REGULARITY OF RADIAL MINIMIZERS
AND EXTREMAL SOLUTIONS OF
SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We consider a special class of radial solutions of semilinear equations \(-\Delta u = g(u)\) in the unit ball of \(\mathbb{R}^n\). It is the class of semi-stable solutions, which includes local minimizers, minimal solutions, and extremal solutions. We establish sharp pointwise, \(L^q\), and \(W^{k,q}\) estimates for semi-stable radial solutions. Our regularity results do not depend on the specific nonlinearity \(g\).

Among other results, we prove that every semi-stable radial weak solution \(u \in H^1_0\) is bounded if \(n \leq 9\) (for every \(g\)), and belongs to \(H^3 = W^{3,2}\) in all dimensions \(n\) (for every \(g\) increasing and convex). The optimal regularity results are strongly related to an explicit exponent which is larger than the critical Sobolev exponent.

1. Introduction

This article is concerned with a special class of radial solutions of semilinear elliptic equations. It is the class of semi-stable solutions, which includes local minimizers, minimal solutions, extremal solutions, and also certain solutions found between a sub and a supersolution. We establish sharp pointwise, \(L^q\), and \(W^{k,q}\) estimates for semi-stable radial solutions. Our regularity results do not depend on the specific nonlinearity in the equation. Some of our bounds hold for every locally Lipschitz nonlinearity, while others hold for every increasing and convex nonlinearity.

The original motivation of our work is the following. Consider the semilinear elliptic problem

\[
\begin{aligned}
-\Delta u &= \lambda f(u) \quad \text{in } \Omega \\
u &\geq 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega, 
\end{aligned}
\]  

(1.1)\(_\lambda\)

where \(\Omega \subset \mathbb{R}^n\) is a smooth bounded domain, \(n \geq 2\), \(\lambda \geq 0\), and the nonlinearity \(f : [0, +\infty) \to \mathbb{R}\) satisfies

\[
f \text{ is } C^1, \text{ nondecreasing and convex, } f(0) > 0, \text{ and } \lim_{u \to +\infty} \frac{f(u)}{u} = +\infty. \quad (1.2)
\]

It is well known that there exists an extremal parameter \(\lambda^*\) such that if \(0 \leq \lambda < \lambda^*\) then (1.1) admits a minimal classical solution \(u_\lambda\). On the other hand, if \(\lambda > \lambda^*\) then (1.1) has no classical solution. Here, classical means
bounded, while minimal means smallest. The set \( \{ u_\lambda : 0 \leq \lambda < \lambda^* \} \) forms a branch of classical solutions increasing in \( \lambda \). Its increasing limit as \( \lambda \to \lambda^* \) is a weak solution \( u^* = u_{\lambda^*} \) of \((1.1_{\lambda^*})\), which is called the extremal solution of \((1.1_\lambda)\).

When \( f(u) = e^u \), it is known that \( u^* \in L^\infty(\Omega) \) if \( n \leq 9 \) (for every \( \Omega \)), while \( u^*(x) = -2\log |x| \) if \( n \geq 10 \) and \( \Omega = B_1 \). A similar phenomenon happens when \( f(u) = (1 + u)^p \) with \( p > 1 \). Brezis and Vázquez [3] raised the question of determining the regularity of \( u^* \), depending on the dimension \( n \), for general nonlinearities \( f \) satisfying (1.2). The best known result is due to Nédélec [16], who proved that, for every \( \Omega \) and nonlinearity \( f \) satisfying (1.2), \( u^* \in L^\infty(\Omega) \) if \( n \leq 3 \), while \( u^* \in H^1_0(\Omega) \) if \( n \leq 5 \).

In this article we establish optimal regularity results for \( u^* \) in the radial case, that is, when \( \Omega = B_1 \) is the unit ball of \( \mathbb{R}^n \). We write \( r = |x| \) for \( x \in \mathbb{R}^n \). Among other results (Theorem 1.10 states all our estimates for \( u^* \)), we prove the following:

**Theorem 1.1.** Assume that \( \Omega = B_1 \), \( n \geq 2 \), and that \( f \) satisfies (1.2). Let \( u^* \) be the extremal solution of \((1.1_\lambda)\). We have that:
(a) If \( n \leq 9 \), then \( u^* \in L^\infty(B_1) \).
(b) If \( n = 10 \), then \( u^*(r) \leq C|\log r| \) in \( B_1 \) for some constant \( C \).
(c) If \( n \geq 11 \), then \( u^* \in L^q(B_1) \) for every \( q < q_0 := 2n/(n - 2\sqrt{n-1} - 4) \). Moreover, for every \( n \geq 11 \) there exists \( p_n > 1 \), given by (1.12), such that \( u^* \notin L^{p_n}(B_1) \) when \( f(u) = (1 + u)^{p_n} \).
(d) \( u^* \in H^3(B_1) = W^{3,2}(B_1) \) for every dimension \( n \).

Statements (a) and (b) are sharp in the sense that the pointwise estimate in (b) is indeed an equality when \( f(u) = e^u \). The statement in (c) makes clear that the exponent \( q_0 \) is also optimal.

The estimates of Theorem 1.1 are consequence of the semi-stability of \( u^* \)—a property which will follow from the minimality of \( u^* \). By semi-stability we mean that the linearized operator of \((1.1_{\lambda^*})\) at \( u^* \) is nonnegative definite (see Definition 1.4). In fact, all our estimates will be based only on the semi-stability of the solution. Hence, they hold not only for extremal solutions as above, but also for local minimizers of the energy, that we describe next.

Consider the energy functional
\[
E_\Omega(u) := \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 - G(u) \right\} \, dx,
\]
where \( G : \mathbb{R} \to \mathbb{R} \) is of class \( C^2 \) and \( \Omega \subset \mathbb{R}^n \) is a smooth bounded domain.

We consider radial functions in \( H^1(B_1) \) (perhaps unbounded) that minimize the energy under small perturbations in the \( C^1_c(B_1 \setminus \{0\}) \) topology. More precisely, we give the following definition:
**Definition 1.2.** We say that a radial function \( u \in H^1(B_1) \) is a radial local minimizer if for every \( \delta > 0 \) there exists \( \varepsilon_\delta > 0 \) such that

\[
E_{B_1 \setminus B_\delta}(u) \leq E_{B_1 \setminus B_\delta}(u + \xi)
\]

for every radial \( C^1 \) function \( \xi \) with compact support in \( B_1 \setminus \overline{B_\delta} \) and with \( \|\xi\|_{C^1} \leq \varepsilon_\delta \). Recall that the energy \( E \) is defined in (1.3).

Note that the energy of \( u \) in the whole \( B_1 \) is a priori not well defined, since \( u \) could be unbounded and we make no growth assumption on \( G \). However, given \( \delta > 0 \), every radial function in \( H^1(B_1) \) also belongs (as a function of \( r = |x| \)) to the Sobolev space \( H^1(\delta, 1) \) in one dimension. Hence, by the Sobolev embedding in one dimension, away from the origin the function is bounded, and thus the energies in Definition 1.2 are well defined. In addition, every radial local minimizer \( u \) is a solution of \(-\Delta u = g'(u) \) in \( B_1 \setminus \{0\} \). Note also that we do not assume \( u \geq u(1) \), nor \( u(1) = 0 \).

The following result states sharp regularity results for the class of radial local minimizers of the energy. Note that no assumption on the potential \( G \) is made besides being of class \( C^2 \).

**Theorem 1.3.** Assume that \( n \geq 2 \) and that \( G : \mathbb{R} \to \mathbb{R} \) is \( C^2 \). Let \( u \in H^1(B_1) \) be a radial local minimizer, in the sense of Definition 1.2. We have that:

(a) If \( n \leq 9 \), then \( u \in L^\infty(B_1) \).

(b) If \( n = 10 \), then \( |u(r)| \leq C\|u\|_{H^1(B_1)}|\log r| \) for \( r < 1/2 \). Here \( C \) is a universal constant.

(c) If \( n \geq 11 \) and \( q < q_0 := 2n/(n - 2\sqrt{n - 4} - 4) \), then \( u \in L^q(B_1) \). Moreover, for some constant \( C_n \) depending only on \( n \),

\[
|u(r)| \leq C_n\|u\|_{H^1(B_1)}r^{-n/2+\sqrt{n-4}/2}|\log r|^{1/2} \quad \text{for } r < 1/2. \tag{1.4}
\]

(d) For every \( n \), \( u \) is either constant, radially decreasing, or radially increasing in \( B_1 \).

The proof of our estimates was inspired by the proof of Simons theorem on the nonexistence of singular minimal cones in \( \mathbb{R}^n \) for \( n \leq 7 \). Here, by singular minimal cone it is meant a cone which is a minimal surface (or “stable minimal surface” in certain literature) and which is not a hyperplane. In Remark 2.2 we explain the strong analogies between both proofs. The connection between semilinear equations modeling phase transitions and minimal surfaces is well known and has been revisited recently in connection with a conjecture of De Giorgi (see [1, 7, 13] and references therein).

Our estimates hold for every bounded semi-stable solution of

\[
\begin{aligned}
-\Delta u &= g(u) \quad \text{in } B_1 \\
u &\geq 0 \quad \text{in } B_1 \\
u &= 0 \quad \text{on } \partial B_1,
\end{aligned} \tag{1.5}
\]
where \( g : [0, +\infty) \to \mathbb{R} \) is locally Lipschitz. To include also some unbounded semi-stable solutions, we give the following definitions.

As in [2], we say that \( u \) is a weak solution of (1.5) if \( u \in L^1(B_1) \) is nonnegative, \( g(u)\delta \in L^1(B_1) \), and

\[
- \int_{B_1} u \Delta \zeta \, dx = \int_{B_1} g(u) \zeta \, dx
\]

for all \( \zeta \in C^2(\overline{B}_1) \) with \( \zeta = 0 \) on \( \partial B_1 \). Here \( \delta(x) = \text{dist}(x, \partial B_1) \) denotes the distance to the boundary of \( B_1 \). A weak solution \( u \) is said to be radially decreasing if and only if \( u(x) = u(r) \) and \( u \) is a decreasing function of the radius \( r \in (0, 1) \). In particular, these solutions satisfy \( u \in L^\infty_{\text{loc}}(B_1 \setminus \{0\}) \). Obviously, every classical solution of (1.5) is radially decreasing, by the Gidas-Ni-Nirenberg symmetry result (see Remark 1.12 for more comments on symmetry).

**Definition 1.4.** Let \( u \in L^\infty_{\text{loc}}(B_1 \setminus \{0\}) \) be a weak solution of (1.5). We say that \( u \) is semi-stable if

\[
Q_u(\xi) := \int_{B_1} \left\{ |\nabla \xi|^2 - g'(u)\xi^2 \right\} \, dx \geq 0
\]

for every \( \xi \in C^\infty_c(B_1 \setminus \{0\}) \), that is, for every \( C^\infty \) function \( \xi \) with compact support in \( B_1 \setminus \{0\} \).

Note that both terms in (1.6) are well defined since \( \xi \) has compact support away from the origin and we assume that \( u \in L^\infty_{\text{loc}}(B_1 \setminus \{0\}) \). For a bounded solution \( u \), semi-stability simply means that the first Dirichlet eigenvalue of the linearized operator \(-\Delta - g'(u)\) in \( B_1 \) is nonnegative. We use the term semi-stability to distinguish it from stability, which would correspond to the first eigenvalue being positive.

The class of semi-stable solutions includes not only minimal and extremal solutions, but also appropriate minimizers — since \( Q_u \) in (1.6) is formally the second variation of energy. See Remark 1.11 for more comments on this direction.

To state our estimates, we define exponents \( q_k \) for \( k \in \{0, 1, 2, 3\} \) by

\[
\begin{cases}
\frac{1}{q_k} = \frac{1}{2} - \frac{\sqrt{n-1}}{n} + \frac{k-2}{n} & \text{for } n \geq 10 \\
q_k = +\infty & \text{for } n \leq 9.
\end{cases}
\]

(1.7)

Note that \( 2 < q_k \leq +\infty \) in all cases.

Concerning pointwise and \( L^p \) estimates, the following is our main result. Here no assumption is made on the nonlinearity \( g \) besides being locally Lipschitz.

**Theorem 1.5.** Let \( n \geq 1 \), \( g : [0, +\infty) \to \mathbb{R} \) be a locally Lipschitz function, and \( u \in H^1_0(B_1) \) be a semi-stable radially decreasing weak solution of (1.5). We have that:
(a) If \( n \leq 9 \), then \( u \in L^\infty(B_1) \). Moreover,
\[
\|u\|_{L^\infty(B_1)} \leq C_n \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\}
\]
for some constant \( C_n \) depending only on \( n \).

(b) If \( n = 10 \), then \( u \in L^q(B_1) \) for all \( q < \infty \). Moreover,
\[
u(r) \leq C \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\} (|\log r| + 1)
\]
in \( B_1 \), \( (1.8) \)

where \( C \) is a universal constant. In particular,
\[
\int_{B_1} \exp \left( \frac{\beta u}{\|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)}} \right) \, dx \leq C
\]
for some universal constants \( \beta > 0 \) and \( C \).

(c) If \( n \geq 11 \) and \( q < q_n \), then \( u \in L^q(B_1) \) and
\[
\|u\|_{L^q(B_1)} \leq C_{q,n} \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\}
\]
(1.9)

where \( C_{q,n} \) is a constant depending only on \( q \) and \( n \). Moreover,
\[
u(r) \leq C_n \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\} r^{-n/2 + \sqrt{n-1} + 2}(|\log r|^{1/2} + 1)
\]
in \( B_1 \), for some constant \( C_n \) depending only on \( n \).

See Remark 1.12 for comments on the verification of the radially decreasing hypothesis made on \( u \).

**Remark 1.6.** In the case that \( g \) is nonnegative, if we multiply \(-\Delta u = g(u)\) by \( 1 - r^2 \) and integrate by parts twice in \( B_1 \), we deduce \( \|g(u)\delta\|_{L^1(B_1)} \leq C_n \|u\|_{L^1(B_1)} \). Hence, when \( g \geq 0 \) all the bounds in Theorem 1.5 can be given only in terms of the \( L^1 \) norm of \( u \).

The following remark shows the sharpness of the estimates in the previous theorem, as well as the necessity of the assumption \( u \in H^1_0 \) for the estimates to hold. The remark also shows that our optimal regularity results are strongly related to an explicit exponent \( p_n \), defined in (1.12) and sometimes called the Joseph-Lundgren exponent, which is larger than the critical Sobolev exponent.

**Remark 1.7.** As mentioned before (see [3] for more details), well known results for problem (1.1.1) with \( f(u) = e^u \) show the optimality of parts (a) and (b) of Theorems 1.1 and 1.5, including the pointwise bound (1.8).

We consider now power nonlinearities. For \( n \geq 3 \) and \( p > n/(n-2) \), the function \( u(r) = r^{-(n-2)/(p-1)} - 1 \) is an unbounded weak solution of
\[
\left\{ \begin{array}{ll}
-\Delta u = \lambda(1 + u)^p & \text{in } B_1 \\
u \geq 0 & \text{in } B_1 \\
u = 0 & \text{on } \partial B_1
\end{array} \right.
\]
(1.11)

with \( \lambda = (2/(p-1))(n - 2 - 2/(p-1)) \).
In [3] it is proved that if

\[ n \geq 11 \quad \text{and} \quad p \geq p_n := \frac{n - 2\sqrt{n - 1}}{n - 2\sqrt{n - 1} - 4} \]  

(1.12)

then the extremal solution of (1.11) is given by \( u^*(r) = r^{-2/(p-1)} - 1 \). Therefore, \( u^* \in L^q(B_1) \) if and only if \( q < n(p - 1)/2 \). To show the sharpness of statements (c) in the previous theorems, we take \( p = p_n \). Then we have that \( n(p_n - 1)/2 = q_0 \) and hence the corresponding \( H_0^1 \) semi-stable solution \( u^* \) does not belong to \( L^q_0(B_1) \). In this case \( p = p_n \), we have \( u^*(r) = r^{-n/2+\sqrt{n-1}+2} - 1 \), which differs from the pointwise power bound (1.10) for the factor \( |\log r|^{1/2} \). It is an open problem to know if this logarithmic factor in (1.10) can be removed.

On the other hand, if we take \( p \) such that

\[ \frac{n}{n - 2} < p \leq \frac{n + 2\sqrt{n - 1}}{n + 2\sqrt{n - 1} - 4} \]  

(1.13)

(which is always possible if \( n \geq 3 \)), then the weak solution \( u(r) = r^{-2/(p-1)} - 1 \) is semi-stable but does not belong to \( H_0^1 \). Since this semi-stable solution is unbounded even in dimension 3, we see the necessity of assuming \( u \in H^1_0 \) in Theorem 1.5. To show the semi-stability of this solution, we simply compute \( \lambda f'(u) = \lambda p(1 + u)^{p-1} = c_{p,n}r^{-2} \) and check that \( c_{p,n} \leq (n - 2)^2/4 \), which is the best constant in Hardy’s inequality

\[ \frac{(n - 2)^2}{4} \int_{B_1} \frac{c^2}{r^2} dx \leq \int_{B_1} |\nabla \xi|^2 dx \quad \text{for all } \xi \in C_c^\infty(B_1). \]

Note that we even have \( c_{p,n} < (n - 2)^2/4 \) if the second inequality in (1.13) is strict. As pointed out in [3], this type of “strange” solutions are apparently isolated objects that can not be obtained as limit of classical solutions.

In our next result, under additional conditions on \( g \), we prove optimal \( W^{k,q} \) estimates, with \( k \leq 3 \), for \( H_0^1 \) semi-stable solutions. Recall (1.7) for the definition of the exponents \( q_k \).

**Theorem 1.8.** Let \( n \geq 1 \), \( g : [0, +\infty) \to \mathbb{R} \) be a locally Lipschitz function, and \( u \in H_0^1(B_1) \) be a semi-stable radially decreasing weak solution of (1.5). We have that:

(a) If \( g \) is nonnegative, then \( u \in W^{1,q}(B_1) \) for every \( q < q_1 \).

(b) If \( g \) and \( g' \) are nonnegative, then \( u \in W^{2,q}(B_1) \) for every \( q < q_2 \).

(c) If \( g \), \( g' \), and \( g'' \) are nonnegative, then \( u \in W^{3,q}(B_1) \) for every \( q < q_3 \). In addition, we have the estimate

\[ g'(u(r)) \leq C_n r^{-2} \quad \text{in } B_1 \]  

(1.14)

for the potential of the linearized operator, where \( C_n \) is a constant depending only on \( n \).
(d) Moreover, under the assumptions of (a) (respectively, (b), (c)), for $k = 1$ (respectively, $k = 2$, $k = 3$) we have:
\[ \|u\|_{W^{q, q_k}(B_1)} \leq C \quad \text{if } q < q_k, \]  
and
\[ |\partial_r^{(k)} u(r)| \leq C_u \|u\|_{L^1(B_1)} r^{-n/2 + \sqrt{n+2-k}} \log r^{1/2} \]  
if $r \leq 1/4$ and $n \geq 10$, where $C$ is a constant depending only on $n$, $q$, and $k$, and upper bounds for $\|u\|_{L^1(B_1)}$, $g$, and $|g'|$, while $C_n$ is a constant depending only on $n$.

Note that for $n \geq 10$, the inclusions
\[ W^{3,q_3} \subset W^{2,q_2} \subset W^{1,q_1} \subset L^g \]
hold and, in addition, correspond to the best Sobolev embeddings. This shows that the exponents $q_k$ in Theorem 1.8 are optimal, since we already know that $q_0$ is optimal in Theorem 1.5.

The bound $Cr^{-2}$ in (1.14) for the potential $g'(u)$ is sharp, in the sense that it is an equality for some constant $C$ when $u = u^*$, $g$ is given by $\lambda^* e^u$ or $\lambda^* (1 + u)^p$, and we consider certain $p$ and $n$.

Remark 1.9. (Open problems) (i) The known regularity results in the nonradial case are very far from the ones in the previous theorems for radial solutions. At least in certain domains, can one prove or disprove some of the radial results? See [3] for more concrete questions in this direction.

(ii) As mentioned before, we do not know if the logarithmic factor in the pointwise bounds (1.10) and (1.16) can be removed or improved.

(iii) Do the estimates of Theorem 1.8 hold for general nonlinearities $g$, without the assumptions on the nonnegativity of $g$, $g'$, and/or $g''$? Recall that in principle, $g$ being decreasing or concave helps to obtain estimates.

The proof of Lemma 2.3, and hence of the estimates of Theorem 1.5, can be carried along for unstable solutions in the case that the first eigenvalue of the linearized problem $-\Delta - g'(u)$ is known to be bounded from below by a negative constant. Of course, this implies to have some apriori control on the potential $g'(u)$.

Our results also lead to estimates for the pure-power problem
\[
\begin{cases}
-\Delta v = \mu^p & \text{in } B_1 \\
v \geq \mu & \text{in } B_1 \\
v = \mu & \text{on } \partial B_1,
\end{cases}
\]
where $\mu$ is a positive constant. This problem reduces to (1.1.1) as follows. Setting $\lambda = \mu^{p-1}$ and $v = \lambda^{1/(p-1)}(1 + u)$, we have that $u$ is nonnegative, vanishes on $\partial B_1$ and satisfies $-\Delta u = \lambda(1 + u)^p$.

We now apply Theorems 1.5 and 1.8 to problem (1.1.1). We can do it since for every $0 \leq \lambda \leq \lambda^*$, the minimality of $u_\lambda$ implies that $u_\lambda$ is a semi-stable
solution (see Remark 1.11). In particular, the extremal solution \( u^* = u_\lambda \) is a weak semi-stable solution. In addition, we will see that \( u^* \in H_0^1(B_1) \) always holds.

Let us first recall the main regularity results known for (1.1.) under assumption (1.2). When \( f(u) = e^u \), it was proved in [10, 15] that \( u^* \in L^\infty(\Omega) \) if \( n \leq 9 \) (for every \( \Omega \)), while \( u^*(x) = -2\log |x| \) if \( n \geq 10 \) and \( \Omega = B_1 \). This last radial result was found by Joseph and Lundgren [14] using phase plane analysis, who also studied radial solutions for \( f(u) = (1 + u)^p \) with \( p > 1 \). In [3] it was proved that if \( \liminf_{u \to -\infty} u f'(u) / f(u) > 1 \), then \( u^* \in H_0^1(\Omega) \) for every \( \Omega \) and \( n \). The best regularity result for general convex \( f \) is due to Nedev [16], who proved that \( u^* \in L^\infty(\Omega) \) if \( n \leq 3 \), while \( u^* \in H_0^2(\Omega) \) if \( n \leq 5 \). In [17], he also proved that \( u^* \in H_0^2(\Omega) \) in every dimension \( n \) if \( \Omega \) is strictly convex. The papers [11, 19, 20] establish further regularity for \( u^* \) in general bounded domains, but assuming additional growth conditions on \( f \). On the other hand, [6, 8] extend some of the radial results in the present paper to reaction equations involving the \( p \)-Laplacian.

In section 5 we explain that the family of minimal solutions \( u_\lambda \) and the extremal solution \( u^* \) of (1.1.) also exist for more general nonlinearities than those satisfying (1.2) —see Proposition 5.1. It suffices to assume:

\[
f \text{ is } C^1, \text{ nondecreasing, } f(0) > 0, \text{ and } \lim_{u \to +\infty} \frac{f(u)}{u} = +\infty. \tag{1.17}\]

The following is the application of Theorems 1.5 and 1.8 to problem (1.1.).

**Theorem 1.10.** Assume that \( \Omega = B_1 \), \( n \geq 2 \), and that \( f \) satisfies (1.17). Let \( u^* \) be the extremal solution of (1.1.). We have that:

(a) If \( n \leq 9 \), then \( u^* \in L^\infty(B_1) \).

(b) If \( n = 10 \), then \( u^*(r) \leq C|\log r| \) in \( B_1 \) for some constant \( C \).

(c) If \( n \geq 11 \), then
\[
u^*(r) \leq Cr^{-n/2+\sqrt{n-1}+2} \log r^{1/2} \quad \text{in } B_1
\]
for some constant \( C \). In particular, \( u^* \in L^q(B_1) \) for every \( q < q_* \).

(d) Assume that, in addition, \( f \) is convex. Then, we have \( u^* \in W^{k,q}(B_1) \) for every \( k \in \{1, 2, 3\} \) and \( q < q_* \). In particular, \( u^* \in H^3(B_1) \) for every \( n \). Moreover, for every \( n \geq 10 \) and \( k \in \{1, 2, 3\} \),
\[
|\partial_r^{k} u^*(r)| \leq Cr^{-n/2+\sqrt{n-1}+2-k} (\log r^{1/2} + 1) \quad \text{in } B_1
\]
for some constant \( C \).

The following are some comments on the class of semi-stable solutions and on the radial symmetry and monotonicity of solutions.

**Remark 1.11.** A radial local minimizer \( u \) as in Definition 1.2 is a solution of \( -\Delta u = g(u) \) in \( B_1 \setminus \{0\} \) (where \( g = G' \)) and, in addition, it is semi-stable in
the sense of Definition 1.4. These statements are easily proved considering the first and second variations of the energy $E$ and using that $G \in C^2$.

Now, assume that $\underline{u} < \overline{u}$ are nonnegative, bounded and, respectively, sub and supersolution of (1.5). Then, the energy functional for (1.5) is well defined in the closed convex set of $H^1_0$ functions $v$ satisfying $\underline{u} \leq v \leq \overline{u}$, and it admits an absolute minimizer $u$ in this convex set. It is well known that $u$ is a classical solution of (1.5). Considering the second variation of energy, it follows that $u$ is a semi-stable solution of (1.5). Indeed, if $u$ is not identically equal to $\underline{u}$, neither to $\overline{u}$, then $\underline{u} < u < \overline{u}$ by the strong maximum principle. In this case, small perturbations of $u$ with compact support lie in the closed convex set where $u$ minimizes the energy, and the second variations give the semi-stability of $u$. Assume now that $u \equiv \overline{u}$ (the case $u \equiv \underline{u}$ is treated similarly). Then, since we assumed $\underline{u} < \overline{u}$, small nonpositive perturbations of $u$ with compact support lie in the closed convex set where $u \equiv \overline{u}$ minimizes the energy. It follows that (1.6) holds for every nonpositive $\xi$ (belonging to $C^1_0$ first, and then to $H^1_0$ by density). Finally, writing every $H^1_0$ function as the difference of its positive and negative parts and using the expression for $Q_u$, we conclude that (1.6) also holds for every $\xi$ in $H^1_0$. All these statements also hold for problem (1.5) posed in a smooth bounded domain $\Omega$ instead of $B_1$.

As a consequence of the previous discussion, the minimal solutions $u_\lambda$ of (1.1.1) for $0 < \lambda < \lambda^*$ are classical semi-stable solutions, since they must agree with the absolute minimizer lying in between 0 and $u_\lambda$ (this follows from the fact that $u_\lambda$ is the minimal or smallest solution). On the other hand, by Fatou’s lemma applied to (1.6) when $\lambda \to \lambda^*$, it follows that $u^*$ is also semi-stable.

Assume now that $g$ is $C^1$ and nondecreasing and that $0 \leq u < \overline{u}$ are $H^1_0(B_1)$ (perhaps unbounded) radial sub and supersolutions of (1.5), respectively. Then, by a result of P. Majer and one of the authors [5], which holds for every nondecreasing nonlinearity (independently of its growth at infinity), there exists an $H^1_0$ absolute minimizer $u$ of the energy lying in between $\underline{u}$ and $\overline{u}$ by considering the second variation of energy, we see that this $H^1_0$ weak solution $u$ is semi-stable. In addition, it will be radially decreasing, by Schwarz symmetrization. As a consequence, $u$ will enjoy the regularity given by Theorems 1.5 and 1.8.

**Remark 1.12.** By the Gidas-Ni-Nirenberg symmetry result, if $u$ is a bounded solution of (1.5) with $g : [0, +\infty) \to \mathbb{R}$ locally Lipschitz, then $u$ is radially decreasing. In the case of weak solutions, if we assume $u \in L^\infty_{\text{loc}}(\overline{B_1} \setminus \{0\})$ and $\lim_{|x| \to 0} u(x) = +\infty$, then we also have that $u$ is radially decreasing. This can be shown with minor modifications of the moving planes method.

However, we point out that there exist nonradial weak solutions of (1.1.1) in $\Omega = B_1$ for $f(u)$ given by $e^u$ and $(1 + u)^p$, for certain dimensions $n$ and exponents $p$. These solutions, which were obtained independently by H. Matano and by Y. Rébaï [18], have a unique isolated singularity near the origin.
The paper is organized as follows. In section 2 we prove the pointwise and $L^q$ estimates of Theorem 1.5. Section 3 deals with the Sobolev estimates of Theorem 1.8. We prove the regularity estimates of Theorem 1.3 for radial local minimizers in section 4. Finally, in section 5 we prove some results regarding minimal and extremal solutions of problem (1.1₃) under hypothesis (1.1₇) on $f$. In this last section we also establish Theorem 1.10, and hence Theorem 1.1.

2. Pointwise and $L^q$ estimates

To prove Theorem 1.5 we need two preliminary results. The following lemma was inspired by the proof of Simons theorem on the nonexistence of singular minimal cones in $\mathbb{R}^n$ for $n \leq 7$ (see Remark 2.2 below).

**Lemma 2.1.** Let $u \in L^\infty_{\text{loc}}(B_1 \setminus \{0\})$ be a radial weak solution of (1.5). Then, for every $\eta \in (H^1 \cap L^\infty)(B_1)$ with compact support in $B_1 \setminus \{0\}$, we have that $r\eta u_r \in (H^1 \cap L^\infty)(B_1 \setminus \{0\})$ has compact support in $B_1 \setminus \{0\}$ and

$$Q_u(r\eta u_r) = \int_{B_1} u_r^2 \left\{|\nabla (r\eta)|^2 - (n-1)\eta^2\right\} dx, \quad (2.1)$$

where $Q_u(\xi)$ is defined by (1.6) for $\xi \in H^1_0(B_1)$ with compact support in $B_1 \setminus \{0\}$.

Note that, in contrast with (1.6), expression (2.1) for the quadratic form $Q_u$ contains no reference to the nonlinearity $g$. This is the reason why our estimates do not depend on the specific nonlinearity $g$.

In [4], we used another version of Lemma 2.1. There, instead of (2.1), we considered the simpler expression

$$Q_u(\eta u_r) = \int_{B_1} u_r^2 \left\{|\nabla \eta|^2 - \frac{n-1}{r^2}\eta^2\right\} dx. \quad (2.2)$$

In this paper we use expression (2.1) since it simplifies the proof of Theorem 1.5 in the case of dimension $n = 2$.

**Remark 2.2.** There is a strong analogy of our proofs with that of Simons theorem on the nonexistence of singular (i.e., different than hyperplanes) minimal cones in $\mathbb{R}^n$ for $n \leq 7$ (see Theorem 10.10 of [12] for details). Indeed, let $E \subset \mathbb{R}^n$ be a open set such that $\partial E$ is a cone with zero mean curvature. Then, the cone $\partial E$ has nonnegative second variation of area (this is sometimes rephrased as “the cone is stable”) if and only if

$$\int_{\partial E} \left\{|\delta \xi|^2 - c^2\xi^2\right\} d\mathcal{H}_{n-1} \geq 0 \quad (2.3)$$

for every $C^1$ function $\xi$ with compact support in $\partial E \setminus \{0\}$. Here, $\delta$ denotes tangential derivatives on the cone $\partial E$ and $c^2$ is the sum of the squares of the $n-1$ principal curvatures of the cone. Note the analogy of (2.3) with (1.6).
Setting $\xi = \eta c$ in (2.3) and using an inequality for $\mathcal{D}c$ (where $\mathcal{D}$ is the Laplace-Beltrami operator on the cone), (2.3) leads to

$$\int_{\partial E} c^2 \left\{ |\delta \eta|^2 - \frac{2}{r^2} \eta^2 \right\} d\mathcal{H}_{n-1} \geq 0, \quad (2.4)$$

which is a similar expression to (2.2). Note that the analogue of $c$ is $u_r$ in the semilinear case and that we will have an equation for $\Delta u_r$ —expression (2.6) below.

Then, the proof of Simons theorem proceeds by using power decay test functions $\eta$ in (2.4) to deduce that $c \equiv 0$ (i.e., the cone is a hyperplane) if $n \leq 7$. We used the same method in [4] to study the stability or instability of radial solutions in all space. Here we use it to get estimates for semi-stable radial solutions in a ball.

**Proof of Lemma 2.1.** Let $\eta \in (H^1 \cap L^\infty)(B_1)$ have compact support in $B_1 \setminus \{0\}$ and let $c$ be any function in $(H^2_{\text{loc}} \cap L^\infty_{\text{loc}})(B_1 \setminus \{0\})$. First, we note that $r\eta c \in (H^1 \cap L^\infty)(B_1 \setminus \{0\})$ and has compact support in $B_1 \setminus \{0\}$.

Next, take $\xi = r\eta c$ in (1.6) to obtain

$$Q_u(r\eta c) = \int_{B_1} \left\{ r^2 \eta^2 |\nabla c|^2 + c^2 |\nabla (r\eta)|^2 + c \nabla c \cdot \nabla (r^2 \eta^2) - g'(u)r^2 \eta^2 c^2 \right\} dx$$

$$= \int_{B_1} \left\{ r^2 \eta^2 |\nabla c|^2 + c^2 |\nabla (r\eta)|^2 - r^2 \eta^2 \nabla \cdot (c \nabla c) - g'(u)r^2 \eta^2 c^2 \right\} dx$$

$$= \int_{B_1} \left\{ c^2 |\nabla (r\eta)|^2 - r^2 \eta^2 (c \Delta c + g'(u)c^2) \right\} dx. \quad (2.5)$$

Differentiating (1.5) with respect to $r$, we have

$$-\Delta u_r + \frac{n-1}{r^2} u_r = g'(u)u_r \quad \text{for } 0 < r < 1. \quad (2.6)$$

By local $W^{2,q}$ estimates for (1.5) and (2.6), we have $u_r \in (H^2_{\text{loc}} \cap L^\infty_{\text{loc}})(B_1 \setminus \{0\})$. Hence, we can take $c := u_r$ in the computations above. Finally, using (2.6) in expression (2.5), we conclude (2.1). \qed

We use now Lemma 2.1, together with the semi-stability assumption, to establish our following result. It is an estimate for the $L^2$ norm of $u_r r^{-\alpha}$ for certain positive exponents $\alpha$ which depend on the dimension $n$. This estimate is the key ingredient in the proof of Theorems 1.5 and 1.8.

**Lemma 2.3.** Let $n \geq 2$, $g : [0, +\infty) \to \mathbb{R}$ be a locally Lipschitz function, and $u \in H^1_0(B_1)$ be a semi-stable radially decreasing weak solution of (1.5). Let $\alpha$ satisfy

$$1 \leq \alpha < 1 + \sqrt{n-1}. \quad (2.7)$$
Then,
\[
\int_{B_{1/2}} u^2 r^{-2\alpha} \, dx \leq \frac{C_n}{(n - 1) - (\alpha - 1)^2} \left\{ \|u\|_{L^1(B_1)}^2 + \|g(u)\delta\|_{L^1(B_1)}^2 \right\},
\]
where \(C_n\) is a constant depending only on \(n\).

Proof. By approximation, the semi-stability of \(u\) implies that \(Q_u(\xi) \geq 0\) for all \(\xi \in H^1(B_1)\) with compact support in \(B_1 \setminus \{0\}\). Hence, Lemma 2.1 leads to
\[
(n - 1) \int_{B_1} u^2 r^2 \eta^2 \, dx \leq \int_{B_1} u^2 |\nabla(r \eta)|^2 \, dx,
\]
for every \(\eta \in (H^1 \cap L^\infty)(B_1)\) with compact support in \(B_1 \setminus \{0\}\).

We now prove that (2.9) also holds for every \(\eta \in (H^1 \cap L^\infty)(B_1)\) with compact support in \(B_1\) (now \(\eta\) does not necessarily vanish around \(0\)) and such that \(|\nabla(r \eta)| \in L^\infty(B_1)\).

Indeed, let \(\eta\) be any \((H^1 \cap L^\infty)(B_1)\) function with compact support in \(B_1\) and such that \(|\nabla(r \eta)| \in L^\infty(B_1)\). Take \(\zeta \in C^\infty(\mathbb{R}^n)\) such that \(0 \leq \zeta \leq 1\), \(\zeta \equiv 0\) in \(B_1\) and \(\zeta \equiv 1\) in \(\mathbb{R}^n \setminus B_2\), and let \(\zeta_\delta(\cdot) = \zeta(\cdot/\delta)\) for \(\delta > 0\). Applying (2.9) with \(\eta\) replaced by \(\eta(\cdot)\zeta_\delta(\cdot)\), we obtain
\[
(n - 1) \int_{B_1} u^2 r^2 \eta^2 \zeta_\delta^2 \, dx \leq \int_{B_1} u^2 |\nabla(r \eta \zeta_\delta)|^2 \, dx.
\]

Now, we find
\[
\int_{B_1} u^2 |\nabla(r \eta \zeta_\delta)|^2 \, dx
\]
\[
= \int_{B_1} u^2 \left\{ |\nabla(r \eta)|^2 \zeta_\delta^2 + 2r^2 \eta^2 |\nabla \zeta_\delta|^2 + \zeta_\delta \nabla \zeta_\delta \cdot \nabla (r^2 \eta^2) \right\} \, dx
\]
\[
\leq \int_{B_1} u^2 |\nabla(r \eta)|^2 \zeta_\delta^2 \, dx + C \int_{B_{3\delta} \setminus B_{\delta}} u^2 r^2 |\eta| \left\{ \frac{r^2}{\delta^2} |\eta| + \frac{r}{\delta} \zeta_\delta |\nabla(r \eta)|\right\} \, dx
\]
\[
\leq \int_{B_1} u^2 |\nabla(r \eta)|^2 \, dx + C \int_{B_{3\delta} \setminus B_{\delta}} u^2 r^2 \, dx,
\]
where \(C\) denotes different positive constants, and we have used that \(\eta\) and \(|\nabla(r \eta)|\) are bounded. Since \(u \in H^1_0(B_1)\), the last term tends to zero as \(\delta \to 0\) (it is here, and only here, where we use the regularity hypothesis that \(u\) is in \(H^1_0\)). By monotone convergence we conclude that (2.9) also holds for every \(\eta \in (H^1 \cap L^\infty)(B_1)\) with compact support in \(B_1\) and such that \(|\nabla(r \eta)| \in L^\infty(B_1)\).

Let \(\epsilon \in (0, 1/2)\). For \(\alpha \geq 1\) satisfying (2.7), apply (2.9) with \(\eta = \eta_\epsilon\) given by
\[
\eta_\epsilon(r) = \begin{cases} 
\epsilon^{-\alpha} - 2^\alpha & \text{if } 0 \leq r \leq \epsilon \\
\epsilon^{-\alpha} - 2^\alpha & \text{if } \epsilon < r \leq 1/2 \\
0 & \text{if } 1/2 < r.
\end{cases}
\]
Note that $\eta$ and $|\nabla (r \eta)|$ are bounded. We obtain
\begin{align*}
(n - 1) \int_{B_{1/2} \setminus B_r} u_r^2 (r^{-\alpha} - 2^\alpha)^2 dx + (n - 1) (e^{-\alpha} - 2^\alpha)^2 \int_{B_r} u_r^2 dx \\
\leq \int_{B_{1/2} \setminus B_r} u_r^2 ((1 - \alpha) r^{-\alpha} - 2^\alpha)^2 dx + (e^{-\alpha} - 2^\alpha)^2 \int_{B_r} u_r^2 dx.
\end{align*}

Since $n \geq 2$, it follows that
\begin{align*}
(n - 1) \int_{B_{1/2} \setminus B_r} u_r^2 (r^{-\alpha} - 2^\alpha)^2 dx \leq \int_{B_{1/2} \setminus B_r} u_r^2 ((1 - \alpha) r^{-\alpha} - 2^\alpha)^2 dx.
\end{align*}

Developing the squares, using $n \geq 2$ and (2.7), we find the estimate
\begin{align*}
\int_{B_{1/2} \setminus B_r} u_r^2 r^{-2\alpha} dx \leq \frac{C_n}{(n - 1) - (\alpha - 1)^2} \int_{B_{1/2} \setminus B_r} u_r^2 r^{-\alpha} dx. \quad (2.10)
\end{align*}

Throughout the proof, $C_n$ (respectively, $C_{\alpha,n}$) denote different positive constants depending only on $n$ (respectively, on $\alpha$ and $n$). Now, choose a positive constant $C_{\alpha,n}$ such that
\begin{align*}
\frac{C_n}{(n - 1) - (\alpha - 1)^2} r^{-\alpha} \leq \frac{1}{2} r^{-2\alpha} + C_{\alpha,n} r^{n-1} \quad \text{for all } r > 0.
\end{align*}

The previous inequality and (2.10) lead to
\begin{align*}
\int_{B_{1/2} \setminus B_r} u_r^2 r^{-2\alpha} dx \leq C_{\alpha,n} \int_{B_{1/2} \setminus B_r} u_r^2 r^{n-1} dx. \quad (2.11)
\end{align*}

Next, we claim that
\begin{align*}
\int_{B_{1/2}} u_r^2 r^{n-1} dx &= |\partial B_1| \int_0^{1/2} u_r^2 r^{2\alpha - 2} dr \\
&\leq C_n \left\{ \| u \|_{L^1(B_1)}^2 + \| g(u) \|_{L^1(B_1)}^2 \right\}. \quad (2.12)
\end{align*}

Assuming this claim for the moment, we complete the proof of the lemma.

We use (2.11) and (2.12), and we let $\epsilon \to 0$ to obtain
\begin{align*}
\int_{B_{1/2}} u_r^2 r^{-2\alpha} dx \leq C_{\alpha,n} \left\{ \| u \|_{L^1(B_1)}^2 + \| g(u) \|_{L^1(B_1)}^2 \right\}. \quad (2.13)
\end{align*}

Note that we want to have a precise expression, depending on $\alpha$, of the previous constant $C_{\alpha,n}$. To obtain it, we apply (2.13) with the special choice $\alpha = (1 + \sqrt{n-1})/2 \in [1, 1 + \sqrt{n-1})$ to deduce
\begin{align*}
\int_{B_{1/2}} u_r^2 r^{-(1+\sqrt{n-1})} dx \leq C_n \left\{ \| u \|_{L^1(B_1)}^2 + \| g(u) \|_{L^1(B_1)}^2 \right\}. \quad (2.14)
\end{align*}

Finally, since $r^{-\alpha} \leq r^{-(1+\sqrt{n-1})}$ in $B_1$, (2.10) and (2.14) lead to the desired estimate (2.8) after letting $\epsilon \to 0$. 

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To finish the proof, we establish claim (2.12). First, since \( u \) is radially decreasing, we have
\[
 u(1/2) \leq C_n \int_{1/4}^{1/2} w r^{n-1} dr \leq C_n \|u\|_{L^1(B_1)}.
\] (2.15)

Let \( \rho \in (1/2, 3/4) \) be chosen such that
\[
 -u_r(\rho) = \frac{u(3/4) - u(1/2)}{1/4} = 4u(1/2) - 4u(3/4) \leq 4u(1/2).
\] (2.16)

For \( s \leq 1/2 \), we integrate \((r^{n-1}u_r)_r = -g(u)r^{n-1}\) with respect to \( r \), from \( s \) to \( \rho \), to obtain
\[
 -u_r(s)s^{n-1} = -u_r(\rho)\rho^{n-1} - \int_s^\rho g(u)r^{n-1} dr \leq C_n \left\{ u(1/2) + \|g(u)\delta\|_{L^1(B_1)} \right\},
\] (2.17)

where we have used (2.16). Thus, combining (2.15) and (2.17), it follows that
\[
 0 \leq -u_r(s)s^{n-1} \leq C_n \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\} \quad \text{for all} \ s \leq 1/2. \] (2.18)

Squaring this inequality and integrating it in \( s \), from 0 to 1/2, we conclude (2.12).

A slight modification of the previous proof leads to the following version of Lemma 2.3. We will use it in section 4 to prove Theorem 1.3 on radial local minimizers.

**Lemma 2.4.** Let \( n \geq 2 \), \( G \in C^2(\mathbb{R}) \), and \( u \in H^1(B_1) \) be a radial solution of \(-\Delta u = G'(u)\) in \( B_1 \setminus \{0\} \). Assume that \( u \) is semi-stable in the sense of Definition 1.4. Let \( \alpha \) satisfy
\[
 1 \leq \alpha < 1 + \sqrt{n-1}.
\]

Then,
\[
 \int_{B_{1/2}} u^2 r^{-2\alpha} dx \leq \frac{C_n}{(n-1) - (\alpha - 1)^2} \|u\|_{H^1(B_1)}^2,
\]

where \( C_n \) is a constant depending only on \( n \).

**Proof.** We simply revise the proof of Lemma 2.3. First it obtains (2.9) by using Lemma 2.1—a lemma that holds in our present situation since all the involved test functions vanish around the origin. At this point (2.9) makes no reference to the nonlinearity \( g = G' \), and what is used to remove the assumption of the test functions vanishing around the origin in (2.9), is that \( u \in H^1 \)—which we assume in Lemma 2.4.

The proof proceeds to estimate (2.11), whose right hand side we bound now by \( \|u\|_{H^1(B_1)}^2 \)—instead of the bound in (2.12). With this bound at hand, the rest of the proof leads, without any change, to the estimate of Lemma 2.4. \( \Box \)
We can now give the:

**Proof of Theorem 1.5.** First, we consider the case \( n = 1 \). Observe that we can proceed exactly as in the proof of Lemma 2.3 to obtain (2.15), (2.16), (2.17), and (2.18). Now, we integrate (2.18) (with \( n = 1 \)) in \( s \), from \( r \) to \( 1/2 \), to deduce

\[
u(r) \leq C \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\} \quad \text{for } 0 < r \leq 1/2,
\]

where we have used (2.15). Since \( u \) is radially decreasing, this is the desired \( L^\infty \) estimate when \( n = 1 \).

Throughout the rest of the proof, we assume that \( n \geq 2 \). Let \( \alpha \) satisfy (2.7). For \( 0 < s \leq 1/2 \), we have

\[
u(s) - \nu(1/2) = \int_s^{1/2} u_r dr = \int_s^{1/2} u_r r^{-\alpha + \frac{n-1}{2}} r^{n-\frac{n-1}{2}} dr \leq C_n \left( \int_{B_{1/2}} \nu^2 r^{-2\alpha} dx \right)^{1/2} \left( \int_s^{1/2} r^{2\alpha+1-n} dr \right)^{1/2}
\]

by Cauchy-Schwarz. Using Lemma 2.3, we deduce

\[
u(s) \leq \nu(1/2) + \frac{C_n}{\sqrt{n - 1 - (\alpha - 1)^2}} \left( \int_{B_{1/2}} \nu^2 r^{-2\alpha+1-n} dr \right)^{1/2} \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\}
\]

for all \( 0 < s \leq 1/2 \). \hspace{1cm} (2.20)

(a) Assume \( n \leq 9 \). The integral in (2.20) is finite with \( s = 0 \) if we take \( 2\alpha + 1 - n > -1 \), i.e.,

\[
(n - 4)/2 < \alpha - 1 \hspace{1cm} (2.21)
\]

Since \( n \leq 9 \), then \( (n - 4)/2 < \sqrt{n - 1} \) and we can choose \( \alpha \) satisfying (2.21) and \( \alpha < 1 + \sqrt{n - 1} \), so that Lemma 2.3 holds. Now, the desired estimate follows from (2.20) and (2.15).

(b) Assume \( n = 10 \). For \( 0 < \varepsilon < 1 \), let \( \alpha = 4 - \varepsilon \) and apply Lemma 2.3 to obtain

\[
\int_{B_{1/2}} \nu^2 r^{-8+2\varepsilon} dx \leq \frac{C}{\varepsilon} \left\{ \|u\|_{L^1(B_1)}^2 + \|g(u)\delta\|_{L^1(B_1)}^2 \right\},
\]

for a universal constant \( C \) (independent of \( \varepsilon \)). This estimate and (2.19) give

\[
u(s) \leq \nu(1/2) + \frac{C}{\sqrt{\varepsilon}} \left( \int_s^{1/2} r^{-1-2\varepsilon} dr \right)^{1/2} \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\}
\]

\[
\leq \nu(1/2) + C \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\} \frac{8-\varepsilon}{\varepsilon}
\]

for \( 0 < s < 1/2 \) and every \( 0 < \varepsilon < 1 \). From this, it follows that \( u \in L^q(B_1) \) for every \( q < \infty \).
In order to prove the pointwise estimate (1.8), given \( s \in (0, 1/2) \) we find the \( \varepsilon \) that minimizes \( s^{-\varepsilon}/\varepsilon \) in (2.22). We obtain

\[
\varepsilon = |\log s|^{-1}. \tag{2.23}
\]

Note that this choice of \( \varepsilon \) belongs to \( (0, 1) \) if \( 0 < s < e^{-1} \). Finally, using (2.22) with \( \varepsilon \) given by (2.23) we obtain

\[
u(s) \leq u(1/2) + C \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\} |\log s| \quad \text{for } 0 < s < e^{-1}.
\]

From this and (2.15), the desired logarithmic bound (1.8) follows.

(c) Assume \( n \geq 11 \). For \( \varepsilon \in (0, 1) \), let \( \alpha = 1 + \sqrt{n - 1} - \varepsilon \) and use (2.20) to obtain

\[
u(s) - u(1/2) \leq \frac{C_n}{\varepsilon} \left( \int_s^{1/2} r^{3-n+2\sqrt{n-1} - 2\varepsilon} dr \right)^{1/2} \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\}
\]

\[
\leq \frac{C_n}{\varepsilon} \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\} s^{2-n/2 + \sqrt{n-1} - \varepsilon} \tag{2.24}
\]

for \( 0 < s < 1/2 \), where we have used that \( 4 - n + 2\sqrt{n - 1} < 0 \) since \( n \geq 11 \).

Now, we use (2.24) to calculate, for \( q \geq 1 \),

\[
\int_{B_{1/2}} (u - u(1/2))^q dx \tag{2.25}
\]

\[
\leq \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\} q \frac{C_n}{\varepsilon q/2} \int_0^{1/2} s^{n-1+q(2-n/2+\sqrt{n-1}-\varepsilon)} ds.
\]

If we set \( q = 2n/(n - 2\sqrt{n - 1} - 4 + 3\varepsilon) \), then the second integral in (2.25) is finite for every \( \varepsilon > 0 \) small enough. Hence, \( u \in L^q(B_{1/2}) \) for every \( q < q_b := 2n/(n - 2\sqrt{n - 1} - 4) \). Bearing in mind (2.15) and using that \( u \) is decreasing, we obtain that \( u \in L^q(B_1) \) and estimate (1.9) for every \( q < q_b \).

Finally, to prove the pointwise estimate (1.10), we consider (2.24) and proceed as in part (b). Now, we need to minimize \( s^{-\varepsilon}/\varepsilon \) for given \( s \). Hence, we take \( \varepsilon = -1/(2 \log s) \), which belongs to \( (0, 1) \) if \( s \in (0, e^{-1/2}) \). With this choice of \( \varepsilon \), (2.24) leads to

\[
u(s) \leq u(1/2) + C_n \left\{ \|u\|_{L^1(B_1)} + \|g(u)\delta\|_{L^1(B_1)} \right\} s^{-n/2 + \sqrt{n-1} + 2} |\log s|^{1/2}
\]

for \( s \in (0, e^{-1/2}) \). Recalling (2.15), the proof is now completed. \( \square \)

3. Sobolev Estimates

This section is devoted to give the:

Proof of Theorem 1.8. We start proving estimate (1.14) of part (c) in every dimension \( n \). First, since \( g \) is convex here, \( g'(u(r)) \) is nonincreasing in \( r \). Hence,
it suffices to prove (1.14) for \( r < 1/2 \). Given \( r \) with \( 2r < 1 \), there exists a radial function \( \xi \in C^\infty_c(B_1 \setminus \{0\}) \) such that \( \xi(s) \equiv 0 \) for \( s \in [0, r/4] \cup [2r, 1] \), \( \xi(s) \equiv 1 \) for \( s \in [r/2, r] \), and \( |\nabla \xi| \leq C r^{-2} \) for a universal constant \( C \). Now, we simply use the semi-stability property (1.6) for \( u \) with the previous test function \( \xi \). Recalling that \( g'(u(s)) \) is nonnegative and nonincreasing in \( s \), we obtain

\[
g'(u(r)) r^n |B_{1/2} \setminus B_{1/2}| \leq \int_{B_r \setminus B_{r/2}} g'(u) dx \leq \int_{B_1} g'(u) \xi^2 dx \leq \int_{B_1} |\nabla \xi|^2 dx
\]

that is, (1.14).

Next, we finish the proof of the theorem when \( n \leq 9 \). By Theorem 1.5, we have that \( u \in L^\infty(B_1) \). Hence, applying standard regularity theory to (1.5), we have \( u \in W^{2,q}(B_1) \) for all \( q < \infty \). Thus, \( g(u) \in W^{1,q}(B_1) \) and therefore \( u \in W^{3,q}(B_1) \) for all \( q < \infty \). From this, all the statements of the theorem, including the bound (1.15), follow in case \( n \leq 9 \).

Thus, throughout the rest of the proof we assume \( n \geq 10 \). First we see that it is enough to prove our estimates in \( B_{1/4} \). Indeed, we have appropriate bounds on the supremum of \( u \) in \( B_1 \setminus B_{1/5} \) by Theorem 1.5. The standard regularity argument used above for \( n \leq 9 \) gives now that \( u \in W^{3,q}(B_1 \setminus \overline{B}_{1/5}) \) for all \( q < \infty \), with appropriate bounds. In particular, we have \( C^2 \) bounds for \( u \) away from \( B_{1/5} \). But from the linearized equation (2.6) we deduce

\[
|u_{rrr}| \leq C_n \left\{ \frac{|u_{rr}|}{r} + \frac{|u_r|}{r^2} + |g'(u)u_r| \right\}
\]

for \( 0 < r < 1 \), (3.2)

and hence \( C^3 \) bounds for \( u \) away from \( B_{1/5} \). We conclude that it suffices to prove the Sobolev estimates in \( B_{1/4} \) (and for \( n \geq 10 \)).

To do so, since \( u \) is radially decreasing, arguing as in (2.15) and (2.16), we can choose \( \bar{\rho} \in (1/4, 1/2) \) such that

\[
0 \leq -u_r(\bar{\rho}) \leq 4u(1/4) \leq C_n \|u\|_{L^1(B_1)}.
\]

(3.3)

For \( 0 < s < 1/4 \), we integrate \( u_{rr} = -(n-1)r^{-1}u_r - g(u) \leq -(n-1)r^{-1}u_r \) (recall that \( g \geq 0 \) by hypothesis) with respect to \( r \), in \((s, \bar{\rho}) \) (note \( s < 1/4 < \bar{\rho} < 1/2 \)) and use (3.3) to obtain

\[
-u_r(s) \leq -u_r(\bar{\rho}) + \int_s^{\bar{\rho}} (n-1)r^{-1}(-u_r) dr
\]

\[
\leq C_n \|u\|_{L^1(B_1)} + \int_s^{1/2} (n-1)(-u_r)r^{-\alpha + \frac{\alpha-1}{\gamma}} r^{\alpha - \frac{\alpha-1}{\gamma} - 1} dr
\]

\[
\leq C_n \|u\|_{L^1(B_1)} + C_n \left( \int_{B_1/2} u_r^2 r^{-2\alpha} dr \right)^{1/2} \left( \int_s^{1/2} r^{2\alpha - n - 1} dr \right)^{1/2},
\]
where we have taken any $\alpha$ satisfying (2.7). This estimate combined with Lemma 2.3 leads to

$$-u_r(s) \leq C_n \|u\|_{L^1(B_1)} + \frac{C_n \|u\|_{L^1(B_1)}}{\sqrt{n-1-(\alpha-1)^2}} \left( \int_s^{1/2} r^{2\alpha-n-1} dr \right)^{1/2}$$

for all $0 < s < 1/4$. We have used $g \geq 0$ and Remark 1.6.

Given any $\varepsilon \in (0,1)$, choose $\alpha = 1 + \sqrt{n-1-\varepsilon}$. Since $-u_r \geq 0$, the previous inequality gives

$$|u_r(s)| \leq \frac{C_n}{\sqrt{\varepsilon}} \|u\|_{L^1(B_1)} s^{-n/2 + \sqrt{n-1+1-\varepsilon}} \quad \text{for all } s \leq 1/4, \quad (3.4)$$

where we have used that $-n/2 + \sqrt{n-1+1} < 0$ since $n \geq 10$.

Part (a) of the theorem follows now easily. We use (3.4) to bound, for $q \geq 1$,

$$\int_{B_{1/4}} |u_r|^q dx \leq \frac{C_n}{\varepsilon q/2} \|u\|_{L^1(B_1)} q \int_0^{1/2} s^{n-1+q(-n/2+\sqrt{n-1+1-\varepsilon})} ds. \quad (3.5)$$

Setting $q = 2n/(n - 2\sqrt{n-1} - 2 + 3\varepsilon)$, then the second integral in (3.5) is finite for every $\varepsilon > 0$ small enough. Hence, $u_r \in L^q(B_{1/4})$ for every $q < q_1 := 2n/(n - 2\sqrt{n-1} - 2)$, and the desired $W^{1,q}(B_1)$ estimate follows.

To establish part (b), assume $g' \geq 0$. Since $u$ radially decreasing, for $s \leq 1/4$ we have

$$g(u(s)) - g(u(1/4)) = -\int_s^{1/4} g'(u) u_r \, dr \leq C_n \int_{B_{1/4} \setminus B_s} g'(u)|u_r|^{1-n} \, dx.$$

Recalling (3.4), we find

$$g(u(s)) \leq g(u(1/4)) + \frac{C_n}{\sqrt{\varepsilon}} \|u\|_{L^1(B_1)} \int_{B_{1/4} \setminus B_s} g'(u) r^{-3n/2 + \sqrt{n-1+1-\varepsilon}} \, dx \quad (3.6)$$

for $s \leq 1/4$. To control the last integral, we use the semi-stability property (1.6) (which, by approximation, holds for every $\xi \in H^1_0(B_1)$; note that $g' \geq 0$ here). We take

$$\xi(r) = \begin{cases} 
    s^{-3n/4 + \sqrt{n-1}/2 + 1-\varepsilon/2} & \text{if } r \leq s \\
    r^{-3n/4 + \sqrt{n-1}/2 + 1-\varepsilon/2} & \text{if } s \leq r \leq 1/4 \\
    (1/4)^{-3n/4 + \sqrt{n-1}/2 - \varepsilon/2} (1-r)/3 & \text{if } 1/4 \leq r \leq 1,
\end{cases}$$

to obtain

$$\int_{B_{1/4} \setminus B_s} g'(u) r^{-3n/2 + \sqrt{n-1+1-\varepsilon}} \, dx \leq \int_{B_1} g'(u) \xi^2 \, dx \leq \int_{B_1} |\nabla \xi|^2 \, dx \leq \int_{B_1} |\nabla \xi|^2 \, dx \leq C_n s^{-n/2 + \sqrt{n-1-\varepsilon}} + C_n$$

$$\leq C_n s^{-n/2 + \sqrt{n-1-\varepsilon}}. \quad (3.7)$$
Since, by Remark 1.6,
\[
\int_{B_{1/4}} g(u)dx \leq \|g(u)\delta\|_{L^1(B_{1})} \leq C_n\|u\|_{L^1(B_{1})}
\]
and \(g(u(r))\) is nonincreasing in \(r\), we have
\[
g(u(1/4)) \leq C_n\|u\|_{L^1(B_{1})}.
\] (3.8)

Combining (3.6), (3.7), and (3.8), we find
\[
0 \leq g(u(s)) \leq \frac{C_n}{\sqrt{\varepsilon}}\|u\|_{L^1(B_{1})} s^{-n/2+\sqrt{n-1}-\varepsilon} \quad \text{for } s \leq 1/4.
\] (3.9)

Recalling (3.4) and (3.9), we deduce
\[
|u_{rr}(s)| = \left| (n-1) \frac{u_r(s)}{s} + g(u(s)) \right| \leq \frac{C_n}{\sqrt{\varepsilon}}\|u\|_{L^1(B_{1})} s^{-n/2+\sqrt{n-1}-\varepsilon}
\] (3.10)
for all \( s \leq 1/4 \). Using (3.10) and proceeding as in part (a) we obtain the desired \( W^{2,q} \) estimate for all \( q < q_3 \).

To establish part (c), we use (3.2), (1.14), (3.4), and (3.10) to obtain
\[
|u_{rrr}(s)| \leq C_n \left( \frac{|u_{rr}(s)|}{s} + \frac{|u_r(s)|}{s^2} + |g'(u(s))u_r(s)| \right)
\leq \frac{C_n}{\sqrt{\varepsilon}}\|u\|_{L^1(B_{1})} s^{-n/2+\sqrt{n-1}-\varepsilon} \quad \text{for } s \leq 1/4.
\] (3.11)

Proceeding as in parts (a) and (b), the \( W^{3,q} \) bounds for \( q < q_3 \) follow from (3.11).

Finally, the pointwise estimates (1.16) follow from (3.4), (3.10), and (3.11) by choosing \( \varepsilon = |\log s|^{-1} \in (0,1) \) for a given \( s \leq 1/4 \).

\[\square\]

4. Regularity of Radial Local Minimizers

In this section we prove Theorem 1.3 on radial local minimizers by modifying slightly the proof of Theorem 1.5.

Proof of Theorem 1.3. Since \( u \in H^1(B_1) \) is a radial function we have that \( u \in L^\infty_{\text{loc}}(\overline{B_1 \setminus \{0\}}) \) (see the comment following Definition 1.2).

Next, from the definition of radial local minimizer we deduce that \( u \) is a solution of \( -\Delta u = g(u) \) in \( B_1 \setminus \{0\} \), where \( g = G' \). We emphasize that \( u \) is a solution away from the origin. Unless one makes an assumption on the sign of \( g \), it is not clear that \( |g(u)| \) should be integrable around the origin.

Since \( u \) is bounded away from the origin, standard \( W^{2,q} \) estimates (applied to the equations satisfied by \( u \) and by its derivatives) give that \( u \in C^{2,\alpha}_{\text{loc}}(\overline{B_1 \setminus \{0\}}) \).

Moreover, the definition of radial local minimizer gives automatically that \( u \) is semi-stable in the sense of Definition 1.4 (see Remark 1.11). Note that the
definition refers only to test functions with compact support away from the origin.

Even that we do not know if \( u \) is a weak solution around the origin, we now check that the proof of Theorem 1.5 still goes through to obtain the desired estimates. Indeed, the proof of Theorem 1.5 is based on (2.19) (whose derivation does not use the radially decreasing assumption on \( u \)). Next, we use Lemma 2.4 instead of Lemma 2.3 to obtain (2.20) with the \( L^1 \) norms in the right hand side replaced by \( \|u\|_{H^1} \). From this point on, the rest of the proof remains the same. Note also that \( |u(1/2)| \) can be controlled by \( \|u\|_{H^1} \) using the Sobolev embedding in one dimension (as in the comment following Definition 1.2). In this way, we obtain statements (a), (b), and (c) of the theorem.

To prove statement (d) on radial monotonicity, it suffices to show that if \( u_r \) is not identically zero, then it never vanishes in \((0,1)\). Arguing by contradiction, assume that \( u_r(r_0) = 0 \) for some \( r_0 \in (0,1) \).

Take \( \zeta \in C^\infty(\mathbb{R}^n) \) to be a radial function such that \( 0 \leq \zeta \leq 1 \), \( \zeta \equiv 0 \) in \( B_1 \) and \( \zeta \equiv 1 \) in \( \mathbb{R}^n \setminus B_2 \), and let \( \zeta_\delta(\cdot) = \zeta(\cdot/\delta) \) for \( 0 < \delta < r_0/2 \). Since \( u_r(r_0) = 0 \), the function \( u_r \zeta_\delta \) vanishes in \( B_\delta \) and belongs to \( H^1_0(B_{r_0} \setminus \{0\}) \). After approximating this function in \( H^1 \) by functions in \( C^\infty_c(B_{r_0} \setminus \{0\}) \), the semi-stability property of Definition 1.4 for \( u \) leads to

\[
\int_{B_{r_0}} \left\{ |\nabla (u_r \zeta_\delta)|^2 - g'(u)(u_r \zeta_\delta)^2 \right\} \, dx \geq 0.
\]

In this expression, we proceed as in the proof of Lemma 2.1. That is, we develop the gradient of the product, integrate by parts (using now that \( u_r(r_0) = 0 \)), and use the linearized equation (2.6). We arrive at

\[
\int_{B_{3\delta} \setminus B_\delta} u_r^2 |\nabla \zeta_\delta|^2 \, dx - \int_{B_{r_0}} \frac{n-1}{r^2} u_r^2 \zeta_\delta^2 \, dx \geq 0. \tag{4.1}
\]

Up to a multiplicative constant, the first of these integrals is bounded by \( \int_{B_{3\delta} \setminus B_\delta} u_r^2 \, dx \), which tends to zero as \( \delta \to 0 \) by Lemma 2.4. Indeed, this lemma applied with \( \alpha = 1 \) gives that \( \int_{B_{1/2}} u_r^2 \, dx < \infty \). Hence, letting \( \delta \to 0 \) in (4.1), we arrive to a contradiction.

\[ \square \]

5. Minimal and extremal solutions

In this section we prove the following result on minimal and extremal solutions of (1.1\_\lambda) under hypothesis (1.17) on \( f \) —where we do not assume \( f \) to be convex as in (1.2). We also establish Theorem 1.10, and hence Theorem 1.1.

**Proposition 5.1.** Assume that \( \Omega \subset \mathbb{R}^n \) is a smooth bounded domain and that \( f \) satisfies (1.17). Then, there exists a parameter \( 0 < \lambda^* < \infty \) such that:

(a) If \( \lambda > \lambda^* \), then there is no classical solution of (1.1\_\lambda).
(b) If \( 0 \leq \lambda < \lambda^* \), then there exists a minimal classical solution \( u_\lambda \) of \((1.1_\lambda)\). Moreover, \( u_\lambda < u_\mu \) if \( \lambda < \mu < \lambda^* \). In addition, \( u_\lambda \) is semi-stable, i.e., the first Dirichlet eigenvalue of the linearized operator at \( u_\lambda \) is nonnegative.

(c) \( u^* = \lim_{\lambda \to \lambda^*} u_\lambda \) is a weak solution of \((1.1_\lambda)\) with \( \lambda = \lambda^* \). In addition, \( u^* \) is semi-stable in the sense that
\[
\int_{\Omega} \lambda^* f'(u^*) \xi^2 \, dx \leq \int_{\Omega} |\nabla \xi|^2 \, dx \quad \text{for every} \ \xi \in C^\infty_c(\Omega).
\]

Note that the first integral is well defined in \([0, +\infty]\) since \( f'(u^*) \geq 0 \).

In this result, since \( f \) is not necessarily convex, the family of minimal solutions \( \{ u_\lambda \} \) may not be continuous as a function of \( \lambda \), that is, it may not be a continuous branch as in the case \( f \) convex. For more details and an example, see the comments and figure following the proof of the proposition.

The ideas in the proof Proposition 5.1 are by now well known. We include them next for the sake of completeness.

**Proof of Proposition 5.1.** First, we prove that there is no classical solution for large \( \lambda \). By \((1.17)\), \( f \) is superlinear at infinity and \( f > 0 \) in all \([0, +\infty)\). It follows that \( \lambda f(u) > \lambda_1 u \) if \( \lambda \) large enough, where \( \lambda_1 \) is the first eigenvalue of \(-\Delta \) in \( H^1_0(\Omega) \) (and \( \varphi_1 > 0 \) a corresponding eigenfunction). Now we argue by contradiction. Assume that \( u \) is a solution of \((1.1_\lambda)\), multiply \((1.1_\lambda)\) by \( \varphi_1 \) and integrate twice by parts in \( \Omega \), to obtain
\[
\lambda_1 \int_{\Omega} u \varphi_1 \, dx = \lambda \int_{\Omega} f(u) \varphi_1 \, dx > \lambda_1 \int_{\Omega} u \varphi_1 \, dx.
\]
Hence, a contradiction.

Next, we prove the existence of a classical solution of \((1.1_\lambda)\) for small \( \lambda \). Since \( f(0) > 0 \), \( \underline{u} = 0 \) is a strict sub-solution of \((1.1_\lambda)\) for every \( \lambda > 0 \). The solution \( \overline{\pi} \) of
\[
\begin{cases}
-\Delta \overline{\pi} &= 1 \quad \text{in} \ \Omega \\
\overline{\pi} &= 0 \quad \text{on} \ \partial \Omega,
\end{cases}
\]
is a bounded supersolution of \((1.1_\lambda)\) for small \( \lambda \), more precisely whenever \( \lambda f(\max \overline{\pi}) < 1 \). For such \( \lambda \)'s, a classical solution \( u_\lambda \) is obtained by monotone iteration starting from \( 0 \). That is, \( u_\lambda \) is the nondecreasing limit of \( u^m \), where \(-\Delta u^m = \lambda f(u^{m-1})\) (with homogeneous Dirichlet data) and \( u^0 \equiv 0 \). Note that, since \( f' \geq 0 \), the \( u^m \)'s are nondecreasing in \( m \) and \( u^m < \overline{\pi} \) for all \( m \).

The extremal parameter \( \lambda^* \) is now defined as the supremum of all \( \lambda > 0 \) for which \((1.1_\lambda)\) admits a classical solution. Hence, both \( 0 < \lambda^* < \infty \) and part (a) of the proposition holds.

(b) Next, if \( \lambda < \lambda^* \) there exists \( \mu \) with \( \lambda < \mu < \lambda^* \) and such that \((1.1_\mu)\) admits a classical solution \( u \). Since \( f > 0 \), \( u \) is a bounded supersolution of \((1.1_\lambda)\), and hence the monotone iteration procedure used above shows that \((1.1_\lambda)\) admits a classical solution \( u_\lambda \) with \( u_\lambda \leq u \). Note that the iteration procedure, and hence the solution that it produces, are independent of the
supersolution \(u\). In addition, we have shown that \(u_\lambda\) is smaller than any classical 

supersolution of (1.1\(\lambda\)). It follows that \(u_\lambda\) is minimal (i.e., the smallest solution) and that \(u_\lambda < u_\mu\).

To show that \(u_\lambda\) is semi-stable, note that the energy functional for (1.1\(\lambda\)) on the set of \(H^1_0(\Omega)\) functions lying in between 0 and \(u_\lambda\) admits an absolute minimizer \(u\). Considering the first and second variation of energy, we see that \(u\) is a semi-stable classical solution of (1.1\(\lambda\)) such that \(u \leq u_\lambda\) (see Remark 1.11 for more details). But, since \(u_\lambda\) is the minimal solution, \(u\) must agree with \(u_\lambda\). Thus \(u_\lambda\) is semi-stable. Part (b) is now proved.

(c) As above, let \(\lambda_1\) be the first eigenvalue of \(-\Delta\) and \(\varphi_1 \geq 0\) a corresponding 

eigenfunction. By (1.17), there exists a constant \(C > 0\) such that \(f(x) \geq (2\lambda_1/\lambda^*)u - C\) for all \(u \geq 0\). Multiply the equation (1.1\(\lambda\)) for \(u_\lambda\) \((\lambda < \lambda^*)\) by \(\varphi_1\) and integrate by parts twice in \(\Omega\), to obtain

\[
\lambda \int_\Omega f(u_\lambda)\varphi_1 dx = \lambda_1 \int_\Omega u_\lambda \varphi_1 dx \leq \frac{\lambda^*}{2} \int_\Omega (f(u_\lambda) + C)\varphi_1 dx.
\]

Taking \(\lambda \geq 3\lambda^*/4\), we see that \(f(u_\lambda)\varphi_1\), and hence \(f(u_\lambda)\delta\), are nondecreasing 
in \(\lambda\) and uniformly bounded in \(L^1(\Omega)\). Multiply (1.1\(\lambda\)) by \(\overline{u}\), the solution of 
(5.1), and integrate by parts twice in \(\Omega\) to conclude

\[
\int_\Omega u_\lambda dx = \lambda \int_\Omega f(u_\lambda)\overline{u} dx \leq C\lambda \int_\Omega f(u_\lambda)\delta dx \leq C,
\]

(5.2)

for some constant \(C\) depending on \(\Omega\) and \(f\). Thus, both sequences \(u_\lambda\) and 
\(\lambda f(u_\lambda)\delta\) are increasing in \(\lambda\) and uniformly bounded in \(L^1(\Omega)\) for \(\lambda < \lambda^*\). By monotone convergence, we conclude that \(u^* \in L^1(\Omega)\) is a weak solution of 
(1.1\(\lambda\)) for \(\lambda = \lambda^*\).

Finally, for \(\lambda < \lambda^*\) we have \(\int_\Omega \lambda f'(u_\lambda)\xi^2 dx \leq \int_\Omega \|
\nabla \xi\|^2 dx\) for all \(\xi \in C^\infty\) with compact support in \(\Omega\). Since \(f' \geq 0\), Fatou’s lemma leads to 

\[
\int_\Omega \lambda^* f'(u^*)\xi^2 dx \leq \int_\Omega \|
\nabla \xi\|^2 dx,
\]

and hence \(u^*\) is semi-stable. \(\Box\)

Next, we make some remarks on the set \(\{u_\lambda : 0 \leq \lambda < \lambda^*\}\) of minimal solutions.

First, if \(f\) is increasing and convex, the first eigenvalue \(\mu_1\{-\Delta - \lambda f'(u_\lambda); \Omega\}\)

of the linearized problem in \(\Omega\) is a decreasing function of \(\lambda\). By semi-stability, it is also a nonnegative function. Hence, if \(\mu_1\{-\Delta - \mu f'(u_\mu); \Omega\} = 0\) for some \(\mu\), then the set \(\{u_\lambda : 0 \leq \lambda < \lambda^*\}\) ends at this \(\mu\) and \(\lambda^* = \mu\). Therefore, for increasing and convex \(f\), the linearized operator has positive first eigenvalue 
for all \(\lambda < \lambda^*\) and, hence, minimal solutions form a continuous branch that can be obtained through the implicit function theorem.

In the case that \(f\) satisfies (1.17) but is not convex, the set of minimal solutions is not necessarily continuous in \(\lambda\). For instance, the \((\lambda, \|u\|_{L^\infty})\) diagram
may have a turning point for some $\mu < \lambda^*$. Since in this case $u_\mu$ is bounded, by the implicit function theorem we have that the first eigenvalue of the linearized problem $\mu_1 \{-\Delta - \mu f(u_\mu); B_1\}$ vanishes. Thus, since $\mu < \lambda^*$ the set \{ $u_\lambda : 0 \leq \lambda < \lambda^*$ \} may have a jump at $\mu$. It is not difficult to show the existence of nonconvex nonlinearities satisfying (1.17) for which this happens. They may be constructed with explicit expressions in three intervals: a first one where the nonlinearity is convex, followed by one where it is concave, and then convex from one value on.

An explicit example of this situation is given by

$$
\begin{align*}
-\Delta u &= \lambda \left\{ \frac{36}{2\pi} \arctan(5u - 15) + \left( \frac{u}{85} \right)^{10} + 10 \right\} \quad \text{in } B_1 \\
u &\geq 0 \quad \text{in } B_1 \\
u &= 0 \quad \text{on } \partial B_1.
\end{align*}
$$

In Figure 1, we show the curve in the $(\lambda, \|u\|_{L^\infty})$ diagram for $(5.3\lambda)$ when $n = 4$. The curve of this diagram has been computed by a finite differences scheme and a Newton method as described in [9]. The set \{ $u_\lambda : 0 \leq \lambda < \lambda^*$ \} of minimal solutions is represented by the solid line. The dotted line corresponds to unstable solutions and the dashed line represents semi-stable solutions that are not minimal solutions. Note that, where the curve moves to the right in $\lambda$, it corresponds to the regions where the solutions are semi-stable, and where the curve moves to the left in $\lambda$, it corresponds to the regions where the solutions are unstable.
Finally, we give the proof of Theorem 1.10, which follows easily from Theorems 1.5 and 1.8. Note also that Theorem 1.1 follows from Theorem 1.10 and Remark 1.7 —this remark is used to prove part (c) of Theorem 1.1.

Proof of Theorem 1.10. By Proposition 5.1 we know that $u^* \in L^1(B_1)$ is a semi-stable weak solution of (1.1, λ) which is the strong limit in $L^1(B_1)$ of $(u_\lambda)$. In order to apply Theorems 1.5 and 1.8 with $u = u^*$ and $g = \lambda^* f$, we need to prove that $u^* \in H^1_0(B_1)$ and that $u^*$ is radially decreasing.

For this, we apply Theorem 1.8 with $\lambda < \lambda^*$ and $u = u_\lambda$ (a smooth solution). Estimate (1.15) applied with $k = 1$ and $q = 2 < q_1$ leads to $\|u_\lambda\|_{H^1_0(B_1)} \leq C$ for some constant $C$ independent of $\lambda$ (since $\|u_\lambda\|_{L^1(B_1)} \leq \|u^*\|_{L^1(B_1)} < \infty$ for all $\lambda$). Thus, a subsequence of $(u_\lambda)$ converges weakly in $H^1_0$ to an $H^1_0$ function $v$, and strongly in $L^1$. It follows that $v = u^*$ and hence $u^* \in H^1_0$. In addition, since the $u_\lambda$ are radially decreasing by the Gidas-Ni-Nirenberg result, $u^*$ is radially nonincreasing. Hence, $u^*$ is smooth away from the origin, superharmonic and nonincreasing. Hopf’s boundary lemma gives that $u^*$ is radially decreasing.

Therefore, we can apply Theorems 1.5 and 1.8 with $u = u^*$. Part (a) of Theorem 1.10 follows from Theorem 1.5. The estimate $u^*(r) \leq -C \log r$ of part (b) follows from (1.8) when $r \leq 1/2$. For $1/2 \leq r \leq 1$ the estimate is consequence of $u^*$ being $C^1$ away from the origin and of $-\partial_r(-\log r) = 1/r \geq 1$ (note that both $u^*$ and $-\log r$ vanish on $r = 1$). Part (c) of the theorem follows in a similar way from part (c) of Theorem 1.5.

Finally, part (d) of the theorem is consequence of Theorem 1.8 together with the $C^3$ estimates away from the origin proved for $u^*$ using (3.2). \hfill \Box

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