Similarity and Dissimilarity Concepts in Machine
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Similarity and Dissimilarity Concepts in Machine Learning

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Abstract

Similarity and dissimilarity are rarely formalized concepts in Artificial Intelligence (AI). Similarity and dissimilarity have a psychological origin, and they have been adapted to AI. In this field, however, similarity and dissimilarity choice is not always dependent on the problem to solve. In this paper, a formalization of similarity and dissimilarity is presented. The purpose of this paper is to contribute to the design and understanding of similarity and dissimilarity in AI, increasing their general utility. A formal definition and some basic properties are introduced. Also, some transformation functions and similarity and dissimilarity operators are presented.

Contents

1   Introduction

From a psychological point of view, human being uses the notion of similarity to solve problems, to search information, to inductively reasoning or to categorize elements. Formally, similarity is defined as the degree of likeliness or analogy between two elements. Opposite to similarity is the concept of dissimilarity, playing an important role. The four main *psychological* similarity models are the geometric model [33, 17] , the feature model[34], the structural model [7, 6, 20] and the transformational model [10, 9]. One of the most well-known psychological theories, Gestalt [37], defines similarity as the result of evaluate if two objects look similar to one another, not if they are the same object. This perception is useful to group objects in classes or groups with common features.

Based from this human point of view, Machine Learning theory has adapted this psychological concept of similarity, becoming one of its essential components. However, there are confusion with definitions and properties of similarity yet. The reason is that similarity and dissimilarity are evasive concepts, rarely formalized. Because of this, in Machine Learning usually the choice is a predefined similarity and not a similarity related to the problem to solve. In this cases, the power and the flexibility of similarity are underused.

With the objective of highlight the utility of similarity and dissimilarity, this work contributes with ideas about designing and understanding similarity and dissimilarity measures, making them useful within different fields of AI. Starting from a formal definition basis, a similarity and dissimilarity theory is constructed. In order to understand better similarity and dissimilarity, their properties are described and analyzed. Then, some transformation tools are introduced to increase similarity and dissimilarity utility and adapt them to specific problems. This tools and their impact on similarity and dissimilarity is also analyzed. These transformations are a way to construct new similarities and dissimilarities based in other ones.

This document is organized as follows. In the next section, similarity and dissimilarity concept is analyzed. Then, in Section 3, a definition of similarity and dissimilarity are introduced, as well as their main properties. In Sections 4.1 and 5, two kinds of transformations are introduced and analyzed. Next, some examples are shown in Section 6. Finally, in Section 7 some aspects of similarity properties are discussed, and in Section 8 are the conclusions about this work.

2   Similarity and dissimilarity scope

Despite of being a widely used concept in AI, similarity and dissimilarity formalization is far from being clear. In general, it is assumed that both evaluate a comparison between two objects. Nevertheless, there are disagrees about their formalization and their properties. Because of this, sometimes similarity is used instead of distance or metric and viceversa.

Based on the original classic psychological work [34] or geometric concepts [33, 17], similarity and dissimilarity are used in several fields to compare many kinds of objects. These theories are based on geometrical distance accepting or rejecting some of the metric axioms. Therefore, this point of view assumes that similarities and dissimilarities are defined in a metric space. However, this assumption is strongly refuted in the literature and there are no agreement about the truthfulness of metric axioms. In fact, some authors like Tversky [34] argued that metric axioms are unnecessarily restrictive. On the contrary, he proposes a *feature contrast* model. Other authors [38, 15, 12, 28], use *fuzzy logic*[4] to model both geometric axioms

and feature axioms of similarity and dissimilarity. At last, *transformational* model considers similarity and dissimilarity between two objects as the transformation cost between them [11].

Independently of the chosen similarity model, either similarity or dissimilarity are used in several fields like Case Based Reasoning (CBR) [24, 22] , in Data Mining [31], in Information Retrieval (IR) [1], in Pattern Matching [2, 35], Fuzzy Logic [13, 36], etc. Likewise, there are some excellent previous works in similarity classification [28] and similarity theory [18].

3   Definitions

3.1   Similarity

In a general sense, similarity expresses the degree of coincidence between two elements. Therefore, it is reasonable to treat them as functions since the objective is to measure or calculate this value between any two elements of the set. Since this can be represented functionally, in this document similarity is considered a function of two arguments. In this section, we will describe similarity properties as well as further notation.

A similarity function is defined as follows:

Let X be a non-empty set where there is defined a equality relation $\overset{x}{=}$. Let s be a function

$$s : X \times X \longrightarrow I_s \subset \mathbb{R}$$

Assume that s is upper bounded, exhaustive and total. This implies that I_s is upper bounded and also that $\sup_{\mathbb{R}} I_s$ exists[1].

Let s_{max} be the maximum value of s (i.e. $s_{max} = \sup_{\mathbb{R}} I_s$). Without loss of generality, consider $s_{max} \geq 0$. In any other case, a non-negative maximum can be obtained applying the transformation $s + |s_{max}|$.

Function s may be required to satisfy the following axioms, for any $x, y, z \in X$:

Property s1 (Reflexivity). $s(x,x) = s_{max}$. This implies $\sup_{\mathbb{R}} I_s \in I_s$.

Property s2 (Strong reflexivity). $s(x,y) = s_{max} \iff x \overset{x}{=} y$.

Property s3 (Symmetry). $s(x,y) = s(y,x)$.

Property s4 (Lower boundedness). A similarity s is *lower bounded* when $\exists a \in \mathbb{R}$ such that $s(x,y) \geq a$, for all $x, y \in X$. This is equivalent to ask that $\inf_{\mathbb{R}} I_s$ exists.

Property s5 (Lower closedness). Given a lower bounded function s, define $s_{min} \equiv \inf_{\mathbb{R}} I_s$. The property ask for existence of $x, y \in X$ such that $s(x,y) = s_{min}$.

Consider now a function $C : X \longrightarrow 2^X$. Each one of the elements in 2^X will be called a *complement of x.*

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[1]In this document we will only focus on similarities whose images are subsets of $\mathbb{R}$. For a more general view see [22, 23, 24].

Property s6 (Complement). A lower closed similarity s defined in X has *complement* function C, where $C(x) = \{x' \in X / s(x, x') = s_{min}\}$, if $\forall x, x' \in X$, $|C(x)| = |C(x')| \neq 0$.

Moreover, if s is also reflexive, necessarily $x \notin C(x)$. On the other hand, s has *unitary complement* function if $\forall x \in X$, $|C(x)| = 1$. In this case, $\forall x \in X$:

$$s(x, y') = s_{max} \iff y' \in C(y), y \in C(x)$$

Let us define a *transitivity operator* in order to introduce transitivity in similarity and dissimilarity functions.

Property s7 (Transitivity). A similarity s defined on X is $\tau_{\Sigma^\star}$-transitive if there is a transitive operator $\tau_{\Sigma^\star}$ such that the following inequality holds:

$$s(x, y) \geq \tau_{\Sigma^\star}(s(x, z), s(z, y)) \quad \forall x, y, z \in X \tag{3.1}$$

Let us to introduce a simple example of similarity.

Example 1. Let $X = \mathbb{Z}^+$, and let s a function defined in X:

$$s(x, y) = 1 - \frac{|x - y|}{|x - y| + 1}$$

Where $I_s \subset \mathbb{Q}$, $\sup_{\mathbb{Q}} I_s = 1$ and $\inf_{\mathbb{Q}} I_s = 0$. Also, the relation $\stackrel{X}{=}$ is the commonly used in $\mathbb{Z}$.

This function satisfies reflexivity, symmetry and strong reflexivity axioms. Moreover, it is lower limited by 0, although it is not lower closed. For this reason, it does not have complement function. Despite of this, it fulfills transitivity, expressed as follows:

$$s(x, y) \geq \max\{s(x, z) + s(z, y) - 1, 0\}$$

This expression will be analyzed further in this document (Example 8).

Consider now another function, defined in X too and expressed as follows:

$$s'(x, y) = \begin{cases} 0 & \text{if } y = -x \\ 1 - \frac{|x - y|}{|x - y| + 1} & \text{otherwise} \end{cases}$$

It is also strongly reflexive, symmetric and transitive. However, this one is lower closed and unitary complement function: for each $x \in X$, $C(x) = \{-x\}$. Further, some examples of the importance of this choice are presented.

Remark 1. The fulfillment of these axioms leads to a *basic semantic* associated with s. This semantic depends on two relations. The first one is between $s(x, y)$ and $s(x, z)$ expressing the way that x is more or less similar to y than it is to z. The other one is the choice of the relation $\stackrel{X}{=}$.

Notation

Along this document we will use the following notation in order to identify the different similarities and their properties.

- The set X of elements will de called *definition set*.

- A *similarity in X* is a function s satisfying axioms s2 and s3. $\Sigma(X)$ denotes the set of all the similarities with a definition set X.

- A similarity is *bounded (closed)* if it satisfies axiom s4 (axiom s5).

- A similarity has *(unitary) complement function* if it satisfies axiom s6.

- A similarity is τ_Σ-*transitive* if it satisfies axiom s7 for a fixed similarity transitive operator τ_Σ.

- A similarity is *strong* if it satisfies axiom s2. Otherwise, it is called *pseudo-similarity*.

3.2   Dissimilarity

Dissimilarity is the opposed concept to similarity since it reflects the degree of unlikeness between two elements. Because of that, both the definition of dissimilarity and the axioms that dissimilarity can fulfill are analogous to those seen in previous section. In fact, there is a direct correspondence between properties.

Again, let X be a non-empty set where it is defined a equality relation $\overset{X}{=}$. Let δ be a function

$$\delta : X \times X \longrightarrow I_\delta \subset \mathbb{R}$$

Assume that δ is *lower* bounded, exhaustive and total. This implies that I_δ is lower bounded and also that $\inf_{\mathbb{R}} I_\delta$ exists.

Let δ_{min} be the minimum value of δ (i.e. $\delta_{min} = \inf_{\mathbb{R}} I_\delta$). Without loss of generality, we will consider $\delta_{min} \geq 0$. In any other case, we can get a non-negative maximum applying the transformation $\delta + |\delta_{min}|$.

The function δ can fulfill the following axioms, for any $x, y, z \in X$:

Property d1 (Reflexivity). $\delta(x, x) = \delta_{min}$. This implies $\inf_{\mathbb{R}} I_\delta \in I_\delta$.

Property d2 (Strong reflexivity). $\delta(x, y) = \delta_{min} \iff x \overset{X}{=} y$.

Property d3 (Symmetry). $\delta(x, y) = \delta(y, x)$.

Property d4 (Upper boundedness). $\exists a \in \mathbb{R}$ such that $\delta(x, y) \leq a$. This implies that $\sup_{\mathbb{R}} I_\delta$ exists.

Property d5 (Upper closedness). Given an upper bounded function δ, let δ_{max} denote $\sup_{\mathbb{R}} I_\delta$. There exist $x, y \in X$ such that $\delta(x, y) = \delta_{max}$. This is equivalent to ask that $\inf_{\mathbb{R}} I_s \in I_s$.

Consider again a function $C : X \longrightarrow 2^X$.

Property d6 (Complement). An upper closed dissimilarity δ defined in X has *complement* function C, where $C(x) = \{x' \in X / \delta(x, x') = \delta_{max}\}$, if $\forall x, x' \in X, |C(x)| = |C(x')| \neq 0$.

Again, if δ is reflexive, necessarily $x \notin C(x)$. On the other hand, δ has *unitary complement* function if $\forall x \in X, |C(x)| = 1$. In this case, $\forall x \in X$:

$$\delta(x, y') = \delta_{min} \iff y' \in C(y), y \in C(x)$$

Property d7 (Transitivity). A dissimilarity δ defined on X is $\tau_{\Delta^\star}$-transitive if there is a transitive operator $\tau_{\Delta^\star}$ such that the following inequality holds:

$$\delta(x,y) \leq \tau_{\Delta^\star}(s(x,z), s(z,y)) \quad \forall x,y,z \in X \tag{3.2}$$

Remark 2. Like similarities do, the fulfillment of these axioms leads to a *basic semantic* associated with δ. This semantic depends on the same two relations: The first one is between $\delta(x,y)$ and $\delta(x,z)$ expressing, in this case, the way that x is more or less dissimilar to y than it is to z. The other one is the choice of the relation $\stackrel{x}{=}$.

Notation

Analogously to similarity, we introduce here the notation that will be used in this document.

- A *dissimilarity* is a function $\delta : X \times X \longrightarrow I_\delta$ fulfilling axioms d2 and d3. Denote $\Delta(X)$ to the set of all the dissimilarities defined on X.

- A dissimilarity is *bounded (closed)* if it satisfies axiom d4 (axiom d5).

- A dissimilarity has *(unitary) complement* if it satisfies axiom d6.

- A dissimilarity is τ_Δ-*transitive* if it satisfies axiom d7 for a fixed dissimilarity transitive operator τ_Δ.

- A dissimilarity is *strong* if it satisfies axiom d2. Otherwise, it is called *pseudo-dissimilarity*.

Next, we introduce an example of two dissimilarities:

Example 2. Let X be the set of vowels: $\{a, A, e, E, i, I, o, O, u, U\}$. Consider the relation $x \stackrel{x}{=} y$ as "x is the same letter than y". For the sake of clarity, denote $C(x,y)$ to the function that calculates the number of vowels between its two arguments plus one, in alphabetical order (e.g. C('a','O')=3,C('a','u')=4,C('a','A')=0).

Define the following dissimilarity with $I_\delta = \{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$:

$$\delta(x,y) = \begin{cases} 0 & \text{if } x \stackrel{x}{=} y \\ \frac{1}{5-C(x,y)} & \text{otherwise} \end{cases}$$

This dissimilarity can be represented as a table:

	a,A	e,E	i,I	o,O	u,U
a,A	0	0.25	0.33	0.5	1
e,E	0.25	0	0.25	0.33	0.5
i,I	0.33	0.25	0	0.25	0.33
o,O	0.5	0.33	0.25	0	0.25
u,U	1	0.5	0.33	0.25	0

Reflexivity and symmetry axioms are fulfilled because the matrix is symmetric and has zeroes in the main diagonal. Moreover, since elements out of the main diagonal are nonzero, strong reflexivity is also fulfilled. Note that the upper bound (1) appears in the matrix, therefore, this dissimilarity is closed.

There are no complement function defined in X but, if it had existed, this dissimilarity would not have complement function because just a pair of elements ('a' and 'u') reach the maximum dissimilarity. Transitivity in this function is given by the next expression. For all $x, y, z \in X$:

$$\delta(x,y) \leq \min \left\{ \delta(x,z) \cdot \delta(z,y) \right\} \delta(x,z) + \delta(z,y) - 5 \cdot \delta(x,z) \cdot \delta(z,y), 4 \}$$

Further in this document, we will analyze the origin of this expression (Example 9).

Note the influence of the choice of $\overset{X}{=}$. If we define it now as "x is the same character than y" (i.e. distinguishing letter 'a' from letter 'A') the values of the previous dissimilarity change. For example, the dissimilarity value between 'a' and 'A', in this case, is:

$$\delta(\text{'a','A'}) = \frac{1}{5 - C(\text{'a','A'})} = \frac{1}{5}$$

while formerly it was 0.

Note that the image of δ has also changed. Now, $I_\delta = \{0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$.

Despite this, δ is still strong reflexive although, in general, not always will be this way.

Thus, if we define dissimilarity δ_2 over the same set X as:

$$\delta_2(x,y) = \frac{C(x,y)}{4}$$

This dissimilarity is again symmetric, bounded and closed. Its image is $I_{\delta_2} = \{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$, and it fulfills the triangle inequality, since $C(x,y)$ is a metric. Note that, using the first definition of $\overset{X}{=}$, δ_2 is strongly reflexive, whereas under the second one is a pseudo-dissimilarity.

4 Transformations between similarities and dissimilarities

There are many links between similarities and dissimilarity. This section introduces some of the similarity and dissimilarity design tools. Using this tools a similarity or dissimilarity can be transformed into another one, fulfilling different properties. Also, a dissimilarity can be converted into a similarity or viceversa. Previously, let us to introduce the concept of *equivalence between similarity functions* and *equivalence between dissimilarity functions*.

4.1 Equivalence Functions

Let X be the definition set. Consider now the set of all the pairs of elements of X and denote it as $X \times X$. For a fixed $s \in \Sigma(X)$, there is a preorder relation in $X \times X$. This preorder is defined as *to belong to a class of equivalence with less or equal similarity value*. This preorder in $X \times X$ depends on s because it is *induced* by s.

Definition 4.1. Given X and $s \in \Sigma(X)$, there exists a preorder, denoted by $\preceq$ in $X \times X$, defined, $\forall x, y, x', y' \in X$, as follows.

$$(x,y) \preceq (x',y') \iff s(x,y) \leq s(x',y')$$

Analogously, given $\delta \in \Delta(X)$, there exists a preorder in $X \times X$ defined as *to belong to a class of equivalence with less or equal dissimilarity value*. Again, this preorder is induced by δ.

This preorder means that, for all $(x, y), (z, w) \in X \times X$

$$(x, y) \preceq (w, z) \wedge (w, z) \preceq (x, y) \text{ does not imply } x \overset{x}{=} w \wedge y \overset{x}{=} z.$$

Therefore, this induces another relation in $X \times X$, denoted by $\prec$, defined as follows:

$$(x, y) \prec (w, z) \iff (x, y) \preceq (w, z) \wedge \neg((w, z) \preceq (x, y))$$

Similarly, denote $(x_1, x_2) \equiv (x_3, x_4)$ when satisfying

$$(x_1, x_2) \preceq (x_3, x_4) \wedge (x_3, x_4) \preceq (x_1, x_2)$$

Consequently, elements of X can be grouped in classes of equivalence using this induced preorder. This classification is done using the similarity value of each pair of elements. Formally expressed, $\forall (x_1, x_2) \in X \times X$:

$$[(x_1, x_2)] = \{(x_3, x_4) \in X \times X \mid s_X(x_1, x_2) = s_X(x_3, x_4)\}$$

Where the square brackets denote *class of equivalence*.

Analogously, the elements in X can also be grouped using a dissimilarity.

Remark 3. Note that the set $X \times X$ is partitioned by the previously classes by means the equivalence relation $\equiv$. This partition is denoted by $X^2/\equiv$.

Definition 4.2 (associate order). Let X be a definition set. Any similarity or dissimilarity induces a preorder in $X \times X$. An *associate order* is the relation in $X^2/\equiv$ defined as follows:

$$[(x, y)] \leq_{/\equiv} [(x', y')] \iff (x, y) \preceq (x', y')$$

Using these concepts the definition of equivalence between similarities and equivalence between dissimilarities is introduced.

Definition 4.3 (equivalent similarities/disimilarities). Two similarities (dissimilarities) with the same definition set X are *equivalent* if they induce the same preorder in $X \times X$.

Note that the equivalence between similarities or between dissimilarities is an equivalence relation. The previous definition 4.3 can be expressed in various ways.

Proposition 4.4. *The three next definitions are analogous. For all $s_X, s'_X \in \Sigma(X)$:*

   i) s_X, s'_X *are equivalent.*

   ii) $\forall x, y, z, w \in X,\ s_X(x, y) < s_X(z, w) \iff s'_X(x, y) < s'_X(z, w)$.

   iii) $\forall x, y, z, w \in X,\ s_X(x, y) > s_X(z, w) \iff s'_X(x, y) > s'_X(z, w)$.

Proof.   **i) to ii)** If s_X, s'_X are equivalent they induce the same preorder in X, so for all $x, y, x', y' \in X$

$$(x, y) \preceq (x', y') \Longleftrightarrow \begin{cases} s_X(x, y) < s_X(x', y') \\ s'_X(x, y) < s'_X(x', y') \end{cases}$$

Consider, for some $x, y, x', y' \in X$, that $s_X(x, y) < s_X(x', y')$. This implies $(x, y) \prec (x', y')$. Thus,

$$(x, y) \prec (x', y') \Longleftrightarrow s'_X(x, y) < s'_X(x', y')$$

ii) to i) Derived from *i) to ii)* since all implications are double.

i) to iii) Analogous to *i) to ii)*, considering, for some $x, y, x', y' \in X$, that $s_X(x, y) > s_X(x', y')$.

iii) to ii) Idem that *ii) to i)*.

$\square$

The main properties of similarities and dissimilarities are kept under equivalence relations. Before, let us to introduce the following lemma about the minimum transitivity operator in similarities and dissimilarities.

Lemma 4.5. *Consider a similarity $s \in \Sigma(X)$ and a dissimilarity $\delta \in \Delta(X)$, both of them strong reflexive.*

If s is lower bounded, $s \in \Sigma(X)$ is τ_Σ-transitive where τ_Σ is

$$\forall a, b \in I_s \quad \tau_\Sigma(a, b) = \begin{cases} a & b = s_{max} \\ b & a = s_{max} \\ s_{min} & otherwise \end{cases}$$

If s is not lower bounded, τ_Σ is

$$\tau_\Sigma(a, b) = \begin{cases} a & b = s_{max} \\ b & a = s_{max} \\ -\infty & otherwise \end{cases}$$

Analogously, if δ is upper bounded, $\delta \in \Delta(X)$ is τ_Δ-transitive, where τ_Δ is

$$\forall a, b \in I_\delta \quad \tau_\Delta(a, b) = \begin{cases} a & b = \delta_{max} \\ b & a = \delta_{max} \\ \delta_{min} & otherwise \end{cases}$$

If δ is not lower bounded, τ_Δ is

$$\tau_\Delta(a, b) = \begin{cases} a & b = \delta_{max} \\ b & a = \delta_{max} \\ +\infty & otherwise \end{cases}$$

Proof.   Let $x, y, z \in X$ be three elements of X.

If $x \overset{X}{=} y$ by means of τ_Σ-transitivity we know that

$$s_{max} \geq \tau_\Sigma(s(x, z), s(z, y))$$

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That is always true.

If $x \overset{X}{=} z$, then $s(x, z) = s_{max}$ and the transitivity is

$$s(x, y) \leq s(z, y)$$

Because $x \overset{X}{=} z$, $s(x, y) = s(z, y)$ so this is also true.

If $y \overset{X}{=} z$, then $s(y, z) = s_{max}$ and the transitivity is

$$s(x, y) \leq s(x, z)$$

Because $y \overset{X}{=} z$, $s(x, y) = s(x, z)$ so, like the previous case, this is also true.

If $x \overset{x}{\neq} y, x \overset{x}{\neq} z$ and $y \overset{x}{\neq} z$, then, using strong reflexive and lower boundedness properties:

$$s(x, y) \geq s_{min}$$

or, if s is not lower bounded

$$s(x, y) \geq -\infty$$

The demonstration using dissimilarities is analogous.

$\square$

Proposition 4.6. *Given two equivalent similarities* $s_1, s_2 \in \Sigma(X)$ *or two equivalent dissimilarities* $\delta_1, \delta_2 \in \Delta(X)$,

- s_1 *(δ_1) is* reflexive *only if* s_2 *(δ_2) is reflexive.*

- s_1 *(δ_1) is* strong reflexive *only if* s_2 *(δ_2) is strong reflexive.*

- s_1 *(δ_1) is* symmetric *only if* s_2 *(δ_2) is symmetric.*

- s_1 *(δ_1) is* lower closed (*upper closed) only if* s_2 *(δ_2) has lower closed (upper closed).*

- s_1 *(δ_1) has* complement *function only if* s_2 *(δ_2) has complement function.*

- s_1 *(δ_1) is* transitive *only if* s_2 *(δ_2) is transitive.*

Proof.   **Reflexivity** Provided that $s_1(x, x) = s_{1_{max}}$, using Definition 4.3,

$$\forall x, y \in X \quad s_2(x, x) > s_2(x, y) \iff s_1(x, x) > s_1(x, y)$$

Therefore denote $s_{2_{max}}$ to the maximum value for similarity s_2.

Strong Reflexivity If s_1 is strong reflexive then

$$s_1(x, y) = s_{1_{max}} \iff x \overset{X}{=} y$$

Suppose that $\exists x, y \in X$ such that $x \overset{x}{\neq} y$ but $s_2(x, y) = s_{2_{max}}$. This means that $s_2(x, x) = s_2(x, y)$ but, using Definition 4.3 we know that this means that

$$s_1(x, x) = s_1(x, y)$$

and this is a contradiction.

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Symmetry Symmetry is trivial, using definition of equivalence.

Lower Boundedness This property cannot be assured.

Lower Closedness Provided that both similarities have lower bound, consider a set of pairs of element of X denoted M_1 such that $\forall (x, y) \in M_1 \; s_1(x, y) = s_{1_{min}}$. Therefore, using Definition 4.3,

$$\forall z, w \in X, \; \forall (x, y) \in M_1 \quad s_1(x, y) < s_1(z, w) \Longleftrightarrow s_2(x, y) < s_2(z, w)$$

Denote $s_{2_{min}} = s_2(x, y)$ for any $(x, y) \in M_1$. This is the lower closure of s_2.

Complement If s_1 has complement function, we know that $\forall x' \in C_1(X) \; s_1(x, x') = s_{1_{min}}$. Therefore, using Definition 4.3,

$$\forall x, y \in X \quad s_1(x, x') < s_1(x, y) \Longleftrightarrow s_2(x, x') < s_2(x, y)$$

Denote $s_2(x, x') = s_{2_{min}}$. Thus, s_2 has complement function and $C_1(X) = C_2(X)$.

Transitivity Analyzing the expression of transitivity

$$\forall x, y, z \in X \quad s_1(x, y) \geq \tau_{1\Sigma}(s_1(x, z), s_1(z, y))$$

we cannot assure the fulfillment of this property by the similarity s_2 because the transitivity operator $\tau_{1\Sigma}$. However, using Proposition 4.5 we can assure that if s_1 fulfils this minimum transitivity, s_2 also fulfils it. To demonstrate this, simply use that both s_1 and s_2 are strong reflexive.

$\square$

Following, we define an equivalence function that allows us to get equivalent similarities or dissimilarities.

Definition 4.7. [Equivalence function] Given two equivalent similarities $s_1, s_2 \in \Sigma(X)$, or two equivalent dissimilarities $\delta_1, \delta_2 \in \Sigma(X)$, an equivalence function is a function $\breve{f} : I_{s_1} \longrightarrow I_{s_2}$, or $I_{\delta_1} \longrightarrow I_{\delta_2}$ such that for all x, y in X,

$$s_2(x, y) = \breve{f} \circ s_1(x, y)$$

or

$$\delta_2(x, y) = \breve{f} \circ \delta_1(x, y)$$

Proposition 4.8. *Any equivalence function $\breve{f}$ fulfills in all its dominium (i.e. I_{s_1}), the following properties:*

- *Strictly increasing monotonicity.*

- *Invertibility.*

Therefore, $\breve{f}$ is a bijection.

Proof. For one side, following the definition 4.7, in order to assure that, for all $x, y, z, w \in X$,

$$s_1(x, y) > s_1(z, w) \Longleftrightarrow \breve{f} \circ s_1(x, y) > \breve{f} \circ s_1(z, w)$$

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function $\breve{f}$ has to be strictly increasing in I_{s_1}.

For other side, using that s_1 and s_2 are equivalent, there exists a function $\breve{f}^{-1}$ such that

$$s_1(x, y) = \breve{f}^{-1} \circ s_2(x, y)$$

Here, $\breve{f}^{-1}$ is the inverse of $\breve{f}$. Therefore, $\breve{f}$ has to be invertible.

$\square$

In this sense, the main properties of similarities and dissimilarities have to be kept by these equivalence functions. The following proposition proves that any equivalence function keeps all the properties expressed in Proposition 4.6.

Remark 4. Note that *lower boundedness* property is it not included in 4.6. This is due to the fact that a *not* lower bounded similarity has a lower bounded equivalent similarity. The analogy between the images of these similarities is that both are open intervals and their lowest value (i.e. $-\infty$ and any real number, respectively) cannot be reached in any case.

The following Property illustrate a case such that the transformation $\breve{f}$ converts a not lower bounded similarity into a bounded one.

Proposition 4.9. *Let s_1 be a similarity in $\Sigma(X)$ and let $\breve{f}$ be an equivalence function such that $\breve{f} : I_{s_1} \longrightarrow [a, b]$. Denote s_2 to the composition of $\breve{f}$ and s_1. Thus, s_2 is a similarity in $\Sigma(X)$ such that $I_{s_2} \subset [a, b]$, keeping the properties of s_1 in the terms of Proposition 4.6 says.*

Proof.   Using that $\breve{f}$ is strictly increasing, s_2 keeps the reflexive and the strong reflexive property by means of considering $b = \breve{f} \circ s_1(x, x) = s_{2_{max}}$. The symmetry, as usual, is trivially kept in this case.

If s_1 has lower closure (therefore, s_2 also has it), consider $a = \breve{f}(s_{1_{min}})$ as the lowest value of s_2. Also, using a as the lowest value assures the fulfilment of the complement property as Proposition 4.6 says.

$\square$

An special case is the following one, where $[a, b] = [0, 1]$.

Corollary 4.10. *Let s_1 be a similarity in $\Sigma(X)$ and let $\breve{f}^{\star}$ be an equivalence function such that $\breve{f}^{\star} : I_{s_1} \longrightarrow [0, 1]$. Denote s_2 to the composition of $\breve{f}^{\star}$ and s_1. Thus, s_2 is a similarity such that $I_{s_2} \subset [0, 1]$.*

In fact,

- $s_{2_{max}} = 1$.

- *if s_1 has lower closure, then $s_{2_{min}} = 0$, and $I_{s_2} = [0, 1]$.*

- *if s_1 does not have lower bound, s_2 does have it (its value is 0) and therefore, $I_{s_2} = (0, 1]$.*

Analogously, let δ_1 be a dissimilarity in $\Delta(X)$ and let $\breve{f}^{\star}$ be an equivalence function such that $\breve{f}^{\star} : I_{\delta_1} \longrightarrow [0, 1]$. Denote δ_2 to the composition of $\breve{f}^{\star}$ and δ_1. Thus, δ_2 is a dissimilarity such that $I_{\delta_2} \subset [0, 1]$.

In fact,

- $\delta_{2_{min}} = 0$.

- *if δ_1 has upper closure, then $\delta_{2_{max}} = 1$, and $I_{\delta_2} = [0, 1]$.*

- *if δ_1 does not have upper bound, δ_2 does have it (its value is 1) and therefore, $I_{\delta_2} = (0, 1]$.*

Proof.   The demonstration is based on setting $a = 0$ and $b = 1$.

$\square$

Remark 5. From here on, denote $\Sigma^\star(X)$ to the set of all the similarities whose images are $[0, 1]$ or $(0, 1]$, depending on if they have lower closure or not. Analogously, denote $\Delta^\star(X)$ to the set of all the dissimilarities whose images are $[0, 1]$ or $[0, 1)$. Also, denote $\tau_{\Sigma^\star}$ and $\tau_{\Delta^\star}$ to the transitivity operators of similarities and dissimilarities in $\Sigma^\star(X)$ and $\Delta^\star(X)$, respectively.

Any similarity in $\Sigma(X)$ or dissimilarity in $\Delta(X)$ has at least one equivalent similarity in $\Sigma^\star(X)$ or dissimilarity in $\Delta^\star(X)$, and all the properties the last ones fulfills are also fulfilled by the one in $\Sigma(X)$ or in $\Delta(X)$ - always under the terms exposed in Proposition 4.6. The two main points to consider when applying an equivalence function to a similarity or dissimilarity are: first, the transformation from an unbounded similarity or dissimilarity onto a bounded one or viceversa. And second, the change of the transitivity operator. In fact, the first one is just considering a scale transformation since the equivalent similarity or dissimilarity image interval is kept open or closed in each case. Therefore, there are no differences apart that the values of similarity or dissimilarity will be condensed into a smaller interval or spread into a wide interval, although this is a relevant fact and it has to be bore in mind. The second one affects even more to the transitive property, since it changes the main operator. This leads to an extensive study of how transitivity works and how these equivalence functions affect it. This issue will discussed further on this document.

Therefore, from here on and using Corollary 4.10 we will restrict our study to similarities and dissimilarities in $\Sigma^\star(X)$ and $\Delta^\star(X)$, respectively. This simplifies the theory because some known results from Probabilistic Metric Spaces[30] and Fuzzy Logic[16] can be used to build up the similarity and dissimilarity theory. However, all the results can be applied in a more general form (i.e. using $\Sigma(X)$ and $\Delta(X)$), taking care about the two points of difference that we mentioned before.

Some examples of equivalence functions are the following, for $s \in \Sigma(X)$ and $\delta \in \Delta(X)$:

- If s has lower bound, a simple lineal transformation can be used.
$$\breve{f}^\star(z) = \frac{z - s_{min}}{s_{max} - s_{min}}$$

  If δ has upper bound:
$$\breve{f}^\star(z) = \frac{z - \delta_{min}}{\delta_{max} - \delta_{min}}$$

- If s does not have lower bound (i.e. $I_s = (-\infty, a]$), the following transformations can be used:
$$\breve{f}^\star(z) = 1 - \frac{a - z}{a - z + 1} \qquad a \geq 0$$

  If δ does not have upper bound (i.e. $I_\delta = [a, +\infty)$):
$$\breve{f}^\star(z) = \frac{z - b}{z - b + 1} \qquad b \geq 0$$

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Transitivity in equivalence

In Proposition 4.6 is stated that transitivity operators of equivalent similarities or dissimilarities can be different. By means of equivalence functions we can obtain equivalent similarities or dissimilarities and also know some of their properties, except transitivity. In this sense, the following theorem state how to obtain the transitivity operator of an equivalent similarity or dissimilarity using the equivalence function.

Theorem 4.11. *Let s_1 be a transitive similarity and δ_1 a transitive dissimilarity. Denote $\tau_{1\Sigma}$ and $\tau_{1\Delta}$ to their respective transitivity operators. Let $\breve{f}$ be an equivalence function. The equivalent similarity $s_2 = \breve{f} \circ s_1$ is $\tau_{2\Sigma}$ transitive, where*

$$\tau_{2\Sigma}(a,b) = \breve{f}(\tau_{1\Sigma}(\breve{f}^{-1}(a), \breve{f}^{-1}(b))) \quad \forall a,b \in I_{s_2}$$

The equivalent dissimilarity $\delta_2 = \breve{f} \circ \delta_1$ is $\tau_{2\Delta}$ transitive, where

$$\tau_{2\Delta}(a,b) = \breve{f}(\tau_{1\Delta}(\breve{f}^{-1}(a), \breve{f}^{-1}(b))) \quad \forall a,b \in I_{\delta_2}$$

Proof. Consider only the similarity case. Therefore $\breve{f} : I_{s_1} \longrightarrow I_{s_2}$.

Using that s_1 is transitive we know that, for all $x, y, z \in X$, $s_1(x,y) \geq \tau_{1\Sigma}(s_1(x,z), s_1(z,y))$. Applying $\breve{f}$ to this inequality we get

$$\breve{f} \circ s_1(x,y) \geq \breve{f} \circ \tau_{1\Sigma}(s_1(x,z), s_1(z,y))$$

Using that $\breve{f}^{-1} \circ s_2 = s_1$,

$$s_2(x,y) \geq \breve{f} \circ \tau_{1\Sigma}(\breve{f}^{-1} \circ s_2(x,z), \breve{f}^{-1} \circ s_2(z,y))$$

Simply defining $\tau_{2\Sigma}$ as it is defined in the Theorem we get the transitivity expression on s_2:

$$s_2(x,y) \geq \tau_{2\Sigma}(s_2(x,z), s_2(z,y))$$

And this proves the Theorem.

$\square$

A particular case of the previous theorem is the one using functions from $\Sigma(X)$ to $\Sigma^{\star}(X)$.

Corollary 4.12. *Let s be a τ_{Σ}-transitive similarity in $\Sigma(X)$ and δ be a τ_{Δ}-transitive dissimilarity in $\Delta(X)$. Consider a similarity $s' \in \Sigma^{\star}(X)$ and a dissimilarity $\delta \in \Delta^{\star}(X)$ such that $s' = \breve{f}_1^{\star} \circ s$ and $\delta' = \breve{f}_2^{\star} \circ \delta$. Both of them are transitive where their respective operators are:*

$$
\begin{aligned}
\tau_{\Sigma^{\star}}(a,b) &= \breve{f}^{\star}(\tau_{\Sigma}(\breve{f}^{\star-1}(a), \breve{f}^{\star-1}(b))) &\quad \forall a,b \in I_{s'} \\
\tau_{\Delta^{\star}}(a,b) &= \breve{f}^{\star}(\tau_{\Delta}(\breve{f}^{\star-1}(a), \breve{f}^{\star-1}(b))) &\quad \forall a,b \in I_{\delta'}
\end{aligned}
$$

Proof. The demonstration is quite simple using that the set of $\breve{f}^{\star}$ functions is a subset of the set of $\breve{f}$ functions.
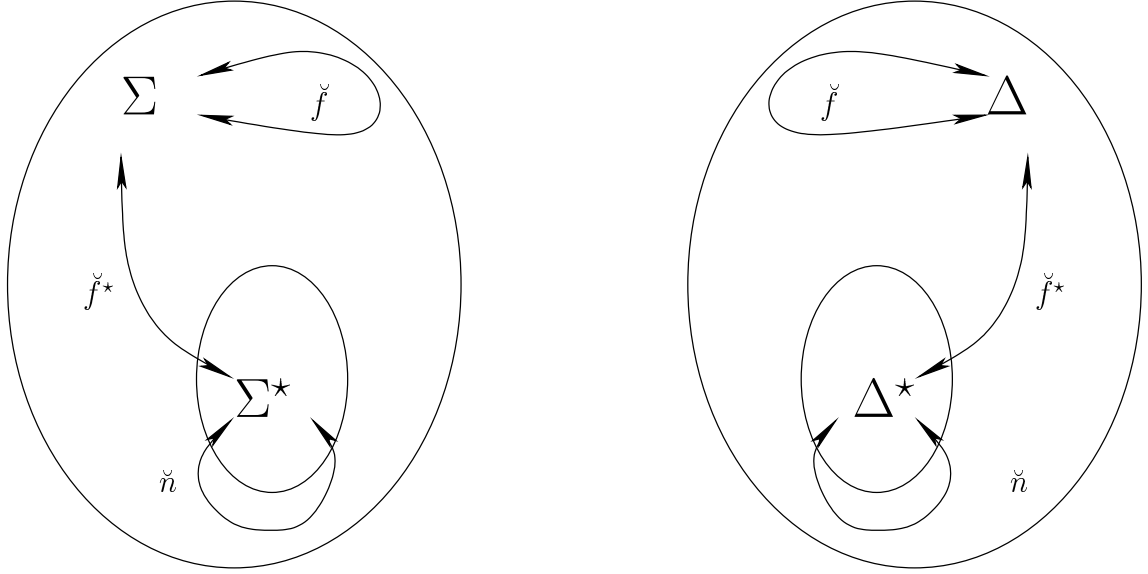
$\square$

Figure 1: Graphical representation of equivalence functions in Σ and Δ

Until now we have seen two classes of equivalence functions: $\breve{f}$ and $\breve{f}^\star$. Let us to introduce more notation. Consider know the equivalence functions between similarities or dissimilarities in $\Sigma^\star(X)$ or $\Delta^\star(X)$ and denote them $\breve{n}$. Therefore, we have the following groups (see Fig. 1):

- $\breve{f} : \Sigma \longrightarrow \Sigma$

- $\breve{f}^\star : \Sigma \longrightarrow \Sigma^\star$

- $\breve{n} : \Sigma^\star \longrightarrow \Sigma^\star$

Note that the second and the third ones are subsets of the first one. Let us to introduce a definition for this functions, although Definition 4.7 also includes this one.

Definition 4.13 (equivalence function on $[0,1]$). An *equivalence function on* $[0,1]$, denoted as $\breve{n}$, is a increasing bijection on $[0,1]$. This implies:

- $\breve{n}(0) = 0$ and $\breve{n}(1) = 1$.

- $\breve{n}$ is continuous on $[0,1]$.

- $\breve{n}$ has inverse on $[0,1]$.

Denote $\breve{\mathcal{N}}$ to the set of all the equivalence functions on $[0,1]$.

Example 3. Let δ be a dissimilarity defined on $\Delta(X)$ given by the following expression for all $x, y \in X$:
$$\delta(x,y) = |x - y|$$
Therefore, $I_\delta = [0, +\infty)$. Besides, δ is transitive by means the transitivity operator $\tau_\delta(a,b) = a + b$, for all $a, b \in I_\delta$.

Applying the following function we get a dissimilarity on $\Delta^\star(X)$. Denote this dissimilarity δ'

$$\breve{f}^\star(z) = \frac{z}{1-z} \quad \forall z \in [0, +\infty)$$

As expected, δ' is also strong reflexive, symmetric and it has not got upper closure, although it has now upper bound. We also know that δ' is transitive, and using Proposition 4.12 this transitivity is given by the following operator, for all $a, b \in [0, 1)$

$$\tau_{\delta'}(a, b) = \frac{a + b - 2 \cdot a \cdot b}{1 - a \cdot b}$$

Converting a similarity into a dissimilarity is another type of transformation. This transformation is studied in the following section.

4.2 Transformation functions between similarities and dissimilarities

Transformation functions introduced so far are concerned separately to similarities or dissimilarities. In this section, it is shown that both concepts are deeply interrelated. Next, let us introduce a way to obtain dissimilarities from similarities or viceversa.

Definition 4.14 (transformation function). A *transformation function* $\hat{n}$ is a decreasing bijection on [0,1]. This implies the following:

- $\hat{n}(0) = 1$ and $\hat{n}(1) = 0$. This is called *limit conditions*.

- $\hat{n}$ is continuous on [0,1].

- $\hat{n}$ has inverse on [0,1].

A transformation function is *involutive* if $\hat{n}^{-1} = \hat{n}$. Denote $\hat{\mathcal{N}}$ to the set of all the functions $\hat{n}$.

Note that this definition is restricted to similarities and dissimilarities in $\Sigma^\star(X)$ and $\Delta^\star(X)$. However, using that both $\breve{f}^\star$ and $\hat{n}$ are bijections an analog transformation function between elements of $\Sigma(X)$ and $\Delta(X)$ is the composition of two or more functions in the following way:

Let $\breve{f}_1^\star : \Sigma(X) \longrightarrow \Sigma^\star(X)$ and $\breve{f}_2^\star : \Delta(X) \longrightarrow \Delta^\star(X)$. Using a transformation function $\hat{n}$ we can build a bijection between $\Sigma(X)$ and $\Delta(X)$ by means of composition:

$$\hat{f} = \breve{f}_1^\star \circ \hat{n} \circ \breve{f}_2^{\star^{-1}}$$

where $\hat{f} : \Sigma(X) \longrightarrow \Delta(X)$.

The following example and Figure 2 illustrates this:

Example 4. Let δ be a dissimilarity having $I_\delta \subseteq [0, +\infty)$. The objective is a equivalent similarity having $I_s \subseteq (0, 1]$. In order to get this, a transformation function is needed. For instance, let g be this function, where $g : [0, +\infty) \longrightarrow (0, 1]$ defined as follows:
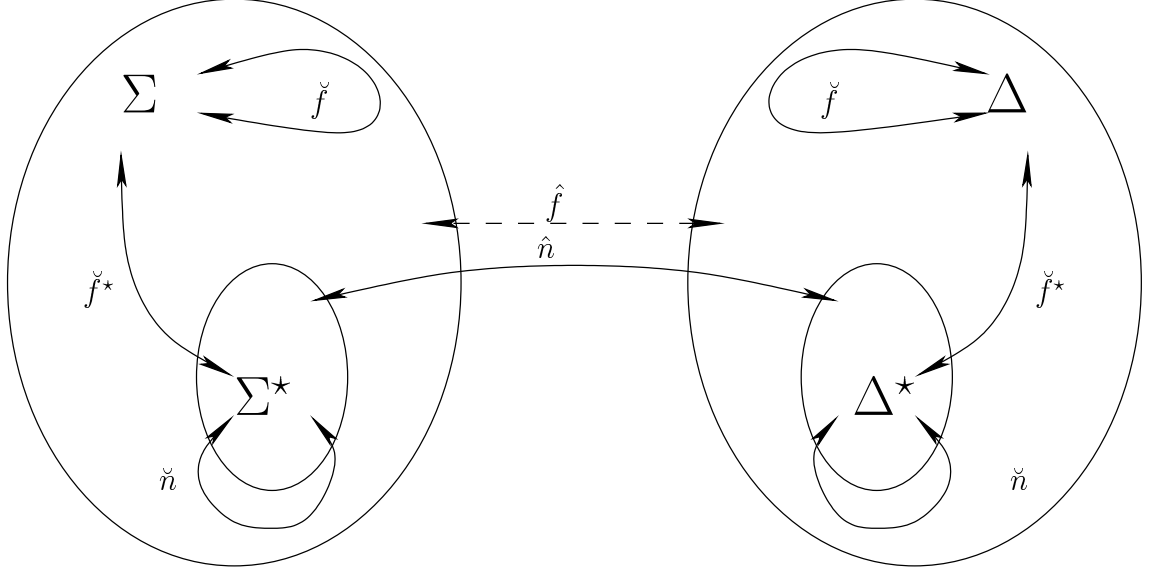
$$g(x) = \frac{1}{x + 1}$$

Figure 2: Graphical representation of equivalence and transformation functions in Σ and Δ

This is a composition of two functions: $\frac{x}{x+1}$ and $1 - x$. Note that the first one is a $\breve{f}^{\star}$-type function and the second one a $\hat{n}$-type function. Applying these functions progressively to the original dissimilarity, we get an equivalent dissimilarity δ' with $I_\delta \subseteq [0, 1)$. Finally, applying $\hat{n}$ to δ' we get a similarity s where $I_s \subseteq (0, 1]$.

Analogously to the Proposition 4.6, it has to be proved that a transformation function $\hat{n}$ gets a similarity from a dissimilarity or viceversa. Nevertheless, we introduce a more general result. Given a fixed function $\hat{n}$, the application over all the similarities in $\Sigma^{\star}(X)$, returns the set of all the dissimilarities in $\Delta^{\star}(X)$. Previously to show this result, let us to introduce more notation. Define the next sets of functions, for all $\hat{n} \in \hat{\mathcal{N}}$:

$$
\begin{aligned}
\Sigma_{\hat{n}}^{\star}(X) &\equiv \{\hat{n} \circ s | s \in \Sigma^{\star}(X)\} \\
\Delta_{\hat{n}}^{\star}(X) &\equiv \{\hat{n} \circ \delta | \delta \in \Delta^{\star}(X)\}
\end{aligned}
$$

It is also needed a little lemma.

Lemma 4.15.
$$
\begin{array}{ll}
\forall \delta \in \Delta^{\star}(X), & \\
\forall \hat{n} \in \hat{\mathcal{N}}, & \breve{n} \circ \delta \in \Sigma^{\star}(X).
\end{array}
$$

Analogously,
$$
\begin{array}{ll}
\forall s \in \Sigma^{\star}(X), & \\
\forall \hat{n} \in \hat{\mathcal{N}}, & \breve{n} \circ s \in \Delta^{\star}(X).
\end{array}
$$

Proof.   Let s be a similarity and $\hat{n}$ a transformation function.

- $\hat{n} \circ s$ is reflexive because $\forall x \in X$,

$$
s(x, x) = 1 \iff (\breve{n} \circ s)(x, x) = 1
$$

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- $\hat{n} \circ s$ is trivially symmetric.

Thus, $\hat{n} \circ s$ is a dissimilarity, this is, belongs to $\Delta^{\star}(X)$.

The demonstration is analogous for a dissimilarity and a $\hat{n}$ function.

$\square$

Theorem 4.16. *Given X, $\forall \hat{n} \in \hat{\mathcal{N}}$,*

i $\Sigma^{\star}_{\hat{n}}(X) = \Delta^{\star}(X)$

ii $\Delta^{\star}_{\hat{n}}(X) = \Sigma^{\star}(X)$

Proof. Let $\hat{n}$ be any function in $\hat{\mathcal{N}}$.

- Let s be a similarity in $\Sigma^{\star}(X)$. Using lemma 4.15, $\hat{n} \circ s \in \Delta^{\star}(X) \implies \Sigma^{\star}_{\hat{n}} \subseteq \Delta^{\star}$.

- Let δ be any dissimilarity in $\Delta^{\star}$. It is trivially true that

$$\delta = (\hat{n} \circ \hat{n}^{-1}) \circ \delta$$

Grouping $\hat{n}^{-1}$ and δ,

$$\delta = \hat{n} \circ (\hat{n}^{-1} \circ \delta)$$

Since $\hat{n}^{-1} \in \hat{\mathcal{N}}$ and using lemma 4.15, it is verified that $(\hat{n}^{-1} \circ \delta) \in \Sigma^{\star}$. Therefore, there exists some $s = \hat{n}^{-1} \circ \delta$, such that $\delta = \hat{n} \circ s$. This implies that $\Delta^{\star} \subseteq \Sigma^{\star}_{\tau_{\Sigma^{\star}}}(X)$.

$\square$

These two corollaries are extracted from the theorem. The first one shows that, effectively, the choice of the $\hat{n}$ function is irrelevant to get all the dissimilarities in $\Delta^{\star}(X)$.

Corollary 4.17. $\forall \hat{n}_1, \hat{n}_2 \in \hat{\mathcal{N}}$,

$$\begin{aligned}
\Sigma^{\star}_{\hat{n}_1}(X) &= \Sigma^{\star}_{\hat{n}_2}(X) \quad and \\
\Delta^{\star}_{\hat{n}_1}(X) &= \Delta^{\star}_{\hat{n}_2}(X)
\end{aligned}$$

Proof. Let $\hat{n}_1$ be a function in $\hat{\mathcal{N}}$, using Theorem 4.16 we get that $\Sigma^{\star}_{\hat{n}_1} = \Delta^{\star}(X)$. This is, for all $\delta \in \Delta^{\star}$ exists $s \in X$ such that $\delta = \hat{n}_1 \circ s$. For any other $\hat{n}_2 \in \hat{\mathcal{N}}$, using Theorem 4.16 we get that $\Sigma^{\star}_{\hat{n}_2} = \Delta^{\star}(X)$. Trivially, $\Sigma^{\star}_{\hat{n}_1} = \Sigma^{\star}_{\hat{n}_2}$.

For $\Delta^{\star}_{\hat{n}_1}$ and $\Delta^{\star}_{\hat{n}_2}$ the demonstration is analogous.

$\square$

Moreover, there no exists any similarity (dissimilarity) that cannot be generated by a $\hat{n}$ function and a dissimilarity (similarity). Formally,

Corollary 4.18. $\forall s \in \Sigma^{\star}(X)$,

$$\exists \hat{n} \in \hat{\mathcal{N}} \wedge \exists \delta \in \Delta^{\star}(X) \text{ such that } s = \hat{n} \circ \delta$$

Analogously, $\forall \delta \in \Delta^{\star}(X)$,

$$\exists \hat{n} \in \hat{\mathcal{N}} \wedge \exists s \in \Sigma^{\star}(X) \text{ such that } \delta = \hat{n} \circ s$$
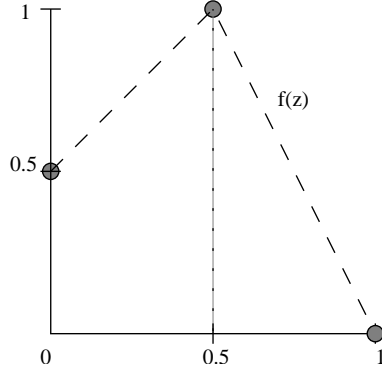
Figure 3: Given a similarity and a dissimilarity, not always exists a $\hat{n}$ function that relates them. For instance, in this case this function relates a similarity and a dissimilarity, but it does not belong to $\hat{\mathcal{N}}$.

Proof.   Suppose that $\exists \delta \in \Delta^\star(X)$ such that $\forall \hat{n} \in \hat{\mathcal{N}}$ such that $\delta \neq \hat{n} \circ s$. Then, $\delta \in DS(X)$ but $\delta \notin \Sigma_{\hat{n}}^\star(X)$. This is impossible, using Theorem 4.16. Thus, the initial supposition is false.

The other case with a similarity is analogous.

$\square$

It has to be notice that, given a similarity s and a dissimilarity δ, it is *not true* that always exists a $\hat{n}$ function that relates them. To show that, consider an example:

Example 5. Let the definition set be $X = \mathbb{N}$ and consider the following similarity in $\Sigma^\star(X)$

$$s(x_1, x_2) = \left\{ \begin{array}{cc} 1 & \text{if } x = y \\ 1/2 & \text{if } x = 2y \text{ ó } y = 2x \\ 0 & \text{otherwise} \end{array} \right.$$

Consider now a dissimilarity in $\Delta^\star(X)$:

$$\delta(x_1, x_2) = \left\{ \begin{array}{cc} 0 & \text{if } x = y \\ 1 & \text{if } x = 2y \text{ ó } y = 2x \\ 1/2 & \text{otherwise} \end{array} \right.$$

Both of them are strongly reflexive, symmetric and both are closed. Nevertheless, the unique function f that fulfills $\delta = f \circ s$ is

$$f(z) = \left\{ \begin{array}{cc} 1/2 & \text{si } z = 0 \\ 1 & \text{si } z = 1/2 \\ 0 & \text{si } z = 1 \end{array} \right.$$

Of course, f is not in $\hat{\mathcal{N}}$. The function f is shown on Figure 3.

4.3 Relations between equivalence and transformation functions

Similarity and dissimilarity functions have an important semantic value, as well as equivalence and transformation functions. Therefore, the choice of the $\hat{n}$ or $\breve{n}$ function is not irrelevant. In fact, an equivalence function can change the behavior of a similarity, and consequently, its semantic meaning. For example, a transformation increasing low values of similarity and decrease high values of similarity, semantically means to potentiate small differences between objects. Moreover, transformations modify similarity and dissimilarity properties. The two transformation functions (i.e $\breve{n}$ and $\hat{n}$) introduced so far are interrelated in this way. In addition, their mutual analogy allows to interrelate them functionally.

Thus, the relations between transformation functions are shown in the following proposition:

Proposition 4.19.

$$
\begin{aligned}
\forall \hat{n}_1, \hat{n}_2 \in \hat{\mathcal{N}}, & \qquad \hat{n}_1 \circ \hat{n}_2 \in \breve{\mathcal{N}} \\
\forall \hat{n}_1 \in \hat{\mathcal{N}}, \breve{n}_1 \in \breve{\mathcal{N}}, & \quad \hat{n}_1 \circ \breve{n}_1 \in \hat{\mathcal{N}} \\
\forall \breve{n}_1 \in \breve{\mathcal{N}}, \hat{n}_1 \in \hat{\mathcal{N}}, & \quad \breve{n}_1 \circ \hat{n}_1 \in \hat{\mathcal{N}} \\
\forall \breve{n}_1, \breve{n}_2 \in \breve{\mathcal{N}}, & \qquad \breve{n}_1 \circ \breve{n}_2 \in \breve{\mathcal{N}}
\end{aligned}
$$

Proof. Using limit conditions of functions in $\hat{\mathcal{N}}$:

$$
\begin{aligned}
(\hat{n}_1 \circ \hat{n}_2)(0) &= \hat{n}_1(1) = 0 \\
(\hat{n}_1 \circ \hat{n}_2)(1) &= \hat{n}_1(0) = 1
\end{aligned}
$$

and viceversa. Using that the composition of decreasing monotonic functions is an increasing monotonic function, it is verified that, $\hat{n}_1 \circ \hat{n}_2 \in \breve{\mathcal{N}}$ and $\hat{n}_2 \circ \hat{n}_1 \in \breve{\mathcal{N}}$. For other hand, using that the composition of increasing monotonic functions is an increasing monotonic function, it is verified that $\breve{n}_1 \circ \breve{n}_2 \in \breve{\mathcal{N}}$ and $\breve{n}_2 \circ \breve{n}_1 \in \breve{\mathcal{N}}$. Finally, the composition of an increasing monotonic function and a decreasing monotonic function is always an decreasing monotonic function.

$\square$

Moreover, due to $\hat{n}$ and $\breve{n}$ functions likeliness, most of the properties are common. Because of that, we introduce the following theorem, that is analogous to Theorem 4.16.

Theorem 4.20. *Given the definition set X, $\forall \breve{n} \in \breve{\mathcal{N}}$,*

i $\Sigma_{\breve{n}}^{\star}(X) = \Sigma^{\star}(X)$

ii $\Delta_{\breve{n}}^{\star}(X) = \Delta^{\star}(X)$

Proof. This demonstration is analogous to the demonstration of Theorem 4.16.

$\square$

Following the analogy, the corollaries 4.17 and 4.18 have their respective equivalent ones.

Corollary 4.21. $\forall \breve{n}_1, \breve{n}_2 \in \breve{\mathcal{N}}$,

$$
\begin{aligned}
\Sigma_{\breve{n}_1}^{\star}(X) &= \Sigma_{\breve{n}_2}^{\star}(X) \quad and \\
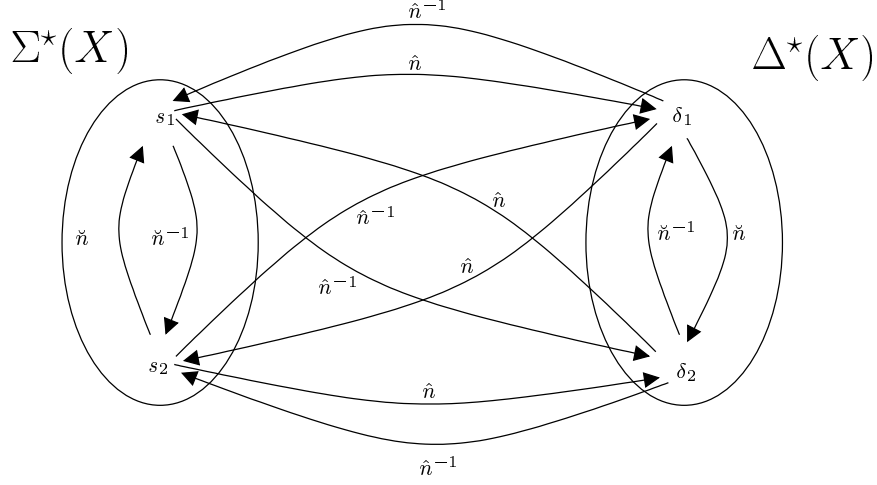\Delta_{\breve{n}_1}^{\star}(X) &= \Delta_{\breve{n}_2}^{\star}(X)
\end{aligned}
$$

Figure 4: This figure shows all the possible transformations for a given similarity. There exist four dual triples $< s_1, \delta_1, \hat{n} >$, $< s_1, \delta_2, \hat{n}^{-1} >$, $< s_2, \delta_1, \hat{n}^{-1} >$ and $< s_2, \delta_2, \hat{n} >$. The transformation $\breve{n}$ can be defined as a composition: $\breve{n} = \hat{n} \circ \hat{n} = \hat{n}^{-1} \circ \hat{n}^{-1}$.

Proof.　　　Analogous to the demonstration of corollary 4.17.

$\square$

Corollary 4.22. $\forall s \in \Sigma^\star(X)$,

$$\exists \breve{n} \in \breve{\mathcal{N}} \wedge \exists s' \in \Sigma^\star(X) \; such \; that \; s = \breve{n} \circ s'$$

Analogously, $\forall \delta \in \Delta^\star(X)$,

$$\exists \breve{n} \in \breve{\mathcal{N}} \wedge \exists \delta' \in \Delta^\star(X) \; such \; that \; \delta = \breve{n} \circ \delta'$$

Proof.　　　Analogous to the demonstration of corollary 4.18.

$\square$

4.4　Duality

Until now, the concepts of similarity and dissimilarity have been introduced as well as $\breve{n}$ and $\hat{n}$ transformations. Their mutual relations have been also showed up. This section introduces the *duality* between similarity and dissimilarity.

Definition 4.23 (duality). Given $s \in \Sigma^\star(X)$, $\delta \in \Delta^\star(X)$ and $\hat{n} \in \hat{\mathcal{N}}$. s, δ are *dual* by means of $\hat{n}$ if

$$\delta = \hat{n} \circ s \tag{4.3}$$

or the equivalent form

$$s = \hat{n}^{-1} \circ \delta \tag{4.4}$$

This duality is expressed by the *triple* $< s, \delta, \hat{n} >$.

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Note that, for a given $s \in \Sigma^\star(X)$ and a $\hat{n} \in \hat{\mathcal{N}}$ there is a relation between two similarities and two dissimilarities (see Figure 4). It depends on the application of $\hat{n}$ or $\hat{n}^{-1}$ in each case. However, if $\hat{n}$ is involutive, the number of triples is reduced to one.

Proposition 4.24. *Given the following triple $< s, \delta, \hat{n} >$. For all $x_1, x_2, x_3, x_4 \in X$:*

1. $s(x_1, x_2) = s(x_3, x_4) \Longleftrightarrow \delta(x_1, x_2) = \delta(x_3, x_4)$

2. $s(x_1, x_2) < s(x_3, x_4) \Longleftrightarrow \delta(x_1, x_2) > \delta(x_3, x_4)$

3. $s(x_1, x_2) > s(x_3, x_4) \Longleftrightarrow \delta(x_1, x_2) < \delta(x_3, x_4)$

Proof. Using the definition of duality, $\delta = \breve{n} \circ s$. Since $\breve{n}$ is a bijective function, (1) is verified. Since it is strictly increasing, (2) and (3) are verified.

$\square$

This duality affect also similarity and dissimilarity properties. Thus, a closed similarity is dual respect to a closed dissimilarity, independently of the chosen $\hat{n}$ function.

Proposition 4.25. *Given a dual triple $< s, \delta, \hat{n} >$, it is true that*

- *δ is strongly reflexive iff s is.*

- *δ is closed iff s is.*

- *δ has (unitary) complement function iff s does.*

- *δ is $\tau_{\Delta^\star}$-transitive iff s is $\tau_{\Sigma^\star}$-transitive, where $\tau_{\Delta^\star}$ can be defined by means of $\tau_{\Sigma^\star}$*

$$\tau_{\Delta^\star}(x, y) = \hat{n}(\tau_{\Sigma^\star}(\hat{n}^{-1}(x), \hat{n}^{-1}(y))) \tag{4.5}$$

Proof. This demonstrates duality in each property for a given $s \in \Sigma^\star(X), \hat{n} \in \hat{\mathcal{N}}(X)$ and $\delta = \hat{n} \circ s$.

- Strong Reflexivity. For all $x, y \in X$ such that $x \overset{x}{\neq} y$, results

$$s(x, y) \neq s_{max}$$

Applying $\hat{n}$ to the previous inequality

$$\delta(x, y) \neq \delta_{min}$$

- Closure. For all $x, y \in X$

$$s(x, y) \geq s_{min}$$

Since $\hat{n}$ is strictly monotonic and decreasing

$$\forall x, y \in X, s(x, y) > s_{min} \Longleftrightarrow (\hat{n} \circ s)(x, y) < \hat{n}(s_{min})$$

Thus, if $\exists x, y \in X$ such that $s(x, y) = s_{min}$, then $(\hat{n} \circ s)(x, y) = \hat{n}(s_{min})$ (i.e. δ is closed)

- (Unitary) complement.
  For all $x, x' \in X$, such that $x' \in C(x)$,
  $$s(x, x') = 0$$
  Applying $\hat{n}$,
  $$(\hat{n} \circ s)(x, x') = \hat{n}(0)$$
  This is,
  $$\delta(x, x') = 1$$
  Therefore, complement property is kept.
- Transitivity. See demonstration in [16, Teorema 3.20, página 84]. It is valid to prove that transitivity property is kept.

$\square$

Proposition 4.26. *Given a dual triple* $< s, \delta, \hat{n} >$, *and given two functions* $s' \in \Sigma^\star(X)$ *and* $\delta' \in \Delta^\star(X)$. s *is equivalent to* s' *and* δ *is equivalent to* δ' *iff exists a function* $\hat{n}' \in \hat{\mathcal{N}}$ *such that* $s' = \hat{n}' \circ \delta'$.

Proof. If s and s' are equivalent, there exists some $\breve{n} \in \breve{\mathcal{N}}$ such that $s' = \breve{n} \circ s$.
Since $s = \hat{n} \circ \delta$ the following is true:
$$s' = \breve{n} \circ \hat{n} \circ \delta$$

Similarly, if δ and δ' are equivalent, for a $\breve{n}' \in \breve{\mathcal{N}}$ this stats
$$\delta = \breve{n}' \circ \delta'$$

Substituting,
$$s' = \breve{n} \circ \hat{n} \circ \breve{n}' \circ \delta'$$

Using Proposition 4.19, it is verified that $s' = \hat{n}' \circ \delta'$ where $\hat{n}' = \breve{n} \circ \hat{n} \circ \breve{n}'$.

For other hand, if exists $\breve{n}' \in \breve{\mathcal{N}}$ such that $s' = \breve{n}' \circ \delta'$, using Proposition 4.19, $\breve{n}'$ is a composition of various functions.

$$\breve{n}' = \breve{n} \circ \hat{n} \circ \breve{n}'$$

Substituting,
$$s' = \breve{n} \circ \hat{n} \circ \breve{n}' \circ \delta'$$

Denote δ to $\breve{n}' \circ \delta'$
$$s' = \breve{n} \circ \hat{n} \circ \delta$$

Knowing that $s = \hat{n} \circ \delta$
$$s' = \breve{n} \circ s$$

Using Definition 4.7, s and s' are equivalent.

$\square$

Consider now the relation between the set of equivalent similarities (or the equivalent dissimilarities) to a given one, and the set of $\breve{n}$ functions. Define this set for a given $s \in \Sigma^\star(X)$.

$$\breve{\Sigma}^\star(s) \equiv \{\breve{n} \circ s | \breve{n} \in \breve{\mathcal{N}}\}$$

And define also this set for a given $\delta \in \Delta^\star(X)$.

$$\breve{\Delta}^\star(\delta) \equiv \{\breve{n} \circ \delta | \breve{n} \in \breve{\mathcal{N}}\}$$

Similarly, for a given $s \in \Sigma^\star(X)$, define the set of all the transformations:

$$\hat{\Delta}^\star(s) \equiv \{\hat{n} \circ s | \hat{n} \in \hat{\mathcal{N}}\}$$

Define the same for a given dissimilarity $\delta \in \Delta^\star(X)$,

$$\hat{\Sigma}^\star(\delta) \equiv \{\hat{n} \circ \delta | \hat{n} \in \hat{\mathcal{N}}\}$$

All this four sets are mutually related. Next theorem shows their relationship.

Theorem 4.27. $\forall s \in \Sigma^\star(X), \delta \in \Delta^\star(X)$,

1. *s and δ are dual by means of a $\hat{n}$ function iff*

$$\breve{\Sigma}^\star(s) = \hat{\Sigma}^\star(\delta)$$

   or equivalently,

$$\breve{\Delta}^\star(\delta) = \hat{\Delta}^\star(s)$$

2. *s and δ are not dual by means of a function s iff*

$$\breve{\Sigma}^\star(s) \cap \hat{\Sigma}^\star(\delta) = \emptyset$$

   or equivalently,

$$\breve{\Delta}^\star(\delta) \cap \hat{\Delta}^\star(s) = \emptyset$$

Proof.   Due to the analogy between the sets $\breve{\Delta}^\star(\delta)$, $\hat{\Delta}^\star(s)$ and $\breve{\Sigma}^\star(\delta), \hat{\Sigma}^\star(s)$, this demonstration uses only the first ones.

1. Let $< s, \delta, \hat{n} >$ be a dual triple. For all $\delta' \in \breve{\Delta}^\star(\delta)$, using Proposition 4.26 and Proposition 4.19, $\exists \hat{n}' \in \hat{\mathcal{N}}$ such that $\delta' = \hat{n}' \circ s$. Therefore, $\delta' \in \hat{\Delta}^\star(s)$ and $\breve{\Delta}^\star(\delta) \subseteq \hat{\Delta}^\star(s)$.
   Inversely, for all $\delta' \in \hat{\Delta}^\star(s)$, using that $\delta = \hat{n} \circ s$ and $\delta' = \hat{n}' \circ s$, we get that

$$\delta' = \hat{n}' \circ \hat{n}^{-1} \circ \delta$$

   Using Proposition 4.19, $\delta' \in \breve{\Delta}^\star(\delta)$. Then, $\hat{\Delta}^\star(\delta) \subseteq \breve{\Delta}^\star(s)$.

2. Let s and δ be a similarity and a dissimilarity such that $s \notin \hat{\Sigma}^\star(\delta)$. Using Proposition 4.19, for all $s' \in \breve{\Sigma}^\star(s)$,

$$s \notin \hat{\Sigma}^\star(\delta) \implies s' \notin \hat{\Sigma}^\star(\delta)$$

   This is, $\breve{\Sigma}^\star(s) \cap \hat{\Sigma}^\star(\delta) = \emptyset$.

$\square$

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4.5 Transitivity and transformations

As we pointed out before, equivalent similarities (or dissimilarities) keep the same properties, even transitivity. However, the transitivity operator changes from one similarity to another. In fact, there exist a hierarchy in transitivity operators, from the most restrictive one to the laxest one.

Definition 4.28. Let $T_{\Sigma^\star}$ be the set of all the similarity transitivity operators. The relation *to be as strong as* is

Given $\tau_{\Sigma^\star}, \tau'_{\Sigma^\star} \in T_{\Sigma^\star}$, $\tau_{\Sigma^\star}$ is as strong as $\tau'_{\Sigma^\star}$ if $\forall\, a, b \in [0,1], \tau_{\Sigma^\star}(a,b) \geq \tau'_{\Sigma^\star}(a,b)$ but $\tau_{\Sigma^\star} \neq \tau'_{\Sigma^\star}$ and it is denoted as $\tau_{\Sigma^\star} \sqsupseteq \tau'_{\Sigma^\star}$.

This relation is a preorder in $T_{\Sigma^\star}$. Similarly, $\tau_{\Sigma^\star}$ *is stronger than* $\tau'_{\Sigma^\star}$ (and it is denoted as $\tau_{\Sigma^\star} \sqsupset \tau'_{\Sigma^\star}$) if $\forall\, a, b \in [0,1] \tau_{\Sigma^\star}(a,b) > \tau'_{\Sigma^\star}(a,b)$.

Analogously, this relation have its dual for dissimilarities:

Definition 4.29. Let $T_{\Delta^\star}$ be the set of all the dissimilarity transitivity operators. The relation *to be as strong as* is

Given $\tau_{\Delta^\star}, \tau'_{\Delta^\star} \in T_{\Delta^\star}$, $\tau_{\Delta^\star}$ is as strong as $\tau'_{\Delta^\star}$ if $\forall\, a, b \in [0,1], \tau_{\Delta^\star}(a,b) \leq \tau'_{\Delta^\star}(a,b)$ but $\tau_{\Delta^\star} \neq \tau'_{\Delta^\star}$ and it is denoted as $\tau_{\Delta^\star} \sqsupseteq \tau'_{\Delta^\star}$.

This relation is a preorder in $T_{\Delta^\star}$. Similarly, $\tau_{\Delta^\star}$ *is stronger than* $\tau'_{\Delta^\star}$ (and it is denoted as $\tau_{\Delta^\star} \sqsupset \tau'_{\Delta^\star}$) if $\forall\, a, b \in [0,1], \tau_{\Delta^\star}(a,b) < \tau'_{\Delta^\star}(a,b)$.

This hierarchy is related to the fulfillment of the transitivity axiom (s7 and d7) in the following way:

Proposition 4.30. *Let s be a $\tau_{\Sigma^\star}$-transitive similarity. For every similarity transitivity operator $\tau'_{\Sigma^\star}$ such that $\tau_{\Sigma^\star} \sqsupseteq \tau'_{\Sigma^\star}$, the similarity s is also $\tau'_{\Sigma^\star}$-transitive.*

Analogously, let δ be a $\tau_{\Delta^\star}$-transitive dissimilarity. For every dissimilarity transitivity operator $\tau'_{\Delta^\star}$ such that $\tau_{\Delta^\star} \sqsupseteq \tau'_{\Delta^\star}$, the dissimilarity δ is also $\tau'_{\Delta^\star}$-transitive.

Proof. Let s be a $\tau_{\Sigma^\star}$-transitive similarity. Then, for all $x, y, z \in X$,

$$s(x,y) \geq \tau_{\Sigma^\star}(s(x,z), s(z,y))$$

Let $\tau'_{\Sigma^\star}$ be a transitivity operator weaker than $\tau_{\Sigma^\star}$. This means that, for all $a, b \in [0,1]$

$$\tau_{\Sigma^\star}(a,b) \geq \tau'_{\Sigma^\star}(a,b)$$

Therefore, for all $x, y, z \in X$,

$$s(x,y) \geq \tau_{\Sigma^\star}(s(x,z), s(z,y)) \geq \tau'_{\Sigma^\star}(s(x,z), s(z,y))$$

Demonstration for $\tau_{\Delta^\star}$ operators is analogous. $\qquad \square$

Next propositions show that transitivity is altered when applying an equivalence function. However, the transitivity operator of the resulting similarity or dissimilarity can be either stronger or weaker that the original one.

Proposition 4.31. *Given $\breve{n} \in \breve{\mathcal{N}}$, $s \in \Sigma^\star(X)$, $\delta \in \Delta^\star(X)$, we state*

a)
  - *Given that s is $\tau_{\Sigma^\star}$-transitive, a sufficient condition that does $\breve{n} \circ s$ at least $\tau_{\Sigma^\star}$-transitive is that, for all $a, b \in I_s$*

$$\breve{n}(\tau_{\Sigma^\star}(a,b)) \geq \tau_{\Sigma^\star}(\breve{n}(a), \breve{n}(b)) \tag{4.6}$$

  - *Given that δ is $\tau_{\Delta^\star}$-transitive, a sufficient condition that does $\breve{n} \circ \delta$ is at least $\tau_{\Delta^\star}$-transitive is that, for all $a, b \in I_\delta$*

$$\breve{n}(\tau_{\Delta^\star}(a,b)) \leq \tau_{\Delta^\star}(\breve{n}(a), \breve{n}(b)) \tag{4.7}$$

b)
  - *If s is $\tau_{\Sigma^\star}$-transitive such that, for all $x, y, z \in X$*

$$s(x,y) = \tau_{\Sigma^\star}(s(x,z), s(z,y)) \tag{4.8}$$

  then Expression 4.6 becomes a sufficient and necessary condition that does $\breve{n} \circ s$ at least $\tau_{\Sigma^\star}$-transitive.
  - *If δ is $\tau_{\Delta^\star}$-transitive such that, for all $x, y, z \in X$*

$$\delta(x,y) = \tau_{\Delta^\star}(\delta(x,z), \delta(z,y)) \tag{4.9}$$

  then Expression 4.7 becomes a sufficient and necessary condition that does $\breve{n} \circ \delta$ at least $\tau_{\Delta^\star}$-transitive.

Proof. In order to prove *(a)*, consider that s is $\tau_{\Sigma^\star}$-transitive. This is, for all $x, y, z \in X$

$$s(x,y) \geq \tau_{\Sigma^\star}(s(x,z), s(z,y))$$

Applying $\breve{n}$ to both sides, we get

$$\breve{n} \circ s(x,y) \geq \breve{n} \circ \tau_{\Sigma^\star}(s(x,z), s(z,y))$$

If Equation 4.6 is fulfilled, then

$$\breve{n} \circ s(x,y) \geq \breve{n} \circ \tau_{\Sigma^\star}(s(x,z), s(z,y)) \geq \tau_{\Sigma^\star}(\breve{n} \circ s(x,z), \breve{n} \circ s(z,y))$$

This shows that $\breve{n} \circ s$ fulfills $\tau_{\Sigma^\star}$-transitivity.

To prove *(b)* consider the fulfillment of Equation 4.8. Then, applying $\breve{n}$ to both sides and using Equation 4.6, we get

$$\breve{n} \circ s(x,y) = \breve{n} \circ \tau_{\Sigma^\star}(s(x,z), s(z,y)) \geq \tau_{\Sigma^\star}(\breve{n} \circ s(x,z), \breve{n} \circ s(z,y))$$

Again, this is analogous for dissimilarities.

$\square$

Corollary 4.32. *Let $\hat{n}_1, \hat{n}_2$ be functions of $\hat{\mathcal{N}}$.*

  - *Given $s \in \Sigma^\star(X)$, $\tau_{\Sigma^\star}$-transitive. $\hat{n}_1 \circ \hat{n}_2 \circ s$ is at least $\tau_{\Sigma^\star}$-transitive if, for all $a, b \in I_s$.*

$$\hat{n}_1 \circ \hat{n}_2 \circ \tau_{\Sigma^\star}(a,b) \geq \tau_{\Sigma^\star}(\hat{n}_1 \circ \hat{n}_2(a), \hat{n}_1 \circ \hat{n}_2(b))$$

- *Given $\delta \in \Delta^\star(X)$, $\tau_{\Delta^\star}$-transitive. $\hat{n}_1 \circ \hat{n}_2 \circ \delta$ is at least $\tau_{\Delta^\star}$-transitive if, for all $a, b \in I_\delta$.*

$$\hat{n}_1 \circ \hat{n}_2 \circ \tau_{\Delta^\star}(a,b) \leq \tau_{\Delta^\star}(\hat{n}_1 \circ \hat{n}_2(a), \hat{n}_1 \circ \hat{n}_2(b))$$

Proof. Simply use that $\forall \hat{n}_1, \hat{n}_2 \in \hat{\mathcal{N}}$, $\hat{n}_1 \circ \hat{n}_2 \in \check{\mathcal{N}}$ and Proposition 4.31. $\qquad \square$

Corollary 4.33. *Given $\check{n} \in \check{\mathcal{N}}$ and $s \in \Sigma^\star(X)$, $\tau_{\Sigma^\star}$-transitive. For all $a, b \in I_s$,*

$$\check{n} \circ \tau_{\Sigma^\star}(a,b) \geq \tau_{\Sigma^\star}(\check{n}(a), \check{n}(b))$$

only if for all $a, b \in I_s$,

$$\check{n}^{-1} \circ \tau_{\Sigma^\star}(a,b) \leq \tau_{\Sigma^\star}(\check{n}^{-1}(a), \check{n}^{-1}(b))$$

Similarly, this is also true for any $\delta \in \Delta^\star(X)$. For all $a, b \in I_\delta$,

$$\check{n} \circ \tau_{\Delta^\star}(a,b) \leq \tau_{\Delta^\star}(\check{n}(a), \check{n}(b))$$

only if for all $a, b \in I_\delta$,

$$\check{n}^{-1} \circ \tau_{\Delta^\star}(a,b) \geq \tau_{\Delta^\star}(\check{n}^{-1}(a), \check{n}^{-1}(b))$$

Proof. Using Proposition 4.31, for all $a, b \in I_s$,

$$\check{n} \circ \tau_{\Sigma^\star}(a,b) \geq \tau_{\Sigma^\star}(\check{n}(a), \check{n}(b))$$

Take $a' = \check{n}(a)$ and $b' = \check{n}(b)$. Using that, for all $a \in I_s$, $\check{n}(a) \in I_s$, we get that for all $a', b' \in I_s$:

$$\check{n} \circ \tau_{\Sigma^\star}(\check{n}^{-1}(a'), \check{n}^{-1}(b')) \geq \tau_{\Sigma^\star}(\check{n}(a), \check{n}(b))$$

Finally, applying $\check{n}^{-1}$ we prove the corollary. $\qquad \square$

Proposition 4.34. *Let $s \in \Sigma^\star(X)$ be a $\tau_{\Sigma^\star}$-transitive similarity, and $\delta \in \Delta^\star(X)$ a $\tau_{\Delta^\star}$-transitive dissimilarity. Given $\check{n}_1, \check{n}_2 \in \check{\mathcal{N}}$, consider the following similarities and dissimilarities:*

- *$s_1 = \check{n}_1 \circ s$, τ_{Σ_1}-transitive.*
- *$s_2 = \check{n}_2 \circ s$, τ_{Σ_2}-transitive.*
- *$\delta_1 = \check{n}_1 \circ \delta$, τ_{Δ_1}-transitive.*
- *$\delta_2 = \check{n}_2 \circ \delta$, τ_{Δ_2}-transitive.*

. For all $a, b \in I_s$:

- *Only if $\check{n}_1^{-1} \circ \check{n}_2(\tau_{\Sigma^\star}(a,b)) > \tau_{\Sigma^\star}(\check{n}_1^{-1} \circ \check{n}_2(a), \check{n}_1^{-1} \circ \check{n}_2(b))$ then $\tau_{\Sigma_2} \sqsupset \tau_{\Sigma_1}$.*
- *Only if $\check{n}_1^{-1} \circ \check{n}_2(\tau_{\Sigma^\star}(a,b)) < \tau_{\Sigma^\star}(\check{n}_1^{-1} \circ \check{n}_2(a), \check{n}_1^{-1} \circ \check{n}_2(b))$ then $\tau_{\Sigma_2} \sqsubset \tau_{\Sigma_1}$.*

- *Only if $\breve{n}_1^{-1} \circ \breve{n}_2(\tau_{\Sigma^\star}(a,b)) = \tau_{\Sigma^\star}(\breve{n}_1^{-1} \circ \breve{n}_2(a), \breve{n}_1^{-1} \circ \breve{n}_2(b))$ then $\tau_{\Sigma_2} = \tau_{\Sigma_1}$.*

For dissimilarities, for all $a, b \in I_\delta$:

- *Only if $\breve{n}_1^{-1} \circ \breve{n}_2(\tau_{\Delta^\star}(a,b)) > \tau_{\Delta^\star}(\breve{n}_1^{-1} \circ \breve{n}_2(a), \breve{n}_1^{-1} \circ \breve{n}_2(b))$ then $\tau_{\Delta_2} \sqsubset \tau_{\Delta_1}$.*

- *Only if $\breve{n}_1^{-1} \circ \breve{n}_2(\tau_{\Delta^\star}(a,b)) < \tau_{\Delta^\star}(\breve{n}_1^{-1} \circ \breve{n}_2(a), \breve{n}_1^{-1} \circ \breve{n}_2(b))$ then $\tau_{\Delta_2} \sqsupset \tau_{\Delta_1}$.*

- *Only if $\breve{n}_1^{-1} \circ \breve{n}_2(\tau_{\Delta^\star}(a,b)) = \tau_{\Delta^\star}(\breve{n}_1^{-1} \circ \breve{n}_2(a), \breve{n}_1^{-1} \circ \breve{n}_2(b))$ then $\tau_{\Delta_2} = \tau_{\Delta_1}$.*

Proof. Remember that $\tau_{\Sigma_2}(a,b) = \breve{n}_2 \circ \tau_{\Sigma^\star}(\breve{n}_2^{-1}(a), \breve{n}_2^{-1}(b))$.

Consider the following:
$$\tau_{\Sigma_2} > \tau_{\Sigma_1}$$

Substituting at both sides,

$$\breve{n}_2 \circ \tau_{\Sigma^\star}(\breve{n}_2^{-1}(a), \breve{n}_2^{-1}(b)) > \breve{n}_1 \circ \tau_{\Sigma^\star}(\breve{n}_1^{-1}(a), \breve{n}_1^{-1}(b))$$

Applying $\breve{n}_1^{-1}$,

$$\breve{n}_1^{-1} \circ \breve{n}_2 \circ \tau_{\Sigma^\star}(\breve{n}_2^{-1}(a), \breve{n}_2^{-1}(b)) > \tau_{\Sigma^\star}(\breve{n}_1^{-1}(a), \breve{n}_1^{-1}(b))$$

Taking $a' = \breve{n}_2^{-1}(a)$ and $b' = \breve{n}_2^{-1}(b)$, and using that $\breve{n}_2^{-1}$ is continuous and $\mathrm{Dom}(\breve{n}_2^{-1}) = \mathrm{Im}(\breve{n}_2^{-1})$, we know that for all $a, b \in I_{s_2}$:

$$\breve{n}_1^{-1} \circ \breve{n}_2 \circ \tau_{\Sigma^\star}(a', b') > \tau_{\Sigma^\star}(\breve{n}^{-1} \circ \breve{n}_2(a'), \breve{n}^{-1} \circ \breve{n}_2(b'))$$

The rest of the cases are analogous.

$\square$

Proposition 4.35. *Let $s \in \Sigma^\star(X)$ be a $\tau_{\Sigma^\star}$-transitive similarity, and $\delta \in \Delta^\star(X)$ a $\tau_{\Delta^\star}$-transitive dissimilarity. Given $\hat{n}_1, \hat{n}_2 \in \hat{\mathcal{N}}$, consider the following similarities and dissimilarities:*

- $\delta_1 = \hat{n}_1 \circ s$, τ_{Δ_1}*-transitive.*

- $\delta_2 = \hat{n}_2 \circ s$, τ_{Δ_2}*-transitive.*

- $s_1 = \hat{n}_1 \circ \delta$, τ_{Σ_1}*-transitive.*

- $s_2 = \hat{n}_2 \circ \delta$, τ_{Σ_2}*-transitive.*

. For all $a, b \in I_s$:

- *Only if $\hat{n}_1^{-1} \circ \hat{n}_2(\tau_{\Sigma^\star}(a,b)) < \tau_{\Sigma^\star}(\hat{n}_1^{-1} \circ \hat{n}_2(a), \hat{n}_1^{-1} \circ \hat{n}_2(b))$ then $\tau_{\Delta_2} \sqsupset \tau_{\Delta_1}$.*

- *Only if $\hat{n}_1^{-1} \circ \hat{n}_2(\tau_{\Sigma^\star}(a,b)) > \tau_{\Sigma^\star}(\hat{n}_1^{-1} \circ \hat{n}_2(a), \hat{n}_1^{-1} \circ \hat{n}_2(b))$ then $\tau_{\Delta_2} \sqsubset \tau_{\Delta_1}$.*

- *Only if $\hat{n}_1^{-1} \circ \hat{n}_2(\tau_{\Sigma^\star}(a,b)) = \tau_{\Sigma^\star}(\hat{n}_1^{-1} \circ \hat{n}_2(a), \hat{n}_1^{-1} \circ \hat{n}_2(b))$ then $\tau_{\Delta_2} = \tau_{\Delta_1}$.*

For dissimilarities, for all $a, b \in I_\delta$:

- *Only if $\hat{n}_1^{-1} \circ \hat{n}_2(\tau_{\Delta^\star}(a,b)) < \tau_{\Delta^\star}(\hat{n}_1^{-1} \circ \hat{n}_2(a), \hat{n}_1^{-1} \circ \hat{n}_2(b))$ then $\tau_{\Sigma_2} \sqsubseteq \tau_{\Sigma_1}$.*

- *Only if $\hat{n}_1^{-1} \circ \hat{n}_2(\tau_{\Delta^\star}(a,b)) > \tau_{\Delta^\star}(\hat{n}_1^{-1} \circ \hat{n}_2(a), \hat{n}_1^{-1} \circ \hat{n}_2(b))$ then $\tau_{\Sigma_2} \sqsupseteq \tau_{\Sigma_1}$.*

- *Only if $\hat{n}_1^{-1} \circ \hat{n}_2(\tau_{\Delta^\star}(a,b)) = \tau_{\Delta^\star}(\hat{n}_1^{-1} \circ \hat{n}_2(a), \hat{n}_1^{-1} \circ \hat{n}_2(b))$ then $\tau_{\Sigma_2} = \tau_{\Sigma_1}$.*

Proof. Remember that $\tau_{\Delta_2}(a,b) = \hat{n}_2 \circ \tau_{\Sigma^\star}(\hat{n}_2^{-1}(a), \hat{n}_2^{-1}(b))$.

Consider the following:
$$\tau_{\Delta_2} > \tau_{\Delta_1}$$

Substituting at both sides,

$$\hat{n}_2 \circ \tau_{\Sigma^\star}(\hat{n}_2^{-1}(a), \hat{n}_2^{-1}(b)) > \hat{n}_1 \circ \tau_{\Sigma^\star}(\hat{n}_1^{-1}(a), \hat{n}_1^{-1}(b))$$

Applying $\hat{n}_1^{-1}$,

$$\hat{n}_1^{-1} \circ \hat{n}_2 \circ \tau_{\Sigma^\star}(\hat{n}_2^{-1}(a), \hat{n}_2^{-1}(b)) < \tau_{\Sigma^\star}(\hat{n}_1^{-1}(a), \hat{n}_1^{-1}(b))$$

Taking $a' = \hat{n}_2^{-1}(a)$ and $b' = \hat{n}_2^{-1}(b)$, and using that $\hat{n}_2^{-1}$ is continuous and $\mathrm{Dom}(\hat{n}_2^{-1}) = \mathrm{Im}(\hat{n}_2^{-1})$, we know that for all $a, b \in I_{\delta_2}$:

$$\hat{n}_1^{-1} \circ \hat{n}_2 \circ \tau_{\Sigma^\star}(a', b') < \tau_{\Sigma^\star}(\hat{n}^{-1} \circ \hat{n}_2(a'), \hat{n}^{-1} \circ \hat{n}_2(b'))$$

The rest of the cases are analogous.

$\square$

Due to the fact that expression in Proposition 4.31 depends on two free parameters (transitivity operator and transformation), in next section we consider one of them fixed, and then evaluate the sufficient conditions that the other one needs to fulfill.

Given that most of the well-known dissimilarities are metrics, consider the set of dissimilarities which transitivity operator is the addition.

Let $\delta \in \Delta^\star(X)$ be $\tau_{\Delta^\star}$-transitive such that $\forall a, b \in I_\delta$, $\tau_{\Delta^\star}(a, b) = a + b$. Therefore, δ is a metric because it fulfills the triangle inequality. Thus, a transformation $\breve{n} \in \breve{\mathcal{N}}$ keeps metric properties if, for all $a, b \in I_\delta$

$$\breve{n}(a+b) \leq \breve{n}(a) + \breve{n}(b) \tag{4.10}$$

This is known as *subadditive condition* and comes directly from Proposition 4.31 as a particular case.

Therefore, we can state that any subadditive transformation keeps metric properties. A great number of functions belong to this group. For instance, *concave* functions do. In order to prove this, let us to introduce the following lemma and a formal definition of convexity:

Definition 4.36. Let f be defined on an interval I that contains neither ∞ nor $-\infty$. Then f is *convex* on I if
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in I$ and all $\lambda \in I$; and f is *concave* on I if

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in I$ and all $\lambda \in I$.

Lemma 4.37. *Let f be defined on $[0, \infty)$. If f is concave and $f(0) = 0$, then f is subadditive. If f is convex and $f(0) = 0$, then f is superadditive.*

Proof.   Let x, y be in $[0, \infty)$. If $x + y = 0$, then $x = y = 0$ and, with $f(0) = 0$, Equation 4.10 is trivial. Hence we can assume that $x + y > 0$, whence

$$x = \left(\frac{x}{x+y} \right)(x+y) + \left(\frac{y}{x+y} \right)0$$

and

$$y = \left(\frac{x}{x+y} \right)0 + \left(\frac{y}{x+y} \right)(x+y)$$

If f is concave and $f(0) = 0$, we therefore obtain

$$f(x) \geq \frac{x}{x+y}f(x+y) + \frac{y}{x+y}f(0) = \frac{x}{x+y}f(x+y)$$

and similarly,

$$f(y) \geq \frac{y}{x+y}f(x+y)$$

Adding the two inequalities yields Equation 4.10.

$\square$

Obviously, $\breve{n}$ fulfills the requirements of Lemma 4.37 and therefore can be concluded that any concave transformation over a metric dissimilarity keeps its transitivity.

Furthermore, in this case, we can show graphically the effects of Proposition 4.34. As a particular case, consider the following corollary:

Corollary 4.38. *Let $\breve{n}_1, \breve{n}_2$ be transformations in $\breve{\mathcal{N}}$ (discarding the identity), and δ a metric dissimilarity. Define two equivalent dissimilarities $\begin{cases} \delta_1 = \breve{n}_1 \circ \delta \\ \delta_2 = \breve{n}_2 \circ \delta \end{cases}$*
Denote τ_{δ_1} to the transitivity operator of δ_1 and denote τ_{δ_2} to the transitivity operator of δ_2. For each interval $[a, b] \subseteq I\delta$,

- *If $\forall z \in [a, b], \breve{n}_1^{-1} \circ \breve{n}_2$ is superadditive then $\tau_{\delta_2} \sqsubset \tau_{\delta_1}$ for all $x, y \in X$ such that $\delta(x, y) = z$.*

- *If $\forall z \in [a, b], \breve{n}_1^{-1} \circ \breve{n}_2$ is subadditive then $\tau_{\delta_2} \sqsupset \tau_{\delta_1}$ for all $x, y \in X$ such that $\delta(x, y) = z$.*

Moreover, if $\forall z \in [a, b], \breve{n}_1(x) = \breve{n}_2(x)$ then $\tau_{\delta_1} = \tau_{\delta_2}$ for all $x, y \in X$ such that $\delta(x, y) = z$.

Proof.   To prove this, simply substitute τ_{Δ^*} in Proposition 4.34 with operator sum.

$\square$

Lemma 4.39. *Let f and g be two concave (convex) functions. $f \circ g$ is concave (convex) if and only if f is non-decreasing.*
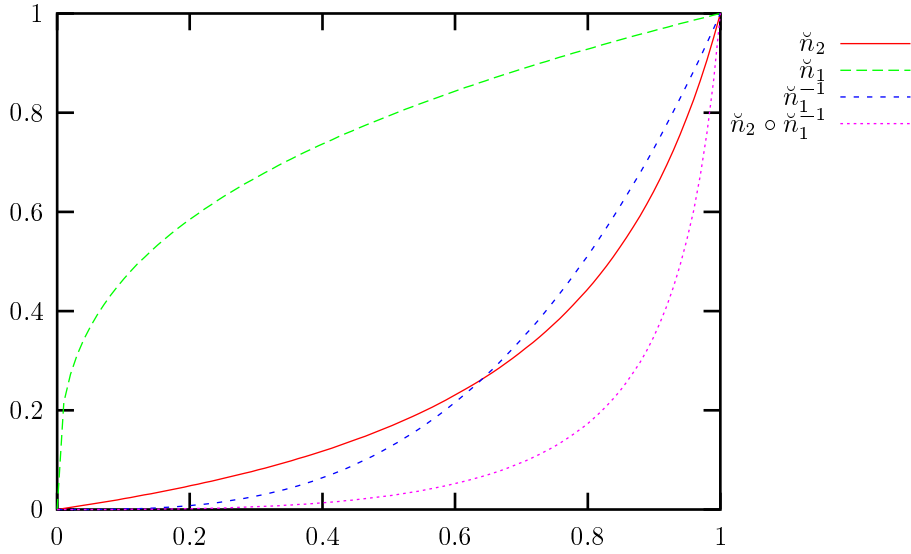
Figure 5: Composition of two transformations $\breve{n}_2(x) = 1 - \frac{1-x}{1-0.8x}$ and $\breve{n}_1(x) = \sqrt[3]{x}$. Note that $\breve{n}_2$ and $\breve{n}_1^{-1}$ are concave.

Proof.   Provided that both g and f are concave, it is true that

$$
\begin{aligned}
f(\lambda x + (1-\lambda)y) &\geq \lambda f(x) + (1-\lambda)f(y) \\
g(\lambda x + (1-\lambda)y) &\geq \lambda g(x) + (1-\lambda)g(y)
\end{aligned}
$$

Thus, applying f non-decreasing to the second expression we get

$$
\begin{aligned}
f \circ g(\lambda x + (1-\lambda)y) &\geq f(\lambda g(x) + (1-\lambda)g(y)) \\
f(\lambda g(x) + (1-\lambda)g(y)) &\geq \lambda f \circ g(x) + (1-\lambda)f \circ g(y) \\
f \circ g(\lambda x + (1-\lambda)y) &\geq \lambda f \circ g(x) + (1-\lambda)f \circ g(y)
\end{aligned}
$$

This is analogous with convexity.

$\square$

Now, using Lemma 4.39, we can state that, for a given metric dissimilarity δ and two transformations $\breve{n}_1$ and $\breve{n}_2$, a sufficient condition to do transitivity of $\breve{n}_1 \circ \delta$ stronger than transitivity of $\breve{n}_2 \circ \delta$ is that $\breve{n}_2$ is concave and $\breve{n}_1^{-1}$ is convex (i.e. $\breve{n}_1^{-1}$ is concave). This is shown graphically on Figure 5.

Similarities also have an equivalent property. Preceding propositions can be modified simply by using the following transitivity operator in place of the additive operator:

$$\tau_{\mathrm{L}}(a,b) = \max\{a + b - 1, 0\} \tag{4.11}$$

For instance, let $s \in \Sigma^\star(X)$ be $\tau_{\Sigma^\star}$-transitive such that $\forall a, b \in I_s$, $\tau_{\Sigma^\star}(a,b) = a + b$. Therefore, s is a metric because it fulfills the triangle inequality. Thus, a transformation $\breve{n} \in \breve{\mathcal{N}}$ keeps

metric properties if, for all $a, b \in I_s$

$$\breve{n}(a + b) \leq \breve{n}(a) + \breve{n}(b) \tag{4.12}$$

This is known as *subadditive condition* and comes directly from Proposition 4.31 as a particular case.

Therefore, an equivalent proposition can be deduced from 4.38

Corollary 4.40. *Let $\breve{n}_1, \breve{n}_2$ be transformations in $\check{\mathcal{N}}$ (discarding the identity), and s a Łukasiewicz similarity (i.e. with transitivity operator like 4.11). Define two equivalent similarities* $\begin{cases} s_1 = \breve{n}_1 \circ s \\ s_2 = \breve{n}_2 \circ s \end{cases}$

Denote τ_{s_1} to the transitivity operator of s_1 and denote τ_{s_2} to the transitivity operator of s_2. For each interval $[a, b] \subseteq Is$,

- *If $\forall z \in [a, b]$,*

$$\breve{n}_1^{-1} \circ \breve{n}_2(\max\{a + b - 1, 0\}) > \breve{n}_1^{-1} \circ \breve{n}_2(a) + \breve{n}_1^{-1} \circ \breve{n}_2(b) - 1$$

  then $\tau_{s_2} \sqsupset \tau_{s_1}$ for all $x, y \in X$ such that $s(x, y) = z$.

- *If $\forall z \in [a, b]$,*

$$\breve{n}_1^{-1} \circ \breve{n}_2(\max\{a + b - 1, 0\}) < \breve{n}_1^{-1} \circ \breve{n}_2(a) + \breve{n}_1^{-1} \circ \breve{n}_2(b) - 1$$

  then $\tau_{s_2} \sqsubset \tau_{s_1}$ for all $x, y \in X$ such that $s(x, y) = z$.

Moreover, if $\forall z \in [a, b]$, $\breve{n}_1(x) = \breve{n}_2(x)$ then $\tau_{s_1} = \tau_{s_2}$ for all $x, y \in X$ such that $s(x, y) = z$.

Proof.   To prove this, simply substitute $\tau_{\Sigma^\star}$ in Proposition 4.34 with $\tau_{\Sigma^\star}(a, b) = \max\{a + b - 1, 0\}$.

$\square$

5 Transformations of the definition set

The power of similarity and dissimilarity functions lays on their several fields of applications. In order to do this, a similarity or a dissimilarity must be flexible and transformable from a context to another. As we have seen so far, there are tools that transform similarities and dissimilarities defined on the same definition set. One step further in this direction is to transform similarities with different definition sets. This is useful because sometimes a similarity value is easier to calculate in a definition set than in others, or it is easier to understand. Even more, some well-known similarities can be used directly in other contexts.

Until now, similarities and dissimilarities were defined in the same definition set. In order to work with different definitions set, we have to extend the definition of *equivalence* (4.3). In fact, the former definition is a particular case of this extended equivalence.

Definition 5.1 (g-equivalence). Given two definition sets X, Y, a function $g : X \longrightarrow Y$, and two similarities or dissimilarities $p_X \in \Pi^\star(X), p_Y \in \Pi^\star(Y)$. The functions p_X, p_Y are *g-equivalent* if the preorders induced by p_X and p_Y in $X \times X$ and $Y \times Y$, respectively, are the

same, where $p_X(x,y) = p_Y(g(x),g(y))$. This means that there are a correspondence between the elements preordered in X^2 and Y^2,

$$(x_1, x_2) \prec (x_3, x_4) \Longleftrightarrow (g(x_1), g(x_2)) \prec (g(x_1), g(x_2))$$

For the sake of shortness, for here on, denote X^2 to $X \times X$. Now, some equivalent definitions to the foregoing one are introduced.

Proposition 5.2. *The following definitions are analogous. For $p_Y \in \Pi^\star(Y)$ and $p_X \in \Pi^\star(X)$.*

 i) p_X, p_Y are g-equivalent.

 ii) p_Y, p'_Y induce the same preorder in Y^2, where $p'_Y(x,y) = p_X(g^{-1}(x), g^{-1}(y))$ and g is a bijection $g : X \longrightarrow Y$.

 iii) $\forall x, y, x', y' \in X$,

- *$p_X(x,y) > p_X(x', y') \Longleftrightarrow p_Y(g(x), g(y)) > p_Y(g(x'), g(y'))$*
- *$g : X \longrightarrow Y$ is bijective.*

Proof.   i) to ii) If p_X and p_Y are g-equivalent, they induce the same preorder in X^2 and Y^2, respectively, and $\forall x, y \in X$, $p_X(x,y) = p_Y(g(x), g(y))$. Similarly, we can define $\forall x, y \in Y, p'_Y(x,y) = p_X(g^{-1}(x), g^{-1}(y))$. Thus, p_Y and p'_Y are the same function and, therefore, they induce the same preorder in Y^2.

 i) to iii) If p_X and p_Y are g-equivalent, they induce the same preorder in X^2 and Y^2. This means that there are a correspondence between the elements preordered in X^2 and Y^2, for all x, y, x', y'

$$(x_1, x_2) \prec (x_3, x_4) \Longleftrightarrow (g(x_1), g(x_2)) \prec (g(x_1), g(x_2))$$

recalling the definition of this preorder we get that

$$p_X(x,y) > p_X(x', y') \Longleftrightarrow p_Y(g(x), g(y)) > p_Y(g(x'), g(y'))$$

Since this is for all $x, y, x', y' \in X$, g must be exhaustive and invertible.

$\square$

Remark 6. Note that if $g(x) = x$ and $X = Y$ the former definition of equivalence is recovered.

Once extended the equivalence, we introduce a way to define similarities and dissimilarities based on other ones defined in a different definition set. Before that, however, let us to introduce the definition of isomorphism between orders, extracted from [25], a lemma and a proposition that will be useful to out main purpose.

Definition 5.3 (isomorphism between orders). Given two ordered sets A and B, an *isomorphism* from A to B, for the orders $\leq_A$ and $\leq_B$, is a bijection f from A to B such that the relations $x_1 \leq_A x_2$ and $f(x_1) \leq_B f(x_2)$ are equivalent. That is, for all $x, y \in A, f(x), f(y) \in B$:

$$x \leq_A y \Longleftrightarrow f(x) \leq_B f(y)$$

Lemma 5.4. *Let X, Y be two definition sets and let g be a function $g : X \longrightarrow Y$. Two similarities, $s_X \in \Sigma^\star(X)$ and $s_Y \in \Sigma^\star(Y)$, are g-equivalent iff exists an isomorphism between the associate orders induced by s_X in X^2 and s_Y in Y^2, respectively.*

Proof. If s_X and s_Y are g-equivalent, we know that, $\forall x, y, x', y' \in X$

$$s_X(x, y) \leq s_X(x', y') \Longleftrightarrow s_Y(g(x), g(y)) \leq s_Y(g(x'), g(y'))$$

Recalling the Definition 4.2

$$s_X(x, y) \leq s_X(x', y') \Longleftrightarrow [(x, y)] \leq_{/\equiv} [(x', y')]$$

since

$$s_Y(g(x), g(y)) \leq s_Y(g(x'), g(y')) \Longleftrightarrow [(g(x), g(y))] \leq_{/\equiv} [(g(x'), g(y'))]$$

we can assure that, if s_X and s_Y are g-equivalent, necessarily,

$$[(x, y)] \leq_{/\equiv} [(x', y')] \Longleftrightarrow [(g(x), g(y))] \leq_{/\equiv} [(g(x'), g(y'))]$$

This is an isomorphism between orders (Definition 5.3).

For other hand, if there exists an isomorphism between the orders, this means that

$$[(x, y)] \leq_{/\equiv} [(x', y')] \Longleftrightarrow [(g(x), g(y))] \leq_{/\equiv} [(g(x'), g(y'))]$$

Following the definition of the partitions produced by the preorders in X and Y, for all $x, y, x', y' \in X$,

$$s_X(x, y) \leq s_X(x', y') \Longleftrightarrow s_Y(x, y) \leq s_Y(x', y')$$

This, following Proposition 4.4, means that s_X and s_Y are g-equivalent.

$\square$

Let us to introduce a particular case of *g*-equivalent transformation. When one of the definition sets is a subset of the other one. (i.e. $Y \subset X$). For the sake of simplicity, consider g the identity. Then, a similarity or a dissimilarity defined in a set X is also defined for any subset of X. Nevertheless, not all the properties are kept.

Proposition 5.5. *Let X be a definition set and let Y be a non-empty subset of X. Given a similarity or dissimilarity $p \in \Pi^\star(X)$. Exists $p' \in \Pi^\star(Y)$ g-equivalent to p defined for all x, y in Y as $p'(x, y) = p(x, y)$. It is strongly reflexive if p' is.*

Proof. Consider that p is a similarity. Firstly, $p' \in \Pi^\star(Y)$ fulfills reflexivity, symmetry and boundedness because $\forall x, y \in Y$, $s'(x, y) = s(x, y)$.

If s is strongly reflexive, $\forall x, y \in X$,

$$s(x, y) = 1 \Longrightarrow x \overset{X}{=} y$$

This trivially includes the particular case $\forall x, y \in Y$. Therefore, s' is also strongly reflexive.

$\square$

As a consequence, for the same similarity or dissimilarity defined in X and, therefore, in a subset of X, the preorder induced by the similarity is the same for the common elements.

Remark 7. Note that in Proposition 5.5 there are not the properties complement and closure. This is because first, the complement of an element can be in $X \setminus Y$ and second, the elements that gets the s_{min} value can be also in $X \setminus Y$.

Theorem 5.6. *Given a similarity or a dissimilarity $p_Y \in \Pi^\star(Y)$ and a injective function $g : X \longrightarrow Y$. Denote $Y_g = Im(g)$ and $X_g = Im(g^{-1})$. Exists a unlimited number of similarities or dissimilarities p_X in X by assimilation of p_Y in Y and by means of function g, fulfilling the following:*

1. *$p_X \in \Pi^\star(X_g)$*

2. *p_X, p_Y are g-equivalent.*

3. *$p_X(x, y) = \breve{n}(p_Y(g(x), g(y))), \ \breve{n} \in \breve{\mathcal{N}}$*

Proof.   Consider that p is a similarity.

Firstly, g is a bijection $X_g \longrightarrow Y_g$. Therefore, using Proposition 5.5 p_Y is also defined in Y_g.

Secondly, p_X is reflexive because, using properties of function $\breve{n} \in \breve{\mathcal{N}}$ (4.13)

$$p_X(x, x) = \breve{n}(p_Y(g(x), g(x))) = \breve{n}(1) = 1$$

Similarly, if $I_{p_Y} \subseteq [0, 1]$ then $I_{p_X} \subseteq [0, 1]$.

Finally, given that p_Y is symmetric, p_X symmetry is trivial to check. Therefore, p_X is a similarity because it is reflexive and symmetric.

Denote $\preceq_Y$ to the induced preorder in Y^2 by p_Y. Also denote $\preceq_{Y/\equiv}$ to its associate order defined on $Y^2/\equiv$ (see Definition 4.2).

Partition X^2 in the following classes:

$$[(x_1, x_2)] = \{(x_3, x_4) \in X^2/(g(x_3), g(x_4)) \in [(g(x_1), g(x_2)]\}$$

Denote this partition $X^2/\equiv$.

Element of $Y^2/\equiv$ and $X^2/\equiv$ are clearly related. Even more, there exists a bijection between both sets denoted by f.

Given that there exists an order in $Y^2/\equiv$ and using that f is a bijection. Transporting this order [25] we can state that there exists an equivalent order in $X^2/\equiv$ (i.e f is an isomorphism between $X^2/\equiv$ and $Y^2/\equiv$).

Given that $X^2/\equiv$ and $Y^2/\equiv$ are the associate orders to the induced preorders by s_X and s_Y on X^2 and on Y^2, using lemma 5.4 we can state that s_Y and s_X are g-equivalent.

$\square$

The Theorem 5.6 says some useful things. Firstly, affirms the existence of a similarity or a dissimilarity in the set X. Obviously, since p_X is related to p_Y by means a $\breve{n}$ function, is also applicable the Proposition 4.9. Therefore, all the known similarities and dissimilarities defined in Y can be used in X.

Proposition 5.7. *Given a similarity or a dissimilarity $p_X \in \Pi^\star(X)$, other similarity or dissimilarity $p_Y \in \Pi^\star(Y)$ and a bijection $g : X \longrightarrow Y$, it is true that*

- p_Y *is* reflexive *only if p_X is reflexive.*

- p_Y *is* strong reflexive *only if p_X is strong reflexive.*

- p_Y *is* symmetric *only if p_X is symmetric.*

- p_Y *is* lower closed (or upper closed) *only if p_X is lower closed (or upper closed).*

- p_Y *has* complement *function only if p_X has complement function.*

- p_Y *is* transitive *only if p_X is transitive.*

Proof. Analogous to Proposition 4.6. Assume p is a similarity for this demonstration.

Reflexivity Provided that $p_X(x,x) = p_{X_{max}}$, using Theorem 5.6 and Definition 4.3,

$$\forall x, y \in X \quad p_Y(g(x), g(x)) > p_Y(g(x), g(y)) \iff p_X(x,x) > p_X(x,y)$$

Therefore denote $p_{Y_{max}}$ to the maximum value for similarity p_Y.

Strong Reflexivity If p_X is strong reflexive then

$$p_X(x,y) = p_{X_{max}} \iff x \overset{X}{=} y$$

Suppose that $\exists x, y \in X$ such that $x \overset{X}{\neq} y$ and, therefore $g(x) \overset{Y}{\neq} g(y)$, but $p_Y(g(x), g(y)) = p_{Y_{max}}$. This means that $p_Y(g(x), g(x)) = p_X(x,y)$ but, using Definition 4.3 we know that this means that

$$p_X(x,x) = p_X(x,y)$$

and this is a contradiction.

Symmetry Symmetry is trivial, using definition of equivalence.

Lower boundedness Again, this property cannot be assured.

Lower closedness Provided that both similarities have lower bound, consider a set of pairs of element of X denoted M_X such that $\forall (x,y) \in M_1\, p_X(x,y) = p_{X_{min}}$. Therefore, using Definition 4.3,

$$\forall z, w \in X, \forall (x,y) \in M_X \quad p_X(x,y) < p_X(z,w) \iff p_Y(g(x), g(y)) < p_Y(g(z), g(w))$$

Denote $p_{Y_{min}} = p_Y(g(x), g(y))$ for any $(x,y) \in M_X$. This is the lower closure of p_Y.

Complement If p_X has complement function, we know that $\forall x' \in C(X)\, p_X(x, x') = p_{X_{min}}$. Therefore, using Definition 4.3 and Theorem 5.6,

$$\forall x, y \in X \quad p_X(x, x') < p_X(x,y) \iff p_Y(g(x), g(x')) < p_Y(g(x), g(y))$$

Denote $p_Y(x, x') = p_{Y_{min}}$. Thus, p_Y has complement.

Transitivity Analyzing the expression of transitivity

$$\forall x, y, z \in X \quad p_X(x, y) \geq \tau_{X\Sigma}(p_X(x, z), p_X(z, y))$$

Again, we cannot assure the fulfillment of this property by the similarity p_Y because the transitivity operator $\tau_{Y\Sigma}$. However, using Proposition 4.5 we can assure that if p_X fulfils this minimum transitivity, p_Y also fulfils it. To demonstrate this, simply use that both p_X and p_Y are strong reflexive.

$\square$

6 Examples

In this section includes some examples of all the concepts introduced in this document, including transitivity operators and transformation functions. Moreover, there are examples of transitivity transformations.

6.1 Operators

Some of the functions described here have been studied in others research fields and can be adapted and used in this context. Thus, for instance, the transformation function concept, denoted by $\hat{n}$, is strongly related with the concept of fuzzy complement. In general, a strictly decreasing and continuous fuzzy complements are in $\hat{\mathcal{N}}$.

In this sense, t-norms and t-conorms, introduced in [29], have been used in fuzzy logic [14, 16]. T-norms can be seen as similarity transitivity operators, while t-conorms can be seen dissimilarity transitivity operators. Therefore, seeking not to repeat work, some of the demonstrations are based in those present in [16].

However, the functions used in [16] are defined over numeric values, while in this document, $\check{n}$ and $\hat{n}$ functions work with *functions*, not values. For other hand, t-norms and t-conorms work with values on [0,1], while transitivity operators work with values in $(0, 1]$ and $[0, 1)$, or in a more general form, on $(-\infty, a]$ and on $[b, +\infty)$, for similarities and dissimilarities, respectively.

Other examples of transitivity operators, added to those widely studied t-norms and t-conorms, are the following:

- Similarity transitivity operators, $\forall a, b \in I_s$

  – Operator $\tau_\Sigma(a, b) = \max\{a + b - k, p\}$, where $I_s = [p, k]$, $k, p \in \mathbb{R}$.
  – Operator $\tau_\Sigma(a, b) = \min(a, b)$, where $I_s = (-\infty, k]$, $k \in \mathbb{R}$.
  – Operator $\tau_\Sigma(a, b) = ab$, where $I_s = [0, 1]$.

- Dissimilarities transitivity operators, $\forall a, b \in I_\delta$

  – Operator $\tau_\Delta(a, b) = a + b$, where $I_\delta = [0, +\infty)$.
  – Operator $\tau_\Delta(a, b) = \max(a, b)$, where $I_\delta = [k, +\infty)$, $k \in \mathbb{R}$.
  – Operator $\tau_\Delta(a, b) = ab$, where $I_\delta = [1, +\infty)$.
  – Operator $\tau_\Delta(a, b) = \sqrt{a^2 + b^2}$, $I_\delta = [0, +\infty)$.

Note that one of the transitivity operators is the operator sum. This expression es known as *triangle inequality*. In fact, a metric is a particular case of a dissimilarity:

Definition 6.1 (metric). A *metric* d is a dissimilarity with $I_d \subseteq [0, +\infty)$ fulfilling the transitivity for the sum operator.

Likewise, by means of the operator max for dissimilarities, we can obtain the ultrametric inequality.

Definition 6.2 (ultrametric). An *ultrametric* d is a dissimilarity with $I_d \subseteq [0, +\infty)$ fulfilling the transitivity for the operator max.

This relation between metrics and dissimilarities is very useful because there are several metrics defined in many fields. Now, this metrics can be used as dissimilarity functions in order to design new similarity or dissimilarity functions.

Remark 8. In the literature there are various transformations that, applied to a metric, keep its properties [8]. In general, this transformations are, among others: monotonic, continuous and subadditive. Actually, this functions are a particular case of dissimilarity equivalence functions. Moreover, keeping metric properties is related to domination.

We introduce now a transformation example:

Example 6. Consider a simple transformation $\breve{f}(x) = x + 1$. It can be seen as a composed transformation. Consider that the image of a similarity is on $[a, b]$. $\breve{f}$ is composed by the following functions:

$$f_1^\star(x) \quad = \quad \frac{x-a}{b-a} \tag{6.13}$$

$$\breve{n}(x) \quad = \quad x \tag{6.14}$$

$$f_2^\star(x) \quad = \quad (a+1) + x(b-a) \tag{6.15}$$

That is, $\breve{f}(x) = f_2^\star \circ \breve{n} \circ f_1^\star(x) = x + 1$. It returns an equivalent similarity in $[a+1, b+1]$.

Two particular cases of transformations are $\breve{n}(z) = z$ and $\hat{n}(z) = 1 - z$. Obviously, the first one does not change transitivity. However, the second one transform a similarity on to a dissimilarity or viceversa. In this case, the similarity transitivity operator and the dissimilarity transitivity operator have the same strength. More precisely, there are a correspondence between the transitivity operator of a similarity and its *dual* by means of the transformation $\hat{n}(z) = 1 - z$. In fuzzy logic, this transformation is called *standard transformation*. For instance, t-norms and t-conorms have a correspondence between them. Here we show the most well-known t-norms and t-conorms:

1. $T_m(a, b) = \begin{cases} a & \text{si } b = 1 \\ b & \text{si } a = 1 \\ 0 & \text{en otro caso} \end{cases}$

2. $T_{\text{Ł}}(a, b) = \max\{0, a + b - 1\}$

3. $T_P(a, b) = ab$

4. $T_M(a, b) = \min(a, b)$

T-conorms:

1. $S_m(a,b) = \begin{cases} a & \text{si } b = 0 \\ b & \text{si } a = 0 \\ 1 & \text{en otro caso} \end{cases}$

2. $S_L(a,b) = \min\{1, a+b\}$

3. $S_P(a,b) = a + b - ab$

4. $S_M(a,b) = \max(a,b)$

There exists a correspondence between each t-norm and each t-conorm (i.e. the first t-norm with the first t-conorm, and so on). This is usually expressed as $T_m = 1 - S_m, T_P = 1 - S_P$, etc. Thus, if a transformation $\hat{n}(z) = 1 - z$ is applied to a similarity or a dissimilarity whose transitivity operator is a t-norm or a t-conorm, the resulting dissimilarity or similarity will be transitive respect to its corresponding t-conorm or t-norm.

6.2 Transitivity and transformations

This examples illustrate the relationship between transitivity and transformation functions.

Example 7. Let δ be defined as follows:

$$\delta(x,y) = e^{|x-y|} - 1$$

It is easy to show that δ is a composition of $\breve{f}(z) = e^z - 1$ and $\delta'(x,y) = |x-y|$. Thus, δ is τ_Δ-transitive with $\tau_\Delta(a,b) = ab + a + b$.

To see this, use Equation 4.5, given that δ' is τ'_Δ-transitive with $\tau'_\Delta(a,b) = a + b$.

Therefore:

$$
\begin{aligned}
\tau_\Delta(a,b) &= \breve{f}(\breve{f}^1(a) + \breve{f}^{-1}(b)) \\
&= e^{|\breve{f}^1(a) + \breve{f}^{-1}(b)|} - 1 \\
&= e^{\ln(1+a) + \ln(1+b)} - 1 \\
&= e^{\ln(1+a)} \cdot e^{\ln(1+b)} - 1 \\
&= (1+a)(1+b) - 1 \\
&= ab + a + b
\end{aligned}
$$

Example 8. Recall the similarity used in Example 1:

$$s(x,y) = 1 - \frac{|x-y|}{|x-y|+1}$$

Its transitivity expression is

$$s(x,y) \geq \max\{s(x,z) + s(z,y) - 1, 0\} \tag{6.16}$$

This expression can be obtained analyzing the similarity. We know that $|x-y|$ is a metric; this is a dissimilarity with sum transitivity.

Consider now this transformation, with $k > 0$:

$$\breve{n}_k(z) = \frac{z}{z + k}$$

Recalling Proposition 4.31 and using the subadditive property of $\breve{n}_k$,

$$\frac{a + b}{a + b + k} < \frac{a}{a + k} + \frac{b}{b + k}$$

we state that $\breve{n}_k(z)$ is also a metric dissimilarity. Therefore, we can affirm that

$$\frac{|x - y|}{|x - y| + 1} \leq \frac{|x - z|}{|x - z| + 1} + \frac{|z - y|}{|z - y| + 1}$$

If we apply now the transformation $\hat{n}(z) = 1 - z$, we will obtain the original expression of the similarity s. Using equation 4.5 the transitivity finally changes to the equation 6.16.

Example 9. In Example 2 the dissimilarity δ is defined in a set of vowels.

$$\delta(x, y) = \begin{cases} 0 & \text{si } x \overset{x}{=} y \\ \frac{1}{5 - C(x,y)} & \text{en otro caso} \end{cases}$$

Again, we can obtain its transitivity expression. It is enough to notice that $C(x, y)$ is a metric dissimilarity and $\frac{1}{5 - C(x,y)}$ is a transformation $f^\star : [0, 4] \longrightarrow (\frac{1}{5}, 1]$. Knowing this, using Theorem 4.11 we have the following transitivity operator:

$$
\begin{aligned}
\tau_{\Delta^\star}(x, y) &= \breve{n}(\min\{\breve{n}^{-1}(x) + \breve{n}^{-1}(y), 4\}) \\
&= \frac{1}{5 - \min\left\{5 - \frac{1}{x} + 5 - \frac{1}{y}, 4\right\}} \\
&= \frac{1}{\max\left\{\frac{1}{x} + \frac{1}{y} - 5, 4\right\}} \\
&= \min\left\{\frac{x \cdot y}{y + x - 5 \cdot x \cdot y}, 4\right\}
\end{aligned}
$$

6.3   Similarities, dissimilarities y metrics

As it has been pointed out previously, there are a connection between dissimilarities and metrics. In the following examples, we show how to pass from a dissimilarity to a metric.

Example 10. Given a metric d in $\mathbb{R}$, we can obtain a similarity by means the following transformation:

$$s(x, y) = \frac{1}{1 + d(x, y)} \tag{6.17}$$

In order to verify that this is a similarity, we have to start from the metric. If we apply to this metric the following function:

$$f^\star(z) = \frac{z}{z + 1}$$

We get a dissimilarity on [0,1). Then, applying the simplest transformation $\hat{n}(z) = 1 - z$. Finally the expression of the similarity is:

$$s(x,y) = 1 - \frac{d(x,y)}{d(x,y)+1} = \frac{1}{1+d(x,y)}$$

Note that the metric is defined on $[0, +\infty)$, while the similarity is defined on $(0, 1]$. Remember that a similarity is closed only if it comes from a closed dissimilarity, and viceversa (see Proposition 4.25).

Example 11. A metric dissimilarity δ_1 is symmetric, bounded and fulfills the triangle inequality. If we apply the function $\hat{n}(z) = 1 - z$ to d_2 we get a similarity s_1 Łukasiewicz-transitive (i.e. $\tau_{s_1}(a,b) = \max\{a+b-1,0\}$). However, if we apply the function $\breve{n}(z) = z^2$ to δ_1 we get a dissimilarity but not a metric (see Corollary 4.38).

Besides, if we apply $\breve{n}$ to s_1, using Corollary 4.40, we get a similarity with a transitivity stronger than s_1. To prove this simply recall Corollary 4.40. If

$$(\max\{a+b-1,0\})^2 \geq (a)^2 + (b)^2 - 1$$

then $\tau_{s_2} \sqsupseteq \tau_{s_1}$. Effectively, the power can be included into the max operator because it is a monotonic operation. Thus,

$$\max\{(a+b-1)^2, 0\} \geq (a)^2 + (b)^2 - 1$$

We can ignore the max operator because $(a+b-1)^2$ will always be greater than 0. Simplifying we get:

$$ab + 1 \geq a + b$$

This is always true if $a, b \in [0,1]$. Denote s_2 to this new similarity.

Now consider the following dissimilarity:

$$\delta_2 = 1 - s_1$$

Which is the relation between δ_1 and δ_2?. Obviously $\delta_2 = \breve{n}_2 \circ \delta_1 = \delta_1(2 - \delta_1)$. However, if $\tau_{s_2} \sqsupseteq \tau_{s_1}$, which is the relation between τ_{δ_1} and τ_{δ_2}?. Using corollary 4.38 and using that $\breve{n}_2$ is subadditive we state that $\tau_{\delta_2} \sqsupseteq \tau_{\delta_1}$, this is τ_{δ_2} is *stronger* than τ_{δ_1}. This means that δ_2 is also a metric dissimilarity.

In the next table are collected the similarities and dissimilarities described here and their respective transitivity operators.

$$
\begin{array}{llll}
\delta_1(x,y) & = & |x-y| & \tau_{\delta_1}(a,b) & = & \min\{a+b,1\} \\
s_1(x,y) & = & 1-|x-y| & \tau_{s_1}(a,b) & = & \max\{a+b-1,0\} \\
s_2(x,y) & = & 1+(x-y)^2 - 2|x-y| & \tau_{s_2}(a,b) & = & (\sqrt{a}+\sqrt{b}-1)^2 \\
\delta_2(x,y) & = & 2|x-y| - (x-y)^2 & \tau_{\delta_2}(a,b) & = & 1-(\sqrt{1-a}+\sqrt{1-b}-1)^2
\end{array}
$$

The next figure illustrates all the process.
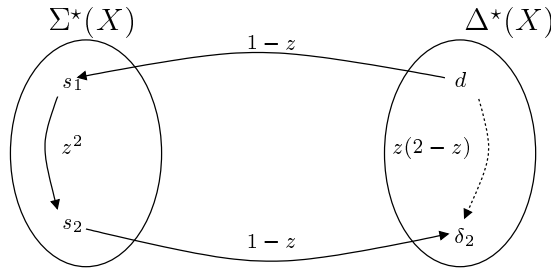
Example 12. Given a complex definition set X where its elements are can share or not common attributes. When comparing two elements $x_1, x_2 \in X$, four values are calculated.

- (a) number of common present attributes.

- (b) number of attributes present in x_1 and absent in x_2.

- (c) number of attributes present in x_2 and absent in x_1.

- (d) number of common absent attributes.

This counters are represented in the next table, where (+) indicates presence and (-) absence.

	$\downarrow x_2$	
$\downarrow x_1$	+	-
+	a	b
-	c	d

A valid similarity function in X is

$$s(x_1, x_2) = \frac{a+d}{a+b+c+d}$$

This is a closed similarity with complement where I_s are the rational numbers on $[0, 1]$. The transitivity operator is $\tau_s(x, y) = \max\{x + y - 1, 0\}$. This is the similarity transitivity operator corresponding to operator sum. It is easy to prove that, in effect, $1 - s$ is a metric.

$$\delta(x_1, x_2) = \frac{b+c}{a+b+c+d}$$

Example 13. Let d_1 be a metric in $[0,1]$. Consider the following similarity:

$$s_1 = 1 - d_1$$

s_1 is τ_{s_1}-transitive where $\tau_{s_1}(a, b) = \max\{a + b - 1, 0\}$.

Consider now the following similarity

$$s_2 = \sqrt{1 - (d_1)^2}$$

Using the equation 4.5 and the properties of transformation functions its transitivity operator is:

$$
\begin{aligned}
\tau_{s_2}(a, b) &= \sqrt{1 - \min\{\sqrt{1 - a^2} + \sqrt{1 - b^2}, 1\}^2} \\
&= \sqrt{1 - \min\{(\sqrt{1 - a^2} + \sqrt{1 - b^2})^2, 1\}} \\
&= \sqrt{\max\{1 - (\sqrt{1 - a^2} + \sqrt{1 - b^2})^2, 0\}} \\
&= \max\{\sqrt{1 - (\sqrt{1 - a^2} + \sqrt{1 - b^2})^2}, 0\} \\
&= \sqrt{1 - (\sqrt{1 - a^2} + \sqrt{1 - b^2})^2}
\end{aligned}
$$

Note that $s_2 = \hat{n}_2 \circ d_1$ where $\hat{n}_2(z) = \sqrt{1 - z^2}$. Using Proposition 4.35 we know that τ_{s_2} is stronger than τ_{s_1} only if $\forall a, b \in [0, 1]$,
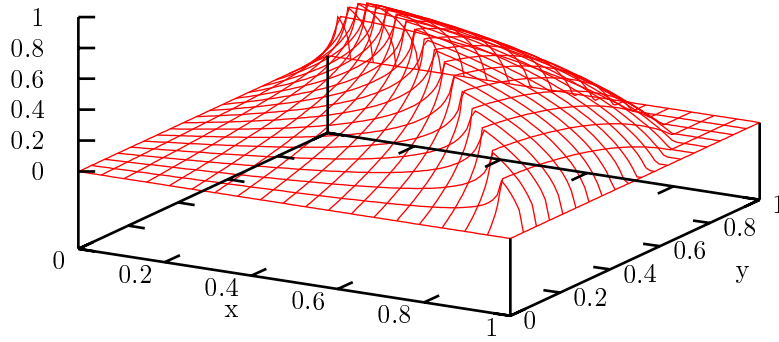
$$\hat{n}_1^{-1} \circ \hat{n}_2 \circ \min\{a + b, 1\}) > \min\{\hat{n}_1^{-1} \circ \hat{n}_2(a) + \hat{n}_1^{-1} \circ \hat{n}_2(b), 1\}$$

This is,

$$1 - \sqrt{1 - (\min\{a + b, 0\})^2} > \min\{2 - \sqrt{1 - a^2} - \sqrt{1 - b^2}, 1\}$$

Graphically, this can be seen in the following figure:

Difference between $\breve{n}_1^{-1} \circ \breve{n}_2 \circ \min\{a + b, 1\}$ and $\min\{\breve{n}_1^{-1} \circ \breve{n}_2(a) + \breve{n}_1^{-1} \circ \breve{n}_2(b), 1\}$ ———

Non-metric transitivity operators

The most restrictive transitivity operator (i.e. the strongest one) is min for similarities and max for dissimilarities. For example, an equivalence relation can be seen as a min-transitivity similarity.

Example 14. Consider any similarity s, min-transitive, where $|I_s| = 2$ (e.g. $I_s = \{0, 1\}$). Similarity s models an equivalence relation $R \subseteq X^2$ as:

$$s(x, y) = \begin{cases} 1 & (x, y) \in R \\ 0 & (x, y) \notin R \end{cases}$$

This similarity is closed, bounded and with complement except in the case that $R = X^2$.

Similarly, we can get a equivalence relation modelled by a dissimilarity. Thus, a dissimilarity δ defined as $\delta = 1 - s$ is max-transitive and models the relation $R^c = X^2 \setminus R$.

6.4   Definition set transformations

Following examples show some definition set transformations. In order to follow this examples, recall Proposition 5.5.
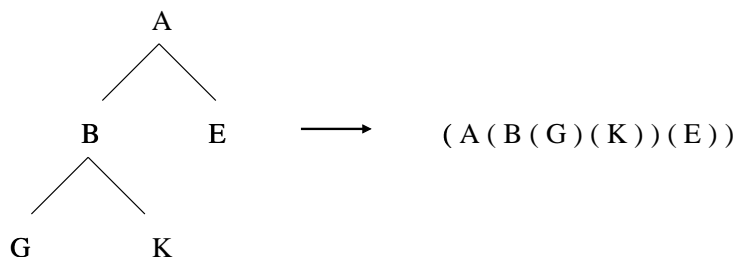
Figure 6: Representation example of trees and lists. The first element of each list is a node, the rest are the branches of that node.

Example 15. This examples show that not all the properties are kept when a similarity or a dissimilarity defined in X is moved to a subset of X.

- $\delta(x, y) = |x| - |y|$ defined on $X = [-1, 1]$ where $\stackrel{x}{=}$ is the usual in $\mathbb{R}$. δ is not strongly reflexive in [-1,1] but, if we consider that δ is defined in $X' = [0, 1]$ it is strongly reflexive.

- The following dissimilarity $\delta'(x, y) = \frac{|x-y|}{2}$ defined on $X = [-1, 1]$, is strongly reflexive on [-1,1]. Of course, on $X' = [0, 1]$ still fulfilling that property.

- $s(x, y) = \frac{\cos(|x-y|)+1}{2}$ defined in $X = [0, 2\pi)$ is closed since $s(\frac{\pi}{2}, \frac{3\pi}{2}) = 0$. However on $X' = [0, \pi)$ it is not closed.

Example 16. Consider a similarity defined over n-ary trees. There are a direct way to transform trees to lists as it is represented in Figure 6. Obviously, this function is not bijective because not all the lists are covered (i.e. it is not exhaustive). However, using Theorem 5.6, a similarity defined for lists is applicable to trees because there exists this bijective function between a subset of lists and trees.

7 Discussion about fulfilling of axioms

Some classic and previous similarity and dissimilarity works affirm that some similarity and dissimilarity axioms are false. Geometrical models of similarity accept symmetry, strong reflexivity and transitivity. However, other authors like Tversky [34] deny this. Tversky affirms that empirical proves demonstrate the falseness of this three axioms. Strong reflexivity is false due to the existence of a different meaning between identity and likeliness when comparing objects. Tversky proposes an example arguing that two complex objects are more similar than two simpler elements based on the fact that they share more features. This is not contradictory with our definition. When comparing two complex objects and two simple objects we are using two different similarities or dissimilarities. Thus, there can achieve different similarity or dissimilarity values.

Tversky also denies the symmetry arguing that say *butchers are like surgeons* is not the same that say *surgeons are like butchers*. Actually, the context is different in each phrase. Because that argument is not valid to refuse the symmetry. In this cases, it is necessary to wonder which attributes are being considered when comparing two complex objects. Anyway, any

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asymmetric function can be transformed into a symmetric function in order to work as a similarity or dissimilarity. An example of this transformation is, for all $x, y \in X$:

$$\frac{1}{2}(s(x,y) + s(y,x))$$

For other hand, Tversky and others also affirm the falseness of transitivity. An example [21]: *If Jamaica is like Cuba and Cuba is like Russia, then Jamaica is like Russia.* At this point, it is essential to differentiate between two things. For one hand, again the context is different in the first phrase than in the second. For the other hand, there is an assumption that Jamaica, Cuba and Russia are either related with a equivalence relation (i.e. Jamaica = Cuba & Cuba = Russia $\implies$ Jamaica = Russia) or it is assumed a geometric model (e.g. a metric). In this document, we introduced a wide definition of transitivity, not restricted to metric and ultrametric relations. In fact, a minimum transitivity is assured for those similarities and dissimilarities strongly reflexive (See Proposition 4.5.

However, there is an exception with operators max and min. In Section 4.5 we showed that the transitivity operator changes when applying $\hat{n}$ and $\check{n}$ functions to similarities and dissimilarities. However, those similarities and dissimilarities min and max transitive, respectively, does not follow this property. Formally,

Proposition 7.1. *Given a* min-*transitive similarity, any equivalent similarity is* min-*transitive. Analogously, given a* max-*transitive dissimilarity, any equivalent dissimilarity is* max-*transitive.*

Proof.   See the analogous demonstration in [16, Teorema 3.19, página 84].

$\square$

A possible undetermined value has no meaning in similarity and dissimilarity framework. This is the reason to require both s and δ to be total. If some definition set does a similarity or dissimilarity not total, Proposition 5.5 allows to use a similarity or dissimilarity on a subset of any definition set. Thus, we can use only the elements of the definition set that does total the similarity or dissimilarity.

With respect to strong reflexivity, in most cases its unfulfillment is due to a bad identification problem with $\stackrel{X}{=}$. This can be resolved redefining this relation.

An alternative to this is to classify the elements of X. This classes, denoted by $E(x)$, where $x \in X$ are defined as follows:

$$E(x) = \{x' \stackrel{X}{=} x | x' \in X\}$$

Finally, define $\stackrel{E}{=}$ as $x \stackrel{E}{=} x' \iff x' \in E(x)$.

Considering the complement axiom, some properties can be extracted:

Proposition 7.2. *If $s \in \Sigma^\star(X)$ is L-transitive and it has complement function, for all $x, y \in X$ and for all $\overline{x} \in C(x), \overline{y} \in C(y)$*

1. $s(x, \overline{y}) + s(x, y) \leq 1$

2. $2s(x, y) + s(y, \overline{x}) + s(\overline{y}, x) \leq 2$

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Proof. If s is Ł-transitive,

$$s(y, \overline{y}) \geq \max\{s(y, x) + s(x, \overline{y}) - 1, 0\}$$

Since $\overline{y} \in C(y)$,

$$0 \geq s(y, x) + s(x, \overline{y}) - 1$$

then we get expression (1).

If we sum to the former transitivity expression the following

$$s(x, \overline{x}) \geq \max\{s(x, y) + s(y, \overline{x}) - 1, 0\}$$

we get

$$
\begin{aligned}
s(x, \overline{x}) + s(y, \overline{y}) &\geq \max\{2s(y, x) + s(\overline{x}, y) + s(\overline{y}, x) - 2, 0\} & (7.18) \\
0 &\geq 2s(y, x) + s(\overline{x}, y) + s(\overline{y}, x) - 2 & (7.19) \\
2 &\geq 2s(y, x) + s(\overline{x}, y) + s(\overline{y}, x) & (7.20)
\end{aligned}
$$

This is the expression (2).

$\square$

Analogously, for dissimilarities:

Proposition 7.3. *If $\delta \in \Delta^{\star}(X)$ is a metric, and it has complement function and $\delta_{max} = 1$, for all $x, y \in X$ and for all $\overline{x} \in C(x), \overline{y} \in C(y)$.*

1. $\delta(x, y) + \delta(x, \overline{y}) \geq 1$

2. $2\delta(x, y) + \delta(y, \overline{x}) + \delta(\overline{y}, x) \geq 2$

Proof. This demonstration is analogue to Proposition 7.2.

$\square$

Based on this properties, some new concepts are introduced:

Definition 7.4 (strict complement). A similarity has a *strict complement* when it verifies

$$s(x, y) + s(x, \overline{y}) = 1$$

Again, a dissimilarity has a *strict complement* when it verifies

$$\delta(x, y) + \delta(x, \overline{y}) = 1$$

Definition 7.5 (symmetric complement). A similarity has a *symmetric complement* when $\forall x, y \in X, s(x, y) = s(\overline{x}, \overline{y})$. A dissimilarity has a *symmetric complement* when $\forall x, y \in X, \delta(x, y) = \delta(\overline{x}, \overline{y})$.

Proposition 7.6. *If $s \in \Sigma^{\star}(X)$ has* strict complement, *then s has* symmetric complement. *If $\delta \in \Delta^{\star}(X)$ has* strict complement, *then δ has* symmetric complement .

Proof. Operating the strict complement condition the demonstration is straight-forward. We know that,

$$s(x, y) + s(x, \overline{y}) \quad = \quad 1 \tag{7.21}$$
$$s(x, y) + s(\overline{x}, y) \quad = \quad 1 \tag{7.22}$$

operating with 7.21 and 7.22:

$$s(x, y) \quad = \quad 1 - s(x, \overline{y}) \tag{7.23}$$
$$s(x, y) \quad = \quad 1 - s(\overline{x}, y) \tag{7.24}$$

It follows that $s(x, \overline{y}) = s(\overline{x}, y)$.

For other hand, dado que $x \in C(\overline{x})$

$$s(\overline{x}, \overline{y}) + s(x, \overline{y}) = 1$$

Thus

$$s(\overline{x}, \overline{y}) = 1 - s(x, \overline{y})$$

Using expression 7.23, it is verified that $s(\overline{x}, \overline{y}) = s(x, y)$.

$\square$

Example 17. An example with complement symmetric similarities and dissimilarities:

- Let $X = \{0, 1\}$ be a definition set and a complement symmetric similarity

$$s(x, y) = 1 - d_H(x, y)$$

(d_H is the *Hamming distance* between bits), where the complement set is $C(x) = \{1 - x\}$.

- Let $X_r = \{\vec{x} \in \mathbb{R}^n | \quad ||X||_2 = r\}$ and a complement symmetric similarity

$$s(\vec{x}, \vec{y}) = \frac{1}{2}(\cos(\vec{x}, \vec{y}) + 1) \in [0, 1]$$

Here, the complement is defined as $C(\vec{x}) = \{-\vec{x}\}$.

8 Conclusions

Similarity or dissimilarity functions are essential to solve most of the problems of AI. The incorrect choice of a standard similarity or dissimilarity can affect to the solution of a problem. Designing or choosing a specific similarity or dissimilarity let to introduce domain knowledge in order to get better outcomes when solving the problem. Therefore, researching on similarity and dissimilarity design can lead to better interpretations and understanding of the results obtained by several Machine Learning applications. Actually, the final objective is to improve these results.

In this document several concept have been formalized. Also, a few theorems and propositions, some of them adapted from other areas, describe similarity and dissimilarity theory. This is the base to advance in the design on order to improve the utility of similarities and dissimilarities. Moreover, concepts like definition set transformations or transitivity are fundamental to keep a semantic value on similarity and dissimilarity. This semantic, for example, let to choose a specific function or another for a specific problem.

Acknowledgements

Notation

Symbol	
s	similarity function
I_s	similarity image
δ	dissimilarity function
I_δ	dissimilarity image
s_{max}	maximum value of a similarity function
δ_{min}	minimum value of a dissimilarity function
$\Sigma(X)$	set of similarity functions defined in X
$\Sigma^\star(X)$	set of similarity functions defined in X where $\mathrm{I}_s \subseteq [0,1]$
$\Delta(X)$	set of dissimilarity functions defined in X
$\Delta^\star(X)$	set of dissimilarity functions defined in X where $\mathrm{I}_\delta \subseteq [0,1]$
p	represents indistinctly similarities and dissimilarities
$\Pi(X)$	represents indistinctly to $\Sigma(X)$ and to $\Delta(X)$
$\Pi^\star(X)$	represents indistinctly to $\Sigma^\star(X)$ and to $\Delta^\star(X)$
τ	transitivity operator
$\tau_{\Sigma^\star}$	transitivity operator for similarities in $\Sigma^\star$
$T_{\Sigma^\star}$	set of transitivity operators for similarity functions
$\tau_{\Delta^\star}$	transitivity operator for dissimilarities in $\Delta^\star$
$T_{\Delta^\star}$	set of transitivity operators for dissimilarity functions
$f^\star$	function $\Sigma \longrightarrow \Sigma^\star$ or $\Delta \longrightarrow \Delta^\star$
$\hat{f}$	function $\Sigma \longrightarrow \Delta$ or $\Delta \longrightarrow \Sigma$
$\breve{f}$	function $\Sigma \longrightarrow \Sigma$ or $\Delta \longrightarrow \Delta$
$\hat{n}$	function $\Sigma^\star \longrightarrow \Delta^\star$ or $\Delta^\star \longrightarrow \Sigma^\star$
$\hat{\mathcal{N}}$	set of $\hat{n}$ functions
$\breve{n}$	functions $\Sigma^\star \longrightarrow \Sigma^\star$ or $\Delta^\star \longrightarrow \Delta^\star$
$\breve{\mathcal{N}}$	set of $\breve{n}$ functions
$\mathcal{P}$	represents indistinctly to a similarity or a dissimilarity set

Bibliography

[1] R. Baeza-Yates and B. Ribeiro-Neto. *Modern Information Retrieval*. ACM Press New York, 1999.

[2] Ll. Belanche, J.J. Valdés, and R. Alquézar. Fuzzy heterogeneous neurons for imprecise classification problems. In John Wiley and Sons, editors, *Intl. Journal of Intelligent Systems*, volume 15, 2000.

[3] Derek Bridge. Defining and combining symmetric and asymmetric similarity measures. In B.Smyth and P.Cunningham, editors, *Advances in Case-Based Reasoning (Procs. of the Fourth European Workshop on Case-Based Reasoning)*, pages 52–63. Springer, 1998.

[4] H. Prade D. Dubois. *Fuzzy Sets And Systems*. Academic Press INC., 1980.

[5] A. Ferguson and D. Bridge. Generalised weighting:a generic combining form for similarity metrics. In Josephine Griffith and Colm O'Riordan, editors, *Procs. of the Eleventh Irish Conference on Artificial Intelligence & Cognitive Science (AICS'2000)*, pages 169–179, 2000.

[6] R. L. Goldstone. The role of similarity in categorization: Providing a groundwork. *Cognition*, (52):125–157, 1994.

[7] R. L. Goldstone. Similarity, interactive activation, and mapping. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, (20):3–28, 1994.

[8] Michiel Hagedoorn. *Pattern Matching using Similarity Measures*. PhD thesis, Utrech University, 2000.

[9] U. Hahn and N. Chater. *Knowledge, Concepts and Categories*, chapter Concepts and Similarity, pages 43–92. Hove, Psychology Press: MIT Press, 1997.

[10] U. Hahn, N. Chater, and L.B.C. Richardson. Similarity and transformation. *Cognition*, (87):1–32, 2003.

[11] U. Hahn, L.B.C. Richardson, and N. Chater. Similarity: A transformational approach. In *In Proceedings of the 23 Annual Meeting of the Cognitive Science Society*, 2001.

[12] F. Klawonn and J. L. Castro. Similarity in fuzzy reasoning. *Mathware & Soft Computing*, 2:197–228, 1995.

[13] Frank Klawonn and Rudolf Kruse. Constructing a fuzzy controller from data. *Fuzzy Sets And Systems*, (85):177–193, 1997.

[14] E. P. Klement. Some mathematical aspects on fuzzy sets: Triangular norms, fuzzy logics and generalized measures. *Fuzzy Sets And Systems*, 90:133–140, 1997.

[15] G. Klir. *Fuzzy Sets:An Overview of Fundamentals, Applications, and Personal Views*, chapter 3. Beijing Normal University Press, 2000.

[16] G. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, chapter 3. Pearson Education POD, 1995.

[17] Carol L. Krumhansl. Concerning the applicability of geometric models to similarity data: The interrelationship between similarity and spatial density. *Psycological Review*, 85, 1978.

[18] Dekang Lin. An information-theoretic definition of similarity. In *Proceedings of International Conference on Machine Learning*, Madison, Wisconsin, July 1998.

[19] J. L. Marichal. *Aggregation Operations for Multicriteria Decision Aid*. PhD thesis, University of Liège, Belgium, 1998.

[20] A. B. Markman and D. Gentner. Structural alignment during similarity comparisons. *Cognitive Psycology*, (25):431–467, 1993.

[21] D.L. Medin, R.L. Goldstone, and D. Gentner. *Respects for similarity*, volume 100, pages 254–278. Psycological Review, 1993.

[22] H. Osborne and D. Bridge. Models of similarity for case-based reasoning. In E.Cambouropolos M.Ramscar, U.Hahn and H.Pain, editors, *Procs. of the Interdisciplinary Workshop on Similarity and Categorisation*, pages 173–179, 1997.

[23] H. Osborne and D. Bridge. Similarity metrics: A formal unification of cardinal and non-cardinal similarity measures. In *ICCBR*, pages 235–244, 1997.

[24] H. Osborne and D. Bridge. We're all going on a summer holiday: An exercise in non-cardinal case base retrieval. In G.Grahen, editor, *Frontiers in Artificial Intelligence and Applications (Procs. of the Sixth Scandinavian Conference on Artificial Intelligence)*, pages 209–219. IOS Press, 1997.

[25] Michel Queysanne. *Álgebra Básica*. Vicens Vives, 1985.

[26] M. Roubens. Fuzzy sets and decision analysis. *Fuzzy Sets And Systems*, 90:199–206, 1997.

[27] S. Saminger, R. Mesiar, and U. Bodenhofer. Domination of aggregation operators and preservation of transitivity. *Internat. J. of Uncertainity in Fuzziness Knowledge-Based Systems*, 10:11–35, 2002.

[28] S. Santini and R. Jain. Similarity measures. In *IEEE Transactions On Pattern Analysis and Machine Intelligence*, 1999.

[29] B. Schweizer and A. Sklar. Statistical metric spaces. *Pacific J. Math*, 1960.

[30] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces*. North-Holland, Amsterdam, 1983.

[31] Shenghuo Zhu Tao Li and Mitsunori Ogihara. A new distributed data mining model based on similarity. In *ACM SAC Data Mining Track*, Melbourne, Florida, USA, March 2003.

[32] R. M. Tardiff. Topologies for probabilistic metric spaces. *Pacific J. Math*, (65):233–251, 1976.

[33] W. S. Torgerson. Multidimensional scaling of similarity. *Psycometrika*, 30, 1965.

[34] A. Tversky. *Features Of Similarity*, volume 84(4), pages 327–352. Psycological Review, July 1977.

[35] R. C. Veltkamp and M. Hagedoorn. Shape similarity measures, properties and constructions. In *Advances in Visual Information Systems, 4th International Conference, VISUAL 2000*, volume 1929 of *Lecture Notes in Computer Science*. Springer, 2000.

[36] J. Williams and N. Steele. Difference, distance and similarity as a basis for fuzzy decision support based on prototypical decision classes. *Fuzzy Sets and Systems*, (131):35–46, 2002.

[37] Gary Yontef. *Awareness, Dialogue, and Process*. The Gestalt Journal Press, 1993.

[38] L. A. Zadeh. Similarity relations and fuzzy orderings. *Inform. Sci.*, 3:177–200, 1971.

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