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**Extension Orderings**

Albert Rubio

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# Extension Orderings

Albert Rubio\*  
Universitat Politècnica de Catalunya  
Pau Gargallo 5  
08028 Barcelona, Spain  
E-mail: rubio@lsi.upc.es.

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## Abstract

In this paper we study how to extend a collection of term orderings on disjoint signatures to a single one, called an *extension ordering*, which preserves (part of) their properties. Apart of its own interest, e.g. in automated deduction, extension orderings turn out to be a new method to obtain simple and constructive proofs for modularity of termination of TRS.

Three different schemes to define extension orderings are given. The first one to deal with reduction orderings, the second one to extend simplification orderings and the last one for total reduction orderings. This provides simpler and more constructive proofs for some known modularity results for (simple and total) termination of rewriting as well as the first —to our knowledge— results for rewriting modulo equational theories.

Finally, our technique is applied to extend an ordering on a given signature to a new one on the signature enlarged with some new symbols. Apart its own interest this extension ordering can be used to prove termination of hierarchical unions of TRS.

## 1 Introduction

General methods for proving termination are crucial for decision procedures and for using rewriting-like methods in theorem proving and programming. A well-known related problem in rewrite theory is to prove the modularity of termination of *term rewrite systems* (TRS's), that is, given two terminating rewrite systems  $R_1$  and  $R_2$ , to show that their union  $R_1 \cup R_2$  is also terminating. This property was shown to be false in general, even when the two rewrite systems do not share any function symbols [Toy87]. Therefore many efforts have been devoted to finding sufficient conditions under which termination is modular. Most existing results impose syntactic (or abstract) conditions on (one of) the two TRS's (e.g. [Rus87, Dro89, Mid89, Gra93, Ohl94, FJ93]), and some other ones

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impose restrictions on the method used to show the termination of the involved TRS's ([KO90, FZ93]).

In this context it is quite surprising that, although usually termination of TRS's is proved by actually building a *reduction ordering* in which the rewrite relation is contained (cf. [DJ90]), in most modularity proofs for termination such an ordering is not explicitly built, that is, in this sense these proofs are not constructive. Hence, for simplicity and to improve the understanding of the problem, it is worth to look at it from a more constructive point of view. This can give information about the weakest conditions ensuring modularity and why they are needed. As a good example for this viewpoint, *total termination* (i.e. the rewrite relation is included in some total reduction ordering) was proved modular provided that one of the TRS's is conservative in [FZ93]. The reason for this requirement seems to be rather technical, and the proof does provide no intuition of its need. Here we prove the same result in a simple way, actually building a total ordering containing the combined system. This gives some intuition about the restriction.

Our main idea is to generalize the combination of TRS's to the combination of orderings. Instead of proving that the ordering induced by the combined rewrite system is well-founded, we build a well-founded ordering and prove that it includes the combined TRS. This is done by means of *extension orderings*. By an extension ordering of a collection of orderings  $\succ_1, \dots, \succ_n$  on disjoint signatures, we mean another ordering  $\succ$  s.t.  $\succ_i \subseteq \succ$  for all  $i$ . Actually, an extension ordering is interesting when it preserves (part of) the properties satisfied by the initial orderings.

In section 3 we define an extension ordering  $\succ_e$  on ground terms, which preserves well-foundedness of the  $\succ_i$ . Moreover, if every  $\succ_i$  is monotonic, then  $\succ_e$  fulfils a property called *quasi-monotonicity* which suffices for termination of rewriting. As expected, due to the non-modularity of termination of *disjoint union* of TRS's,  $\succ_e$  cannot be lifted, in general, to terms with variables. However, lifting is possible for right-linear systems. This gives us a simple constructive proof for modularity of right-linear TRS's.

Section 4 is on simple termination (i.e. the rewrite relation is included in some simplification ordering). We define an extension ordering  $\succ_s$ , preserving monotonicity and the subterm property to prove the modularity of simple termination (this was first proved in a non-constructive way by [KO90]).

In section 5 we give an extension ordering preserving well-foundedness, monotonicity and totality on ground terms, which allows us to prove the results of [FZ93].

In section 6 these extension orderings are proved to preserve also *E*-compatibility for certain classes of equational theories *E*. Therefore we can obtain the the first —as far as we know— modularity results for rewriting modulo equational theories. This is a first example of new results that can be derived from our extension orderings approach.

In section 7 we extend any ordering, to allow new function symbols. These results can sometimes be used to prove termination of *hierarchical unions* of TRS's [Der93, Rao94].

In section 8 some possible improvements to the present results are commented and the general application of the method in termination proofs is discussed.

## 2 Preliminaries

We adopt the standard notations and definitions for term rewriting given in [DJ90]. Here we recall some of them.

The set of terms is denoted by  $\mathcal{T}(\mathcal{F}, \mathcal{X})$  for a given set of fixed-arity function symbols  $\mathcal{F}$  and set of variables  $\mathcal{X}$ . Variable-free terms, denoted  $\mathcal{T}(\mathcal{F})$ , are called *ground*. By  $|t|_f$  we mean the number of occurrences of the function symbol  $f$  in the term  $t$ , and the size of a term  $t$ , denoted by  $|t|$ , is the number of function symbols occurring in  $t$ . By  $\text{Vars}(t)$  and  $\text{Mvars}(t)$  we denote respectively the set and the multiset of variables occurring in  $t$ . The subterm of  $t$  at position  $p$  is denoted by  $t|_p$ , and the replacement of  $t|_p$  by a term  $s$  is denoted by  $t[s]_p$ . By  $\text{Head}(t)$  we mean the top-most function symbol in  $t$ . Since we denote ambiguously some operations on sets and multisets by the same symbols, it should be clear from the context whether we operate on sets or on multisets.

We denote by  $=_E$  the congruence generated on  $\mathcal{T}(\mathcal{F})$  by the equational theory  $E$ . Let  $s$ ,  $t$ ,  $s'$  and  $t'$  be arbitrary terms in  $\mathcal{T}(\mathcal{F})$ , and let  $f$  be a function symbol in  $\mathcal{F}$ . Then a (strict partial) ordering on  $\mathcal{T}(\mathcal{F})$  (a transitive irreflexive relation)  $\succ$  fulfills the *subterm property* if  $f(\dots t \dots) \succ t$ . Furthermore, it is *monotonic* if  $s \succ t$  implies  $f(\dots s \dots) \succ f(\dots t \dots)$ . A monotonic ordering that fulfills the subterm property is called a *simplification ordering* and for finite signatures is *well-founded*: there are no infinite sequences  $t_1 \succ t_2 \succ \dots$ . An ordering  $\succ$  is *E-compatible* if  $s' =_E s \succ t =_E t'$  implies  $s' \succ t'$ . Finally we say that an ordering is *total* on (the  $E$ -congruence classes of)  $\mathcal{T}(\mathcal{F})$  if  $s \neq t$  ( $s \neq_E t$ ) implies  $s \succ t$  or  $t \succ s$  for all terms  $s, t \in \mathcal{T}(\mathcal{F})$ . Any well-founded monotonic ordering  $\succ$  total on  $\mathcal{T}(\mathcal{F})$  is a simplification ordering.

An ordering on  $\mathcal{T}(\mathcal{F}, \mathcal{X})$  can also be required to be *stable* under substitutions, i.e. if  $s \succ t$  implies  $s\sigma \succ t\sigma$  for all terms  $s$  and  $t$  and substitutions  $\sigma$ . A well-founded monotonic ordering stable under substitutions is called a *reduction ordering*.

An ordering  $\succ$  on  $\mathcal{T}(\mathcal{F})$  can always be lifted to an ordering  $\succ_\nu$  (or simply  $\succ$ ) on  $\mathcal{T}(\mathcal{F}, \mathcal{X})$ , which is stable under substitutions in the following way:

**Definition 2.1** Let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F}, \mathcal{X})$ . Then  $s \succ t$  if  $s\sigma \succ t\sigma$  for all substitutions  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ .

Let  $\succ$  be a relation on terms. Then  $\text{max}_\succ(S)$  is the set of all maximal terms in the set of terms  $S$ , i.e. the set  $S'$  s.t.  $t \in S'$  iff there is no  $s$  in  $S$  such that  $s \succ t$ . When working modulo an equational theory  $E$ ,  $\text{max}_\succ(S)$  is any set obtained from  $S'$  as defined above by choosing one element in each  $E$ -congruence class.

Given an ordering  $\succ$  on terms and a congruence relation  $=$ , the lexicographic extension  $\succ^{lex}$  of  $\succ$  wrt.  $=$  for  $n$ -tuples is defined as:

$$\langle s_1, \dots, s_n \rangle \succ^{lex} \langle t_1, \dots, t_n \rangle \quad \text{iff} \quad s_1 = t_1, \dots, s_{k-1} = t_{k-1} \text{ and } s_k \succ t_k$$

for some  $k$  in  $1 \dots n$ .

The extension to finite multisets of a congruence  $=$  is defined as:  $\{s_1, \dots, s_m\} = \{t_1, \dots, t_n\}$  if  $m = n$  and there exists some permutation  $\pi$  of  $1 \dots m$ , s.t.  $s_{\pi(i)} = t_i$  for all  $i: 1 \dots n$ .

The extension, wrt. a congruence  $=$ , of an ordering  $\succ$  to finite multisets, denoted by  $\succ$ , is defined by:

$M = \{s_1, \dots, s_m\} \succ \{t_1, \dots, t_n\} = N$  if

- $M \neq \emptyset$  and  $N = \emptyset$  or
- $s_i = t_j$  and  $M \setminus \{s_i\} \succ N \setminus \{t_j\}$ , for some  $i = 1 \dots m$  and  $j = 1 \dots n$  or
- $s_i \succ t_{j_1} \wedge \dots \wedge s_i \succ t_{j_k}$  and  
 $(M \setminus \{s_i\}) \not\succeq N \setminus \{t_{j_1}, \dots, t_{j_k}\}$  for some  $i = 1 \dots m$  and  $1 \leq j_1 < \dots < j_k \leq n$   
 $(k = 0 \dots n)$ .

where  $M \not\succeq N$  means  $M \succ N$  or  $M = N$ .

If  $\succ$  is a well-founded ordering on terms then  $\succ^{lex}$  and  $\succ$  are well-founded orderings on  $n$ -tuples of terms and on finite multisets of terms respectively, more precisely  $\succ^{lex}$  and  $\succ$  preserve irreflexivity, transitivity and well-foundedness.

A term rewrite system (TRS) is a set of rules  $l \rightarrow r$  where  $l$  and  $r$  are terms in  $\mathcal{T}(\mathcal{F}, \mathcal{X})$  with  $\text{Vars}(l) \supseteq \text{Vars}(r)$ . For a given TRS  $R$  and terms  $s$  and  $t$  in  $\mathcal{T}(\mathcal{F}, \mathcal{X})$ , we say that  $s$  rewrites into  $t$ , written  $s \rightarrow_R t$  if  $s|_p = l\sigma$  and  $t = s[r\sigma]_p$  for some rule  $l \rightarrow r \in R$ , position  $p$  in  $s$  and substitution  $\sigma$ . Moreover,  $s$  rewrites modulo  $E$  into  $t$ , denoted by  $s \rightarrow_{R/E} t$  if  $s =_E s'$ ,  $s'|_p = l\sigma$  and  $t = s'[r\sigma]$ , for some rule  $l \rightarrow r \in R$ , term  $s'$ , position  $p$  in  $s'$  and substitution  $\sigma$ .

A rewrite system  $R$  is *terminating* over  $\mathcal{T}(\mathcal{F})$  if there exists no infinite sequence  $t_1 \rightarrow_R t_2 \rightarrow_R \dots$  of terms in  $\mathcal{T}(\mathcal{F})$ . We speak about *E-termination* when we are considering rewriting modulo an equational theory  $E$ .

**Proposition 2.2** Let  $R$  be a TRS with left and right hand sides in  $\mathcal{T}(\mathcal{F}, \mathcal{X})$ . Then  $R$  is ( $E$ -)terminating over  $\mathcal{T}(\mathcal{F})$  if there is an ( $E$ -compatible) well-founded monotonic ordering  $\succ$  s.t.  $l\sigma \succ r\sigma$  for all  $l \rightarrow r \in R$  and ground substitution  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ .

Therefore if  $\succ$  is an ( $E$ -compatible) reduction ordering it suffices to prove  $l \succ r$  for all rule  $l \rightarrow r$  in  $R$ . In fact, if  $R$  is ( $E$ -)terminating then  $\rightarrow_R^\dagger$  is itself an ( $E$ -comparable) reduction ordering.

**Proposition 2.3** Let  $R$  be a TRS with left and right hand sides in  $\mathcal{T}(\mathcal{F}, \mathcal{X})$ . If  $R$  is ( $E$ -)terminating over  $\mathcal{T}(\mathcal{F})$  (and  $\mathcal{F}$  contains at least one constant symbol) then  $R$  is ( $E$ -)terminating over  $\mathcal{T}(\mathcal{F} \cup \mathcal{F}')$  for any set of function symbols  $\mathcal{F}'$ .

This means that we can, in general, speak about termination without mentioning over which signature.

If  $\rightarrow_R$  is ( $E$ -)terminating then  $\rightarrow_R^\dagger$  is an ( $E$ -compatible) reduction ordering.

The embedding relation  $\succ_{emb}$  is the ordering generated by the rewrite system  $\{f(x_1, \dots, x_n) \rightarrow x_i \mid \forall f \in \mathcal{F} \wedge i : 1 \dots n\}$ .

The union of two rewrite systems  $R_1$  and  $R_2$  is called a *disjoint union* (or *direct sum*), and denoted by  $R_1 \uplus R_2$ , if  $R_1$  and  $R_2$  do not share any symbol.

From now on we suppose that we have  $n$  relations  $\succ_1, \dots, \succ_n$  on  $\mathcal{T}(\mathcal{F}_1), \dots, \mathcal{T}(\mathcal{F}_n)$  respectively, where all  $\mathcal{F}_i$  are disjoint and contain at least one constant symbol.

### 3 Modularity of termination of TRS's

This section is devoted to extend general orderings on terms and hence to study modularity of termination of TRS's. First the ground case is considered. Then the applicability of the defined ordering to terms with variables and its utility for termination proofs of TRS's is analyzed.

#### 3.1 A transformation on terms

Here we define a transformation from terms to multisets of terms that will be used in the following to define an extension ordering. The mapping  $N_i$  below eliminates from terms in  $\mathcal{T}(\mathcal{F})$  those function symbols that are not in  $\mathcal{F}_i$ :

**Definition 3.1** Let  $N_i$  be a mapping from terms in  $\mathcal{T}(\mathcal{F})$  to multisets of terms in  $\mathcal{T}(\mathcal{F}_i)$  recursively defined as:

$$N_i(f(t_1, \dots, t_m)) = \begin{cases} \{f(t'_1, \dots, t'_m) \mid t'_j \in N_i(t_j) \cup Z_i(t_j)\} & \text{if } f \in \mathcal{F}_i \\ N_i(t_1) \cup \dots \cup N_i(t_m) & \text{otherwise} \end{cases}$$

where  $0_i$  is some (fixed) constant symbol in  $\mathcal{F}_i$  and  $Z_i(t) = \{0_i\}$  if  $t \notin \mathcal{T}(\mathcal{F}_i)$  and  $Z_i(t) = \emptyset$  otherwise.

**Proposition 3.2** For all  $i, j : 1 \dots n$  and for all terms  $t$  in  $\mathcal{T}(\mathcal{F}_i)$ , if  $j \neq i$  then  $N_j(t) = \emptyset$  and if  $j = i$  then  $N_j(t) = \{t\}$ .

**Proposition 3.3** For all  $i : 1 \dots n$ , ground terms  $s, t \in \mathcal{T}(\mathcal{F})$  and function symbol  $f \in \mathcal{F}$ , if  $t \in \mathcal{T}(\mathcal{F}_i)$  whenever  $s \in \mathcal{T}(\mathcal{F}_i)$  then  $N_i(s) = N_i(t)$  implies  $N_i(f(\dots s \dots)) \supseteq N_i(f(\dots t \dots))$ .

**Proof** If  $f \notin \mathcal{F}_i$  then  $N_i(f(\dots s \dots)) = X \cup N_i(s)$  and  $N_i(f(\dots t \dots)) = X \cup N_i(t)$  for some  $X$  obtained from the transformations of the other top level arguments, and therefore  $N_i(f(\dots s \dots)) = N_i(f(\dots t \dots))$ .

If  $f \in \mathcal{F}_i$  then  $N_i(f(\dots s \dots)) = \{f(\dots s' \dots) \mid s' \in N_i(s) \cup Z_i(s)\}$  and  $N_i(f(\dots t \dots)) = \{f(\dots t' \dots) \mid t' \in N_i(t) \cup Z_i(t)\}$ . Since  $t \in \mathcal{T}(\mathcal{F}_i)$  whenever  $s \in \mathcal{T}(\mathcal{F}_i)$ , either  $Z_i(s) = Z_i(t) = \emptyset$  or  $Z_i(s) \neq \emptyset$ . Therefore  $N_i(f(\dots s \dots)) \supseteq N_i(f(\dots t \dots))$ .  $\square$

#### 3.2 Well-founded extension orderings

Here a method for extending a collection of orderings on ground terms built over disjoint signatures is given. This method produces an extension ordering preserving well-foundedness, i.e. we have a well-founded extension ordering.

**Definition 3.4** The relation  $\succ_e$  on  $\mathcal{T}(\mathcal{F})$  extending  $\succ_i$  for all  $i : 1 \dots n$  is defined as:

$$s \succ_e t \text{ iff } N_1(s) = N_1(t) \wedge \dots \wedge N_{i-1}(s) = N_{i-1}(t) \wedge N_i(s) \succ_i N_i(t) \text{ for some } i : 1 \dots n$$

In other words, given relations  $\succ_i$  on  $\mathcal{F}_i$  and a linear ordering among them (in this case the ordering on the subindexes), we build the relation  $\succ_e$ .

Of course, the roles of all  $\succ_i$  (and  $N_i$ ) are symmetric here, and can be exchanged, obtaining in fact different orderings  $\succ_e$ .

**Lemma 3.5**  $\succ_e$  extends  $\succ_i$  for all  $i : 1 \dots n$ .

**Proof** By proposition 3.2, for all  $i, j : 1 \dots n$  and for all  $s, t \in \mathcal{T}(\mathcal{F}_i)$  we have  $N_j(s) = N_j(t) = \emptyset$  if  $j < i$ , and  $N_j(s) = \{s\}$  and  $N_j(t) = \{t\}$  otherwise. Then  $s \succ_i t$  implies  $s \succ_e t$ .  $\square$

**Lemma 3.6** If  $\succ_i$  is an ordering for all  $i : 1 \dots n$  then  $\succ_e$  is an ordering.

**Proof** Irreflexivity: since  $N_i(s) = N_i(s)$ , by irreflexivity of  $\succ_i$  (and hence of  $\succ_i$ ) for all  $i : 1 \dots n$ , we have  $N_i(s) \not\succ_i N_i(s)$  and therefore  $s \not\succ_e s$ .

Transitivity: suppose  $s \succ_e t$  and  $t \succ_e u$  with  $N_1(s) = N_1(t), \dots, N_{j-1}(s) = N_{j-1}(t), N_j(s) \succ_j N_j(t)$  and  $N_1(t) = N_1(u), \dots, N_{k-1}(t) = N_{k-1}(u), N_k(t) \succ_k N_k(u)$ , for some  $j, k : 1 \dots n$ . Then, by transitivity of  $\succ_j$  and  $\succ_k$  (and hence of  $\succ_j$  and  $\succ_k$ ), we have  $N_1(s) = N_1(u), \dots, N_{m-1}(s) = N_{m-1}(u), N_m(s) \succ_k N_m(u)$ , for  $m = \min(j, k)$ .  $\square$

**Lemma 3.7** If  $\succ_i$  is well-founded for all  $i : 1 \dots n$  then  $\succ_e$  is well-founded.

**Proof** Suppose that there exists an infinite sequence  $t_1 \succ_t t_2 \succ_t \dots$ . By well-foundedness of  $\succ_1$  and hence of  $\succ_1$ , there must be some  $t_j$  in the sequence s.t.  $N_1(t_j) = N_1(t_k)$  for all  $k > j$ . Let  $t_1^1 \succ_t t_2^1 \succ_t \dots$  the infinite subsequence starting from  $t_j$ . Repeating this reasoning  $n$  times it follows that there must be an infinite subsequence  $t_1^n \succ_t t_2^n \succ_t \dots$  s.t.  $N_i(t_1^n) = N_i(t_2^n) = \dots$  for all  $i : 1 \dots n$ , which contradicts the existence of such an infinite decreasing sequence.  $\square$

Although  $\succ_e$  is not monotonic in general it fulfils the following property which we call *quasi-monotonicity* and which, as we will see, is enough to ensure termination of TRS's.

**Lemma 3.8** Let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F})$ . If  $\succ_i$  is monotonic and  $t \in \mathcal{T}(\mathcal{F}_i)$  whenever  $s \in \mathcal{T}(\mathcal{F}_i)$  for all  $i : 1 \dots n$  then  $s \succ_e t$  implies  $f(\dots s \dots) \succ_e f(\dots t \dots)$ , i.e.  $\succ_e$  is *quasi-monotonic*.

**Proof** Assume  $N_1(s) = N_1(t) \wedge \dots \wedge N_{j-1}(s) = N_{j-1}(t) \wedge N_j(s) \succ_j N_j(t)$ , for some  $j : 1 \dots n$ . We have to show  $f(\dots s \dots) \succ_e f(\dots t \dots)$ . By proposition 3.3, since  $t \in \mathcal{T}(\mathcal{F}_i)$  whenever  $s \in \mathcal{T}(\mathcal{F}_i)$  for all  $i : 1 \dots n$ , we have  $N_1(f(\dots s \dots)) \succ_1 N_1(f(\dots t \dots)) \wedge \dots \wedge N_{j-1}(f(\dots s \dots)) \succ_{j-1} N_{j-1}(f(\dots t \dots))$ . Then assume  $N_1(f(\dots s \dots)) = N_1(f(\dots t \dots)) \wedge \dots \wedge N_{j-1}(f(\dots s \dots)) = N_{j-1}(f(\dots t \dots))$  (otherwise the lemma already holds):

- If  $f \notin \mathcal{F}_j$  then  $N_j(f(\dots s \dots)) = N_j(s) \cup X$  and  $N_j(f(\dots t \dots)) = N_j(t) \cup X$  for some  $X$  obtained from the transformations of the other top level arguments, and therefore  $N_j(f(\dots s \dots)) \succ_j N_j(f(\dots t \dots))$ .

- If  $f \in \mathcal{F}_j$  then  $N_j(f(\dots s \dots)) = \{f(\dots s' \dots) \mid s' \in N_j(s) \cup Z_j(s)\}$  and  $N_j(f(\dots t \dots)) = \{f(\dots t' \dots) \mid t' \in N_j(t) \cup Z_j(t)\}$ . Since  $t \in \mathcal{T}(\mathcal{F}_j)$  whenever  $s \in \mathcal{T}(\mathcal{F}_j)$ , either  $Z_j(s) = Z_j(t) = \emptyset$  or  $Z_j(s) \neq \emptyset$ . Therefore, by monotonicity of  $\succ_j$ ,  $N_j(f(\dots s \dots)) \succ_j N_j(f(\dots t \dots))$ .

Since in both cases,  $N_j(f(\dots s \dots)) \succ_j N_j(f(\dots t \dots))$ , it holds that  $f(\dots s \dots) \succ_e f(\dots t \dots)$ .  $\square$

### 3.3 Terms with variables and TRS's

Let  $R$  be  $R_1 \uplus \dots \uplus R_n$  and let  $\succ_i$  be reduction orderings on  $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$  containing the terminating rewrite relation  $\rightarrow_{R_i}^+$  for all  $i : 1 \dots n$ . In the following consider the ordering  $\succ_e$  lifted to terms with variables as in definition 2.1, which is stable under substitutions.

A possible way to prove termination of  $R$  over  $\mathcal{T}(\mathcal{F})$  (see e.g. [Der87]) is to show that there exists a well-founded ordering  $\succ$  on  $\mathcal{T}(\mathcal{F})$  s.t.:

1.  $l\sigma \succ r\sigma$  for all rules  $l \rightarrow r \in R$  and all substitutions  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ .
2.  $s \rightarrow_R t$  and  $s \succ t$  implies  $f(\dots s \dots) \succ f(\dots t \dots)$  for all  $s, t \in \mathcal{T}(\mathcal{F})$  and  $f \in \mathcal{F}$ .

From these two properties we can derive that  $s \succ t$  whenever  $s \rightarrow_R t$  for all terms  $s, t \in \mathcal{T}(\mathcal{F})$ , i.e.  $R$  is terminating over  $\mathcal{T}(\mathcal{F})$ . Property (2) is fulfilled by  $\succ_e$ :

**Lemma 3.9** Let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F})$ . Then  $s \rightarrow_R t$  and  $s \succ_e t$  implies  $f(\dots s \dots) \succ_e f(\dots t \dots)$  for all  $f \in \mathcal{F}$ .

**Proof** Since if  $s \rightarrow_R t$  then  $t \in \mathcal{T}(\mathcal{F}_i)$  whenever  $s \in \mathcal{T}(\mathcal{F}_i)$  for all  $i : 1 \dots n$  and all  $\succ_i$  are monotonic orderings, by lemma 3.8  $s \succ_e t$  implies  $f(\dots s \dots) \succ_e f(\dots t \dots)$  for all  $f \in \mathcal{F}$ .  $\square$

However, property (1) does not hold in general for  $\succ_e$ . This is not surprising since this result would imply the termination of the direct sum of any two terminating rewriting systems, which is well-known to be false, as shown by Toyama's counter example ([Toy87]). If we apply our method to this example, indeed the resulting ordering does not fulfil the property:

**Example 3.10** Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be  $\{f, 1, 0_1\}$  and  $\{g, 0_2\}$  respectively and let  $\succ_1$  and  $\succ_2$  be the orderings generated by the TRS's  $R_1 = \{f(1, 0_1, x) \rightarrow f(x, x, x)\}$  and  $R_2 = \{g(x, y) \rightarrow x, g(x, y) \rightarrow y\}$  respectively. Then with the substitution  $\{x \mapsto g(1, 0_1)\}$  we have:

$N_1(f(1, 0_1, g(1, 0_1))) = \{f(1, 0_1, 1), f(1, 0_1, 0_1), f(1, 0_1, 0_1)\}$  and  
 $N_1(f(g(1, 0_1), g(1, 0_1), g(1, 0_1))) = \{f(1, 1, 1), \dots, f(1, 0_1, 1), f(1, 0_1, 0_1), f(1, 0_1, 0_1), \dots, f(0_1, 0_1, 0_1)\}$ , which implies  $f(g(1, 0_1), g(1, 0_1), g(1, 0_1)) \succ_e f(1, 0_1, g(1, 0_1))$ , contradicting the rule of  $R_1$ .  $\square$

In fact, the problem is that although  $\succ_e$  extends each  $\succ_i$  at the ground level, this becomes false at the variable level. Nevertheless, by imposing some syntactic conditions on the TRS's, we can prove that  $\succ_e$  includes the rewrite relation  $\rightarrow_R$  on  $\mathcal{T}(\mathcal{F})$ . We consider the case where all rules in  $R$  are *right linear*:



**Definition 3.11** A term  $t$  is *linear* if all variables in  $\text{Vars}(t)$  occur only once in  $t$ . A TRS  $R$  is *right linear* if the right hand sides of all rules in  $R$  are linear.

**Proposition 3.12** Let  $R$  be a right-linear TRS over a signature  $\mathcal{F}$  and let  $0$  be a new constant symbol (not occurring in  $R$ ). Then  $R$  is terminating iff  $R'' = R \cup R'$  is terminating, where  $R' = \{f(\dots 0 \dots) \rightarrow 0 \mid f \in \mathcal{F}\}$ .

**Proof** The right to left implication is trivial. For the the left to right implication, let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F} \cup \{0\})$  and let  $\succ$  be a well-founded ordering defined as:  $s \succ t$  iff  $|s|_0 > |t|_0$  or  $|s|_0 = |t|_0$  and  $s \rightarrow_R^+ t$  (note that if  $R$  is terminating then in particular  $R$  is terminating over  $\mathcal{T}(\mathcal{F} \cup \{0\})$ ). We will prove that  $R''$  is terminating over  $\mathcal{T}(\mathcal{F} \cup \{0\})$ , and hence terminating. Assume that  $t_1$  is a minimal (wrt.  $\succ$ ) term s.t. there is an infinite rewriting sequence with  $\rightarrow_{R''}$  starting from  $t_1$ . Then since  $R'$  is obviously terminating the sequence must be of the form  $t_1 \rightarrow_{R'}^* t_2 \rightarrow_R t_3 \rightarrow_{R''} \dots$ . Now we have  $|t_1|_0 \geq |t_2|_0$  and by right linearity of  $R$  also  $|t_2|_0 \geq |t_3|_0$ . Assume  $|t_1|_0 = |t_2|_0 = |t_3|_0$ , otherwise it contradicts the minimality (wrt.  $\succ$ ) of  $t_1$ . Let  $l \rightarrow r$  be the rewrite rule in  $R$  applied to  $t_2$ , i.e.  $t_2|_p = l\sigma$  and  $t_3 = t_2[r\sigma]_p$  for some substitution  $\sigma$ . Then, as  $r$  is linear, for all  $x \in \text{Vars}(l)$  if  $0$  occurs in  $x\sigma$  then  $x$  occurs only once in  $l$  and  $x$  occurs in  $r$ . Therefore there exists some substitution  $\sigma'$  s.t.  $t_1|_p = l\sigma'$  and  $t_1[r\sigma'] = t_2' \rightarrow_{R'}^* t_3$ , which since  $t_1 \succ t_2'$  contradicts the minimality (wrt.  $\succ$ ) of  $t_1$ .  $\square$

Note that this property do not hold for non right linear TRS:

**Example 3.13** The rewrite system  $R = \{f(x, x) \rightarrow f(g(x), h(x))\}$  is terminating. But  $R'' = R \cup \{f(0, x) \rightarrow 0, f(x, 0) \rightarrow 0, g(0) \rightarrow 0, h(0) \rightarrow 0\}$  is not, since we have the following infinite sequence:

$$f(0, 0) \rightarrow_{R''} f(g(0), h(0)) \rightarrow_{R''} f(0, h(0)) \rightarrow_{R''} f(0, 0) \rightarrow_{R''} \dots \quad \square$$

By this proposition we can consider that if  $R_i$  is right linear and terminating then there is a reduction ordering  $\succ_i$  on  $\mathcal{T}(\mathcal{F}_i)$  including  $\rightarrow_{R_i}^+$ , where  $\mathcal{F}_i$  contains a constant symbol, which is taken as  $0_i$ , s.t.  $f(\dots 0_i \dots) \succ_i 0_i$ .

**Proposition 3.14** Let  $s$  be a term in  $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$  for some  $i : 1 \dots n$ .  $N_j(s\sigma) = \bigcup_{x \in \mathcal{M}\text{vars}(s)} N_j(x\sigma)$  for all  $j : 1 \dots n$  with  $i \neq j$  and all ground substitutions  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$  (remind that  $\mathcal{M}\text{vars}(s)$  denotes the multiset of variables occurring in  $s$ ).

**Proof** By induction on the size of  $s$ . If  $s$  is a variable then the property trivially holds. Otherwise, let  $s$  be  $f(s_1, \dots, s_p)$ . Then, as  $f \notin \mathcal{F}_j$ , we have  $N_j(f(s_1, \dots, s_p)) = N_j(s_1) \cup \dots \cup N_j(s_p)$ . Now by induction hypothesis  $N_j(s_k) = \bigcup_{x \in \mathcal{M}\text{vars}(s_k)} N_j(x\sigma)$  for all  $k : 1 \dots p$ , which, since  $\mathcal{M}\text{vars}(s) = \bigcup_{k=1}^p \mathcal{M}\text{vars}(s_k)$ , implies  $N_j(s\sigma) = \bigcup_{x \in \mathcal{M}\text{vars}(s)} N_j(x\sigma)$ .  $\square$

**Proposition 3.15** Let  $t$  be a term in  $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$  for some  $i : 1 \dots n$ . If  $t$  is not a variable then  $N_i(t\sigma) \supseteq \{t\sigma' \mid x\sigma' \in N_i(x\sigma) \cup Z_i(x\sigma) \wedge x \in \text{Vars}(t)\}$  for all ground substitutions  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ . Moreover if  $t$  is linear then  $\max_{\succ_i}(N_i(t\sigma)) \subseteq \{t\sigma' \mid x\sigma' \in N_i(x\sigma) \cup Z_i(x\sigma) \wedge x \in \text{Vars}(t)\}$  for all ground substitutions  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ .

**Proof** The first part is obvious. For the second part we proceed by induction on the size of  $t$ . If  $t$  is a variable it holds trivially. Otherwise let  $t$  be  $f(t_1, \dots, t_n)$ . Then  $N_i(f(t_1\sigma, \dots, t_n\sigma)) = \{f(t'_1, \dots, t'_n) \mid t'_i \in N_i(t_j\sigma) \cup Z_i(t_j\sigma)\}$ .

If  $t_j$  is not a variable then if  $t_j\sigma \in \mathcal{T}(\mathcal{F}_i)$  then  $Z_i(t_j\sigma) = \emptyset$ . Otherwise (if  $t_j\sigma \in \mathcal{T}(\mathcal{F}_i)$ ) we have  $Z_i(t_j\sigma) = \{0_j\}$  but then, since  $t_j$  is not a variable  $\text{Head}(t_j\sigma) \in \mathcal{F}_i$ , and as  $t_j\sigma \notin \mathcal{T}(\mathcal{F}_i)$ , there must be an immediate subterm of  $t_j\sigma$  which is not in  $\mathcal{T}(\mathcal{F}_i)$ . Therefore, there is a term  $t'$  in  $N_i(t_j\sigma)$  s.t.  $0_j$  is a proper subterm of  $t'$ , i.e.  $t' \succ_i 0_j$ . Therefore, since by induction hypothesis  $\max_{\succ_i}(N_i(t_j\sigma)) \subseteq \{t_j\sigma' \mid x\sigma' \in N_i(x\sigma) \cup Z_i(x\sigma) \wedge x \in \text{Vars}(t)\}$ , we have  $\max_{\succ_i}(N_i(t_j\sigma) \cup Z_i(t_j\sigma)) \subseteq \{t_j\sigma' \mid x\sigma' \in N_i(x\sigma) \cup Z_i(x\sigma) \wedge x \in \text{Vars}(t)\}$  (note that if  $t_j$  is a variable this trivially holds).

Since  $t$  is linear, variables occurring in some  $t_j$  do not occur elsewhere, therefore, by monotonicity of  $\succ_i$ ,  $\max_{\succ_i}(N_i(f(t_1\sigma, \dots, t_n\sigma))) \subseteq \{t\sigma' \mid x\sigma' \in N_i(x\sigma) \cup Z_i(x\sigma) \wedge x \in \text{Vars}(t)\}$ .  $\square$

**Lemma 3.16** Let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$  for some  $i : 1 \dots n$ . If  $\succ_i$  is stable under substitutions and  $t$  is linear then  $s \succ_i t$  implies  $s\sigma \succ_e t\sigma$  for all substitutions  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ , i.e.  $s \succ_e t$ .

**Proof** Since  $t$  is linear and  $\text{Vars}(s) \supseteq \text{Vars}(t)$ , by proposition 3.14 we have  $N_j(s\sigma) \supseteq N_j(t\sigma)$  for all  $j : 1 \dots i - 1$ . Therefore either  $N_j(s\sigma) \succ_j N_j(t\sigma)$  for some  $j : 1 \dots i - 1$  which implies  $s\sigma \succ_e t\sigma$ , or  $N_j(s\sigma) = N_j(t\sigma)$  for all  $j : 1 \dots i - 1$ . But then by proposition 3.15 we have  $N_i(s\sigma) \supseteq \{s\sigma' \mid x\sigma' \in N_i(x\sigma) \cup Z_i(x\sigma) \wedge x \in \text{Vars}(s)\}$  and  $\max_{\succ_i}(N_i(t\sigma)) \subseteq \{t\sigma' \mid x\sigma' \in N_i(x\sigma) \cup Z_i(x\sigma) \wedge x \in \text{Vars}(t)\}$ . Therefore, by stability under substitutions of  $\succ_i$  we have that for each  $t'$  in  $\max_{\succ_i}(N_i(t\sigma))$  there is a term  $s'$  in  $N_i(s\sigma)$  s.t.  $s' \succ_i t'$ . Therefore  $N_i(s\sigma) \succ_i N_i(t\sigma)$ , and hence  $s\sigma \succ_e t\sigma$ .  $\square$

This lemma leads to a constructive proof of modularity of termination for direct sums of right linear TRS's:

**Theorem 3.17** Let  $R_1, \dots, R_n$  be right linear terminating TRS's over  $\mathcal{T}(\mathcal{F}_1), \dots, \mathcal{T}(\mathcal{F}_n)$  respectively. Then  $R = R_1 \uplus \dots \uplus R_n$  is terminating.

**Proof** If  $R_1, \dots, R_n$  are terminating then  $\rightarrow_{R_1} \dots \rightarrow_{R_n}$  are respectively included in some reduction orderings  $\succ_1 \dots \succ_n$ . Then, by lemmas 3.7, 3.6, 3.9 and 3.16 the extension ordering  $\succ_e$  of  $\succ_1 \dots \succ_n$  is a well-founded ordering s.t.  $s \succ_e t$  for all ground terms  $s, t \in \mathcal{T}(\mathcal{F})$  with  $s \rightarrow_R t$ , i.e.  $R_1 \uplus \dots \uplus R_n$  is terminating.  $\square$

## 4 Modularity of Simple termination of TRS's

In this section *simple termination* (which for finite signatures implies termination) is considered:

**Definition 4.1** A TRS  $R$  (with left and right hand sides in  $\mathcal{T}(\mathcal{F})$ ) is *simply terminating* over  $\mathcal{T}(\mathcal{F})$  iff there exists some simplification ordering  $\succ$  s.t.  $l\sigma \succ r\sigma$  for all  $l \rightarrow r$  in  $R$  and substitutions  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ . We say that  $R$  is simply terminating (in general) if  $R$  is simply terminating over  $\mathcal{T}(\mathcal{F} \cup \mathcal{F}')$  for any set of new function symbols  $\mathcal{F}'$ .

At the end of this section we will see that if a TRS (with left and right hand sides in  $\mathcal{T}(\mathcal{F})$ ) is simply terminating over  $\mathcal{T}(\mathcal{F})$  then it is simply terminating.

Although the ordering  $\succ_e$  can be shown to preserve the subterm property, due to its restrictions on the applicability to terms with variables it is not to be powerful enough to prove modularity of simple termination. Here we present a new extension ordering which extends any collection of simplification orderings at the ground level and at the variable level. Therefore, the modularity of simple termination follows.

#### 4.1 A transformation on terms

**Definition 4.2** Let  $S_i$  be a mapping from terms in  $\mathcal{T}(\mathcal{F})$  to terms in  $\mathcal{T}(\mathcal{F}_i)$  recursively defined as:

$$S_i(f(t_1, \dots, t_m)) = \begin{cases} 0_i & \text{if } f \notin \mathcal{F}_i \\ f(S_i(t_1), \dots, S_i(t_m)) & \text{otherwise} \end{cases}$$

where  $0_i$  is some (fixed) constant symbol in  $\mathcal{F}_i$ .

Let  $S$  be a mapping from terms in  $\mathcal{T}(\mathcal{F})$  to terms in  $\cup_{i=1}^n \mathcal{T}(\mathcal{F}_i)$  defined as:

$$S(t) = S_i(t) \quad \text{if } \text{Head}(t) \in \mathcal{F}_i$$

Let  $S_E$  be a mapping from terms in  $\mathcal{T}(\mathcal{F})$  to multisets of terms in  $\cup_{i=1}^n \mathcal{T}(\mathcal{F}_i)$  defined as:

$$S_E(t) = \{S(t') \mid t \succeq_{emb} t'\}$$

**Proposition 4.3** For all  $i : 1 \dots n$  and for all ground terms  $t$  in  $\mathcal{T}(\mathcal{F}_i)$ ,  $S(t) = \{t\}$  and  $S_E(t) = \{t' \mid t \succeq_{emb} t'\}$ .

**Proposition 4.4** Let  $t_1, \dots, t_p$  be ground terms in  $\mathcal{T}(\mathcal{F})$  and let  $f$  be a function symbol in  $\mathcal{F}$ . Then  $S_E(f(t_1, \dots, t_p)) = S_E(t_1) \cup \dots \cup S_E(t_p) \cup \{S(f(t'_1, \dots, t'_p)) \mid t_j \succeq_{emb} t'_j\}$ .

#### 4.2 Simplification extension orderings

Here a method for extending simplification orderings on ground terms built over disjoint signatures is given. This method produces an extension ordering which preserves monotonicity and subterm property, i.e. we have a simplification extension ordering.

**Definition 4.5** Let  $s, t$  be terms in  $\mathcal{T}(\mathcal{F})$ . Then  $s = f(s_1, \dots, s_p) \succ_s g(t_1, \dots, t_q) = t$  iff

- $\max_{\succ_u}(S_E(s)) \succ_u \max_{\succ_u}(S_E(t))$  or
- $\max_{\succ_u}(S_E(s)) = \max_{\succ_u}(S_E(t))$  and  $f = g$  and  $\{s_1, \dots, s_p\} \succ_s \{t_1, \dots, t_p\}$

where the relation  $\succ_u$  is defined as  $\cup_{i=1}^n \succ_i$ .

Note that, since all  $\gamma_i$  are orderings on  $\mathcal{T}(\mathcal{F}_i)$ , if  $s \gamma_u t$  then  $s, t \in \mathcal{T}(\mathcal{F}_i)$  and  $s \gamma_i t$  for some  $i : 1 \dots n$ , which implies that  $\gamma_u$  is an ordering whenever all  $\gamma_i$  also are.

**Proposition 4.6** If  $\gamma_i$  is a monotonic ordering on  $\mathcal{T}(\mathcal{F}_i)$  fulfilling the subterm property for some  $i : 1 \dots n$  then  $\gamma_s$  extends  $\gamma_i$ .

**Proof** Assume  $s \gamma_i t$  for some  $s, t \in \mathcal{T}(\mathcal{F}_i)$ . By proposition 4.3,  $S_E(s) = \{s' \mid s \succeq_{emb} s'\}$  and  $S_E(t) = \{t' \mid t \succeq_{emb} t'\}$ . Then since  $\gamma_i$  is a monotonic ordering fulfilling the subterm property we have  $max_{\gamma_u}(S_E(s)) = \{s\}$  and  $max_{\gamma_u}(S_E(t)) = \{t\}$  and therefore  $max_{\gamma_u}(S_E(s)) \succ_u max_{\gamma_u}(S_E(t))$ , which implies  $s \gamma_s t$ .

**Lemma 4.7** If  $\gamma_i$  are orderings on  $\mathcal{T}(\mathcal{F}_i)$  for all  $i : 1 \dots n$  then  $\gamma_s$  is an ordering on  $\mathcal{T}(\mathcal{F})$ .

**Proof** Irreflexivity:  $s \not\gamma_s s$ . By induction on the size of  $s$ . By irreflexivity of  $\gamma_i$  for all  $i : 1 \dots n$ , we have that  $\gamma_u$  is irreflexive and hence  $\not\gamma_u$ . Therefore, since  $S_E(s) = S_E(s)$  and hence  $max_{\gamma_u}(S_E(s)) = max_{\gamma_u}(S_E(s))$ , it holds  $max_{\gamma_u}(S_E(s)) \not\gamma_u max_{\gamma_u}(S_E(s))$ . If  $s$  is of the form  $f(s_1, \dots, s_p)$ , by induction hypothesis, we have  $\{s_1, \dots, s_p\} \not\gamma_s \{s_1, \dots, s_p\}$ . Therefore  $s \not\gamma_s s$ .

Transitivity:  $s \gamma_s t$  and  $t \gamma_s u$  implies  $s \gamma_s u$ . By induction on the size of  $s, t$  and  $u$ . Since all  $\gamma_i$  are transitive,  $\gamma_u$  is also transitive and hence  $\succ_u$ . Therefore if  $max_{\gamma_u}(S_E(s)) \succ_u max_{\gamma_u}(S_E(t))$  or  $max_{\gamma_u}(S_E(t)) \succ_u max_{\gamma_u}(S_E(u))$  then  $max_{\gamma_u}(S_E(s)) \succ_u max_{\gamma_u}(S_E(u))$ . Otherwise,  $max_{\gamma_u}(S_E(s)) = max_{\gamma_u}(S_E(t))$  and  $max_{\gamma_u}(S_E(t)) = max_{\gamma_u}(S_E(u))$ . Then  $s, t$  and  $u$  are of the form  $s = f(s_1, \dots, s_p)$ ,  $t = f(t_1, \dots, t_p)$  and  $u = f(u_1, \dots, u_p)$ , with  $\{s_1, \dots, s_p\} \succ_s \{t_1, \dots, t_p\}$  and  $\{t_1, \dots, t_p\} \succ_s \{u_1, \dots, u_p\}$ . Therefore, by induction hypothesis  $\{s_1, \dots, s_p\} \succ_s \{u_1, \dots, u_p\}$  and hence  $s \gamma_s u$ .  $\square$

**Lemma 4.8** If  $\gamma_i$  are orderings on  $\mathcal{T}(\mathcal{F}_i)$  fulfilling the subterm property for all  $i : 1 \dots n$  then  $\gamma_s$  fulfils the subterm property on  $\mathcal{T}(\mathcal{F})$ , i.e.  $f(\dots s \dots) \gamma_s s$  for all  $s \in \mathcal{T}(\mathcal{F})$  and  $f \in \mathcal{F}$ .

**Proof** Assume  $f \in \mathcal{F}_i$ . Since obviously  $S_E(f(\dots s \dots)) \supset S_E(s)$  (note that all terms embedded in  $s$  are also embedded in  $f(\dots s \dots)$ ), we have that  $max_{\gamma_u}(S_E(f(\dots s \dots))) \supseteq max_{\gamma_u}(S_E(s))$ , which implies  $max_{\gamma_u}(S_E(f(\dots s \dots))) \not\prec_u max_{\gamma_u}(S_E(s))$ . Now we will show that  $max_{\gamma_u}(S_E(f(\dots s \dots))) = max_{\gamma_u}(S_E(s))$  is not possible. Suppose it is. Then, since there are terms headed by  $f$  in  $S_E(f(\dots s \dots))$  (and hence there are terms in  $max_{\gamma_u}(S_E(f(\dots s \dots)))$  headed by symbols in  $\mathcal{F}_i$ ), there should be a term  $s_1$  headed by a symbol  $g$  in  $\mathcal{F}_i$ , s.t.  $s \succeq_{emb} s_1$  and  $S_i(s_1) \in max_{\gamma_u}(S_E(s))$ . Therefore we have  $f(\dots s \dots) \succeq_{emb} f(\dots s_1 \dots)$  and  $S(f(\dots s_1 \dots)) = S_i(f(\dots s_1 \dots)) = f(\dots S_i(s_1) \dots) \gamma_i S_i(s_1)$  (as  $f, g \in \mathcal{F}_i$  and  $\gamma_i$  fulfils the subterm property). This, by transitivity of all  $\gamma_i$ , leads to a contradiction since there is a term in  $S_E(f(\dots s \dots))$  which is greater (wrt.  $\gamma_i$  and hence wrt.  $\gamma_u$ ) than a term in  $max_{\gamma_u}(S_E(s))$ , i.e.  $max_{\gamma_u}(S_E(f(\dots s \dots))) \neq max_{\gamma_u}(S_E(s))$ .  $\square$

**Proposition 4.9** If  $\max_{\gamma_u}(S_E(s)) \succeq_u \max_{\gamma_u}(S_E(t))$  then for all  $t'$  with  $t \succeq_{emb} t'$  there is some  $s$  with  $s \succeq_{emb} s'$  s.t.  $S_i(s') \succeq_i S_i(t')$  for all  $i : 1 \dots n$ .

**Proof** If  $\max_{\gamma_u}(S_E(s)) \succeq_u \max_{\gamma_u}(S_E(t))$  then for all  $t'' \in S_E(t)$  there exists a  $s'' \in S_E(s)$  s.t.  $s'' \succeq_u t''$ . Therefore, since  $S_E(s) = \{S(s') \mid s \succeq_{emb} s'\}$  and  $S_E(t) = \{S(t') \mid t \succeq_{emb} t'\}$ , for all  $t'$  with  $t \succeq_{emb} t'$  there is some  $s$  with  $s \succeq_{emb} s'$  s.t.  $S(s') \succeq_u S(t')$ . Now, since  $\gamma_u$  is the union of all  $\gamma_i$  for  $i : 1 \dots n$  and each  $\gamma_i$  can only compare terms in  $\mathcal{T}(\mathcal{F}_i)$ , if  $S(s') \succeq_u S(t')$  then  $\text{Head}(s)$  and  $\text{Head}(t)$  must be in  $\mathcal{F}_j$  for some  $j : 1 \dots n$  and  $S_j(s') \succeq_j S_j(t')$ . Moreover  $S_k(s') = S_k(t') = 0_k$  for all  $k : 1 \dots n$  and  $k \neq j$ . Therefore,  $S_i(s') \succeq_i S_i(t')$  for all  $i : 1 \dots n$ .  $\square$

**Proposition 4.10** If  $\gamma_i$  are monotonic orderings on  $\mathcal{T}(\mathcal{F}_i)$  for all  $i : 1 \dots n$  then  $\max_{\gamma_u}(S_E(s)) \succeq_u \max_{\gamma_u}(S_E(t))$  implies  $\max_{\gamma_u}(S_E(f(\dots s \dots))) \succeq_u \max_{\gamma_u}(S_E(f(\dots t \dots)))$  for all  $s, t \in \mathcal{T}(\mathcal{F})$  and  $f \in \mathcal{F}$ .

**Proof** We will prove instead that for all  $t'' \in S_E(f(u_1 \dots u_p, t, v_1 \dots v_q))$  there exists some  $s'' \in S_E(f(u_1 \dots u_p, s, v_1 \dots v_q))$  s.t.  $s'' \succeq_u t''$ , since by transitivity of  $\gamma_u$  (as all  $\gamma_i$  are transitive), we have that  $\max_{\gamma_u}(X) \succeq_u \max_{\gamma_u}(Y)$  iff for all  $y \in Y$  there exists some  $x \in X$  s.t.  $x \succeq_u y$  for all multisets  $X$  and  $Y$  with terms in  $\mathcal{T}(\mathcal{F})$ . By proposition 4.4 we have:

$$\begin{aligned} S_E(f(u_1 \dots u_p, s, v_1 \dots v_q)) &= S_E(u_1) \cup \dots \cup S_E(u_p) \cup S_E(s) \cup S_E(v_1) \cup \dots \cup S_E(v_q) \cup \\ &\{S(f(u'_1 \dots u'_p, s', v'_1 \dots v'_q)) \mid u_j \succeq_{emb} u'_j \wedge s \succeq_{emb} s' \wedge v_k \succeq_{emb} v'_k\} \text{ and} \\ S_E(f(u_1 \dots u_p, t, v_1 \dots v_q)) &= S_E(u_1) \cup \dots \cup S_E(u_p) \cup S_E(t) \cup S_E(v_1) \cup \dots \cup S_E(v_q) \cup \\ &\{S(f(u'_1 \dots u'_p, t', v'_1 \dots v'_q)) \mid u_j \succeq_{emb} u'_j \wedge t \succeq_{emb} t' \wedge v_k \succeq_{emb} v'_k\}. \end{aligned}$$

Then on one hand, since  $\max_{\gamma_u}(S_E(s)) \succeq_u \max_{\gamma_u}(S_E(t))$ , we have that for all  $t'' \in S_E(t)$  there is some  $s'' \in S_E(s)$  s.t.  $s'' \succeq_u t''$ . On the other hand, by proposition 4.9, we have for all  $t'$  with  $t \succeq_{emb} t'$  there is some  $s$  with  $s \succeq_{emb} s'$  s.t.  $S_i(s') \succeq_i S_i(t')$  for all  $i : 1 \dots n$ , which by monotonicity of each  $\gamma_i$  implies that for all  $t'' \in \{S(f(u'_1 \dots u'_p, t', v'_1 \dots v'_q)) \mid u_j \succeq_{emb} u'_j \wedge t \succeq_{emb} t' \wedge v_k \succeq_{emb} v'_k\}$  there is some  $s'' \in \{S(f(u'_1 \dots u'_p, s', v'_1 \dots v'_q)) \mid u_j \succeq_{emb} u'_j \wedge s \succeq_{emb} s' \wedge v_k \succeq_{emb} v'_k\}$  s.t.  $s'' \succeq_u t''$ : note that if  $f \in \mathcal{F}_i$  then

$$\begin{aligned} S(f(u'_1 \dots u'_p, s', v'_1 \dots v'_q)) \succeq_u S(f(u'_1 \dots u'_p, t', v'_1 \dots v'_q)) &\text{ is equivalent to} \\ S_i(f(u'_1 \dots u'_p, s', v'_1 \dots v'_q)) = f(S_i(u'_1) \dots S_i(u'_p), S_i(s'), S_i(v'_1) \dots S_i(v'_q)) &\succeq_i \\ f(S_i(u'_1) \dots S_i(u'_p), S_i(t'), S_i(v'_1) \dots S_i(v'_q)) = S_i(f(u'_1 \dots u'_p, t', v'_1 \dots v'_q)). & \end{aligned}$$

All together ensures that for all  $t'' \in S_E(f(u_1 \dots u_p, t, v_1 \dots v_q))$  there exists some  $s'' \in S_E(f(u_1 \dots u_p, s, v_1 \dots v_q))$  s.t.  $s'' \succeq_u t''$ .  $\square$

**Lemma 4.11** If  $\gamma_i$  are monotonic orderings on  $\mathcal{T}(\mathcal{F}_i)$  for all  $i : 1 \dots n$  then  $\gamma_s$  is monotonic on  $\mathcal{T}(\mathcal{F})$ .

**Proof** We have to prove that  $s \succ_s t$  implies  $f(\dots s \dots) \succ_s f(\dots t \dots)$  for all  $s, t \in \mathcal{T}(\mathcal{F})$  and  $f \in \mathcal{F}$ . If  $s \succ_s t$  then  $\max_{\gamma_u}(S_E(s)) \succeq_u \max_{\gamma_u}(S_E(t))$ , which by proposition 4.10 implies  $\max_{\gamma_u}(S_E(f(\dots s \dots))) \succeq_u \max_{\gamma_u}(S_E(f(\dots t \dots)))$ . Assume  $\max_{\gamma_u}(S_E(f(\dots s \dots))) = \max_{\gamma_u}(S_E(f(\dots t \dots)))$  (otherwise the property already holds). But then, since  $\{\dots s \dots\} \succ_{\gamma_s} \{\dots t \dots\}$ , we have  $f(\dots s \dots) \succ_s f(\dots t \dots)$ .  $\square$

From the previous lemmas, the following theorem holds:

**Theorem 4.12** If  $\succ_i$  are simplification orderings on  $\mathcal{T}(\mathcal{F}_i)$  for all  $i : 1 \dots n$ . Then  $\succ_s$  is a simplification ordering on  $\mathcal{T}(\mathcal{F})$  extending  $\succ_i$  for all  $i : 1 \dots n$ .

### 4.3 Terms with variables and TRS's

Let  $R_i$  be simply terminating TRS's over  $\mathcal{T}(\mathcal{F}_i)$  for all  $i : 1 \dots n$ , and let  $R$  be  $R_1 \uplus \dots \uplus R_n$ . In the following consider the ordering  $\succ_s$  lifted to terms with variables as in definition 2.1, which is stable under substitutions.

**Proposition 4.13** Let  $s$  be a term in  $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$  for some  $i : 1 \dots n$ . If  $\succ_i$  is a simplification ordering and  $s$  is not a variable then  $\max_{\succ_u}(S_E(s\sigma)) = \max_{\succ_u}(\cup_{x \in \text{Vars}(s)} S_E(x\sigma) \cup \{s\sigma' \mid (x\sigma' \in \{S_i(s') \mid x\sigma \succeq_{emb} s'\}) \wedge x \in \text{Vars}(s)\})$  for all substitutions  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ .

**Proof** By induction on the size of  $s$ , assuming  $s = f(s_1, \dots, s_p)$ . By proposition 4.4, we have  $S_E(f(s_1\sigma, \dots, s_p\sigma)) = S_E(s_1\sigma) \cup \dots \cup S_E(s_p\sigma) \cup \{f(S_i(s'_1), \dots, S_i(s'_p)) \mid s_j\sigma \succeq_{emb} s'_j\}$  (note that  $S(f(\dots)) = S_i(f(\dots))$ , as  $f \in \mathcal{F}_i$ ).

First, as  $\succ_i$  is a simplification ordering,  $\max_{\succ_u}(\{f(S_i(s'_1), \dots, S_i(s'_p)) \mid s_j\sigma \succeq_{emb} s'_j\}) = \max_{\succ_u}(\{s\sigma' \mid (x\sigma' \in \{S_i(s') \mid x\sigma \succeq_{emb} s'\}) \wedge x \in \text{Vars}(s)\})$  (note that every term in the first set is embedded by or equal to someone in the second set).

Second, if  $s_j$  is not a variable then by induction hypothesis we have  $\max_{\succ_u}(S_E(s_j\sigma)) = \max_{\succ_u}(\cup_{x \in \text{Vars}(s_j)} S_E(x\sigma) \cup \{s_j\sigma' \mid (x\sigma' \in \{S_i(s') \mid x\sigma \succeq_{emb} s'\}) \wedge x \in \text{Vars}(s)\})$ , but by subterm property of  $\succ_i$  for all  $s''_j \in \{s_j\sigma' \mid (x\sigma' \in \{S_i(s') \mid x\sigma \succeq_{emb} s'\}) \wedge x \in \text{Vars}(s)\}$  there is some  $s'' \in \{s\sigma' \mid (x\sigma' \in \{S_i(s') \mid x\sigma \succeq_{emb} s'\}) \wedge x \in \text{Vars}(s)\}$ , s.t.  $s'' \succ_i s''_j$ . Therefore the part which has to be considered in  $\max_{\succ_u}(S_E(s_j\sigma))$  for all  $s_j$  (even if it is a variable) is  $\max_{\succ_u} \cup_{x \in \text{Vars}(s_j)} S_E(x\sigma)$ . Finally we have  $\max_{\succ_u}(S_E(f(s_1\sigma, \dots, s_p\sigma))) = \max_{\succ_u}(\cup_{j=1}^p \cup_{x \in \text{Vars}(s_j)} S_E(x\sigma) \cup \{s\sigma' \mid (x\sigma' \in \{S_i(s') \mid x\sigma \succeq_{emb} s'\}) \wedge x \in \text{Vars}(s)\})$ , which, since the result of  $\max_{\succ_u}$  do not change if we remove some repetitions, implies  $\max_{\succ_u}(S_E(s\sigma)) = \max_{\succ_u}(\cup_{x \in \text{Vars}(s)} S_E(x\sigma) \cup \{s\sigma' \mid (x\sigma' \in \{S_i(s') \mid x\sigma \succeq_{emb} s'\}) \wedge x \in \text{Vars}(s)\})$ .  $\square$

**Lemma 4.14** Let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$  for some  $i : 1 \dots n$ . If  $\succ_i$  is a simplification ordering and  $s\sigma' \succ_i t\sigma'$  for all substitutions  $\sigma'$  with range in  $\mathcal{T}(\mathcal{F}_i)$ , then  $s\sigma \succ_s t\sigma$  for all ground substitutions with range in  $\mathcal{T}(\mathcal{F})$ , i.e.  $s \succ_s t$ .

**Proof** If  $t$  is a variable then, by subterm property of  $\succ_s$ , the lemma holds. Otherwise by proposition 4.13, we have

$\max_{\succ_u}(S_E(s\sigma)) = \max_{\succ_u}(\cup_{x \in \text{Vars}(s)} S_E(x\sigma) \cup \{s\sigma' \mid (x\sigma' \in \{S_i(s') \mid x\sigma \succeq_{emb} s'\}) \wedge x \in \text{Vars}(s)\})$  and

$\max_{\succ_u}(S_E(t\sigma)) = \max_{\succ_u}(\cup_{x \in \text{Vars}(t)} S_E(x\sigma) \cup \{t\sigma' \mid (x\sigma' \in \{S_i(s') \mid x\sigma \succeq_{emb} s'\}) \wedge x \in \text{Vars}(t)\})$ . Since  $\text{Vars}(s) \supseteq \text{Vars}(t)$  and  $s\sigma' \succ_i t\sigma'$  for all substitutions  $\sigma'$  with range in  $\mathcal{T}(\mathcal{F}_i)$ , it holds that  $\max_{\succ_u}(S_E(s\sigma)) \succ_u \max_{\succ_u}(S_E(t\sigma))$  and hence  $s\sigma \succ_s t\sigma$ .  $\square$

**Theorem 4.15** Let  $R_1, \dots, R_n$  be simply terminating TRS's over  $\mathcal{T}(\mathcal{F}_1), \dots, \mathcal{T}(\mathcal{F}_n)$  respectively. Then  $R = R_1 \uplus \dots \uplus R_n$  is simply terminating over  $\mathcal{T}(\mathcal{F})$ .

**Proof** If  $R_i$  is simply terminating over  $\mathcal{T}(\mathcal{F}_i)$  for all  $i : 1 \dots n$  then, by definition, for each  $R_i$ , there exists some simplification ordering  $\succ_i$  on  $\mathcal{T}(\mathcal{F}_i)$ , s.t.  $l\sigma \succ_i r\sigma$  for all rules  $l \rightarrow r \in R_i$  and all substitution  $\sigma$  with range in  $\mathcal{T}(\mathcal{F}_i)$ . Therefore, by theorem 4.12 and lemma 4.14 it follows that  $\succ_s$  is a simplification ordering on  $\mathcal{T}(\mathcal{F})$ , s.t.  $l\sigma \succ_s r\sigma$  for all rule  $l \rightarrow r \in R$  and substitutions  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ , i.e.  $R$  is simply terminating over  $\mathcal{T}(\mathcal{F})$ .  $\square$

**Corollary 4.16** Let  $R$  be a TRS with left and right hand sides in  $\mathcal{T}(\mathcal{F}, \mathcal{X})$ . If  $R$  is simply terminating over  $\mathcal{T}(\mathcal{F})$  then  $R$  is simply terminating over  $\mathcal{T}(\mathcal{F} \cup \mathcal{F}')$  for any set of new function symbols  $\mathcal{F}'$ , i.e.  $R$  is simply terminating.

**Proof** Consider  $R' = \{f(\dots, x, \dots) \rightarrow x \mid f \in \mathcal{F}'\}$ . Then obviously  $R'$  is simply terminating over  $\mathcal{T}(\mathcal{F}')$ . Therefore by previous theorem  $R \uplus R'$  is simply terminating over  $\mathcal{T}(\mathcal{F} \cup \mathcal{F}')$ , and hence, in particular,  $R$  is simply terminating over  $\mathcal{T}(\mathcal{F} \cup \mathcal{F}')$ .  $\square$

From the theorem and the corollary it follows that simple termination is modular (note that, obviously, if  $R_i$  is simply terminating then  $R_i$  is simply terminating over  $\mathcal{T}(\mathcal{F}_i)$ ).

## 5 Modularity of Total termination of TRS's

In this section *total termination* (which implies simple termination) is considered:

**Definition 5.1** A TRS  $R$  (with left and right hand sides in  $\mathcal{T}(\mathcal{F}, \mathcal{X})$ ) is *totally terminating* iff there exists some well-founded monotonic ordering  $\succ$  total on  $\mathcal{T}(\mathcal{F})$  s.t.  $l\sigma \succ r\sigma$  for all  $l \rightarrow r$  in  $R$  and substitutions  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ .

Therefore the orderings  $\succ_e$  and  $\succ_s$  (defined in section 3.2 and 4.2) can not be used since they do not preserve totality of the initial orderings  $\succ_i$ , i.e. the orderings  $\succ_e$  and  $\succ_s$  are not necessarily total on  $\mathcal{T}(\mathcal{F})$  even if all  $\succ_i$  are total on  $\mathcal{T}(\mathcal{F}_i)$ . However, it is quite easy to adapt  $\succ_e$  to obtain an extension ordering  $\succ_t$ , which also preserves totality (and monotonicity). Moreover, the resulting extension ordering  $\succ_t$  is proved to need weaker sufficient conditions to extend orderings on terms with variables. Therefore, we obtain better results on modularity of total termination.

### 5.1 A transformation on terms

First we need to slightly modify the transformation defined in section 3.1 in order to obtain an ordering preserving monotonicity:

**Definition 5.2** Let  $T_i$  be a mapping from terms in  $\mathcal{T}(\mathcal{F})$  to multisets of terms in  $\mathcal{T}(\mathcal{F}_i)$  recursively defined as:

$$T_i(f(t_1, \dots, t_m)) = \begin{cases} \{f(t'_1, \dots, t'_m) \mid t'_j \in \{0_i\} \cup T_i(t_j)\} & \text{if } f \in \mathcal{F}_i \\ T_i(t_1) \cup \dots \cup T_i(t_m) & \text{otherwise} \end{cases}$$

where  $0_i$  is some (fixed) minimal wrt.  $\succ_i$  constant symbol in  $\mathcal{F}_i$ . Note that if  $\succ_i$  is a total ordering then there is only one minimal constant, and if  $\succ_i$  is also monotonic then  $0_i$  is the minimal term wrt.  $\succ_i$  in  $\mathcal{T}(\mathcal{F}_i)$ .

**Proposition 5.3** Let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F})$ . Then  $T_i(s) = T_i(t)$  implies  $T_i(f(\dots s \dots)) = T_i(f(\dots t \dots))$  for all  $i : 1 \dots n$  and  $f \in \mathcal{F}$ .

## 5.2 Total extension orderings

Now we describe a method for extending a collection of total orderings on ground terms built over disjoint signatures. This method allow us to obtain an extension ordering which preserves totality (in addition to monotonicity and well-foundedness), i.e. we have a total extension ordering.

**Definition 5.4** Let  $\succ_r$  be an arbitrary relation on  $\mathcal{T}(\mathcal{F})$  and let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F})$ . Then  $s \succ_t t$  iff

- $T_1(s) = T_1(t) \wedge \dots \wedge T_{i-1}(s) = T_{i-1}(t) \wedge T_i(s) \succ_i T_i(t)$  for some  $i : 1 \dots n$  or
- $T_i(s) = T_i(t)$  for all  $i : 1 \dots n$  and  $s \succ_r t$

**Proposition 5.5** If  $\succ_i$  is a monotonic ordering total on  $\mathcal{T}(\mathcal{F}_i)$  for some  $i : 1 \dots n$  then  $t \succeq_i t'$  for all  $t' \in T_i(t)$  and  $t \in \mathcal{T}(\mathcal{F}_i)$ .

**Proof** By induction on the size of  $t$ . Assume  $t = f(t_1, \dots, t_p)$ . Then  $T_i(t) = \{f(t'_1, \dots, t'_p) \mid t'_j \in \{0_i\} \cup T_i(t_j)\}$ . By induction hypothesis  $t_j \succeq_i t'_j$  for all  $t'_j \in T_i(t_j)$  and since  $\succ_j$  is a monotonic total ordering  $0_j$  is the minimal term wrt.  $\succ_j$ , i.e.  $t_j \succeq_i 0_j$ . Therefore by monotonicity  $t = f(t_1, \dots, t_p) \succeq_i t' = f(t'_1, \dots, t'_p)$  for all  $f(t'_1, \dots, t'_p) \in T_i(t)$ .  $\square$

**Proposition 5.6** If  $\succ_i$  is a monotonic ordering total on  $\mathcal{T}(\mathcal{F}_i)$  for some  $i : 1 \dots n$  then  $\succ_t$  extends  $\succ_i$ .

**Proof** Since for all  $i, j : 1 \dots n$  and for all ground terms  $u$  in  $\mathcal{T}(\mathcal{F}_i)$ ,  $T_j(u) = \emptyset$  if  $j \neq i$ , we have  $T_j(s) = T_j(t) = \emptyset$  for all  $j : 1 \dots i - 1$ . By proposition 5.5 we have  $s \succeq_i s'$  for all  $s' \in T_i(s)$  and  $t \succeq_i t'$  for all  $t' \in T_i(t)$ . Therefore, by transitivity,  $s \succ_i t$  implies  $T_i(s) \succ_i T_i(t)$ , and hence  $s \succ_t t$ .  $\square$

Note that if all  $\succ_i$  are not monotonic total orderings then the relation  $\succ_t$  do not extend each  $\succ_i$ . That is the reason why the interpretations are different in the definition of  $\succ_e$ . On the other hand with this new interpretations, as we will see,  $\succ_t$  also preserves monotonicity (which is not the case with  $\succ_e$ ).

**Lemma 5.7** If  $\succ_i$  for all  $i : 1 \dots n$  and  $\succ_r$  are orderings then  $\succ_t$  is an ordering.



**Proof Irreflexivity:** since  $T_i(s)$  is unique for all  $s \in \mathcal{T}(\mathcal{F})$  and all  $i : 1 \dots n$ , by irreflexivity of all  $\succ_i$  (and hence of  $\succ_i$ ), we have  $T_i(s) \not\succeq_i T_i(s)$  and by irreflexivity of  $\succ_r$  we have  $s \not\succeq_r s$ . Therefore  $s \not\succeq_t s$ .

**Transitivity:** let  $s, t$  and  $u$  be terms in  $\mathcal{T}(\mathcal{F})$  s.t.  $s \succ_t t$  and  $t \succ_t u$ . Let  $j$  and  $k$  be the minimal indexes in  $1 \dots n$  s.t.  $T_1(s) = T_1(t) \wedge \dots \wedge T_{j-1}(s) = T_{j-1}(t)$  and  $T_1(t) = T_1(u) \wedge \dots \wedge T_{k-1}(t) = T_{k-1}(u)$ . And let  $m = \min(j, k)$ . If  $m \leq n$  then, by transitivity of all  $\succ_i$  (and hence of  $\succ_i$ ), we have  $T_1(s) = T_1(u) \wedge \dots \wedge T_{m-1}(s) = T_{m-1}(u) \wedge T_m(s) \succ_m T_m(u)$ , and therefore  $s \succ_t u$ . Otherwise,  $T_i(s) = T_i(t) = T_i(u)$  for all  $i : 1 \dots n$  and  $s \succ_r t$  and  $t \succ_r u$ , which by transitivity of  $\succ_r$  implies  $s \succ_r u$  and hence  $s \succ_t u$ .  $\square$

**Lemma 5.8** If  $\succ_i$  for all  $i : 1 \dots n$  and  $\succ_r$  are monotonic then  $\succ_t$  is monotonic.

**Proof** We have to prove that  $s \succ_t t$  implies  $f(\dots s \dots) \succ_t f(\dots t \dots)$  for all  $s, t \in \mathcal{T}(\mathcal{F})$  and  $f \in \mathcal{F}$ . Let  $j$  be the minimal index in  $1 \dots n$  s.t.  $T_k(s) = T_k(t)$  for all  $k = 1 \dots j - 1$ . By proposition 5.3 we have  $T_k(f(\dots s \dots)) = T_k(f(\dots t \dots))$  for all  $k = 1 \dots j - 1$ .

If  $j \leq n$  then  $T_j(s) \succ_j T_j(t)$ . There are two cases two to be distinguished:

1.  $f \notin \mathcal{F}_j$ . Then  $T_j(f(\dots s \dots)) = X \cup T_j(s) \succ_j X \cup T_j(t) = T_j(f(\dots t \dots))$ , where  $X$  is the union of the transformations of the all other immediate subterms.
2.  $f \in \mathcal{F}_j$ . Then, by monotonicity of  $\succ_j$ ,  $T_j(f(\dots s \dots)) = \{f(\dots s' \dots) \mid s' \in \{0_j\} \cup T_j(s)\} \succ_j \{f(\dots t' \dots) \mid t' \in \{0_j\} \cup T_j(t)\} = T_j(f(\dots t \dots))$ .

Otherwise (i.e.  $j = n + 1$ ), we have  $T_k(s) = T_k(t)$  and  $T_k(f(\dots s \dots)) = T_k(f(\dots t \dots))$  for all  $k = 1 \dots n$ . Then, by monotonicity of  $\succ_r$ , we have that  $s \succ_r t$  implies  $f(\dots s \dots) \succ_r f(\dots t \dots)$ .  $\square$

**Lemma 5.9** If  $\succ_i$  for all  $i : 1 \dots n$  and  $\succ_r$  are well-founded then  $\succ_t$  is well-founded.

**Proof** Suppose that there exists an infinite sequence  $t_1 \succ_t t_2 \succ_t \dots$ . By well-foundedness of  $\succ_1$  and hence of  $\succ_1$ , there must be some  $t_j$  in the sequence s.t.  $T_1(t_j) = T_1(t_k)$  for all  $k > j$ . Let  $t_1^1 \succ_t t_2^1 \succ_t \dots$  the infinite subsequence starting from  $t_j$ . Repeating this reasoning  $n$  times it follows that there must be an infinite subsequence  $t_1^n \succ_t t_2^n \succ_t \dots$  s.t.  $T_i(t_1^n) = T_i(t_2^n) = \dots$  for all  $i : 1 \dots n$ , which implies that there is an infinite sequence  $t_1^n \succ_r t_2^n \succ_r \dots$ , contradicting the well-foundedness of  $\succ_r$ .  $\square$

**Lemma 5.10** If  $\succ_i$  for all  $i : 1 \dots n$  and  $\succ_r$  are total orderings on  $\mathcal{T}(\mathcal{F}_i)$  and  $\mathcal{T}(\mathcal{F})$  respectively then  $\succ_t$  is a total ordering on  $\mathcal{T}(\mathcal{F})$ .

**Proof** By totality of  $\succ_i$  (and hence of  $\succ_i$ ) we have either  $T_i(s) \succ_i T_i(t)$  or  $T_i(t) \succ_i T_i(s)$  or  $T_i(s) = T_i(t)$ , for all  $i : 1 \dots n$  and  $s, t \in \mathcal{T}(\mathcal{F})$ . Then either  $s \succ_t t$  or  $t \succ_t s$  or  $T_i(s) = T_i(t)$  for all  $i : 1 \dots n$ , and in the latter, since  $\succ_r$  is total, either  $s \succ_r t$  or  $t \succ_r s$  or  $s = t$ . Therefore, by definition of  $\succ_t$ , it holds that either  $s \succ_t t$  or  $t \succ_t s$  or  $s = t$ , i.e.  $\succ_t$  is total on  $\mathcal{T}(\mathcal{F})$ .  $\square$

From the previous lemmas, the following theorem holds:

**Theorem 5.11** If  $\succ_i$  are monotonic well-founded orderings total on  $\mathcal{T}(\mathcal{F}_i)$  for all  $i : 1 \dots n$ . Then  $\succ_t$  is a monotonic well-founded ordering total on  $\mathcal{T}(\mathcal{F})$  extending  $\succ_i$  for all  $i : 1 \dots n$ .

### 5.3 Terms with variables and TRS's

Let  $R_i$  be totally terminating TRS's for all  $i : 1 \dots n$ , and let  $R$  be  $R_1 \uplus \dots \uplus R_n$ . As before, in the following we consider the ordering  $\succ_t$  lifted to terms with variables as in definition 2.1.

**Definition 5.12** A term  $t$  is said to be conservative wrt. a term  $s$  if for all variable  $x \in \mathcal{V}ars(t)$  the number of occurrences of  $x$  in  $t$  is smaller or equal to the number of occurrences of  $x$  in  $s$ . A TRS  $R$  is conservative  $r$  is conservative wrt.  $l$  for all rule  $l \rightarrow r \in R$ .

**Proposition 5.13** Let  $s$  be a term in  $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$  for some  $i : 1 \dots n$ .  $T_j(s\sigma) = \cup_{x \in \mathcal{M}vars(s)} T_j(x\sigma)$  for all  $j : 1 \dots n$  with  $i \neq j$  and all ground substitutions  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ .

**Proof** By induction on the size of  $s$ . If  $s$  is a variable then the property trivially holds. Otherwise, let  $s$  be  $f(s_1, \dots, s_p)$ . Then, as  $f \notin \mathcal{F}_j$ , we have  $T_j(f(s_1, \dots, s_p)) = T_j(s_1) \cup \dots \cup T_j(s_p)$ . Now by induction hypothesis  $T_j(s_k) = \cup_{x \in \mathcal{M}vars(s_k)} T_j(x\sigma)$  for all  $k : 1 \dots p$ , which, since  $\mathcal{M}vars(s) = \cup_{k=1}^p \mathcal{M}vars(s_k)$ , implies  $T_j(s\sigma) = \cup_{x \in \mathcal{M}vars(s)} T_j(x\sigma)$ .  $\square$

**Proposition 5.14** Let  $\succ_i$  be a monotonic ordering total on  $\mathcal{T}(\mathcal{F}_i)$  and let  $t$  be a term in  $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$ . If  $t$  is not a variable then for all ground substitutions  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ , the maximal (wrt.  $\succ_i$ ) term in  $T_i(t\sigma)$  is  $t\sigma'$ , where  $x\sigma'$  is the maximal (wrt.  $\succ_i$ ) term in  $\{0_i\} \cup T_i(x\sigma)$  for all variable  $x \in \mathcal{V}ars(t)$ .

**Proof** By induction on the size of  $t$ . Let  $t$  be  $f(t_1, \dots, t_p)$ . Then  $T_i(f(t_1\sigma, \dots, t_p\sigma)) = \{f(t'_1, \dots, t'_n) \mid t'_j \in \{0_i\} \cup T_i(t_j\sigma)\}$ . Now, by induction hypothesis (if  $t_j$  is not a variable) and by minimality wrt.  $\succ_i$  of  $0_i$ , we have that the maximal (wrt.  $\succ_i$ ) term in  $\{0_i\} \cup T_i(t_j\sigma)$  is  $t_j\sigma'_j$ , where  $x\sigma'_j$  is the maximal (wrt.  $\succ_i$ ) term in  $\{0_i\} \cup T_i(x\sigma)$  for all variable  $x \in \mathcal{V}ars(t_j)$ . Since  $\succ_i$  is total the maximal is unique and hence  $x\sigma'_j$  is always the same term for all  $j : 1 \dots p$  s.t.  $x \in \mathcal{V}ars(t_j)$ . Therefore taking  $\sigma'$  as the union of all  $\sigma'_j$ , by monotonicity we have that the maximal (wrt.  $\succ_i$ ) term in  $T_i(f(t_1\sigma, \dots, t_p\sigma))$  is  $t\sigma' = f(t_1\sigma', \dots, t_p\sigma')$ , where  $x\sigma'$  is the maximal (wrt.  $\succ_i$ ) term in  $\{0_i\} \cup T_i(x\sigma)$  for all variable  $x \in \mathcal{V}ars(t)$ .  $\square$

**Lemma 5.15** Let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F}_i, \mathcal{X})$  for some  $i : 1 \dots n$  and let  $\succ_i$  be a monotonic ordering total on  $\mathcal{T}(\mathcal{F}_i)$ . If  $i = 1$  or  $t$  is conservative wrt.  $s$  then  $s\sigma' \succ_i t\sigma'$  for all substitutions  $\sigma'$  with range in  $\mathcal{T}(\mathcal{F}_i)$  implies  $s\sigma \succ_t t\sigma$  for all substitutions  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ , i.e.  $s \succ_t t$ .

**Proof** If  $t$  is conservative wrt.  $s$  then, by proposition 5.13,  $T_j(s\sigma) \supseteq T_j(t\sigma)$  for all  $j : 1 \dots i - 1$ . Therefore either  $T_j(s\sigma) \succ_j T_j(t\sigma)$  for some  $j : 1 \dots i - 1$  which implies  $s\sigma \succ_t t\sigma$ , or  $T_j(s\sigma) = T_j(t\sigma)$  for all  $j : 1 \dots i - 1$ . Consider the later case (otherwise the lemma already holds). If  $t$  is a variable, since any total well-founded monotonic ordering

fulfils the subterm property, we have  $s\sigma \succ_i t\sigma$  (note that  $s$  cannot be a variable and that  $\text{Vars}(s) \supseteq \text{Vars}(t)$ ). Otherwise, by proposition 5.14 we have that the maximal (wrt.  $\succ_i$ ) terms in  $T_i(s\sigma)$  and  $T_i(t\sigma)$  are  $s\sigma'$  and  $t\sigma'$  respectively, where  $x\sigma'$  is the maximal (wrt.  $\succ_i$ ) term in  $T_i(x\sigma)$  for all variable  $x \in \text{Vars}(s) \supseteq \text{Vars}(t)$ . Then, since  $s\sigma' \succ_i t\sigma'$  for all substitutions  $\sigma'$  with range in  $\mathcal{T}(\mathcal{F}_i)$ ,  $T_i(s\sigma) \succ_i T_i(t\sigma)$ . Finally, obviously if  $i = 1$  then the conservative condition is not needed.

Again, by theorem 5.11 and lemma 5.15, it leads to a result on modularity of termination:

**Theorem 5.16** Let  $R_1, \dots, R_n$  be totally terminating TRS's. If  $R_2, \dots, R_n$  are conservative then  $R = R_1 \uplus \dots \uplus R_n$  is totally terminating.

## 6 $E$ -compatible extension orderings and modularity of $E$ -termination

Here we will study how to deal with the extension orderings when we have  $E$ -compatible orderings. We will apply the same techniques as in previous sections to obtain easily some results for certain equational theories  $E$ , more precisely for *permutative* theories:

**Definition 6.1** An equation  $s = t$ , with  $s = f(s_1, \dots, s_m)$  and  $t = f(t_1, \dots, t_m)$ , is *permutative* if  $\{s_1, \dots, s_m\} = \{t_1, \dots, t_m\}$ .

An equational theory  $E$  is permutative if it has a presentation in which all the axioms are permutative.

**Example 6.2** The axiom  $f(g(a), x, x) = f(x, g(a), x)$  is permutative since  $\{g(a), x, x\} = \{x, g(a), x\}$ . The commutativity axiom is also permutative.  $\square$

**Proposition 6.3** Let  $E$  be a permutative theory. If  $f(s_1, \dots, s_m) =_E g(t_1, \dots, t_r)$  then  $f = g$  (and hence  $m = r$ ) and  $\{s_1, \dots, s_m\} =_E \{t_1, \dots, t_m\}$ .

In the following the same extension orderings  $\succ_e$ ,  $\succ_s$  and  $\succ_t$  defined in sections 3,4 and 5 respectively allow us to prove (i) modularity of  $E$ -termination of right linear TRS's, (ii) modularity of simple  $E$ -termination and (iii) modularity of total  $E$ -termination if one TRS is conservative, for any permutative theory  $E$ .

Note that in fact we have each TRS  $R_i$  being  $E_i$ -terminating for some equational theory  $E_i$  on  $\mathcal{T}(\mathcal{F}_i)$ , but since we consider only disjoint signatures it is equivalent to speak about  $E$ -termination for all  $\succ_i$  where  $E = E_1 \cup \dots \cup E_n$ .

### 6.1 Modularity of $E$ -termination

Now we will prove that  $\succ_e$  defined as in section 3.2, but replacing  $=$  by  $=_E$ , is  $E$ -compatible for any permutative theory  $E$  if  $\succ_i$  are  $E$ -compatible for all  $i : 1 \dots n$ , i.e.  $\succ_e$  preserves  $E$ -compatibility.

First we will see that the transformations  $N_i$  are compatible wrt. any permutative theory  $E$  for all  $i : 1 \dots n$ .

**Lemma 6.4** Let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F})$  and let  $E$  be a permutative theory. If  $s =_E t$  then  $N_i(s) =_E N_i(t)$  for all  $i : 1 \dots n$ .

**Proof** We will proceed by induction on the size of  $s$  and  $t$ . Since  $s =_E t$ , assume  $s = f(s_1, \dots, s_m)$  and  $t = f(t_1, \dots, t_m)$  we have  $\text{Head}(s) = \text{Head}(t) = f$  with  $\{s_1, \dots, s_m\} =_E \{t_1, \dots, t_m\}$ . Therefore by induction hypothesis  $\{N_i(s_1), \dots, N_i(s_m)\} =_E \{N_i(t_1), \dots, N_i(t_m)\}$ . We consider two cases:

1.  $f \notin \mathcal{F}_i$ . Then by definition of  $N_i$ , we have  $N_i(s) = N_i(s_1) \cup \dots \cup N_i(s_m)$  and  $N_i(t) = N_i(t_1) \cup \dots \cup N_i(t_m)$ , therefore by induction hypothesis it holds.
2.  $f \in \mathcal{F}_i$ . Then by definition of  $N_i$ , we have  $N_i(s) = \{f(s'_1, \dots, s'_m) \mid s'_j \in N_i(s_j) \cup Z_i(s_j)\}$  and  $N_i(t) = \{f(t'_1, \dots, t'_m) \mid t'_k \in N_i(t_k) \cup Z_i(t_k)\}$ , then by induction hypothesis  $N_i(s) =_E N_i(t)$  (note that if  $s_j =_E t_k$  then  $Z_i(s_j) = Z_i(t_j)$ ).  $\square$

From this lemma the  $E$ -compatibility of  $\succ_e$  follows:

**Lemma 6.5** If  $\succ_i$  for all  $i : 1 \dots n$  are resp.  $E$ -compatible orderings then  $\succ_e$  is an  $E$ -compatible ordering for any permutative theory  $E$ .

**Proof** We have to prove that  $s' =_E s \succ_e t =_E t'$  implies  $s' \succ_e t'$  for all  $s, s', t, t' \in \mathcal{T}(\mathcal{F})$ . If  $s \succ_e t$  then there is some  $i : 1 \dots n$  s.t.  $N_j(s) =_E N_j(t)$  for all  $j : 1 \dots i - 1$  and  $N_i(s) \succ_i N_i(t)$ . Then by lemma 6.4  $N_j(s') =_E N_j(s) =_E N_j(t) =_E N_j(t')$  for all  $j : 1 \dots i - 1$  and  $N_i(s') =_E N_i(s)$  and  $N_i(t) =_E N_i(t')$ . Therefore by  $E$ -compatibility of  $\succ_i$ ,  $N_i(s') =_E N_i(s) \succ_i N_i(t) =_E N_i(t')$  implies  $N_i(s') \succ_i N_i(t')$ , and hence  $s' \succ_e t'$ .  $\square$

From the  $E$ -compatibility of all  $\succ_i$  it is straightforward to adapt all results of section 3.2 when considering equality modulo  $E$ . Therefore we have that if  $\succ_i$  are  $E$ -compatible monotonic well-founded orderings for all  $i : 1 \dots n$  then  $\succ_e$  is an  $E$ -compatible quasi-monotonic well-founded ordering.

Similarly, the results of section 3.3 also hold. Note that proposition 3.12 is still true for rewriting modulo permutative theories, i.e. we can consider that in all signatures there is a constant symbol, which is taken as  $0_i$ , s.t.  $f(\dots, 0_i, \dots) \succ_i 0_i$  for all  $E$ -compatible reduction ordering  $\succ_i$ . Therefore this leads to the following result on modularity of termination of rewriting modulo an equational theory:

**Theorem 6.6** Let  $R_1, \dots, R_n$  be right linear  $E$ -terminating TRS's over  $\mathcal{T}(\mathcal{F}_1), \dots, \mathcal{T}(\mathcal{F}_n)$  respectively and let  $E$  be a permutative theory. Then  $R = R_1 \uplus \dots \uplus R_n$  is  $E$ -terminating.

## 6.2 Modularity of Simple $E$ -termination

Here we show that the ordering  $\succ_s$  defined as in section 4, but considering  $=_E$  instead of  $=$ , is  $E$ -compatible for any permutative theory  $E$  if  $\succ_i$  are  $E$ -compatible for all  $i : 1 \dots n$ , i.e.  $\succ_s$  preserves  $E$ -compatibility.

Now we will prove that the transformation  $S_E$  is compatible wrt. any permutative theory  $E$ :

**Proposition 6.7** Let  $E$  be a permutative theory  $E$ . If  $s =_E t$  then  $\{s' \mid s \succeq_{emb} s'\} =_E \{t' \mid t \succeq_{emb} t'\}$ .

**Proposition 6.8** Let  $E$  be a permutative theory  $E$ . If  $s =_E t$  then  $S_i(s) =_E S_i(t)$  for all  $i : 1 \dots n$  and  $S(s) =_E S(t)$ .

**Proof** If  $s =_E t$  then  $Head(s) = Head(t) = f$ . Assume  $f \in \mathcal{F}_j$  for some  $j : 1 \dots n$  and  $s = f(s_1, \dots, s_m)$  and  $t = f(t_1, \dots, t_m)$  with  $\{s_1, \dots, s_m\} =_E \{t_1, \dots, t_m\}$ .

If  $i \neq j$  then  $S_i(s) =_E S_i(t) = 0_i$ . Otherwise,  $S_i(s) = f(S_i(s_1), \dots, S_i(s_m))$  and  $S_i(t) = f(S_i(t_1), \dots, S_i(t_m))$ , which by induction hypothesis implies  $S_i(s) =_E S_i(t)$ . Finally, since  $S_j(s) =_E S_j(t)$ , we have  $S(s) =_E S(t)$ .  $\square$

**Proposition 6.9** Let  $E$  be a permutative theory  $E$ . If  $s =_E t$  then  $S_E(s) =_E S_E(t)$ .

**Proof** If  $s =_E t$ , by proposition 6.7 we have  $\{s' \mid s \succeq_{emb} s'\} =_E \{t' \mid t \succeq_{emb} t'\}$ . Therefore, since  $S_E(s) = \{S(s') \mid s \succeq_{emb} s'\}$  and  $S_E(t) = \{S(t') \mid t \succeq_{emb} t'\}$ , by proposition 6.8 it follows that  $S_E(s) =_E S_E(t)$ .  $\square$

This lemma allows us to prove the  $E$ -compatibility of  $\succ_s$ :

**Lemma 6.10** If  $\succ_i$  for all  $i : 1 \dots n$  are resp.  $E$ -compatible orderings then  $\succ_s$  is an  $E$ -compatible ordering for any permutative theory  $E$ .

**Proof** We have to show that  $s' =_E s \succ_s t =_E t'$  implies  $s' \succ_s t'$  for all terms  $s, s', t, t' \in \mathcal{T}(\mathcal{F})$ . We proceed by induction on the size of  $s$ . By proposition 6.8, if  $s' =_E s$  and  $t' =_E t$  then  $S_E(s) =_E S_E(s')$  and  $S_E(t) =_E S_E(t')$ . Therefore, by  $E$ -compatibility of  $\succ_i$  (and hence of  $\succ_u$ ), we have  $max_{\succ_u}(S_E(s)) =_E max_{\succ_u}(S_E(s'))$  and  $max_{\succ_u}(S_E(t)) =_E max_{\succ_u}(S_E(t'))$ .

If  $max_{\succ_u}(S_E(s)) \succ_u max_{\succ_u}(S_E(t))$  then, by  $E$ -compatibility of  $\succ_i$  (and hence of  $\succ_u$ ), we have that  $max_{\succ_u}(S_E(s')) =_E max_{\succ_u}(S_E(s)) \succ_u max_{\succ_u}(S_E(t)) =_E max_{\succ_u}(S_E(t'))$  implies  $max_{\succ_u}(S_E(s')) \succ_u max_{\succ_u}(S_E(t'))$ , and hence  $s' \succ_s t'$ .

If  $max_{\succ_u}(S_E(s)) =_E max_{\succ_u}(S_E(t))$  then  $max_{\succ_u}(S_E(s)) =_E max_{\succ_u}(S_E(t))$ . Suppose  $s = f(s_1, \dots, s_m)$ ,  $s' = f(s'_1, \dots, s'_m)$ ,  $t = g(t_1, \dots, t_r)$  and  $t' = g(t'_1, \dots, t'_r)$ . Then  $f = g$  (and hence  $m = r$ ) and  $\{s_1, \dots, s_m\} \succ_s \{t_1, \dots, t_m\}$ . Moreover  $\{s_1, \dots, s_m\} =_E \{s'_1, \dots, s'_m\}$  and  $\{t_1, \dots, t_m\} =_E \{t'_1, \dots, t'_m\}$ . Therefore, by induction hypothesis,  $\{s'_1, \dots, s'_m\} =_E \{s_1, \dots, s_m\} \succ_s \{t_1, \dots, t_m\} =_E \{t'_1, \dots, t'_m\}$  implies  $\{s'_1, \dots, s'_m\} \succ_s \{t'_1, \dots, t'_m\}$ , and hence  $s' \succ_s t'$ .  $\square$

From the  $E$ -compatibility of all  $\succ_i$  and  $\succ_s$  it is straightforward to adapt all results of section 4 when considering equality modulo  $E$ . Therefore we have that if  $\succ_i$  are  $E$ -compatible monotonic well-founded orderings for all  $i : 1 \dots n$  then  $\succ_e$  is an  $E$ -compatible quasi-monotonic well-founded ordering, s.t. if  $s \succ_i t$  then  $s\sigma \succ_s t\sigma$  for all substitution  $\sigma$  with range in  $\mathcal{T}(\mathcal{F})$ . This leads to the following result on modularity of simple termination of rewriting modulo an equational theory:

**Theorem 6.11** Let  $R_1, \dots, R_n$  be simply terminating TRS's over  $\mathcal{T}(\mathcal{F}_1), \dots, \mathcal{T}(\mathcal{F}_n)$  respectively and let  $E$  be a permutative theory. Then  $R = R_1 \uplus \dots \uplus R_n$  is simply terminating over  $\mathcal{T}(\mathcal{F})$ .

### 6.3 Modularity of Total $E$ -termination

Finally we consider total  $E$ -compatible orderings. We will show that the ordering  $\succ_t$  defined as in section 5, but replacing  $=_E$  by  $=$ , preserves  $E$ -compatibility for any permutative theory  $E$ .

The following lemma (which is proved very similarly to lemma 6.4) shows that the transformations  $T_i$  are compatible wrt. any permutative theory  $E$  for all  $i : 1 \dots n$ .

**Lemma 6.12** Let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F})$  and let  $E$  be a permutative theory. If  $s =_E t$  then  $T_i(s) =_E T_i(t)$  for all  $i : 1 \dots n$ .

From this lemma it follows the  $E$ -compatibility of  $\succ_t$ :

**Lemma 6.13** If  $\succ_i$  for all  $i : 1 \dots n$  and  $\succ_r$  are resp.  $E$ -compatible orderings then  $\succ_t$  is an  $E$ -compatible ordering for any permutative theory  $E$ .

**Proof** We have to prove that  $s' =_E s \succ_t t =_E t'$  implies  $s' \succ_t t'$  for all  $s, s', t, t' \in \mathcal{T}(\mathcal{F})$ . If  $s \succ_t t$  then there are two possible cases:

If there is some  $i : 1 \dots n$  s.t.  $T_j(s) =_E T_j(t)$  for all  $j : 1 \dots i-1$  and  $T_i(s) \succ_i T_i(t)$ . Then by lemma 6.12  $T_j(s') =_E T_j(s) =_E T_j(t) =_E T_j(t')$  for all  $j : 1 \dots i-1$  and  $T_i(s) =_E T_i(s')$  and  $T_i(t) =_E T_i(t')$ . Therefore, by  $E$ -compatibility of  $\succ_i$ ,  $T_i(s') =_E T_i(s) \succ_i T_i(t) =_E T_i(t')$  implies  $T_i(s') \succ_i T_i(t')$  and hence  $s' \succ_t t'$ .

If there is some  $T_i(s) =_E T_i(t)$  for all  $i : 1 \dots n$  then by lemma 6.12  $T_i(s') =_E T_i(s) =_E T_i(t) =_E T_i(t')$  for all  $i : 1 \dots n$  and by  $E$ -compatibility of  $\succ_r$  we have that  $s' =_E s \succ_r t =_E t'$  implies  $s' \succ_r t'$ , and hence  $s \succ_t t$ .  $\square$

The following lemma states the totality up to  $=_E$  of  $\succ_t$  (which is proved as lemma 5.10).

**Lemma 6.14** If  $\succ_i$  for all  $i : 1 \dots n$  and  $\succ_r$  are total on the  $E$ -congruence classes of terms in  $\mathcal{T}(\mathcal{F}_i)$  and  $\mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2)$  respectively, then  $\succ_t$  is total on the  $E$ -congruence classes of terms in  $\mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2)$ .

Again adapting the results of section 5 (using the  $E$ -compatibility of all  $\succ_i$  and  $\succ_r$ ) we obtain the following modularity result:

**Theorem 6.15** Let  $R_1, \dots, R_n$  be totally  $E$ -terminating TRS's and let  $E$  be a permutative theory. If  $R_2, \dots, R_n$  are conservative then  $R = R_1 \uplus \dots \uplus R_n$  is totally  $E$ -terminating.

## 7 Enlarging the signature

Here we present a new extension ordering, which allow us to exten an ordering to deal with new function symbols. This extension ordering is base on a precedence on the set of function symbols. Although when dealing with terms with variables this extension ordering impose strong syntactic condition on the comapared terms, in the case of total extensions it has a better behaviour.

This results can be applied (in a similar way to the previous sections) to prove the termination of *hierarchical unions* [Der93, Rao94]:  $R_1 \cup R_2$  where the defined symbols of  $R_2$ , denoted  $\mathcal{D}_2$  do not occur in  $R_1$ ; We can consider simply that we are enlarging the signature  $\mathcal{F}_1$  with the new symbols of  $\mathcal{F}_2$  which are at least  $\mathcal{D}_2$ . Then  $R_1 \cup R_2$  is terminating if it is included in one of the extension ordering we will define below.

## 7.1 Recursive Path Extension Orderings

We first show how to extend an ordering  $\succ$  on  $\mathcal{T}(\mathcal{F})$ , with  $\mathcal{F}$  finite, to an ordering  $\succ_p$  on  $\mathcal{T}(\mathcal{F}')$ , where  $\mathcal{F}' = \mathcal{F} \cup \mathcal{F}_0$  and  $\mathcal{F}_0$  is a set of new function symbols.

**Definition 7.1** Let  $P$  be a mapping from terms in  $\mathcal{T}(\mathcal{F}')$  to multisets of terms in  $\mathcal{T}(\mathcal{F})$  recursively defined as:

$$P(f(t_1, \dots, t_m)) = \begin{cases} \{0\} & \text{if } m = 0 \text{ and } f \notin \mathcal{F} \\ \{f(t'_1, \dots, t'_m) \mid t'_i \in P(t_i)\} & \text{if } f \in \mathcal{F} \\ P(t_1) \cup \dots \cup P(t_m) & \text{otherwise} \end{cases}$$

where 0 is some (fixed) minimal (wrt.  $\succ$ ) representative constant symbol in  $\mathcal{T}(\mathcal{F})$ .

**Definition 7.2** Let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F}')$ . Then  $s = f(s_1, \dots, s_m) \succ_p g(t_1, \dots, t_n) = t$  iff

$P(s) \succ P(t)$  or  $P(s) = P(t)$  for all  $i : 1 \dots n$  and

- $s_i \succeq_p t$ , for some  $i : 1 \dots m$
- $f \succ_{\mathcal{F}} g$  and  $s \succ_p t_j$ , for all  $j : 1 \dots n$
- $f = g$  and  $\{s_1, \dots, s_m\} \succ_p \{t_1, \dots, t_n\}$

where  $\succ_{\mathcal{F}}$  is a precedence on  $\mathcal{F}'$  (although, in fact, relations in  $\succ_{\mathcal{F}}$  between symbols in any  $\mathcal{F}$  are not needed, since if  $P(s) = P(t)$  and  $f \neq g$  then  $f \in \mathcal{F}' - \mathcal{F}$  or  $g \in \mathcal{F}' - \mathcal{F}$ ).

**Proposition 7.3**  $\succ_p$  extends  $\succ$  on  $\mathcal{T}(\mathcal{F})$ .

**Lemma 7.4** If  $\succ$  is an ordering on  $\mathcal{T}(\mathcal{F}')$  and  $\succ_{\mathcal{F}}$  is an ordering on  $\mathcal{F}'$  then  $\succ_p$  is an ordering on  $\mathcal{T}(\mathcal{F}')$ .

**Proof** Irreflexivity: We will prove instead  $s \not\prec_p s'$  for all subterms  $s'$  of  $s$ . In particular, since  $s$  is a subterm of  $s$  irreflexivity holds. By induction on (the sum of) the sizes of  $s$  and  $s'$ . Since  $P(s') \subseteq P(s)$ , by irreflexivity of  $\succ$ , we have  $P(s') \not\prec_p P(s)$ . Consider  $P(s') = P(s)$ ,  $s' = f(s'_1, \dots, s'_n)$  and  $s = g(s_1, \dots, s_m)$ : By induction hypothesis  $s'_i \not\prec_p s$  for all  $i : 1 \dots n$  and obviously  $s'_i \neq s$ . If  $f \succ_{\mathcal{F}} g$  then  $s'$  is a proper subterm of  $s$  and therefore  $s' \not\prec_p s_i$  for some  $i : 1 \dots m$  (not that  $s'$  is a subterm of some  $s_i$ ). If  $f = g$  then  $m = n$  and then if  $s' = s$  then by induction hypothesis (remind that irreflexivity is a particular case of the property we are proving)  $\{s_1, \dots, s_n\} \not\prec_p \{s_1, \dots, s_n\}$  Otherwise

$s'$  is a proper subterm of  $s$  and there is some  $j : 1 \dots n$ , s.t.  $s'_i$  is a subterm of  $s_j$  for all  $i : 1 \dots n$ . Therefore by induction hypothesis  $s'_i \not\prec_p s_j$  for all  $i : 1 \dots n$  and hence  $\{s'_1, \dots, s'_n\} \not\prec_p \{s_1, \dots, s_n\}$ .

Transitivity: suppose  $s \succ_p t$  and  $t \succ_p u$ , we proceed by induction on the size of  $s$ ,  $t$  and  $u$ .

By transitivity of  $\succ$  if  $s \succ t$  and  $t \succ u$  then  $s \succ u$ . Also if  $P(s) = P(t)$  (or  $s \succ t$  and  $P(t) = P(u)$ ) we have  $s \succ u$ . Otherwise if  $P(s) = P(t) = P(u)$ , assuming  $s = f(s_1 \dots s_m)$ ,  $t = g(t_1 \dots t_q)$  and  $u = h(u_1 \dots u_r)$ , we can distinguish the following cases:

1. If  $s_i \succeq_p t$  for some  $i : 1 \dots m$  (i.e.  $s \succ_p t$  applying the first branch) then, since  $t \succ_p u$ , by induction hypothesis  $s_i \succ_p u$  and hence  $s \succ_p u$ .
2. If  $f \succ_{\mathcal{F}} g$  and  $t_j \succeq_p u$  for some  $j : 1 \dots q$  then, by definition we have  $s \succ_p t_j$  for every  $j : 1 \dots q$ , and therefore, by induction hypothesis, it holds that  $s \succ_p u$ .
3. If  $f = g$  and  $t_j \succeq_p u$  for some  $j : 1 \dots q$  then, by definition we have  $s_k \succeq_p t_j$  for some  $k : 1 \dots m$  (note that we are comparing the multisets of the arguments of  $s$  and  $t$ ), and therefore, by induction hypothesis, it holds that  $s \succ_p u$ .
4. If  $f \succ_{\mathcal{F}} g$  or  $f = g$  and  $g \succ_{\mathcal{F}} h$  then we have  $s \succ_p t$  and  $t \succ_p u_k$  for every  $k : 1 \dots r$ , and by induction hypothesis  $s \succ_p u_k$  for every  $k : 1 \dots r$ . Since (by transitivity of  $\succ_{\mathcal{F}}$ )  $f \succ_{\mathcal{F}} h$ ,  $s \succ_p u$  holds.
5. If  $f = g = h$  then  $\{s_1, \dots, s_m\} \succ_p \{t_1, \dots, t_n\}$  and  $\{t_1, \dots, t_n\} \succ_p \{u_1, \dots, u_r\}$ . Therefore, by induction hypothesis, we have  $\{s_1, \dots, s_m\} \succ_p \{u_1, \dots, u_r\}$ , which implies  $s \succ_e u$ .  $\square$

**Lemma 7.5** If  $\succ$  is monotonic then  $\succ_p$  also is.

**Proof** Assume  $s \succ_p t$ . We have to show that  $f(\dots s \dots) \succ_p f(\dots t \dots)$ . By monotonicity of  $\succ$  and definition of  $P$  we have that if  $P(s) \succ P(t)$  then  $P(f(\dots s \dots)) \succ P(f(\dots t \dots))$  and hence  $f(\dots s \dots) \succ_p f(\dots t \dots)$ . Otherwise if  $P(s) = P(t)$  then, by proposition 3.3,  $P(f(\dots s \dots)) = P(f(\dots t \dots))$  and, since  $\{\dots s \dots\} \succ_p \{\dots t \dots\}$ , it follows  $f(\dots s \dots) \succ_p f(\dots t \dots)$ .  $\square$

**Proposition 7.6** Let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F}')$ . If  $s \succeq_{emb} t$  then  $P(s) \succeq_{emb} P(t)$ .

**Lemma 7.7** Let  $s$  and  $t$  be terms in  $\mathcal{T}(\mathcal{F}')$ . If  $P(s) = P(t)$  and  $t$  is embedded in  $s$  (i.e.  $s \succ_{emb} t$ ) then  $s \succ_p t$ .

**Proof** We proceed by induction on the size of  $s$  and  $t$ . Assume  $s = f(s_1, \dots, s_m)$  and  $t = g(t_1, \dots, t_n)$ .

If  $s_i \succeq_{emb} t$  for some  $i : 1 \dots m$  then  $P(s_i) \succeq_{emb} P(t)$  and, since  $P(s) = P(t)$ , we have  $P(s_i) = P(t)$  and then by induction  $s_i \succ_p t$  and hence  $s \succ_p t$ .

Otherwise  $f = g$  and  $s_i \succeq_{emb} t_i$  for all  $i : 1 \dots m$  and  $s_j \succ_{emb} t_j$  for some  $j : 1 \dots m$ . Then  $P(s_i) \succeq_{emb} P(t_i)$  and, since  $P(s) = P(t)$ , we have  $P(s_i) = P(t_i)$  and by induction hypothesis  $s_i \succeq_p t_i$  for all  $i : 1 \dots m$  and  $s_j \succ_p t_j$  for some  $j : 1 \dots m$  and hence,  $\{s_1, \dots, s_m\} \succ_p \{t_1, \dots, t_m\}$  implying  $s \succ_p t$ .  $\square$



**Lemma 7.8** If  $\succ$  is well-founded then  $\succ_p$  also is.

**Proof** Since  $\succ$  is well-founded, any infinite decreasing sequence must contain a infinite subsequence  $t_1 \succ_p t_2 \succ_p \dots \succ_p \dots \succ_p t_j \succ_p \dots$  s.t.  $P(t_1) = P(t_2) = \dots = P(t_j) = \dots$ . Then by lemma 7.7, applying Kruskal's theorem (note that we are dealing with a finite signature) we have that such a sequence cannot exist.  $\square$

As in section 3.2 we can only deal with terms with variables under certain linearity conditions. So in what follows consider the transformation  $P$  enlarged with  $P(x) = \{x\}$  for all variable  $x$ :

**Lemma 7.9** If  $\succ$  is stable under substitutions then if  $t$  is linear  $s \succ t$  implies  $s\sigma \succ_p t\sigma$  for all substitution  $\sigma$  with range in  $\mathcal{T}(\mathcal{F}')$ .

In this case with this transformation we can ensure stability under substitutions for linear terms:

**Lemma 7.10** If  $\succ$  is stable under substitutions then  $\succ_p$  is stable under substitutions when comparing linear terms.

**Proof** By induction on the size of the terms, taking into account that if  $P(t) = \{t_1, \dots, t_n\}$  then  $P(t\sigma) = P(t_1\sigma) \cup \dots \cup P(t_n\sigma)$  and that if  $t$  is linear then  $P(t_i\sigma) = \{t_i\sigma' \mid x\sigma' \in P(x\sigma) \wedge x \in \text{Vars}(t_i)\}$  for all substitutions  $\sigma$ .

## 7.2 Total Path Extension Orderings

These restrictions can be overcome if we consider total reduction orderings. Then we can increase the power of the ordering  $\succ_p$  defined above by associating a status to each function symbol and adding the following branches to the ordering:

- $f = g$  and  $\text{Stat}(f) = \text{mul}$  and  $\{s_1, \dots, s_m\} \succ_p \{t_1, \dots, t_n\}$
- $f = g$  and  $\text{Stat}(f) = \text{lex}$  and  $\langle s_1, \dots, s_m \rangle \succ_p^{\text{lex}} \langle t_1, \dots, t_n \rangle$  and  $s \succ t_i$  for all  $i : 1 \dots m$ .

Note that as a consequence of totality we have that  $\succ$  is a simplification ordering. Then it is easy to adapt the proofs of the previous section to obtain the following theorem:

**Theorem 7.11** If  $\succ$  is a simplification ordering then  $\succ_p$  is a simplification ordering on  $\mathcal{T}(\mathcal{F}')$ .

**Proof** Irreflexivity, transitivity and monotonicity are proved as in lemmas 7.4 and 7.5. And for the subterm property we have that either  $P(f(\dots s \dots)) \succ P(s)$  or  $P(f(\dots s \dots)) = P(s)$  and then by definition of  $\succ_p$  it follows that  $f(\dots s \dots) \succ_p s$ .

**Lemma 7.12** If  $\succ$  is total on  $\mathcal{T}(\mathcal{F})$  and  $\text{Stat}(f) = \text{lex}$  for all symbols  $f \in \mathcal{F}'$  and  $\succ_{\mathcal{F}}$  is a total precedence (in fact as said symbols in  $\mathcal{F}$  are not need to be comparable) then  $\succ_p$  is total on  $\mathcal{T}(\mathcal{F}')$ .

**Proof** Totality. Since  $\succ$  is total on  $\mathcal{T}(\mathcal{F})$ , for all terms  $s$  and  $t$  in  $\mathcal{T}(\mathcal{F}')$  either  $P(s) \succ P(t)$  or  $P(t) \succ P(s)$  or  $P(s) = P(t)$ . If  $P(s) = P(t)$  (otherwise terms are already comparable) then by induction hypothesis either  $s_i \succeq_p t$  for some  $i : 1 \dots m$  (and hence  $s \succ_p t$ ) or  $t \succeq_p s_i$  for all  $i : 1 \dots m$ . Similarly, either  $t_j \succeq_p s$  for some  $j : 1 \dots n$  (and hence  $t \succ_p s$ ) or  $s \succeq_p t_j$  for all  $j : 1 \dots n$ . Then suppose  $t \succeq_p s_i$  and  $s \succeq_p t_j$  (otherwise  $s$  and  $t$  are already comparable). If  $P(s) = P(t)$  either  $f = g$  or  $f \notin \mathcal{F}$  or  $f \notin \mathcal{F}$ . So if  $f = g$  then, by induction hypothesis, either  $\langle s_1, \dots, s_m \rangle \succ_p^{lex} \langle t_1, \dots, t_n \rangle$  or  $\langle t_1, \dots, t_n \rangle \succ_p^{lex} \langle s_1, \dots, s_m \rangle$  and hence  $s \succ_p t$  or  $t \succ_p s$ . If  $f \neq g$  by totality of  $\succ_{\mathcal{F}}$  either  $f \succ_{\mathcal{F}} g$  or  $g \succ_{\mathcal{F}} f$  and hence  $s \succ_p t$  or  $t \succ_p s$ .  $\square$

Now, if we consider total orderings we can prove that the ordering  $\succ_p$  is stable under substitutions (without any syntactic restriction):

**Lemma 7.13** If  $\succ$  is stable under substitutions and  $\succ_p$  is a reduction ordering total on  $\mathcal{T}(\mathcal{F}')$  then  $\succ_p$  is stable under substitutions and  $s \succ t$  implies  $s \succ_p t$  for all terms  $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ .

**Proof** By induction on the size of the terms and using the same ideas of the proofs of lemmas 5.15 and 7.10.

This results improve the ones given in [BDP89], where only enlargements with new constants were considered. Finally, note that in section 5 we could have used this path extension instead of using the ordering  $\succ_r$  as last component.

### 7.3 $E$ -compatible orderings

All results in this section can be obtained for  $E$ -compatible ordering for some equational theories  $E$ . In the case of recursive path extension orderings we can deal with theories  $E$  containing associativity and commutativity axioms, provided that the new function symbols are free (or commutative) and that in the precedence the new function symbols are smaller than the old ones (in fact it is only required the new symbols to be smaller than the associative old ones). Then a slightly modification of the ordering defined here is  $E$ -compatible (see [Rub94] for details).

## 8 Improvements and conclusions

It seems possible to improve the results of this paper by defining other extension orderings. For instance, we can easily relax the conditions of section 3 by considering  $\succ_2$  as an ordering on  $\mathcal{T}(\mathcal{F})$ , i.e. it is an ordering on  $\mathcal{T}(\mathcal{F}_2, \mathcal{X})$  stable under substitutions with range in  $\mathcal{T}(\mathcal{F})$ . Then the extension ordering:  $s \succ_c t$  iff  $N_1(s) \succ_1 N_1(t)$  or  $N_1(s) = N_1(t)$  and  $s \succ_2 t$  allows us to prove that if  $R_1$  is right linear and  $R_2$  is conservative and both are terminating then  $R_1 \uplus R_2$  is terminating. Therefore we believe that it is possible to prove most of the known results in combination of rewrite systems using extension ordering (and perhaps to improve them, e.g. to more general equational theories). But this requires a

deeper study of transformations (note that in this paper we already use four different kind of transformations) and their combination.

Also the applicability of the method to modularity of termination of *constructor-sharing* or *composable* TRS's ([Gra93, Ohl94, FJ93, KO92]) and conditional term rewrite systems ([Mid93]) has to be further investigated.

On the other hand, extension orderings can also be used to prove the termination of the union of particular TRS's. For instance, we can build an extension ordering for proving that  $R \uplus \{g(x, y) \rightarrow x\}$  is terminating for any terminating TRS  $R$ . In fact this is true for any single projection rule  $g(\dots x \dots) \rightarrow x$ . This gives a very simple proof for the fact that adding pairing to (e.g. simply) typed lambda calculus preserves strong normalization.

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