenth century.

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Q2
References

Angeli, S D, *Miscellaneum hyperbolicum*, Bologna, 1659.

Baroncini, G, and Cavazza, M, *La corrispondenza di Pietro Mengoli*, Olschki Florence, 1986.

Bos, H J M, 'On the representation of curves in Descartes *Géométrie*', *Archive for the History of Exact Sciences*, 24 (1981), 295–338.

Bos, H J M, *Redefining geometrical exactness. Descartes' transformation of the early modern concept of construction*, Berlin: Springer-Verlag, 2001.

Bosmans, H, 'Sur l'oeuvre mathématique de Blaise Pascal', *Mathesis*, 38 (1924), (Supplément), 1–59.

Cavalieri, Bonaventura, *Geometria degli indivisibili di Bonaventura Cavalieri*, (trans L Lombardo Radice), Torino: Unione Tipografico-Editrice Torinese, 1966.

Clagett, M, *Nicole Oresme and the medieval geometry of qualities and motion*, Madison, WI: Wisconsin Madison, 1968.

Crombie, A C, *Historia de la ciencia: de San Agustin a Galileo, Siglos XIII–XVII*, (Alianza Universidad), Madrid: Alianza Editorial, 1980.

Descartes, René, in D E Smith and M L Latham (eds), *The geometry of René Descartes*, New York: Dover, 1954.

Edwards, A W F, *Pascal's arithmetical triangle. The story of a mathematical idea*, Oxford: Oxford University Press, 2002 (1st edition, 1987).

Fermat, Pierre de, in Tannery, P and Charles, H (eds), *Oeuvres*, 4 vols &spp, Paris: Gauthier-Villars, 1891–1922.

Giusti, Enrico, 'La "Géométrie" di Descartes tra numeri e grandezze', *Giornale Critico Della Filosofia Italiana*, 66 (1987), 409–432.

Giusti, Enrico, 'Le prime ricerche di Pietro Mengoli: la soma delle serie', in S Coen (ed), *Geometry and complex variables: proceedings of the international meeting on the Occasion on the IX Centennial of the University of Bologna*, New York: Marcel Dekker, 1991.

Giusti, Enrico, 'Algebra and geometry in Bombelli and Viete', *Bollettino di Storia delle Scienze Matematiche*, 12 (1992), 303–328.

Hérigone, Pierre, *Cursus Mathematicus nova, brevi et clara método demonstratus, Per NOTAS reales & universales, citra usum cuiuscumque idiomatis, intellectu, faciles/ Cours Mathematique demonstréd' une nouvelle briefve et Claire methode. Par notes reelles & universelles, qui peuvent ester entendues sans l'usaged'aucune langue*. 5 vols. (1634, 1637) plus a supplement (1642), for the author and Henry Le Gras Paris, 1634/1637/1642.

Hofmann, Joseph, *Leibniz in Paris 1672–1676*, Cambridge: Cambridge University Press, 1974.

Knobloch, Eberhard, 'Leibniz's rigorous foundation of infinitesimal geometry by means of Riemannian sums', *Synthese*, 133 (2002), 59–73.

Leibniz, Gottfried Wilhelm, *Sämtliche Schriften und Briefe* (series III): *Mathematischer, naturwissenschaftlicher und technischer Briefwechsel* (vol 1), 1672–76, II edition, Berlin-Brandenburgischen Akademie der Wissenschaften, 1988; series VII: *Mathematische Schriften* (vol 1), 1672–76: *Geometrie – Zahlentheorie – Algebra* (1st part), Berlin, 1990; series VII: *Mathematische Schriften* (vol 3), 1672–76: *Differenzen – Folgen – Reihen*, Berlin: Akademie-Verlag, 2003; series VII: *MathematischeSchriften* (vol 6), 1673–76: *Arithmetische Kreisquadratur*, Berlin: Akademie-Verlag, 2012.

Leibniz, Gottfried Wilhelm, *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis* (critical edition with commentary by E Knobloch), Göttingen: Vandenhoeck & Ruprecht, 1993.

Mahoney, M S, *The mathematical career of Pierre de Fermat*, Princeton, NJ: Princeton University Press, 1973.

Mahoney, M S, 'The beginnings of algebraic thought in the seventeenth century', in S Gaukroger (ed), *Descartes' philosophy, mathematics and physics*, Totowa, NJ/Brighton: Totowa Barnes and Noble/Harvester Brighton, 1980.

Mancosu, P, *Philosophy of mathematics and mathematical practice in the seventeenth century*, Oxford: Oxford University Press, 1996.

Massa, Ma Rosa, 'Mengoli on "Quasi Proportions"', *Historia Mathematica*, 24 (1997), 257–280.

Massa-Esteve, Ma Rosa, *Estudis matemàtics de Pietro Mengoli (1625–1686): Taules triangulars i quasi proporcions com a desenvolupament de l'àlgebra de Viète*, Tesi Doctoral, Universitat Autònoma de Barcelona, 1998 (http://hdl.handle.net/10803/3103).

Massa-Esteve, Ma Rosa, 'Algebra and geometry in Pietro Mengoli (1625–1686)', *Historia Mathematica*, 33 (2006a), 82–112.

Massa-Esteve, Ma Rosa, *L'algebrització de les matemàtiques: Pietro Mengoli*, Barcelona: Institut d'Estudis Catalans, 2006b.

Massa-Esteve, Ma Rosa, 'Symbolic language in early modern mathematics: the algebra of Pierre Hérigone (1580–1643)', *Historia Mathematica*, 35 (2008), 285–301.

Massa-Esteve, Ma Rosa, 'The symbolic treatment of Euclid's, *Elements* in Hérigone's *Cursus mathematicus* (1634, 1637, 1642)', in A Heeffer and M Van Dyck (eds), *Philosophical aspects of symbolic reasoning in early modern mathematics*, London: King's College London, 2010.

Massa-Esteve, Ma Rosa, 'The role of symbolic language in the transformation of mathematics', *Philosophica*, 87 (2012), 153–193.

Massa-Esteve, Ma Rosa, 'The role of indivisibles in Mengoli's quadratures', in V Julien (ed), *Seventeenth-century indivisibles revisited* (Science Networks: Historical Studies 49), Berlin: Springer, 2015.

Massa-Esteve, Ma Rosa, 'Nous resultats i procediments en les matemàtiques del segle XVII: càlcul de maxims a Pietro Mengoli (1626/1627–1686)', *Butlletí de la Societat Catalana de Matemàtiques*, 31 (2016), 51–71.

Massa-Esteve, Ma Rosa, and Delshams, Amadeu, 'Euler's beta integral in Pietro Mengoli's works', *Archive for History of Exact Sciences*, 63 (2009), 325–356.

Mengoli, Pietro, *Geometriae speciosae elementa*, Bologna, 1659.

Mengoli, Pietro, *Circolo*, Bologna, 1672.

Natucci, A, 'Mengoli', in C C Gillispie (ed), *Dictionary of scientific biography* (vol 9), New York: Scribner's, 1970–91.

Oldenburg, Henry, in R Hall and M B Hall (eds), *The correspondence of Henry Oldenburg* (13 vols), Madison, WI: University of Wisconsin Press, 1695–1975, London: Mansell, 1975–1977, Abingdon: Taylor & Francis, 1986.

Pascal, Blaise, *Oeuvres completes*, Paris: Gallimard, 1954.

Probst, Siegmund, 'Leibniz as reader and second inventor: the cases of Barrow and Mengoli', in N B Goethe, P Beeley, and D Rabouin (eds), *G.W. Leibniz, interrelations between mathematics and philosophy* (Archimedes vol 41), Berlin: Springer, 2015.

Rabouin, David, 'Leibniz's rigorous foundations of the method of indivisibles. Or how to reason with impossible notions', in V Julien (ed), *Seventeenth-century indivisibles revisited*, (Science Networks: Historical Studies 49), Berlin: Springer, 2015.

Rivaud, A, et al. *Catalogue critique des manuscrits de Leibniz. Fascicule II (Mars 1672–Novembre 1676)*, SFIL Poitiers, 1914–24.

Stedall, J, *The greate invention of algebra: Thomas Harriot's treatise on equations*, Oxford: Oxford University Press, 2003.

Stedall, J, 'Symbolism, combinations, and visual imagery in the work of Thomas Harriot', *Historia Mathematica*, 34 (2007), 380–401.

Stedall, J, *From Cardano's great art to Lagrange's reflections: filling a gap in the history of algebra*, Zurich: European Mathematical Society Publishing House, 2011.

Torricelli, E, in G Loria et al. (eds), *Opere* (3 vols), Faenza, 1919.

Viète, F, *The analytic art* (trans T R Witmer), Kent, OH: Kent State University Press, 1983.