Isospectral Polyhedra
and
Monotone Boolean Formulae

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Isothetic Polyhedra
and
Monotone Boolean Formulae

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Abstract

We consider the problem of converting boundary representations of
isothetic polyhedra into constructive solid geometry (CSG) representa-
tions. The CSG representation is a boolean formula based on the
halfspaces supporting the faces of the polyhedron. This boolean for-
mula exhibits two important features: no term is complemented (it is
monotone) and each supporting halfspace appears in the formula once
and only once. In this work we prove that such a representation exists
for those cyclic isothetic polyhedra such that for each cyclic deficiency
set in the polyhedron it is possible to find out at least either a convex
or a concave path of extremal edges which splits the cyclic deficiency
set into two subsets.
1 Introduction

Peterson [7] considered the problem of obtaining a Constructive Solid Geometry (CSG) representation for polyhedral solids not necessarily convex such as extrusions and pyramids. These solids are simple in the sense that we they can be described by the two-dimensional base plus a parameter giving the third dimension. In [7] Peterson proved that every polyhedra belonging to these two families admits of a representation by a boolean formula based on the halfspaces supporting its faces. This boolean formula exhibits two important features: no term is complemented (it is monotone) and each supporting halfspace appears in the formula once and only once. Figure 1 gives an example.

Dobkin et al. [2] proved that not all the polyhedra have a CSG representation of the style given above. Dobkin et al. called this kind of formula a Peterson-style formula. In this work two questions were left open: 1) Can the interior of a polyhedron with \( n \) faces be represented by a formula using \( O(n) \) literals? 2) Is it possible to characterize polyhedra, other than extrusions and pyramids, that can be represented by a Peterson-style formula?

Juan-Arinyo considered the second open question in [3]. There, the convex hull decomposable isothetic polyhedra were defined and it was proved that every polyhedron in this family has a Peterson-style formula. It is easy to see that the algorithm given in [3] is a simplification of the Alternating Sum of Volumes (ASV) Decomposition algorithm reported in [9] and that it trivially applies to every convex hull decomposable polyhedron.

The major problem of the method reported in [3] is that it does not always converge. Polyhedra for which the decomposition does not converge are called cyclic polyhedra. The non-convergence severely limits the domain of polyhedra that the method can handle. In this work we propose a modification of the algorithm reported in [3] that allows to extend the polyhedral domain for which a Peterson-style formula exists. The domain is extended with those cyclic isothetic polyhedra such that for each cyclic deficiency it is possible to find out at least either a convex or a concave path of pseudo extremal edges that splits the cyclic
deficiency set into several subsets.

We will prove, by giving an example, that a Peterson-style formula does not exist for every isothetic cyclic polyhedron.

2 Mathematical Fundamentals

We shall be dealing with simple polyhedra which are defined by a finite set of adjacent plane polygons such that every edge of a polygon is shared by exactly one other polygon and no subset of polygons has the same property [8] that is, the polyhedra are two-manifolds in the Euclidean space \( \mathbb{E}^3 \).

A polyhedron \( P \) may be represented by a specification of its faces, \( f_i \) which define its boundary, and it may be written as

\[
P = \{f_1, f_2, \ldots, f_i, \ldots, f_n\}
\]

Each face \( f_i \) is defined by an ordered set of edges \( e_j \),

\[
f_i = \{e_0, e_1, \ldots, e_j, \ldots, e_m\}
\]

and each edge is defined by two ordered vertices \( \{v_1, v_2\} \). Then, a face \( f_i \) can be defined by an ordered set of vertices,

\[
f_i = \{v_0, v_1, \ldots, v_j, \ldots, v_m\}
\]

where \( \{v_i, v_j\} \) with \( i \in \{0, m\} \) and \( j = i+1 \mod (m+1) \) defines edge \( e_i \). \( V(P), E(P) \) and \( F(P) \) respectively denote the set of vertices, the set of edges and the set of faces of a polyhedron \( P \). The bounded component of \( \mathbb{E}^3 \) determined by the polyhedron \( P \) will be denoted by \( R(P) \).

A point set \( S \) in \( \mathbb{E}^3 \) is convex if, for any two possible points \( p_1 \) and \( p_2 \) in \( S \), the segment \( \overline{p_1p_2} \) is entirely contained in \( S \). A halfspace \( H \) is a supporting halfspace of a closed subset \( S \) of \( \mathbb{E}^3 \) if the intersection of \( H \) and \( S \) is not empty and \( S \) is contained in \( H \). Given a plane \( h \) it is a supporting plane of \( S \) if and only if \( h \) bounds a supporting halfspace of \( S \), [1]. We shall call \( h_{f_i} \) The supporting halfspace of face \( f_i \) and we shall use the expression \( h_{f_i} \) or \( h_f \) or \( h_i \) to denote it.

The convex hull \( CH(S) \) of a set of points \( S \) in \( \mathbb{E}^3 \) is the boundary of the smallest convex domain in \( \mathbb{E}^3 \) containing \( S \). Given a set \( S \), its convex hull \( CH(S) \) is unique, [1]. Note that the convex hull \( CH(P) \) of polyhedron \( P \) is the same as the convex hull of \( V(P) \), i.e., \( CH(P) = CH(V(P)) \).

With each edge of a polyhedron \( P \) we shall associate the internal dihedral angle defined as the angle through the interior of the polyhedron between the two polyhedron faces incident to the edge. An edge is defined to be convex if and only if the associated angle is less or equal to 180°; otherwise it is said to be concave.

According to the existence of supporting halfspaces each vertex of a polyhedron can be classified as follows [5]:

1. If the vertex has a supporting halfspace, it is called locally supportable. Furthermore, if the supporting halfspace of the vertex also supports the polyhedron, the vertex is said to be globally supportable.
2. If the complement of the vertex has a supporting halfspace, the vertex is said to be \textit{complementary supportable}.

3. If neither the vertex nor the complement of the vertex has a supporting halfspace, the vertex is said to be \textit{non-supportable}.

3 Isothetic Polyhedra

A polyhedron is said to be \textit{isothetic} if all of its edges (faces) are oriented in three orthogonal directions.

The number of faces converging to a vertex of an isothetic polyhedron can be either three, four or six. Figure 2 shows the two possible ways in which three halfspaces can be arranged to generate an isothetic vertex. Depending on the side of the faces where the polyhedron inside is, the vertex in Figure 2 is convex or concave. If the vertex is convex the enclosed region can be represented by \( h_1 \cap h_2 \cap h_3 \) and it is locally supportable by the supporting halfspace of each incident face. If the vertex is concave the enclosed region can be represented by \( h_1 \cup h_2 \cup h_3 \) and it is complementary supportable by the supporting halfspace of each incident face.

Similarly, if the inside of the vertex in Figure 2 is on the other side of the drawing plane with respect to the reader the enclosed region is given by \( (h_1 \cap h_2) \cup h_3 \) and the vertex is complementary supportable by the halfspace \( h_3 \). When the vertex inside is in the reader’s eye side the enclosed region is given by \( (h_1 \cap h_2) \cup h_3 \) and it is locally supportable by \( h_3 \).

Figure 3 shows the arrangement of four halfspaces to generate a four faces isothetic vertex. Depending on where the vertex inside is, the enclosed region is given by \( (h_1 \cap h_2) \cup (h_3 \cap h_4) \) or by \( (h_1 \cup h_2) \cap (h_3 \cup h_4) \). In both cases the vertex is non-supportable.

Finally, Figure 4 depicts the isothetic vertex made up of six faces. The region enclosed by the vertex is given by

\[ (h_1 \cap h_2) \cup (h_3 \cap h_4) \cup (h_5 \cap h_6) \]

or

\[ (h_1 \cup h_3) \cap (h_2 \cup h_5) \cap (h_4 \cup h_6) \]
Figure 3: Isothetic vertex with four faces.

when the inside is in the back of the drawing plane and the region is given by

\[(h_1 \cup h_2) \cap (h_3 \cup h_4) \cap (h_5 \cup h_6)\]

or

\[(h_1 \cap h_3) \cup (h_2 \cap h_5) \cup (h_4 \cap h_6)\]

when the inside is in the same side of the reader's eye. The six faces isothetic vertices are non-supportable. Table 1 summarizes for each isothetic vertex the number of faces and edges converging to the vertex, the number of convex and concave edges, and the supportability.

Hereafter we shall refer to the boolean formula that defines the inside part of a given vertex as the \textit{vertex configuration} or just the \textit{configuration}.

To close this section we have a definition and a theorem the result of which we shall make use of. Figure 5 illustrates the discussion with a two-dimensional example.

**Definition 3.1** Let \( P \) be an isothetic polyhedron. Then the isothetic hull of \( P \), \( IH(P) \), is defined as the bounding box of \( P \) such that the faces of \( IH(P) \) and the faces of \( P \) are iso-oriented.

Figure 4: Isothetic vertex with six faces.
Table 1: Isothetic vertices taxonomy.

<table>
<thead>
<tr>
<th># faces</th>
<th># edges</th>
<th># convex edges</th>
<th># concave edges</th>
<th>supportability</th>
<th>path</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>-</td>
<td>local</td>
<td>convex</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>-</td>
<td>-</td>
<td>local</td>
<td>convex</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>complementary</td>
<td>concave</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td>complementary</td>
<td>concave</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>non supportable</td>
<td>convex/concave</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>non supportable</td>
<td>convex/concave</td>
</tr>
</tbody>
</table>

**Theorem 3.1** Let $P$ be an isothetic polyhedron and $IH(P)$ its isothetic hull. Let $A \subseteq F(P)$ be the set of faces of $P$ such that belong to $IH(P)$. Let $B \subseteq F(P)$ be the set of faces of $P$ such that belong to $CH(P)$. Then $A = B$.

**Proof** Let $f$ be an arbitrary face in $A$ and let $h_f$ be its supporting halfplane. By definition of $IH(P)$, $f$ is isothetic and polyhedron $P$ is in one side of $h_f$. Then $h_f \cap^* F(P) \subseteq CH(P)$, that is, $f \subseteq CH(P)$; consequently $f \in B$.

Conversely, let $f$ be an arbitrary face in $B \subseteq F(P)$. Obviously, $f$ is isothetic and is in $CH(P)$. Assume that $f$ is not in $A$, that is, $f$ is not in $IH(P)$. But $f \in CH(P)$ implies that $P$ is in one side of $h_f$. Then $f$ should not be isothetic. This contradiction completes the proof. $\square$

4 Decomposition Methods

Decomposition methods for non-convex polyhedra that use the convex hull have been widely used in form feature recognition [4, 6, 9]. The application of this technique yields a boolean representation of the polyhedron in terms of convex components with alternating addition and subtraction of convex components. In [9] Woo called this technique *Alternate Sum of Volumes* (ASV) decomposition.

A decomposition method also based on convex hull computation is used in [3]. There the goal is to convert boundary representations of a restricted class of isothetic polyhedra into

![Figure 5: Convex Hull and Isothetic Hull.](image)
equivalent CSG representations. The representation generated by this method is a Peterson-style representation and consists of a CSG tree whose leaves are the halfspaces supporting the faces of the polyhedron and whose interior nodes are the regularized boolean operations intersection and union. We shall call this method the Convex Hull (CH) decomposition method.

4.1 Alternating Sum of Volumes Method

The Alternating Sum of Volumes (ASV) method computes first the convex hull of the given polyhedron then finds the regularized set difference between the polyhedron and its convex hull. If the difference is empty, the polyhedron is convex and is representable as the intersection of the halfspaces supporting the polyhedron faces. If the difference is not empty each of its connected component defines a solid deficiency set. Now the polyhedron can be represented as the regularized difference of its convex hull and its deficiencies. Then ASV method is applied recursively to each deficiency set until it is convex. Figure 6 shows an example of ASV decomposition process where \( D_i \) is a solid deficiency.

From the point of view of solid modelling, the ASV decomposition method has two major problems. One problem is that it does not always converge. The other problem is the need of computing explicitly the whole set of faces that define the deficiencies: those faces belonging to the polyhedron and those belonging to the polyhedron convex hull but not the polyhedron itself; as a result, the verbosity in the final representation is higher than needed.

4.2 Convex Hull Method

The Convex Hull (CH) method is similar to the ASV method. The most important difference is that now there is no need of computing faces that do not belong to the polyhedron itself. As in the ASV method, first the convex hull \( CH(P) \) of a given isothetic polyhedron \( P \) is computed. If all faces of \( P \) belong to \( CH(P) \) then \( P \) is convex and is representable as the intersection of the halfspaces supporting the polyhedron faces. Otherwise polyhedron faces are split into several subsets. One set is made of the faces such that their supporting halfplanes belong to \( CH(P) \). The faces in this set are not necessarily connected. Each of the other sets contains a maximal connected set of faces such that their supporting halfplanes do not belong to \( CH(P) \) and defines a deficiency set.

Now the polyhedron can be represented as the regularized intersection of the halfspaces supporting those faces of \( P \) in \( CH(P) \) and the generalized halfspaces defined in \( \mathbb{E}^3 \) by deficiency sets. The CH method is applied recursively to each deficiency set until all the faces in the set belong to the deficiency convex hull. The boolean operations applied are intersection and union alternatively. In Figure 7 the CH-decomposition method is applied to the polyhedron in Figure 6. \( P_i \) is the set of faces of \( P \) in \( CH(P) \) and \( D_i \) is a deficiency set of faces of \( P \) not in \( CH(P) \).

Note that the CH-method naturally applies to general polyhedra. Anyway, CH-decomposition method also suffers from the non-convergence problem. Figure 8 shows a polyhedron for which both ASV and CH decomposition methods do not converge. We shall refer to these polyhedra as cyclic polyhedra.
$$P = CH_0 - CH_1 + CH_2$$

Figure 6: ASV decomposition.
$P = P_1 \cap^* (P_2 \cup^* P_3)$

Figure 7: CH decomposition.
Figure 8: Cyclic polyhedron.

The non-convergence severely limits the domain of polyhedra that the method can handle. We shall devote next section to extend this domain.

5 Extending the Domain

In this work we extend the polyhedral domain for whith a Peterson-style formula exists with those cyclic isothetic polyhedra such that for each cyclic deficiency it is possible to find out at least either a convex or a concave path of extremal edges that splits the cyclic deficiency set.

Taken into account theorem 3.1, it is clear that when applying the CH-decomposition method to an isothetic polyhedron $P$, computing deficiency sets with respect $CH(P)$ is equivalent to computing deficiency sets with respect $IH(P)$. Therefore in what follows we will use $IH(P)$, instead of the $CH(P)$.

Any given non-convex isothetic polyhedron $P$ always has at least six faces on its isothetic hull, $IH(P)$, and $IH(P)$ always splits the set of faces of $P$ into several subsets. One of these sets is made of faces such that their supporting halfplanes belong to $IH(P)$. Faces in this set are not necessarily connected. Each of the other sets contains a maximal connected set of faces such that their supporting planes do not belong to $IH(P)$. Each of these sets is called a deficiency set of $P$. Let $\{h_i, 1 \leq i \leq 6\}$ denote the supporting halfspaces of the faces of $P$ belonging to $IH(P)$ and let $D_j, 1 \leq j \leq m$ denote the deficiencies of $P$. Furthermore, let $R(D_j)$ be the generalized halfspace defined in $E^3$ by deficiency $D_j$. Then the expression

$$\bigcap_i h_i \bigcap_j R(D_j)$$

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is a boolean formula that properly represents polyhedron $P$. Obviously, if, for every $j$, $R(D_j)$ has a Peterson-style formula, then the formula above is also a Peterson-style formula. Hence, what we have to do is to build the Peterson-style formula for a cyclic deficiency set $D_j$, [3].

5.1 Face extensions

Let us introduce first some definitions. Figure 9 illustrates these definitions.

**Definition 5.1** Let $f$ be a face of a deficiency $D$, $h_f$ its supporting plane and $e$ an edge of face $f$ with $l$ being the supporting line of $e$. Furthermore, let $\vec{n}$ be the normal to $e$ in $h_f$ pointing outwards $f$. The extension of face $f$ through $e$ is the set of points on the supporting plane of $f$ swept by $l$ while moving in the direction given by $\vec{n}$ and keeping parallel to itself.

**Definition 5.2** Given the conditions in Definition 5.1, the restricted extension of $f$ through $e$ is the set points on the supporting plane of $f$ swept by $e$ while moving in the direction given by $\vec{n}$ and keeping parallel to itself.

5.2 Extremality

Now we shall define the property of extremality for edges in a deficiency set $D$.

**Definition 5.3** Let $e$ be a convex edge defined by faces $f_1$ and $f_2$ both belonging to deficiency $D$. Let the configuration of the two vertices of $e$ be different from each other. Then, edge $e$ is said to be extremal if and only if the intersections of the restricted extensions of $f_1$ and $f_2$ through edge $e$ with the interior of $R(D)$ are empty.

Edges $e_1, e_2$ and $e_3$ in Figure 10 are extremal convex edges.

**Definition 5.4** Let $e$ be a convex edge defined by faces $f_1$ and $f_2$ both belonging to deficiency $D$. Let the configuration of the two vertices of $e$ be the same. Then, edge $e$ is said to be extremal if and only if the intersections of the extensions of $f_1$ and $f_2$ through edge $e$ with the interior of $R(D)$ are empty.

Extremality definitions for concave edges are,
Defnition 5.5 Let $e$ be a concave edge defined by faces $f_1$ and $f_2$ both belonging to deficiency $D$. Let the configuration of the two vertices of $e$ be different from each other. Then, edge $e$ is said to be extremal if and only if the intersections of the restricted extensions of $f_1$ and $f_2$ through edge $e$ with the exterior of $R(D)$ are empty.

Defnition 5.6 Let $e$ be a concave edge defined by faces $f_1$ and $f_2$ both belonging to deficiency $D$. Let the configuration of the two vertices of $e$ be the same. Then, edge $e$ is said to be extremal if and only if the intersections of the extensions of $f_1$ and $f_2$ through edge $e$ with the exterior of $R(D)$ are empty.

Edges $e_4, e_5, e_6, e_7$ and $e_8$ in Figure 10 are extremal concave edges.

Let us now define extremal paths of edges on a deficiency set $D$.

Defnition 5.7 An extremal convex path on deficiency $D$ is a set of connected edges of $D$ such that: 1) The first vertex of the first edge and the last vertex of the last edge are on $IH(D)$; 2) No vertex in the set of edges is shared by more than two edges; and 3) All the edges in the set are extremal and convex.

An extremal concave path is defined analogously. Edges $e_1$ and $e_2$ in Figure 10 define an extremal convex path while edges $e_4, e_5, e_7$ and $e_8$ define an extremal concave path. Given an extremal path on a deficiency set its path faces will be the set of faces in the deficiency set such that at least an edge of each face belongs to the path. Notice that an extremal path on a deficiency set $D$ defines a partition of the faces of $D$.

5.3 The Algorithm

Theorem 5.1 Let $D$ be a deficiency set of faces and $\gamma$ an extremal path on $D$. Let $D_1$ and $D_2$ be the two subsets of faces induced by $\gamma$ in $D$ and let $R(D_1)$ and $R(D_2)$ be the regions they define in $E^3$. Then, the region $R(D)$ defined in $E^3$ by $D$ is given by $R(D) = R(D_1) \cap R(D_2)$ when $\gamma$ is convex and by $R(D) = R(D_1) \cup R(D_2)$ when $\gamma$ is concave.

Proof It is sufficient to see that the extensions of path faces in $D_1$ and in $D_2$ induced in $D$ by the extremal path $\gamma$ do not intersect $D$ anywhere. □
Let $\text{OB}(0)$ stand for the regularized boolean operation $\land^*$, $\text{OB}(1)$ stand for the regularized boolean operation $\lor^*$ and let $r$ be the recursion level. Then we can rewrite the algorithm reported in [3] to compute a Peterson-style formula for isothetic polyhedra such that either they are CH-decomposable or all the deficiency sets at the recursion level where cyclicity appears have at least an extremal path.

PROCEDURE Write_Boolean (OB)
  /* Adds to the CSG tree an internal node representing
    boolean operation OB */
ENDPROCEDURE

PROCEDURE Write_Halfspace (H)
  /* Adds to the CSG tree a leaf pointing to the
    halfspace H */
ENDPROCEDURE

PROCEDURE Split_Cyclic_Deficiency (deficiencies, r)
  /* Splits a connected set of faces that defines a cyclic
   deficiency into two or more subsets induced by maximal
   paths. All Maximal paths are either convex or concave.
   Recursion level $r$ is reset according to maximal paths type */
ENDPROCEDURE

PROCEDURE Compute_CSG_Tree (D, IHD, r)
  Compute_IHConnected_Sets (D, IHD, convexes, deficiencies)
i := i + 1
WHILE convexes(i) <> NIL DO
  FOR all faces f in convexes(i) DO
    IF f is not the first face THEN Write_Boolean (OB(r)) ENDF
    Write_Halfspace (Hf)
  ENDFOR
  i := i + 1
ENDWHILE

IF convexes(i) = NIL and deficiencies has just one set THEN
  Split_Cyclic_Deficiency (deficiencies, r)
i := 1
WHILE deficiencies(i) <> NIL DO
  IF i > 1 THEN Write_Boolean (OB(r)) ENDF
  Compute_IsotheticHull (deficiencies(i), IH)
  Compute_CSG_Tree (deficiencies(i), IH, r+1 mod 2)
i := i + 1
ENDWHILE
ELSE
  i := 1
WHILE deficiencies(i) <> NIL DO
  Write_Boolean (left_parenthesis)
Compute_IsotheticHull (deficiencies(i), IH)
Compute_CSG_Tree (deficiencies(i), IH, r + 1 mod 2)
Write_Boolean (right_parenthesis)
i := i + 1
ENDWHILE
ENDIF
ENDPROCEDURE

PROCEDURE Compute_IHconnected_Sets (D, IHD, convexes, deficiencies)
i := 0
j := 0
First_face (D, f)
REPEAT
IF face in IHD THEN
i := i + 1
convexes(i) := NIL
Connect_Condex_Sets (f, D, IHD, convexes(i))
ELSE
j := j + 1
deficiencies(j) := NIL
Connect_Deficiency_Sets (f, D, IHD, deficiencies(j))
ENDIF
Search_Nonconnected_Face (D, found, face)
UNTIL not found
convexes(i + 1) := NIL
deficiencies(j + 1) := NIL
ENDPROCEDURE

Let CSG be a binary tree that will store the computed Peterson-style formula for polyhedron P and let p be the pointer to the next tree node. If CSG and p are held in static variables, the initial call to the recursive procedure is,

PROCEDURE Compute_CSG_from_BR (P, CSG)
Compute_IsotheticHull (P, IHP)
p := 0
Compute_CSG_tree (P, IHP, 0)
ENDPROCEDURE

Figure 11 shows how the algorithm works. Let P be the polyhedron. IH(P) splits the set of faces of P into one convex connected set \{a, b, c, d, e, f\} plus one connected cyclic deficiency set \{g, h, i, j, k, l, m, n\}. Then

\[ R(P) = a \cap^* b \cap^* c \cap^* d \cap^* e \cap^* f \cap^* R(g, h, i, j, k, l, m, n) \]

The cyclic deficiency set can be split by either a convex or a concave maximal path. If we chose the convex paths the deficiency set is split into three sets, \{g, h\}, \{i, j\} and \{k, l, m, n\}.  

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Figure 11: Computation process.
\[ R(g, h, i, j, k, l, m, n) = R(g, h) \cap^* \\
R(i, j) \cap^* \\
R(k, l, m, n) \]

Recursive application of the algorithm to each of the sets above gives,

\[ R(g, h) = g \cup^* h \]
\[ R(i, j) = i \cup^* j \]
\[ R(k, l, m, n) = m \cup^* n \cup^* R(k, l) \]
\[ R(k, l) = k \cap^* l \]

Then the CSG binary tree stores for polyhedron \( P \) the following Peterson-style formula,

\[ R(P) = a \cap^* b \cap^* c \cap^* d \cap^* e \cap^* f \cap^* ((g \cup^* h) \cap^* (i \cup^* j) \cap^* (m \cup^* n \cup^* (k \cap^* l))) \]

It is worth to note that a given polyhedron has, in general, several valid Peterson-style formulas. The formula computed by the procedure presented here depends on the choice of the type and number of maximal paths used in splitting cyclic deficiency sets.

Figure 12 shows a deficiency set for which the given procedure does not compute a Peterson-style formula and it still exists.

\[ R(D) = R(a, b, c) \cup^* R(d, e, f, g, h) \]
\[ = (a \cap^* b \cap^* c) \cup^* (d \cap^* (e \cup^* f \cup^* g \cup^* h)) \]

The problem is that the extremality definition given above does not define a extremal path in this case.

On the other hand, not all isothetic polyhedra have a Peterson-style formula. The deficiency set shown in Figure 13 is a simple case for which it is not possible to find such a formula. The simplest formula we can write is

\[ R(D) = (a \cap^* b \cap^* (c \cup^* d)) \cup^* ((f \cup^* g \cup^* e) \cap^* (a \cup^* h)) \]

where the supporting halfspace of face \( a \) should appear twice.

6 Conclusions

First we have defined the extremality concept. Then we have extended the domain of isothetic polyhedra that admits of a CSG representation in the form of a Peterson-style formula. We have given a procedure that computes such a representation for those isothetic cyclic polyhedra that have cyclic deficiencies for which at least an extremal path can be found. Finally we have seen, by giving an example, that not all the isothetic cyclic polyhedra have a CSG representation in the form of a Peterson-style formula.

A question that is left open is: can we find a characterization of isothetic cyclic polyhedra that admits of a Peterson-style formula?
Figure 12: Cyclic deficiency for which the given procedure does not compute a Peterson-style formula.

Figure 13: Cyclic deficiency with no Peterson-style formula.
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