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PhD Thesis

Diffusion through non-transverse heteroclinic chains: A long-time instability for the NLS

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Contents

1 Introduction, State of the Art and Main Results

1.1 Global Instability

The existence of global instability in Dynamical Systems, and in Hamiltonian Systems in particular, is one of the main problems to understand the global behavior of a dynamical system. It originated in Hamiltonian systems with the consideration by H. Poincaré [Poi67] that the study and comprehension of the orbits of nearly integrable Hamiltonian systems is the *fundamental problem of dynamics* and was conjectured by V.I. Arnold [Arn64] in his very first famous example of the so-called *Arnold diffusion*. Arnold introduced a constructive method to detect global instability, based on the existence of a chain of *transverse* heteroclinic orbits associated to *whiskered* invariant tori. For any such *transition chain*, shadowing results relying on suitable local normal forms of whiskered tori plus the transversality of the whiskers along the heteroclinic connections between different invariant tori provide the existence of true trajectories that *shadow* the transition chain, that is, go arbitrarily close to it.

This has been one of the most powerful constructive mechanism for proving global instability of diffusion in Hamiltonian systems, or in more general dynamical systems. To "design a travel" along trajectories between different regions of the phase space, one searches first for several *landmarks*, which are invariant objects of saddle type, and then search for transverse heteroclinic connections between different landmarks. Typically these landmarks are invariant tori, but larger objects, like Normally Hyperbolic Invariant manifold (NHIM) can be even more useful. Here the word "constructive" is really well used from a practical way, since such mechanism has been used for the design of several spatial missions, like the Genesis mission (http://genesismission.jpl.nasa.gov/) and others.

The requirement of a transverse intersection between invariant manifolds imply that such manifolds have large dimension (say at least half of the dimension of the phase space). In systems with not a large dimension, like Hamiltonian systems with 3 degrees of freedom, the construction of "long enough" transverse heteroclinic may be not too difficult, but clearly the hope of such construction or detection of transverse heteroclinic orbits between different invariant objects in systems with larger dimensions is clearly unrealistic, and even impossible with systems of infinite dimensions, like evolution PDEs.

What one can encounter in systems with large dimension are invariant objects of saddle type of, say, "modest" dimension, with also a modest number of saddle component which may give rise to *non-transverse* heteroclinic connections between some of them. That is, there are only some low-dimensional landmarks (equilibria, periodic orbits or invariant tori of low dimension) with some few saddle directions in their local linear part which give rise to some connections between them. Think, for instance, in what happens in local resonant forms of elliptic equilibria of Hamiltonian systems with many degrees of freedom.

The main problem with such non-transverse transition chain, is that, up to our knowledge, there are no results about non-transverse shadowing, so it does not seem clear at all the existence of nearby shadowing trajectories. Indeed, it is possible to create examples (see Chapter 2) where such non-transverse transition chains do *not* give rise to any shadowing orbit.

The main motivation for us to study this kind of global instability comes from the remarkable paper by Colliander et al. [CKS$^+$10] on global instability in the cubic defocusing Non-Linear Schrödinger (NLS) equation with periodic boundary conditions. As explained in Section 1.2, for arbitrary $N > 0$, the authors approximate the dynamics of the NLS by a complex Hamiltonian system with N degrees of freedom, called the *Toy Model System*. On each level of the energy, this Toy Model System possesses N invariant circles (periodic orbits) $\mathcal{T}_j$, $j = 1, \ldots, N$, with a 4-dimensional saddle behavior, more precisely, $\pm\sqrt{3}$ are (double) characteristic exponents of any invariant circle. Counting the neutral characteristic exponent, each invariant circle possesses 3-dimensional stable and unstable invariant manifolds which intersect between consecutive invariant circles in 2-dimensional heteroclinic manifolds. Clearly, such intersection is *non-transverse*, and we have in this system a non-transverse chain between the first and the last invariant circle. By means of a lot of (remarkable and Gronwall-like) quantitative estimates, Colliander et al. manage to prove the existence of a true trajectory of the Toy Model System connecting arbitrarily small neighborhoods of the first and last invariant circle. Coming back to the NLS system, they can prove the existence of global unstable solutions of the PDE for any $s > 1$-Sobolev norm.

Nevertheless, in [CKS$^+$10] there is no explanation neither of the dynamical reason of the existence of this shadowing trajectory, nor on the possibility of the existence of such shadowing propety for other systems.

Much more dynamics are present in the paper by Guàrdia and Kaloshin [GK15], also devoted to the same NLS equation. In this paper, the authors show explicitly the step to PDE resonant normal form required in the NLS equation to get afterwards the Toy Model System, and using in the Toy Model System a resonant normal form close to each invariant circle, they manage to find shadowing trajectories which are further from the unstable and stable invariant manifolds of the invariant circles than the ones found in [CKS+10], and thanks to this they improve (enshorten) the time estimate. Again, there is no dynamical explanation of the existence of such shadowing trajectories, to provide further possibility of application to other systems.

The main goal of this work is to provide a geometrical method to ensure the shadowing of trajectories close to non-transverse heteroclinic orbits, and to apply topological methods to justify such shadowing. Some examples where such non-transverse shadowing can or cannot be applied are presented in Chapter 2, and finally such methods are applied in Chapter 3 to the Toy Model System. The geometrical method is based on controlling all the saddle directions used along a transition chain. The topological method relies on covering relations applied to singular transitions—close to the invariant objects or landmarks—where adequate Shilnikov coordinates are introduced, and regular transitions—close to the heteroclinic connections between the landmarks. We introduce this methodology along this Introduction.

1.2 Instability in NLS

In a remarkable paper [CKS+10], Colliander et al. consider the cubic defocusing NLS

$$\begin{cases} -i\partial_t u + \Delta u &= |u|^2 u, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{T}^2 \end{cases} \tag{1.1}$$

and prove the so-called *long-time strong instability (of the flow near* 0*)* (terminology introduced by Hani in his thesis [Han11]):

Theorem 1. *Let $s > 1$, $K \gg 1$ and $0 < \delta \ll 1$ be given parameters. Then there exists a global smooth solution $u(t,x)$ to (1.1) and a time $T > 0$ with*

$$\|u(0)\|_{H^s} \leq \delta \qquad \textit{and} \qquad \|u(T)\|_{H^s} \geq K.$$

Let us remark that on the one-dimensional torus, equation (1.1) is completely integrable due to the famous result of Zakharov-Shabat [ZS71]. As a corollary $\|u(t)\|_{H^s(\mathbb{T})} \leq C\|u(0)\|_{H^s(\mathbb{T})}$, $s \geq 1$, for all $t > 0$. Even if Theorem 1 is stated for (1.1) in the two torus, it can be applied to the d-dimensional torus with $d > 2$, since the solution obtained is also a solution for equation (1.1) in $\mathbb{T}^d$ setting all the other harmonics to zero.

Using the techniques from [GK15], Guàrdia proves in [Gua14] the same result for the cubic defocusing NLS with some external convolution potential.

Haus and Procesi [HP14] consider the quintic defocusing NLS on the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$

$$-i\partial_t u + \Delta u = |u|^4 u, \tag{1.2}$$

and prove, following [CKS$^+$10] as closely as possible, exactly Theorem 1 for (1.2). Actually, Guàrdia, Haus and Procesi in the recent paper [GHP15] prove the same result for the defocusing NLS with any odd power for the nonlinearity.

Let us remark that in finite dimensional systems such kind of diffusive orbits are usually constructed by proving that the stable and unstable manifolds of a chain of unstable tori intersect. Usually this is done with tori of co-dimension one so that the manifolds should intersect for dimensional reasons. Unfortunately in the infinite dimensional case one is not able to prove the existence of codimension one tori.

In [CKS$^+$10, GK15, HP14] this problem is avoided by taking advantage of the specific form of the equation. First one reduces to an approximate equation, i.e. the first order Birkhoff normal form. Then for this dynamical system one proves directly the existence of chains of one dimensional unstable tori (periodic orbits) together with their heteroclinic connections. Next one proves the existence of a slider solution which shadows the heteroclinic chain in a finite time. Finally, one proves the persistence of the slider solution for the full NLS.

Concerning velocity of diffusion, let us mention that Bourgain [Bou00a] proved a Nekhoroshev type theorem for a perturbation of the cubic NLS. Namely, for s large and a typical initial datum $u(0) \in H^s(\mathbb{T})$ of small size $\|u(0)\|_s \leq \varepsilon$ he proved

$$\sup_{t \leq T} \|u(t)\|_s \leq C\varepsilon, \qquad |t| < T, \qquad T \leq \varepsilon^A$$

with $A = A(s) \to 0$ as $s \to \infty$. Similar upper bounds on the growth have been obtained also for the NLS equation on $\mathbb{R}$ and $\mathbb{R}^2$ as well as on compact manifolds. This is an indication of absence of a polynomial growth and motivated Bourgain [Bou00b] to pose the following question:

> Are there solutions in dimension 2 or higher with unbounded growth of H^s-norm for $s > 1$?

Moreover, he conjectured, that in case this is true, the growth should be subpolynomial in time, that is,

$$\|u(t)\|_{H^s} \ll t^{\varepsilon} \|u(0)\|_{H^s} \qquad \text{for} \qquad t \to \infty.$$

Note that Theorem 1 does not contradict Bourgain conjecture about the subpolynomial growth. Indeed, Theorem 1 only obtains solutions with arbitrarily large but finite growth in the Sobolev norms whereas Bourgain conjecture refers to unbounded growth.

Note that the initial data are *small* in H^s, in contrast with a previous work by Kuksin [Kuk97] about growth of the higher Sobolev norms of solutions of (1.1), and in contrast with the mentioned paper by Guardia and Kaloshin [GK15], whose diffusing solutions have small L^2-norm, but not initial small H^s-norm.

Indeed, the paper [GK15] follows the same general strategy of [CKS+10] and constructs orbits whose Sobolev norm grows (by an arbitrary factor) in a time which is polynomial in the growth factor. This is done by a careful analysis of the equation and using in a clever way various tools from diffusion in finite dimensional systems. It is worth remarking that the solutions in [GK15] differ from solutions studied in [CKS+10] in a substantial way, because the authors apply to information about dynamics contained in [CKS+10] a finitely smooth resonant normal form [BK92] (see also [BdlLW96, BK96]) and adequate initial conditions to achieve a lot of cancellations. Nevertheless, these solutions do not have initial small H^s-norm.

Hani [Han14] has achieved a remarkable progress towards the existence of unbounded Sobolev orbits: for a class of cubic NLS equations with non-polynomial nonlinearity, the combination of a result like Theorem 1 with some clever topological arguments leads to the existence of solutions with diverging Sobolev norm.

In broadest outline, the proof of Theorem 1 in [CKS+10] proceeds by first viewing (1.1) as an infinite dimensional system of O.D.E.'s in $a_n(t)$ for $n \in \mathbb{Z}^2$, where $a_n(t)$ is closely related to the Fourier mode $\hat{u}(t, n)$ of the solution. The authors identify a related system, which they call the resonant system, that they use as an approximation to the full system, which is very much related to normal forms (a difference is that the nonresonant terms are removed there using perturbation theory directly, rather than by first transforming the Hamiltonian and then using perturbation theory to handle the resulting higher order terms). The goal then is to build a solution $r_n(t)$ $(n \in \mathbb{Z}^2)$ to the resonant system which grows in time. This is accomplished by choosing the initial data $r_n(0)$ $(n \in \mathbb{Z}^2)$ to be supported on a certain frequency set $\Lambda \subset \mathbb{Z}^2$ in such a way that the resonant system of O.D.E.'s collapses to an even simpler, finite dimensional system that they call the *Toy Model System*, and whose solution is denoted by $b(t) = (b_1(t), b_2(t), \ldots, b_N(t))$. Each variable $b_i(t)$ represents how a certain subset of the $r_n(t)$ $(n \in \mathbb{Z}^2)$ evolves in time.

There are two independent but related ingredients which complete the proof of the main Theorem. First, they show the existence of the frequency set Λ, which is defined in terms of the desired Sobolev norm growth and according to a wish-list of geometric and combinatorial properties aimed at simplifying the resonant system. Second, they show that the Toy Model System exhibits unstable orbits that travel from an arbitrarily small neighborhood of one invariant manifold to near a distant invariant manifold. It is this instability which is ultimately responsible for the support of the solutions energy moving to higher frequencies.

Instabilities like this have been remarked on at least as far back as Poincaré, but have been studied with increasing interest since the seminal paper of Arnold [Arn64]. The authors in [CKS+10] claim that their construction has similarities with previous work on so-called "Arnold Diffusion". At the same time, however, they say that the instability they observe in the Toy Model System differs in some respects from the original phenomenon observed by Arnold. Their analysis of the instability seems to be different than arguments presently in the literature and might be of independent interest. More specifically, they say that it may be possible to prove the instability by *softer* methods that do not require as many quantitative estimates as the arguments in [CKS+10], although the presence of secular modes in the dynamics may complicate such a task. While such a soft proof would be simpler, they believe that the approach there is also of interest, as it provides a rather precise description of the orbits.

Arnold diffusion is quoted as related to the work of [CKS+10], saying that there are several definitions of Arnold diffusion. It is also just mentioned in a loose way in [HP14].

1.2.1 Diffusion in the Toy Model System

As we have said, the instability Theorem 1 form [CKS+10] is based on a reduction a a finite dimensional system of ordinary differential equation, the Toy Model System, where they prove a diffusing Theorem. We start outlining the steps for the reduction. However, we are going to follow the reduction from [GK15] since we think it is more understandable in terms of the Dynamical Systems.

Consider the Fourier expansion of the solution of (1.1)

$$u(t,x) = \sum a_n(t)\mathrm{e}^{\mathrm{i}nx}, \quad a_n(t) := \hat{u}(t,n).$$

If we compute the differential equation that the Fourier modes satisfy we obtain an infinite system of ODE's:

$$-\mathrm{i}\frac{\mathrm{d}a_n}{\mathrm{d}t} = |n|^2 a_n + \sum_{\substack{n_1-n_2+n_3=n \\ n_1,n_2,n_3\in\mathbb{Z}}} a_{n_1}\overline{a_{n_2}}a_{n_3},$$

that, as the the original PDE, can be seen as a Hamiltonian system. For this system one can compute its *Resonant Birkhoff Normal Form* near the origin by removing nonresonant terms. Using gauge freedom, one can remove linear and some non-linear terms.

Truncating the normal form up to order four (in terms of the Hamiltonian) we are reduced to a complex N-dimensional system given by a Hamiltonian

$$H_N(b_1,\dots,b_N) = \frac{1}{4}\sum_{j=1}^{N}|b_j|^4 - \frac{1}{2}\sum_{j=2}^{N-1}\left(b_j^2\overline{b_{j-1}}^2 + \overline{b_j}^2 b_{j-1}^2\right)$$

where each b_j is complex valued, with associated equations $\mathrm{d}b_j/\mathrm{d}t = 2\mathrm{i}\partial h_N/\partial\overline{b_j}$ $(j=1,\dots,N)$, that is:

$$\frac{\mathrm{d}b_j}{\mathrm{d}t} = -\mathrm{i}|b_j|^2 b_j + 2\mathrm{i}\overline{b_j}\left(b_{j-1}^2 + b_{j+1}^2\right), \quad j=1,\dots,N \tag{1.3}$$

with the convention $b_0 = b_{N+1} = 0$.

Once we have introduced the problem, now we state the diffusing Theorem that will give the instability in the NLS:

Theorem 2. *Given $N > 1$, $\epsilon \ll 1$, there is initial data $b(0) = (b_1(0), \dots, b_N(0)) \in \mathbb{C}^N$ for (3.1) and there is a time $T = T(N, \epsilon)$ so that*

$$\begin{aligned} |b_3(0)| \geq 1 - \epsilon, \quad |b_j(0)| \leq \epsilon, \quad j \neq 3 \\ |b_{N-2}(T)| \geq 1 - \epsilon, \quad |b_j(T)| \leq \epsilon, \quad j \neq N-2. \end{aligned}$$

We start analyzing the properties of the Toy Model System. It is important to notice that the total mass

$$M_N(b_1, \dots, b_N) = \sum_{j=1}^{N} |b_j|^2$$

is a first integral of the Hamiltonian system (1.3), and will be taken equal to 1, that is, we will restrict ourselves to

$$\mathcal{S}_N = \{b \in \mathbb{C}^n \,:\, M_N(b) = 1\}\,.$$

It is very important to notice that for any set of indexes $i \in I \subset \{1, \dots, N\}$ the subspace $L_I = \{b = (b_1, \dots, b_N) : b_i = 0 \text{ for } i \notin I\}$ is an invariant subspace of the Toy Model System. That gives us a very useful invariance, that we will call the *mode invariance*, when one considers $I = \{1, \dots, i-1, i+1, \dots, N\}$. That is, when some mode b_k is set at zero for $t = 0$, it will remain at zero for all time.

Consider now, for any index $j \in \{1, \dots, N\}$, the subspace L_j. It is a 1D-complex subspace (2D real space) and

$$\mathcal{T}_j = L_j \cap \mathcal{S}_N = \{b \in \mathcal{S}_N \,:\, |b_j| = 1,\, b_k = 0\,\forall\, k \neq j\},$$

is an invariant circle. If we compute the inner dynamics of $\mathcal{T}_j$, we have:

$$\frac{\mathrm{d}b_j}{\mathrm{d}t} = -\mathrm{i}|b_j|^2 b_j = -\mathrm{i}b_j,$$

that means that $\mathcal{T}_j$ is a periodic orbit.

If we, now, take a look again to Theorem 1.3 we can reinterpret it: we look for a solution that initially is close to the periodic orbit $\mathcal{T}_3$ and after some time T it is close to periodic orbit $\mathcal{T}_{N-2}$.

This result relies on the fact that these periodic orbits are not isolated but connected through heteroclinic connections. The existence of such connections can be obtained when one considers the invariant subspace, L_I, for two consecutive subindexes $I = \{j, j+1\}$. The restricted Hamiltonian, $H_2(b_j, b_{j+1})$, is a Hamiltonian with two degrees of freedom with the total mass M_2 as a first integral. Then the system is integrable. This system contains the periodic orbits $\mathcal{T}_j$ and $\mathcal{T}_{j+1}$ and also a family of heteroclinic connections between them (forward and backwards in time):

$$\gamma^+_{j,j+1} : \{\mathcal{T}_j \to \mathcal{T}_{j+1}\} \qquad \gamma^-_{j+1,j} : \{\mathcal{T}_{j+1} \to \mathcal{T}_j\}.$$

Indeed, we have the following explicit solutions:

$$\gamma^+_{j,j+1} : b^+(t) = \left(b^+_j(t), b^+_{j+1}(t)\right) = \left(\frac{1}{\sqrt{1+e^{2\sqrt{3}t}}}e^{-i(t+\alpha)}, \frac{1}{\sqrt{1+e^{-2\sqrt{3}t}}}e^{-i(t+\alpha)}e^{i\pi/3}\right)$$

$$\gamma^-_{j+1,j} : b^-(t) = \left(b^-_j(t), b^(_{j+1}t)\right) = \left(\frac{1}{\sqrt{1+e^{-2\sqrt{3}t}}}e^{-i(t+\alpha)}, \frac{1}{\sqrt{1+e^{2\sqrt{3}t}}}e^{-i(t+\alpha)}e^{2i\pi/3}\right),$$

with $\alpha \in \mathbb{T}$. The solution $b^+(t)$ connects $\mathcal{T}_j$ with $\mathcal{T}_{j+1}$ forward in time while $b^-(t)$ does it backwards. The most important fact to point out is that we do not have a single heteroclinic but a continuous family of heteroclinic connections.

As a conclusion, we can say that to prove Theorem 2 we must shadow the heteroclinic chain. That means that we have to connect two consecutive heteroclinics in a neighborhood of each periodic orbit and, after that, to follow close to the known heteroclinic connection.

Since we will need to analyze system (1.3) close to the $\mathcal{T}_j$ it is useful to use the following change of coordinates:

$$b_{j-1} = \left(\omega^2 x_- + \omega y_-\right) \mathrm{e}^{\mathrm{i}\theta}, \quad b_j = r\mathrm{e}^{\mathrm{i}\theta}, \quad b_{j+1} = \left(\omega^2 x_+ + \omega y_+\right) \mathrm{e}^{\mathrm{i}\theta},$$

where $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$ and for $k \neq j-1, j, j+1$,

$$b_k = c_k \mathrm{e}^{\mathrm{i}\theta}.$$

Writing $c = (c_1, \ldots, c_{j-2}, c_{j+2}, \ldots, c_N)$, the periodic orbit $\mathcal{T}_j$ has, in these coordinates, the following expression:

$$\mathcal{T}_j = \{r = 1,\, x_- = y_- = x_+ = y_+ = 0,\, c = 0\}.$$

Using the mass conservation law we can get rid of the coordinate r and obtain a system of equations for $(\theta, x_-, y_-, x_+, y_+, c)$. If we only look at the equations for (x_-, y_-, x_+, y_+, c), we realize that they do not depend on θ and so, after the previous reduction, we can see $\mathcal{T}_j$ as an equilibrium point located at the origin for the coordinates (x_-, y_-, x_+, y_+, c).

We have made this changes because if we linearize the system around the origin we obtain that:

- The coordinates x_-, x_+ are linearly unstable with the same characteristic exponent $\sqrt{3}$.

- The coordinates y_-, y_+ are linearly stable with the same characteristic exponent $-\sqrt{3}$.
- The coordinates c_k for $k \neq j-1, j, j+1$ are linearly centers with characteristic exponent i.

Remark. The characteristic exponents do not depend on j.

This implies that to study the flow close to the periodic orbit, $\mathcal{T}_j$, corresponds to study the flow close to a partially hyperbolic partially elliptic equilibrium point.

The above heteroclinics correspond to the four hyperbolic axis.

1.3 Outline and main results of the thesis

To end this Introduction we are going to explain the main parts in which the thesis is divided. We will also announce the main results obtained.

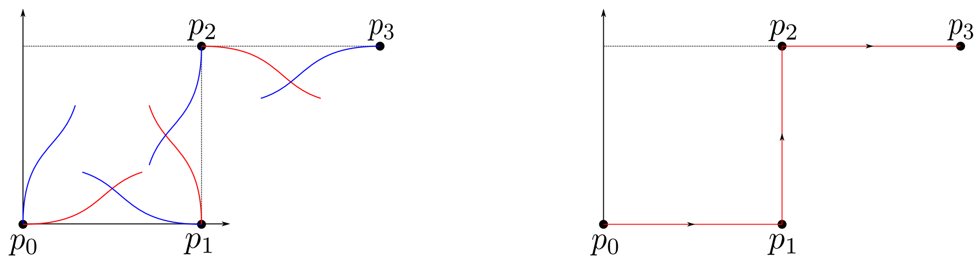
1.3.1 Non-transverse shadowing

In Chapter 2 we introduce a new scheme that will allow to find orbit that shadow a heteroclinic chain when the intersections of the invariant manifolds are not transverse.

In Section 2.1 we show that this is precisely the situation in the heteroclinic chain for the Toy Model System (1.3). To prove that, we write (1.3) in action-angle coordinates. After the reduction of the total mass and the restriction to a Hamiltonian of 2-degrees of freedom we can take a Poincaré section for an angle and define a Poincaré map. For this map we will not obtain a single heteroclinic connection, but an invariant curve such that each point on it is heteroclinic. With this we can be sure that the intersection between the invariant manifolds is not transverse.

The deduction of the system in these new coordinates will be useful for Section 2.3 to generate easy examples where the diffusion takes place.

In Section 2.2 we show the possible differences that could give rise a non-transverse heteroclinic chain with respect to a standard transverse one. We will see that the non-transverse situation can create geometric obstructions that could impede the continuation of the shadowing of the heteroclinic chain. More concretely, we are going to give an schematic argument considering a two dimensional map with four fixed points:



The question is if we are able to connect a vicinity of the first point, p_0, with a vicinity of the last point p_3 in both situations: the transverse case (left picture) and the non-transverse case (right picture). It seems clear that for the non-transverse situation a geometric obstruction could appear. These geometric obstructions do not appear in the transverse situation and this is the reason one can define arbitrary transitions in the problems where the Arnold diffusion is detected. Since, as we have said, the situation for the Toy Model System is the non-transverse one, we need to find a justification that relies on different arguments from transversality to prove the shadowing of the heteroclinic chain. The justification will be the large dimension of the system and the fact that each connection *takes place in a new direction*, this is, a direction that has not been used before.

This is the first evidence that makes us think that the instability in the Toy Model System is not given by an Arnold diffusion mechanism.

To illustrate that this kind of non-transverse transition is feasible we are going to generate, in Section 2.3 other examples inspired in the Toy Model System for which we can detect such a diffusion.

Taking into account the expression of the Toy Model System in the action-angle coordinates from Section (2.1) we can modify the original Hamiltonian (keeping invariant all the important objects as the periodic orbits and the heteroclinic connections) and obtain one for which there exists an invariant configuration of the angles. That means that we can reduce the original N degrees of freedom Hamiltonian to a system of equations of dimension N. The interesting property of the example is that for each equilibrium point (that corresponds to a periodic orbit) we only have two hyperbolic directions (instead of four), one stable (that corresponds to the incoming heteroclinic from the previous equilibrium point) and one unstable (that corresponds to the outgoing heteroclinic to the next equilibrium point).

Even though the system is simple, it is not simple enough to ensure the shadowing of the heteroclinic chain. However, its structure will inspire us to construct a very easy example where we can ensure the shadowing. Let $n \geq 1$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ a vector field. Consider the system:

$$\dot{x} = F(x)$$

with

$$\begin{cases} F_1(x) &=& \lambda_1 x_1 - \lambda_1 x_1^2 \\ F_i(x) &=& (\lambda_i - \mu_i)x_i - \lambda_i x_i^2 + \mu_i x_i x_{i-1} \qquad \text{for } 1 < i \le n \end{cases} \tag{1.4}$$

Notice that:

- The system has $n+1$ equilibrium points:

  $$p_j = (1, \overset{j}{\ldots}, 1, 0, \overset{n-j}{\ldots}, 0)$$

  for $j = 0 \ldots n$.

- The segments that connect two consecutive equilibrium points,

  $$C_{j-1,j} = \left\{ \left(1, \overset{j-1}{\ldots}, 1, x_j, 0, \overset{n-j}{\ldots}, 0\right), \ 0 \le x_j \le 1 \right\}$$

  for $j = 1 \ldots n$ are invariant and heteroclinic, that is, for all $x \in C_{j-1,j}$

  $$\lim_{t\to\infty} \Phi_t(x) = p_j \qquad \lim_{t\to-\infty} \Phi_t(x) = p_{j-1},$$

  where $\Phi_t(x)$ is the flow of the system.

- If $\lambda_1 > 0$ and $\mu_i > \lambda_i > 0$ for $i = 2, \ldots, n$ each fixed point has only one unstable direction, while all the others are linearly stable.

The main important feature of the system (1.4) is that is a triangular system. That means that it is integrable by quadratures.

If we set, for example $n = 4$, $\lambda_i = 1$ and $\mu_i = 2$ for $i = 1, \ldots, 4$ we obtain the following numerical solution:

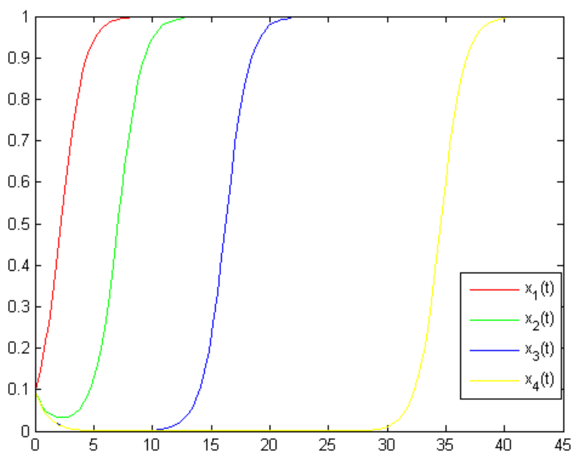


Figure 1.1: Solution of system (1.4) for $\lambda = 1$ and $\mu = 2$.

Notice that the system behaves in the desired way: we are visiting the fixed points consecutively.

After that, we can say that, in contrast with the transverse shadowing, which is closely related to non-integrability or existence of quasi-random motion, the global instability for non-transverse shadowing has nothing to do with integrability or non-integrability of the system, as is shown in the previous example.

In Section 2.4 we propose a result that guarantees the existence of an orbit that shadows the non-transverse heteroclinic chain and is the following. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism with the following properties:

1. The points $p_i = (1, \overset{i}{\ldots}, 1, 0, \overset{n-i}{\ldots}, 0)$ are fixed under f for $i = 0 \ldots n$.

2. The segments C_i that connect the points p_{i-1} and p_i,
$$C_i = \{(1, \overset{i-1}{\ldots}, 1, t, 0, \overset{n-i}{\ldots}, 0), 0 \leq t \leq 1\}$$
for $1 \leq i \leq n$ are invariant under f and, for all $x \in C_i$:
$$\lim_{k \to \infty} f^k(x) = p_i \qquad \lim_{k \to -\infty} f^k(x) = p_{i-1}.$$

3. At each point p_i the i-th direction is stable and the $(i+1)$-th is unstable. This means:
$$Df(p_i)e_i = \mu_i e_i, \quad |\mu_i| < 1$$
$$Df(p_i)e_{i+1} = \lambda_i e_{i+1}, \quad |\lambda_i| > 1$$

4. The past directions, defined by $\vec{e_1}, \ldots, \vec{e}_{i-1}$, are contracting directions around the fixed point p_i but with a lower rate than μ_i. The future directions, defined by $\vec{e}_{i+2}, \ldots, \vec{e}_n$, are expanding directions around the fixed point p_i but with a lower rate than λ_i.

Theorem 3. *Under the previous assumptions, for all $\epsilon > 0$ there exist a point x_0 and a sequence of integers $0 = k_0 < k_1 < \cdots < k_n$ such that:*
$$||f^{k_i}(x_0) - p_i|| < \epsilon \quad i = 0, \ldots, n.$$

We would like to prove Theorem 3 in a rigorous way, that is, for any map f satisfying the previous hypothesis. However we are going to present only the proof when the map is linear close to the fixed points and an affine map represents the heteroclinic connection. To do so, we are going to use the language of h-sets and covering relations. All the definitions and results concerning these tools can be found in Appendix A.

The key point for the proof of this theorem is, again, that each new connections takes place in a direction that has not used before. We show how we lose one direction at each connection.

We have to point out that the class of system for which we can apply Theorem 3 does not contain, for instance the Toy Model System: we are dealing with systems such that have only two dominant hyperbolic directions (one for the incoming heteroclinic and one for the outgoing heteroclinic).

However, what we can do is to prepare, using the language of covering relations again, the wider class of systems for which one can prove that to shadow the heteroclinic chain is feasible. This is done in Subsection 2.4.2. After the construction of the class and the definition of the h-sets we are going to use them to prove the same result but assuming that some covering relations between the defined h-sets hold.

1.3.2 Applying the new scheme to the Toy Model System

After the detection of the geometric mechanism that explains that the shadowing of a non-transverse heteroclinic chain between invariant objects is feasible, in Chapter 3 we prove Theorem 2 using the scheme of Chapter 2. Actually, we are able to prove a slightly different Theorem that is the following:

Theorem 4. *Let $N > 1$, $\delta \ll 1$. There exists a time, $T_0^* = T_0^*(N, \delta)$ such that, for all time $T^* \geq T_0^*$, there exists an initial data $b(0) = (b_1(0), \dots, b_N(0)) \in \mathbb{C}^N$ for* (1.3) *such that*

$$\begin{aligned} |b_3(0)| \geq 1 - \delta, \quad & |b_j(0)| \leq \delta, \quad j \neq 3 \\ |b_{N-2}(T^*)| \geq 1 - \delta, \quad & |b_j(T^*)| \leq \delta, \quad j \neq N-2. \end{aligned}$$

In terms of the periodic orbits, we can connect the initial periodic orbit $\mathcal{T}_3$ with the final one $\mathcal{T}_{N-2}$ but, also, the result is valid for a larger transition time. That means that, if we want, we can obtain a diffusing orbit that passes closer to the heteroclinic connections.

To prove this result we are not going to use the h-sets and covering relations in order to give a more understandable argument. We are going to consider disks of decreasing dimension at each step. In the j-th step we consider a disk of dimension equal to the number of future directions, defined by the modes $b_{j+1}, \dots, b_N$. This means that we are considering all the past coordinates, $b_1, \dots, b_{j-1}$ fixed at this time. Recall that the central mode, b_j, does not play a role since it is eliminated through the mass reduction. Around the $\mathcal{T}_j$ periodic orbit, since we want to escape through the direction defined by the outgoing heteroclinic, we are going restrict our disk to a subdisk that behaves in the desired way, that is, we consider the part of the disk that will continue close to the heteroclinic chain. It is, precisely, in this reduction where we have lost two dimensions (or one complex) and it will be performed by a modified version of the standard Shilnikov Theorem ([Den89], [Shi67]). To end the step in this inductive argument we just need to prove that our subdisk approaches the vicinity of the next periodic orbit $\mathcal{T}_{j+1}$ just following the heteroclinic connection.

2 Non-transverse diffusion

The main problem in [CKS+10] consists on finding an orbit which visits the neighborhoods of N invariant 1-dimensional objects in a N-dimensional complex system. Each object is connected with the previous and the following one with heteroclinic connections, so the authors look for a solution that concatenates these connections. This kind of scheme seems similar to Arnold diffusion, but we plan to explain that it is another kind of phenomenon since we do not have a transverse intersection between the invariant manifolds. Actually, we can find some examples similar to The Toy Model for which this intersection is not transverse but coincident. In addition, these examples could be integrable systems in contrast to Arnold diffusion, a phenomenon that only takes place in non integrable systems.

We are also going to present a general scheme for detection of this new kind of shadowing argument that can be applied, precisely, to the Toy Model System.

2.1 Non-transverse intersection in the Toy Model System

We want to illustrate that the transition chain used for proving Theorem 2 is non-transverse. In fact, we will show it for the first connection between $\mathcal{T}_1$ and $\mathcal{T}_2$. The argument for the other transitions is analogous. To do so, we are going to write the Toy Model System (3.1) in another coordinates. In the whole discussion we are going to work with the Hamiltonian instead of the associated equations. Recall that the Hamiltonian function and the symplectic 2-form that defines the Toy Model System is:

$$H_N(b,\bar{b}) = \frac{1}{4}\sum_{j=1}^{N}|b_j|^4 - \frac{1}{2}\sum_{j=2}^{N-1}\left(b_j^2\overline{b_{j-1}}^2 + \overline{b_j}^2 b_{j-1}^2\right), \qquad \Omega = \frac{i}{2}\sum_{j=1}^{N}\mathrm{d}b_j\wedge\mathrm{d}\overline{b_j}, \quad (2.1)$$

where $b=(b_1,\dots,b_N)\in\mathbb{C}^N$.

First of all, we write the Hamiltonian (2.1) in symplectic polar coordinates. That is:

$$b_j = \sqrt{2I_j}\mathrm{e}^{i\theta_j} \qquad \overline{b_j} = \sqrt{2I_j}\mathrm{e}^{-i\theta_j}. \quad (2.2)$$

Lemma 1. *After doing the change of coordinates defined in* (2.2)*, the Hamiltonian* (2.1) *becomes:*

$$H(I,\theta) = \sum_{j=1}^{N} \left(I_j^2 - 4I_jI_{j-1}\cos 2\left(\theta_j - \theta_{j-1}\right)\right) \qquad \Omega = \sum_{j=1}^{N} dI_j \wedge d\theta_j. \tag{2.3}$$

From now on, we will call actions to I_j and angles to θ_j. Recall that the periodic orbits $\mathcal{T}_j$ for $1 \le j \le N$ can be seen in these coordinates as:

$$\mathcal{T}_j = \left\{(I,\theta) \in \mathbb{R}^N \times \mathbb{T}^N,\ I_j = \frac{1}{2},\ I_k = 0\,\forall k \neq j\right\}.$$

It seems that we are obtaining objects of larger dimension: since we do not have any restriction on the angles θ, $\mathcal{T}_j$ seems to be an n-dimensional torus. However, we recall that the angular coordinate θ_k is not well defined whenever $I_k = 0$, so we can think of $\mathcal{T}_j$ as a 1-dimensional torus, parameterized by θ_j and forget about the rest of angular coordinates.

As we know, there is a conserved quantity in the system, the total mass:

$$M_N(b) = \sum_{k=1}^{N} |b_k|^2 = 2\sum_{k=1}^{N} I_k,$$

and the idea is to use this fact performing a change of coordinates that will treat this quantity as a variable, obtaining a system with 1 degree of freedom less, since it will be costant. So, we write:

$$\left\{\begin{array}{lcl} J_1 & = & \sum_{j=1}^{N} I_j \\ J_i & = & I_i \text{ for } i \neq 1, \end{array}\right.$$

and using the Mathieu transformations [MHO92], we can compute the change on the angles for which the complete change is symplectic. This change is:

$$\left\{\begin{array}{lcl} I_1 & = & J_1 - \sum_{j=2}^{N} J_j \\ I_i & = & J_i \text{ for } i \neq 1 \end{array}\right. \qquad \left\{\begin{array}{lcl} \theta_1 & = & \varphi_1 \\ \theta_i & = & \varphi_1 + \varphi_i \text{ for } i \neq 1 \end{array}\right. \tag{2.4}$$

$$\left\{\begin{array}{lcl} J_1 & = & \sum_{j=1}^{N} I_j \\ J_i & = & I_i \text{ for } i \neq 1 \end{array}\right. \qquad \left\{\begin{array}{lcl} \varphi_1 & = & \theta_1 \\ \varphi_i & = & \theta_i - \theta_1 \text{ for } i \neq 1. \end{array}\right. \tag{2.5}$$

Notice that we have chosen exactly φ_1 for the conjugated angle for J_1. This is because we are going to focus in the first connection.

Lemma 2. *After doing the change of coordinates defined in* (2.4) *and* (2.5) *the Hamiltonian* (2.3) *becomes:*

$$\begin{aligned} H(J,\varphi) = & J_1^2 + 2\sum_{j=2}^{N} {J_j}^2 + \sum_{i=3}^{N}\sum_{j=2}^{i-1} J_i J_j - 2J_1 \sum_{j=2}^{N} J_j \\ & - 4J_1 J_2 \cos 2\varphi_2 + 4J_2 \cos 2\varphi_2 \sum_{j=2}^{N} J_j \\ & - 4\sum_{j=3}^{N} J_j J_{j-1} \cos 2(\varphi_j - \varphi_{j-1}). \end{aligned} \tag{2.6}$$

As we expected, φ_1 does not appear in the Hamiltonian (it is a cyclic coordinate), so the conjugated action, J_1, is constant. Taking into account that the periodic orbits live in the mass level $M_N(b) = 1$, we will take $J_1 = 1/2$. After this we have reduced by one degree of freedom the original Hamiltonian system (2.1). We must point out that, although the cyclic coordinates φ_1 does not play a role in the system, it has its own equation, that is:

$$\dot{\varphi}_1 = -1 + 2\sum_{k=2}^{N} J_k + 4J_2 \cos 2\varphi_2.$$

In these new coordinates, the periodic orbits $\mathcal{T}_j$ become:

$$\mathcal{T}_1 = \left\{ (J,\varphi) \in \left[0, \frac{1}{2}\right]^{N-1} \times \mathbb{T}^N, \;\; J_k = 0 \, \forall k \neq j \right\}$$
$$\mathcal{T}_j = \left\{ (J,\varphi) \in \left[0, \frac{1}{2}\right]^{N-1} \times \mathbb{T}^N, \;\; J_j = \frac{1}{2}, \; J_k = 0 \, \forall k \neq j \right\}.$$

Again we are obtaining larger dimensional objects due to the degeneracy of the change of coordinates in, precisely, the periodic orbits.

The last change of coordinates that we will perform is motivated by the special dependence that appears between the angular variables. We put:

$$\left\{ \begin{array}{rcl} J_j & = & K_j - K_{j+1} \text{ for } j \neq N \\ J_N & = & K_N \end{array} \right. \qquad \varphi_j = \sum_{i=2}^{j} \psi_i \tag{2.7}$$

$$K_j = \sum_{i=j}^{N} J_i \qquad \left\{ \begin{array}{rcl} \psi_2 & = & \varphi_2 \\ \psi_j & = & \varphi_j - \varphi_{j-1} \text{ for } j \neq 2. \end{array} \right. \tag{2.8}$$

Lemma 3. *After doing the change of coordinates defined in* (2.7) *and* (2.8)*, the Hamiltonian* (2.6) *becomes:*

$$\begin{aligned}H(K,\psi) =& \frac{1}{4} - K_2(1+2\cos 2\psi_2) + 2K_3\cos 2\psi_2 \\ &+ 2\sum_{j=2}^{N} K_j^2\,(1+2\cos 2\psi_j) - 2\sum_{j=3}^{N} K_jK_{j-1}\,(1+2\cos 2\psi_j + 2\cos 2\psi_{j-1}) \\ &+ 4\sum_{j=3}^{N-1} K_{j+1}K_{j-1}\cos 2\psi_j. \end{aligned} \tag{2.9}$$

We refer to Appendix B for the proofs of these Lemmas.

With these new coordinates the equation for the cyclic coordinate φ_1 becomes:

$$\dot{\varphi}_1 = -1 + 2K_2 + 4(K_2 - K_3)\cos 2\psi_2. \tag{2.10}$$

Now we compute the periodic orbits $\mathcal{T}_j$ in the new coordinates and we obtain objects which only depend on a configuration of the actions:

$$\mathcal{A}_1 = \left\{ (K,\psi) \in \left[0, \frac{1}{2}\right]^{N-1} \times \mathcal{T}^{N-1} \,:\, K = 0 \right\}$$

$$\mathcal{A}_j = \left\{ (K,\psi) \in \left[0, \frac{1}{2}\right]^{N-1} \times \mathcal{T}^{N-1} \,:\, K_i = \frac{1}{2}\,\forall i \le j,\ K_i = 0\,\forall i > j \right\}.$$

So they are $(N-1)$-dimensional tori, objects with larger dimension than the original periodic orbits. This is due to the fact that the polar coordinates that we have introduced are not well defined, precisely, in the periodic orbits. For this reason we will start considering:

$$\mathcal{T}_j \subset \mathcal{A}_j \text{ for } 1 \le j \le N.$$

The first thing that we see is that although $\mathcal{A}_j$ are invariant, only some subsets have stable and unstable invariant manifolds that will lead to the connections between them. So we must find these subsets.

To this end, we write the equations of motion in the subspace:

$$\Gamma_{j-1,j} = \left\{ (K,\psi) \in \left[0, \frac{1}{2}\right]^{N-1} \times \mathcal{T}^{N-1} \,:\, K_i = \frac{1}{2}\,\forall i < j,\ K_i = 0\,\forall i > j \right\},$$

which contains the tori $\mathcal{A}_{j-1}$ and $\mathcal{A}_j$:

$$\left\{ \begin{array}{rcl} \dot{K}_i &=& 0 \text{ for } 2 \le i \le j-1 \\ \dot{K}_j &=& 8K_j \sin 2\psi_j \left(\frac{1}{2} - K_j\right) \\ \dot{K}_i &=& 0 \text{ for } j+1 \le i \le N. \end{array} \right.$$

On the one hand, as we expected, we can see that $\Gamma_{j-1,j}$ is invariant by the flow and it is clear that the heteroclinic connections live in $\Gamma_{j-1,j}$. On the other hand, the equations for the angles are:

$$\begin{cases} \dot{\psi}_i &= 0 \text{ for } 2 \leq i \leq j-3 \\ \dot{\psi}_{j-2} &= 4\cos 2\psi_{j-1}\left(\frac{1}{2} - K_j\right) \\ \dot{\psi}_{j-1} &= -2\left(1 + 2\cos 2\psi_{j-1}\right)\left(\frac{1}{2} - K_j\right) + 4K_j \cos 2\psi_j \\ \dot{\psi}_j &= 2\left(1 + 2\cos 2\psi_j\right)\left(\frac{1}{2} - K_j\right) - 2K_j\left(1 + 2\cos 2\psi_j\right) \\ \dot{\psi}_{j+1} &= 2K_j\left(1 + 2\cos 2\psi_{j+1}\right) - 4\cos 2\psi_j\left(\frac{1}{2} - K_j\right) \\ \dot{\psi}_{j+2} &= -4K_j \cos 2\psi_{j+1} \\ \dot{\psi}_i &= 0 \text{ for } j+3 \leq i \leq N \end{cases}$$

Notice that the system formed by (K_j, ψ_j) is uncoupled and its phase portrait is:

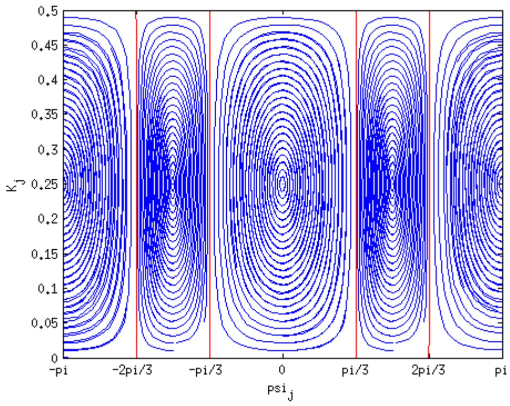


Figure 2.1: Phase portrait in the plane (K_j, ψ_j)

For this reason, in order to increase the K_j variable from 0 to 1/2 (that is the heteroclinic), we must take the subspace:

$$\gamma_{j-1,j} = \left\{(K, \psi) \in \Gamma_{j-1,j} \,:\, \psi_j = \frac{\pi}{3}\right\}.$$

Using the same argument in $\Gamma_{j,j+1}$ we see that we must take:

$$\gamma_{j,j+1} = \left\{(K, \psi) \in \Gamma_{j,j+1} \,:\, \psi_{j+1} = \frac{\pi}{3}\right\}.$$

Finally we define:

$$\mathcal{T}_j = \gamma_{j-1,j} \cap \gamma_{j,j+1} = \left\{(K, \psi) \in \mathcal{A}_j \,:\, \psi_j = \frac{\pi}{3},\, \psi_{j+1} = \frac{\pi}{3}\right\}.$$

Thus, in these coordinates we obtain $(N-3)$-dimensional tori but we must always recall that some of the solutions that we will generate may have no sense in the original coordinates.

This is the Toy Model System seen in these new coordinates and it is the basic motivation for considering a new scheme for diffusion that differs from Arnold's.

Let us focus in the system formed by (K_2, ψ_2) in $\Gamma_{1,2}$ but now including the cyclic coordinate φ_1 from equation (2.10) particularized in $\Gamma_{1,2}$. Recall that φ_1 is the angle that corresponds to the first mode b_1, so, since we are working in the first transition, this coordinate makes sense. The system formed by (φ_1, K_2, ψ_2) has the following equations:

$$\begin{aligned}
\dot{\varphi}_1 &= -1 + 2K_2\left(1 + 2\cos 2\psi_2\right)\\
\dot{K}_2 &= 8K_2 \sin 2\psi_2 \left(\frac{1}{2} - K_2\right)\\
\dot{\psi}_2 &= 2\left(1 + 2\cos 2\psi_2\right)\left(\frac{1}{2} - K_2\right) - 2K_2\left(1 + 2\cos 2\psi_2\right).
\end{aligned}$$

Notice that if we restrict the system to $\gamma_{1,2}$, we obtain:

$$\begin{aligned}
\dot{\varphi}_1 &= -1\\
\dot{K}_2 &= 4\sqrt{3}K_2\left(\frac{1}{2} - K_2\right)\\
\dot{\psi}_2 &= 0.
\end{aligned}$$

Then, we can give an explicit expression for the dynamics of the heteroclinic:

$$\gamma_{1,2}(t) = (\varphi_1(t), K_2(t), \psi_2(t)) = \left(\varphi_1(0) - t, \frac{1}{2 + \frac{1-2K_2(0)}{K_2(0)}\mathrm{e}^{-2\sqrt{3}t}}, \frac{\pi}{3}\right).$$

We are going to compute a Poincaré map that will reduce the dimension of the system in $\gamma_{1,2}$. Let $\Sigma_0 = \{\varphi_1 = \varphi_1(0)\}$. It is clear hat if we start with a point in $\Sigma_0 \cap \gamma_{1,2}$, the action of the flow will return us back to $\Sigma_0 \cap \gamma_{1,2}$ after some time, $t^*(K_2)$, that will only depend on K_2 since $\psi_2 = \pi/3$. Actually, this time is common for all the points in $\gamma_{1,2}$ and it is equal to 2π. So we have defined a Poincaré map $P : \Sigma_0 \cap \gamma_{1,2} \to \Sigma_0 \cap \gamma_{1,2}$ that has the following expression:

$$P(K_2) = \frac{1}{2 + \frac{1-2K_2}{K_2}\mathrm{e}^{-4\pi\sqrt{3}}}.$$

Notice that its positive (negative) iterates tend to $K_2 = 1/2$ ($K_2 = 0$) for any point K_2 in the segment, that means that each point is heteroclinic.

As a conclusion, we have obtained a map, P, for which the intersection of the invariant manifolds of two consecutive fixed points coincide, that is, it is not transverse. That means that if we recover the flow, we will have that the intersection between the periodic orbits $\mathcal{T}_1$ and $\mathcal{T}_2$ is not transverse.

As we have said, the argument is analogous for any pair of periodic orbits $\mathcal{T}_j$ and $\mathcal{T}_{j+1}$.

2.2 Transverse versus Non-Transverse

Once we have convinced ourselves that the Toy Model System does not present transverse intersection in the heteroclinic chain just like in Arnold Diffusion, we want to explain the difference between the transverse and the non-transverse situation.

To do so, we are going to consider a two dimensional map with four fixed points, located at the points:

$$p_0 = (0,0) \quad p_1 = (1,0) \quad p_2 = (1,1) \quad p_3 = (2,1).$$

We are going to assume also that each point, p_i, has a one dimensional stable manifold, $\mathcal{W}^s(p_i)$, and a one dimensional unstable manifold, $\mathcal{W}^u(p_i)$ both tangent to some linear subspaces. That is

- $\mathcal{W}^s(p_0)$ is tangent to the subspace generated by $\vec{e}_2$ at p_0 and $\mathcal{W}^u(p_0)$ is tangent to the subspace generated by $\vec{e}_1$ at p_0.
- $\mathcal{W}^s(p_1)$ is tangent to the subspace generated by $\vec{e}_1$ at p_1 and $\mathcal{W}^u(p_1)$ is tangent to the subspace generated by $\vec{e}_2$ at p_1.
- $\mathcal{W}^s(p_2)$ is tangent to the subspace generated by $\vec{e}_2$ at p_2 and $\mathcal{W}^u(p_2)$ is tangent to the subspace generated by $\vec{e}_1$ at p_2.
- $\mathcal{W}^s(p_3)$ is tangent to the subspace generated by $\vec{e}_1$ at p_3 and $\mathcal{W}^u(p_3)$ is tangent to the subspace generated by $\vec{e}_2$ at p_3.

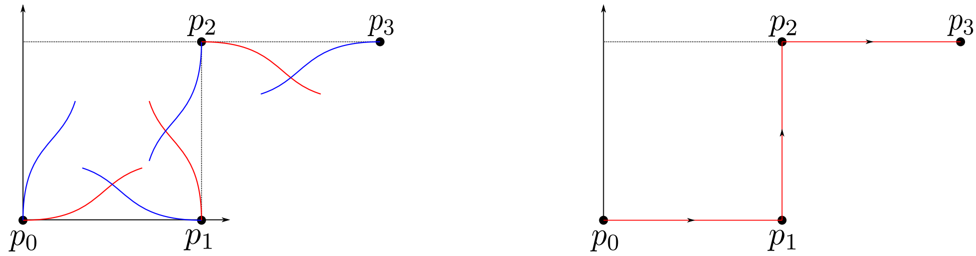
Now we are going to consider two different scenarios. The first one consists on assuming that the unstable manifold of a point p_i intersects transversally with the stable manifold of the following point p_{i+1}, that is:

$$\begin{gathered} \mathcal{W}^u(p_i) \cap \mathcal{W}^s(p_{i+1}) \neq \emptyset \\ \forall q \in \mathcal{W}^u(p_i) \cap \mathcal{W}^s(p_{i+1}) \Rightarrow T_q\mathcal{W}^u(p_i) + T_q\mathcal{W}^s(p_{i+1}) = \mathbb{R}^2. \end{gathered}$$

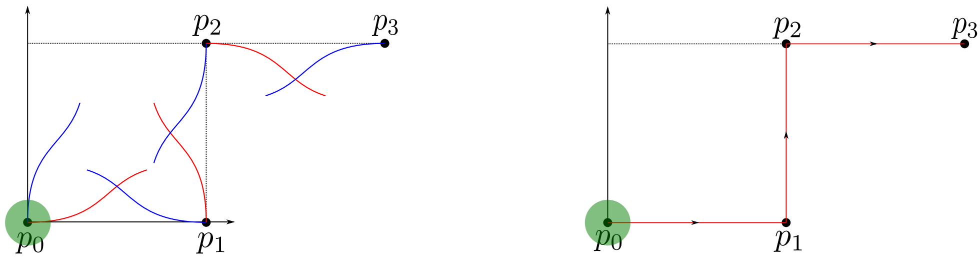
The second will be given by a non-transverse intersection of the manifolds and, since we are dealing with a low dimensional system, that will mean that those manifolds coincide in a branch:

$$\begin{gathered} \mathcal{W}^u(p_i) \cap \mathcal{W}^s(p_{i+1}) \neq \emptyset \\ \forall q \in \mathcal{W}^u(p_i) \cap \mathcal{W}^s(p_{i+1}) \Rightarrow T_q\mathcal{W}^u(p_i) + T_q\mathcal{W}^s(p_{i+1}) = \mathbb{R}. \end{gathered}$$

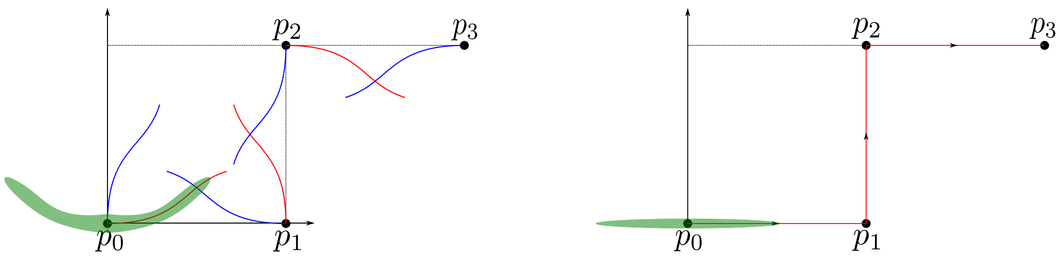
The schematic situation is the following:



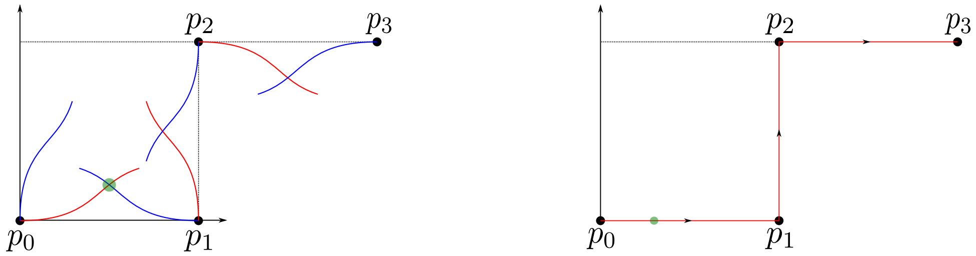
We wonder if it is possible to connect p_0 with p_3 through the map, in both situations. To do so, we consider a ball containing the first fixed point p_0:



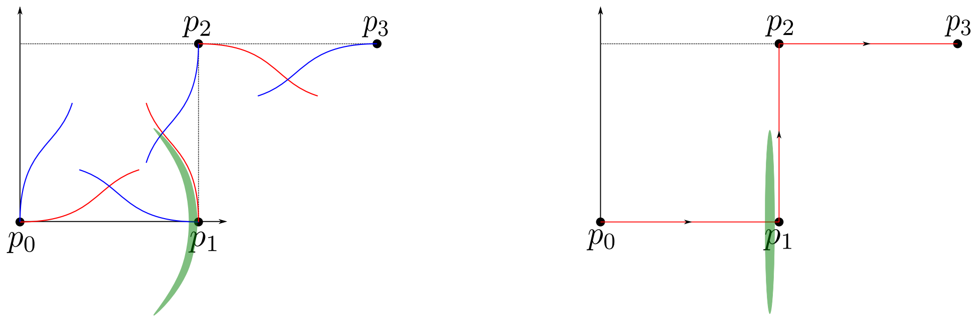
If we compute iterates of the ball through the maps we can expect that it is expanded in the unstable direction and contracted in the stable direction:



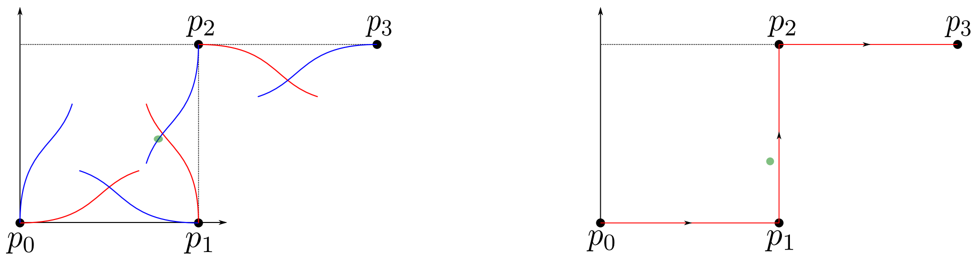
Notice that in both cases the domain intersects the stable manifold of the following fixed point, p_1. So, in the transverse case the domain contains a heteroclinic point. In the non-transverse situation this is obvious because the manifolds are coincident. We can now restrict our domain precisely around that intersection point for the transverse case and at some place in the right-hand side of the fixed point p_0 for the non-transverse case.



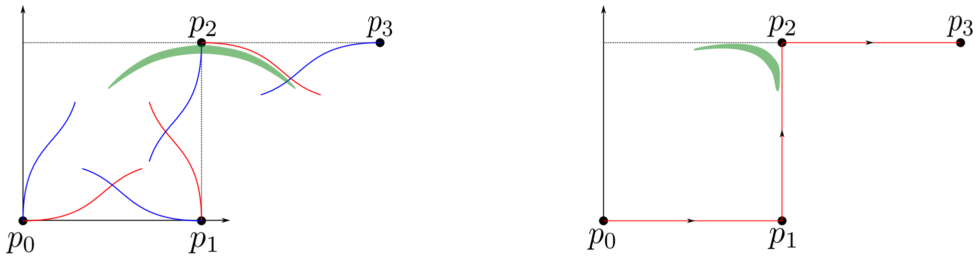
If we compute forward iterates of the restricted domain we are going to approach the fixed point p_2 since, in both cases, our domains contain heteroclinic points. The domains will not only approach p_1 but also, after some iterates, will spread to the unstable manifold of p_1:



Here we find the first big difference between the transverse and the non-transverse case. In the transverse case, since everything tends to the unstable manifold of p_1, it is clear that our domain will intersect the stable manifold of p_2. In the non-transverse situation, our domain will never cross the stable manifold of p_2 since it corresponds to an invariant curve. Then, we restrict our domain in the intersection for the transverse case and in the upper part of p_1 since we want to reach p_3:



Using, in the transverse case, the same argument as before, since our domain contains a heteroclinic point in the transverse situation, forward iterates will spread our domain on the unstable manifold of p_2. For the non transverse case we will reach the proximity of p_2 after some iterates, but our domain will be trapped and could not visit the following fixed point, p_3:



In the transverse case, we could continue and see that the domain will visit p_3.

After this schematic approach, we can see that, on the one hand, in the transverse situation there are no geometric obstructions in shadowing the heteroclinic chain. On the other hand, in the non-transverse case, we can see that, in general, we cannot visit as many invariant objects as we want. So, now, we wonder why the authors of [CKS+10] can connect N periodic orbits in the Toy Model System. The main reason is the large dimension of the system and the fact that each connection takes place in a direction that has not been used in the past.

Notice that, if in the non-transverse example the last fixed point p_3 is located in a new dimension (that means that the system is three dimensional) we could continue with the argument and visit p_3.

In the next subsection we are going to generate examples for which it is clear that one can shadow a non-transverse heteroclinic chain.

2.3 Examples with diffusion in a non-transverse situation

Let us try to simplify the Toy Model System. Consider the Hamiltonian (2.9). It corresponds to the Toy Model System reduced in one degree of freedom using the mass conservation law. If we add some terms to this Hamiltonian that only depend on the actions, K_j, we will modify the equations for the angles keeping the same equations for K_j. Than means that we will not change the geometry for the periodic orbits or the heteroclinic connections. The terms that we will add will create an invariant manifold for the angles. Indeed, if H is the Hamiltonian from (2.9), we take:

$$\tilde{H}(K,\psi) = H(H,\psi) + K_3 - 2\sum_{j=3}^{N} K_j K_{j-1} + 2\sum_{j=3}^{N-1} K_{j+1}K_{j-1}.$$

As we have said we obtain the same equations for the actions (and the same invariant subspaces $\Gamma_{j-1,j}$, $\mathcal{A}_j$) while the equations for the angles are:

$$\begin{aligned}
\dot{\psi}_2 =& 2\left(1+2\cos 2\psi_2\right)\left(\frac{1}{2}-K_2\right) - 2\left(1+2\cos 2\psi_2\right)\left(K_2-K_3\right) \\
&+ 2\left(1+2\cos 2\psi_3\right)\left(K_3-K_4\right) \\
\dot{\psi}_3 =& 2\left(1+2\cos 2\psi_3\right)\left(K_2-K_3\right) - 2\left(1+2\cos 2\psi_3\right)\left(K_3-K_4\right) \\
&- 2\left(1+2\cos 2\psi_2\right)\left(\frac{1}{2}-K_2\right) + 2\left(1+2\cos 2\psi_4\right)\left(K_4-K_5\right) \\
\dot{\psi}_i =& 2\left(1+2\cos 2\psi_i\right)\left(K_{i-1}-K_i\right) - 2\left(1+2\cos 2\psi_i\right)\left(K_i-K_{i+1}\right) \\
&- 2\left(1+2\cos 2\psi_{i-1}\right)\left(K_{i-2}-K_{i-1}\right) \\
&+ 2\left(1+2\cos 2\psi_{i+1}\right)\left(K_{i+1}-K_{i+2}\right) \text{ for } 4\le i\le N-1 \\
\dot{\psi}_N =& 2\left(1+2\cos 2\psi_N\right)\left(K_{N-1}-K_i\right) - 2K_N\left(1+2\cos 2\psi_N\right) \\
&- 2\left(1+2\cos 2\psi_{N-1}\right)\left(K_{N-2}-K_{N-1}\right).
\end{aligned}$$

Notice that if we take, for instance, $\psi_i = \pi/3$ for all $i = 2\ldots N$ all the equations vanish, so we have identified an invariant subspace. In this subspace the equations for the actions are:

$$\left\{\begin{array}{rcl}
\dot{K}_2 &=& 4\sqrt{3}\left(\frac{1}{2}-K_2\right)\left(K_2-K_3\right) \\
\dot{K}_i &=& 4\sqrt{3}\left(K_{i-1}-K_i\right)\left(K_i-K_{i+1}\right) \text{ for } i\neq 2,N \\
\dot{K}_N &=& 4\sqrt{3}K_N\left(K_{N-1}-K_N\right).
\end{array}\right. \tag{2.11}$$

Let us analyze this system:

- It has N equilibrium points:

  $$p_1 = (0,\ldots,0) \qquad p_j = \left(\frac{1}{2},\overset{j-1}{\ldots},\frac{1}{2},0,\overset{N-j}{\ldots},0\right)$$

  for $j = 2\ldots N$.

- The segments that connect two consecutive equilibrium points,

  $$C_{j-1,j} = \left\{\left(\frac{1}{2},\overset{j-2}{\ldots},\frac{1}{2},K_j,0,\overset{N-j}{\ldots},0\right),\ 0\le K_j\le\frac{1}{2}\right\}$$

  for $j = 2\ldots N$ are invariant and heteroclinic, that is, for all $K\in C_{j-1,j}$

  $$\lim_{t\to\infty}\Phi_t(K) = p_i \qquad \lim_{t\to-\infty}\Phi_t(K) = p_{i-1},$$

  where $\Phi_t(K)$ is the flow of the system.

Notice that each connection takes place in a new direction, not used before.

We have now obtained a simpler example for which we could prove the same scheme of diffusion. However, we want to construct an even easier example where we can guarantee the diffusion.

Inspired by the structure of system (2.11), we impose the previous conditions in a generic system defined by a polynomial of degree two. Simplifying as much as possible, we obtain the following system:

$$\dot{x} = F(x)$$

with

$$\begin{cases} F_1(x) &= \lambda_1 x_1 - \lambda_1 x_1^2 \\ F_i(x) &= (\lambda_i - \mu_i)x_i - \lambda_i x_i^2 + \mu_i x_i x_{i-1} \qquad \text{for } 1 < i \leq n \end{cases} \tag{2.12}$$

with $\lambda_i > 0$ for $1 \leq i \leq n$ and $\mu_i \in \mathbb{R}$ for $1 < i < n$.

Remark. In order to get a simpler system, we have set the distance between two consecutive equilibrium points at one.

Note that this is a triangular system and we can integrate each equation, since

$$\dot{x}_i = f_i(t)x_i + \beta x_i^2 \Rightarrow x_i(t) = \frac{e^{\int_0^t f_i(s)ds}}{\frac{1}{x_i(0)} - \beta_i \int_0^t e^{\int_0^s f_i(r)dr} ds},$$

with $f_i(t) = \lambda_i - \mu_i + \mu_i x_{i-1}(t)$ and $\beta_i = -\lambda_i$.

We can check now the linear behavior around the equilibrium points computing the derivative of the vector field:

$$DF(p_0) = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 - \mu_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_i - \mu_i & & \\ & & & & \ddots & \\ & & & & & \lambda_n - \mu_n \end{pmatrix}$$

$$DF(p_i) = \begin{pmatrix} -\lambda_1 & & & & & & & \\ \mu_2 & -\lambda_i & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \mu_i & -\lambda_i & & & & \\ & & & & \lambda_{i+1} & & & \\ & & & & & \lambda_{i+2} - \mu_{i+2} & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_n - \mu_n \end{pmatrix}$$

so we have different possibilities of choosing the parameters.

- If $\mu_i > \lambda_i$ for all $i = 1 \ldots N$, each point, p_j, has only one unstable direction defined by $\vec{e}_{j+1}$, while the rest of the directions are stable.

- If $\mu_i = \lambda_i$ for all $i = 1 \ldots N$, each point, p_j, has only one unstable direction, defined by $\vec{e}_{j+1}$. All the "past" directions, defined by $\{\vec{e}_1, \ldots, \vec{e}_j\}$ are stable while all the "future" directions, defined by $\{\vec{e}_{j+2}, \ldots, \vec{e}_n\}$, are linearly neutral.

- If $\mu_i < \lambda_i$ for all $i = 1 \ldots N$, at each point, p_j, all the "past" directions, defined by $\{\vec{e}_1, \ldots, \vec{e}_j\}$, are stable while all the "future" directions, defined by $\{\vec{e}_{j+2}, \ldots, \vec{e}_n)\}$, are linearly unstable.

Notice that we could have a mixed situation but it will not introduce any significant difference. Let us show the numerical integration of the system for these three possibilities. For the sake of concreteness we are going to assume $\lambda = \lambda_i$ and $\mu = \mu_i$ for all $i = 1, \ldots, n$ in a four dimensional system. In the three cases, we are going to take the same initial condition:

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = \frac{1}{10}.$$

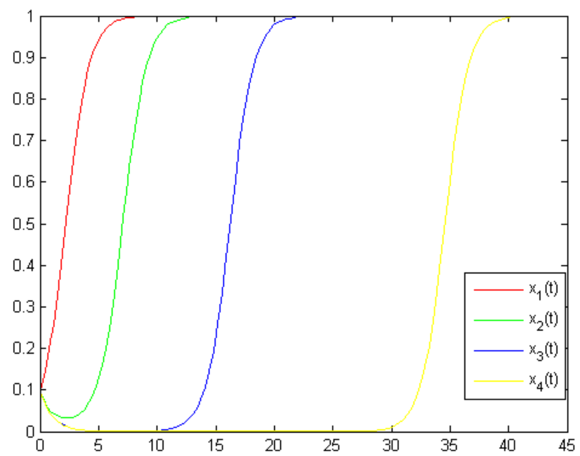


Figure 2.2: Solution of system (2.12) for $\lambda = 1$ and $\mu = 2$.

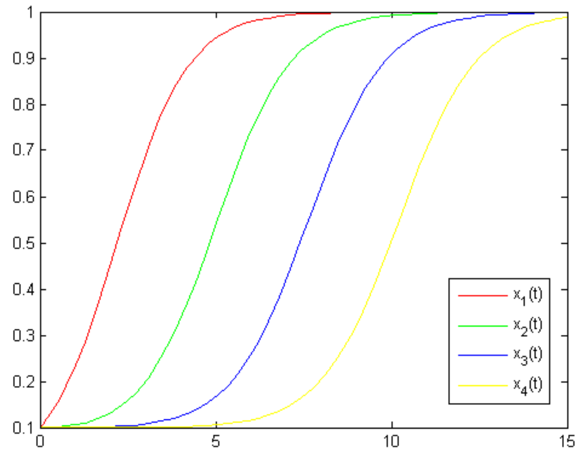


Figure 2.3: Solution of system (2.12) for $\lambda = 1$ and $\mu = 1$.

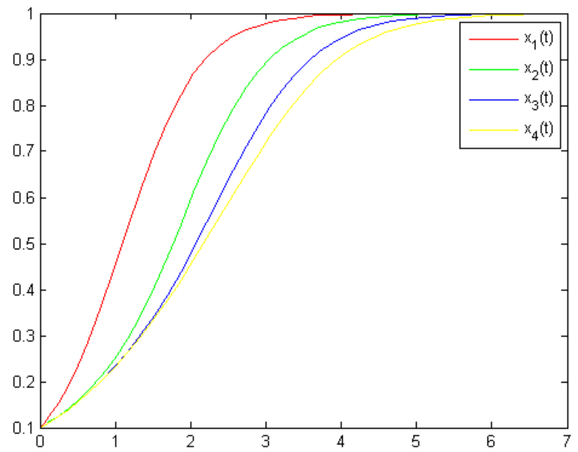


Figure 2.4: Solution of system (2.12) for $\lambda = 2$ and $\mu = 1$.

We can see that in the three cases we achieved our goal: to visit $p_4 = (1, 1, 1, 1)$ starting close to $p_0 = (0, 0, 0, 0)$. However it is only clear that we visit all the intermediate points $p_1 = (1, 0, 0, 0)$, $p_2 = (1, 1, 0, 0)$ and $p_3 = (1, 1, 1, 0)$ in the first situation, when $\mu > \lambda$. In this case, all the directions are stable in each point, and are only activated at its turn. In the other cases, all the future directions are unstable at each point so they are activated at the very beginning although the characteristic exponent, $\lambda - \mu$, is lower than the one in the heteroclinic, λ, in the considered cases. So, if we want to visit all the intermediate points in the two last situations, we have to decrease the initial condition for the future directions. Indeed, if we take:

$$x_1(0) = \frac{1}{10}, \quad x_2(0) = x_3(0) = x_4(0) = \frac{1}{100},$$

in the case when $\mu = \lambda = 1$, we obtain:

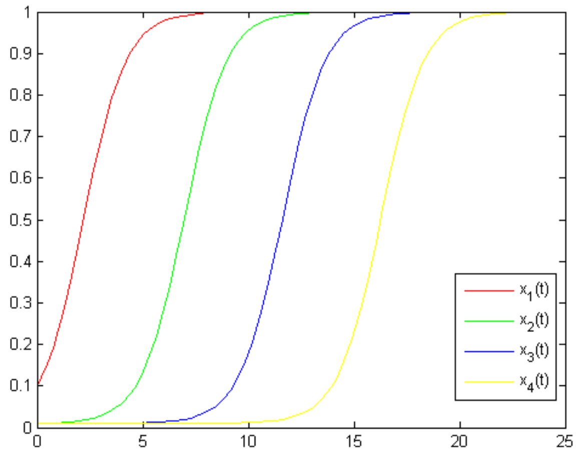


Figure 2.5: Solution of system (2.12) for $\lambda = 1$ and $\mu = 1$.

Notice that it is enough to distinguish only the first component. The weak coupling activates the component in order, since the first equation that notices the growth of x_1 is the one for x_2.

For the third case, when $\lambda > \mu$, we recall that all the future coordinates are unstable in p_0 and the linear part almost dominates in front of the coupling that would have made increase the coordinates in order. So, considering the following decreasing sequence of initial condition:

$$x_1(0) = 10^{-1}, \quad x_2(0) = 10^{-2} \quad x_3(0) = 10^{-3} \quad x_4(0) = 10^{-4},$$

we obtain:

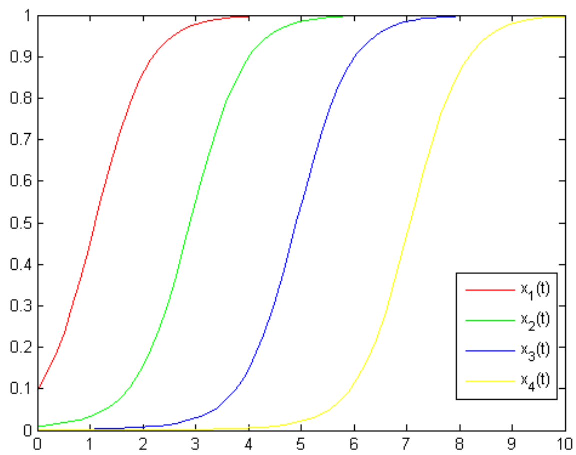


Figure 2.6: Solution of system (2.12) for $\lambda = 2$ and $\mu = 1$.

Remark. Notice that the diffusing times are very different in the three situations.

Although there are numerical evidences that the system behaves in the desired way, we could expect that just looking at the equations. It is clear that

$$\lim_{t\to\infty} x_1(t) = 1.$$

By induction, assuming

$$l_{i-1} = \lim_{t\to\infty} x_{i-1}(t)$$

and looking for equilibria of $x_i(t)$ for $t \to \infty$, it has to satisfy

$$\lim_{t\to\infty} \dot{x}_i(t) = 0,$$

i.e.

$$0 = \lambda_i l_i(1 - l_i) - \mu_i l_i(1 - l_{i-1}).$$

So, we get $l_i = 0$ or $l_i = 1$ but since $\dot{x}_i(t) > 0$ for t large enough we can conclude $l_i = 1$.

The main conclusion of this part is that we have obtained an easy example for which we can ensure the transition chain even the intersection between the invariant manifolds is not transverse, regarding the high dimension and the disposition of the equilibrium points and the heteroclinic connections: each one in a new direction not used before. In addition the system is integrable by quadratures which goes against the notion of Arnold's diffusion, completely prohibited in integrable systems.

To end this section we can relate both examples (2.11) and (2.12). Scaling the variables in (2.11), $y_i = 2K_i$ for $2 \le i \le N$, the system becomes:

$$\begin{cases} \dot{y}_2 &= 2\sqrt{3}\,(1 - y_2)\,(y_2 - y_3) \\ \dot{y}_i &= 2\sqrt{3}\,(y_{i-1} - y_i)\,(y_i - y_{i+1}) \;\text{ for } i \ne 2, N \\ \dot{y}_N &= 2\sqrt{3} y_N\,(y_{N-1} - y_N) \end{cases}$$

that can be seen as a modification of (2.12) with $\lambda_i = \mu_i = 2\sqrt{3}$ for all i and adding some terms:

$$\dot{y} = G(y)$$

with

$$\begin{cases} G_2(y) &= F_2(y) - 2\sqrt{3} y_3 + 2\sqrt{3} y_2 y_3 \\ G_i(y) &= F_i(y) - 2\sqrt{3} y_{i-1} y_{i+1} + 2\sqrt{3} y_i y_{i+1} \;\text{for } i \ne 2, N \\ G_N(y) &= F_N(y) \end{cases}$$

Recall that system (2.11) comes from a Hamiltonian system that has an invariant subspace in some configurations of the angles. Due to the similarity between these systems we can think in finding a Hamiltonian for which system (2.12) corresponds to a subsystem in a concrete invariant subspace. Indeed, take:

$$\hat{H}(x, \theta) = (\lambda_1 x_1 - \lambda_1 x_1^2)\cos\theta_1 + \sum_{i=2}^{n} ((\lambda_i - \mu_i)x_i - \lambda_i x_i^2 + \mu_i x_i x_{i-1})\cos\theta_i.$$

Then, if we compute the equations of motion for x_i we recover exactly the equations (2.12) but multiplied by $-\sin\theta_i$. The system for the angles is

$$\begin{cases} \dot\theta_1 &= -(\lambda_1 - 2\lambda_1 x_1)\cos\theta_1 - \mu_2 x_2\cos\theta_2 \\ \dot\theta_i &= -(\lambda_i - \mu_i - 2\lambda_i x_i + \mu_i x_{i-1})\cos\theta_i - \mu_{i+1}x_{i+1}\cos\theta_{i+1} \qquad \text{for } 1 < i < n \\ \dot\theta_n &= -(\lambda_n - \mu_n - 2\lambda_n x_n + \mu_n x_{n-1})\cos\theta_n \end{cases}$$

Then, if we set all the angles in the subspace $\{\theta_i = -\pi/2,\, 1 \le n\}$, we obtain that all the angles are fixed and the equations for x_i become exactly the ones in (2.12) since $-\sin\left(-\frac{\pi}{2}\right) = 1$.

As a conclusion of the first Sections of the present Chapter we can say that, for the Toy Model System, the intersection between the invariant manifolds are not transverse. With this fact we can almost discard that the mechanism for diffusion relies on the Arnold diffusion mechanism so, then, there is no geometric explanation of why the invariant objects can be connected. In addition, we have shown through examples that the lack of transversality can forbid the connection in a transition chain. However, we have detected the reason why the connection could be possible in the Toy Model System. The geometric mechanism relies on the fact that we are dealing with a high dimensional system and that each new connection is defined by a direction that *has not been used before*. We have presented, also, an integrable example for which the diffusion is feasible, totally discarding, then, the Arnold diffusion as a mechanism since it is typical for non integrable systems.

2.4 A proposal for a new scheme of diffusion

The main goal of this section would be to establish a global Theorem trough which we can prove this kind of diffusion for a general system that has a non-transverse heteroclinic chain. However this is not what we have achieved and we have to resign ourselves with a Theorem for concrete cases. From now on we are going to work with maps instead of flows.

2.4.1 A Theorem for an easy situation

Let us start with a simple situation. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism with the following properties:

1. The points $p_i = (1, \overset{i}{\ldots}, 1, 0, \overset{n-i}{\ldots}, 0)$ are fixed under f for $i = 0 \ldots n$.

2. The segments C_i that connect the points p_{i-1} and p_i,

$$C_i = \{(1, \overset{i-1}{\ldots}, 1, t, 0, \overset{n-i}{\ldots}, 0), 0 \le t \le 1\}$$

for $1 \leq i \leq n$ are invariant under f and, for all $x \in C_i$:

$$\lim_{k\to\infty} f^k(x) = p_i \qquad \lim_{k\to-\infty} f^k(x) = p_{i-1}.$$

3. At each point p_i the i-th direction is stable and the $(i+1)$-th is unstable. This means:

$$Df(p_i)e_i = \mu_i e_i, \quad |\mu_i| < 1$$
$$Df(p_i)e_{i+1} = \lambda_i e_{i+1}, \quad |\lambda_i| > 1$$

4. The past directions, defined by $\vec{e_1}, \dots, \vec{e}_{i-1}$, are contracting directions around the fixed point p_i but with a lower rate than μ_i. The future directions, defined by $\vec{e}_{i+2}, \dots, \vec{e}_n$, are expanding directions around the fixed point p_i but with a lower rate than λ_i.

Theorem 5. *Under the previous assumptions, for all $\epsilon > 0$ there exist a point x_0 and a sequence of integers $0 = k_0 < k_1 < \cdots < k_n$ such that:*

$$||f^{k_i}(x_0) - p_i|| < \epsilon \quad i = 0, \dots, n.$$

Remark. Notice that we connect $n+1$ points in a n dimensional space. We cannot guarantee that the result is valid for more points. This Theorem is designed to be applied in high dimensional systems.

Remark. This is a very simple version of the Theorem that can be widely generalized. We are assuming that there are only two dominant coordinates around each fixed point. That means that this could not be applied to the Toy Model System, where we have four dominant directions. We can think that this simplified situation would correspond for systems such that the transition chain is defined in a subspace of half dimension, just like (2.11).

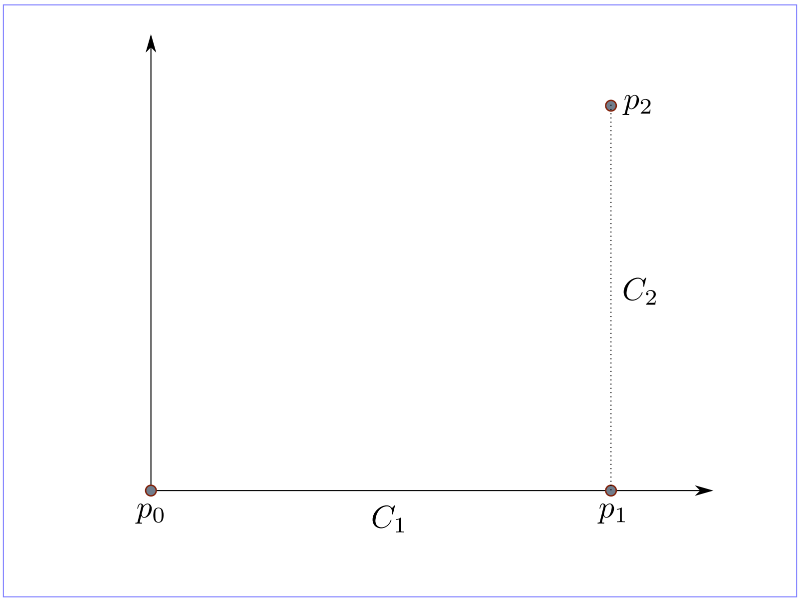
2.4.1.1 Sketch of the proof: loosing dimensions

Here we present a sketch of the proof of Theorem 5 with some pictures. We are going to consider only a two dimensional map. So, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ with three fixed points:

$$p_0 = (0,0) \quad p_1 = (1,0) \quad p_2 = (1,1),$$

with invariant segments C_1 and C_2 defined as

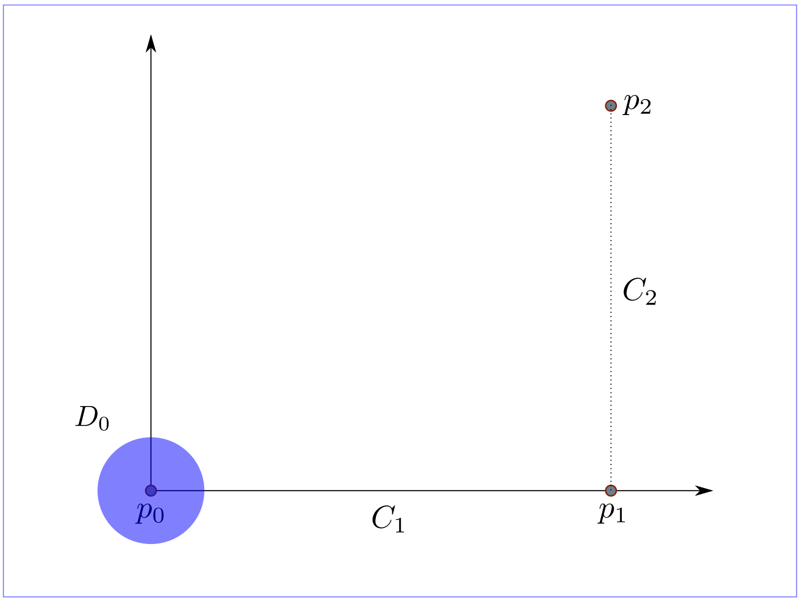
$$C_1 = \{(x_1, x_2) : 0 \leq x_1 \leq 1, \, x_2 = 0\} \quad C_2 = \{(x_1, x_2) : x_1 = 0, \, 0 \leq x_2 \leq 1\}$$



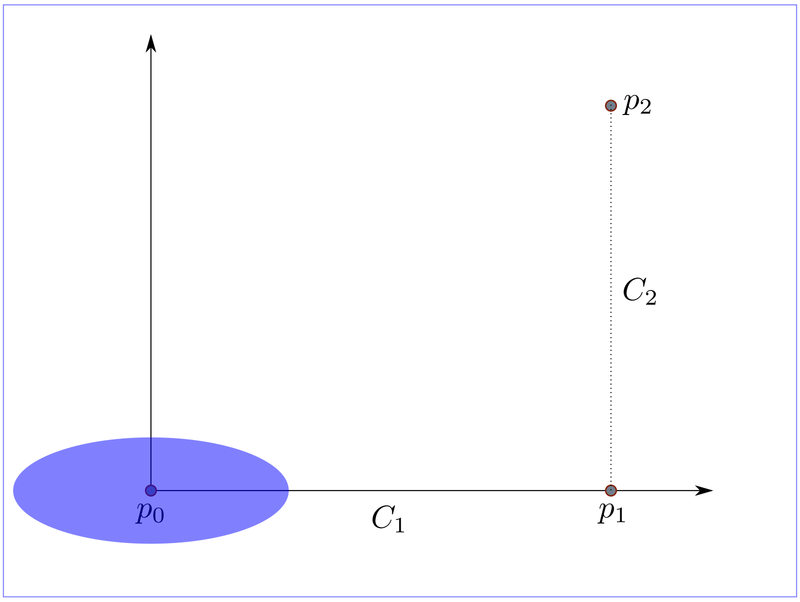
Assume also that the derivatives of the map around the fixed points have the following structure:

$$Df(p_0) = \begin{pmatrix} \lambda_{0,1} & 0 \\ 0 & \lambda_{0,2} \end{pmatrix} \quad Df(p_1) = \begin{pmatrix} \mu_{1,1} & 0 \\ 0 & \lambda_{1,2} \end{pmatrix} \quad Df(p_2) = \begin{pmatrix} \mu_{2,1} & 0 \\ 0 & \mu_{2,2} \end{pmatrix} \tag{2.13}$$

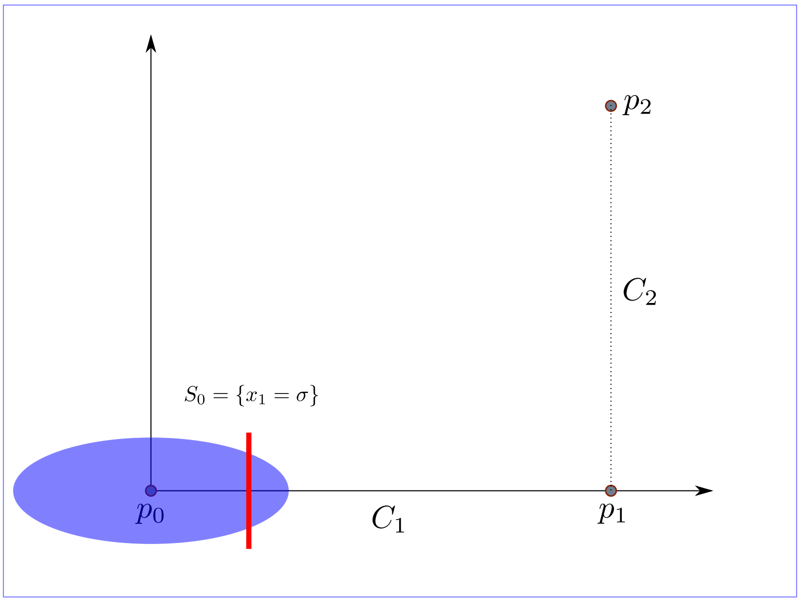
where $\lambda_{0,1}, \lambda_{0,2}, \lambda_{12} > 1$ and $0 < \mu_{1,1}, \mu_{2,1}, \mu_{2,2} < 1$. We start considering a domain (ball) of full dimension centered around p_0, D_0.



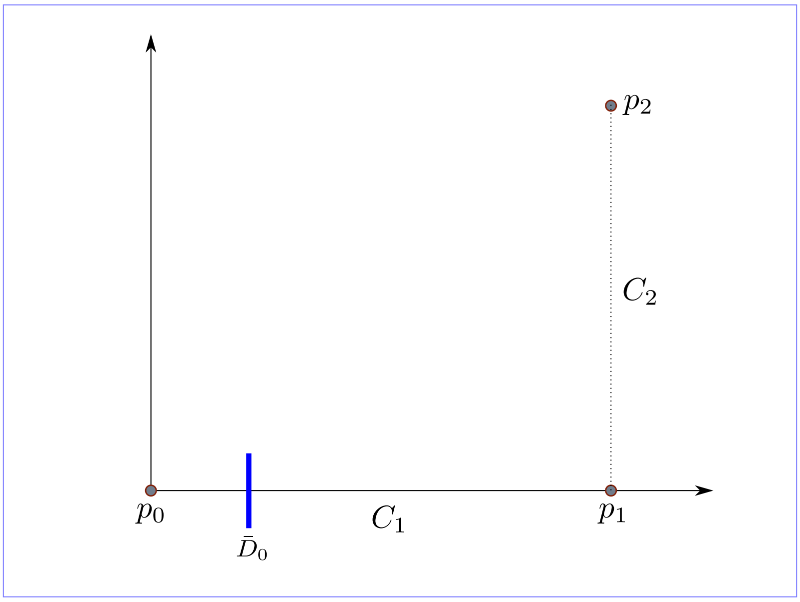
Given the linear stability from (2.13), we can assume the after one iteration of the map, our initial ball D_0 will be expanded in both directions:



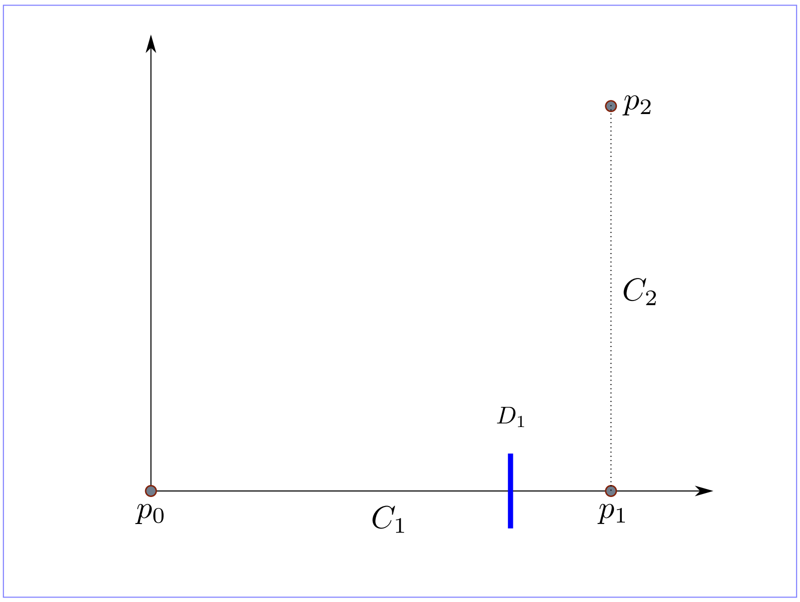
It is now time to make a call: from all the possible directions, we are only interested in the one defined by the outgoing heteroclinic, that is, the segment C_1. Then we consider a section $S_0 = \{x_1 = \sigma\}$ where σ is some small parameter:



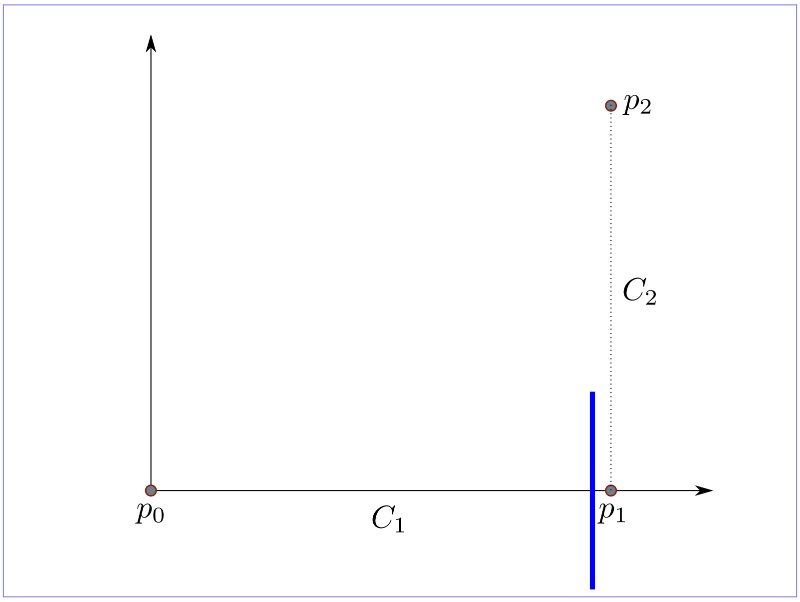
Since we are only interested in the points of our ball close to the heteroclinic, we intersect the domain with the section S_0. We say that we have lost the x_1 direction. We could not use this direction in future steps. Our domain, $\bar{D}_0$ has one dimension less than D_0, that is, dimension one.



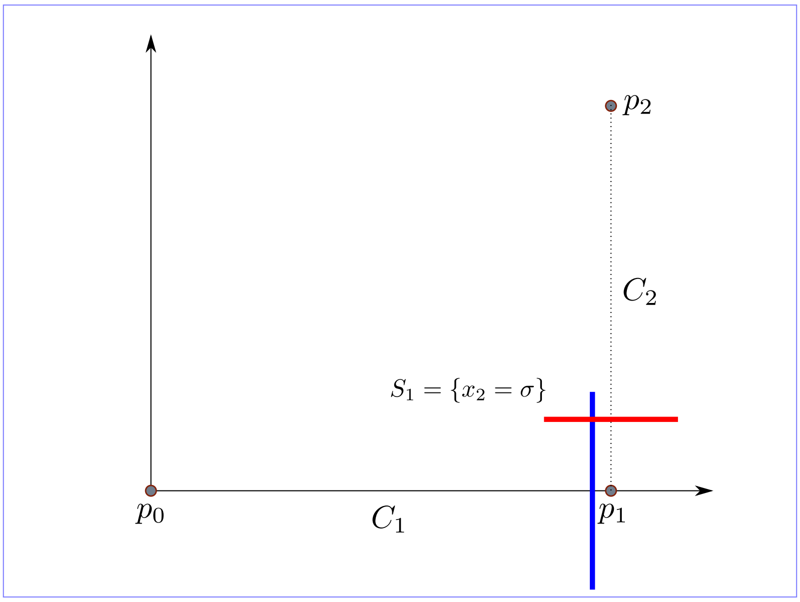
Then we continue with this domain. After several iterations of the map since the domain is close to the heteroclinic connection, we can ensure that $\bar{D}_0$ will approach p_1 and get a domain D_1:



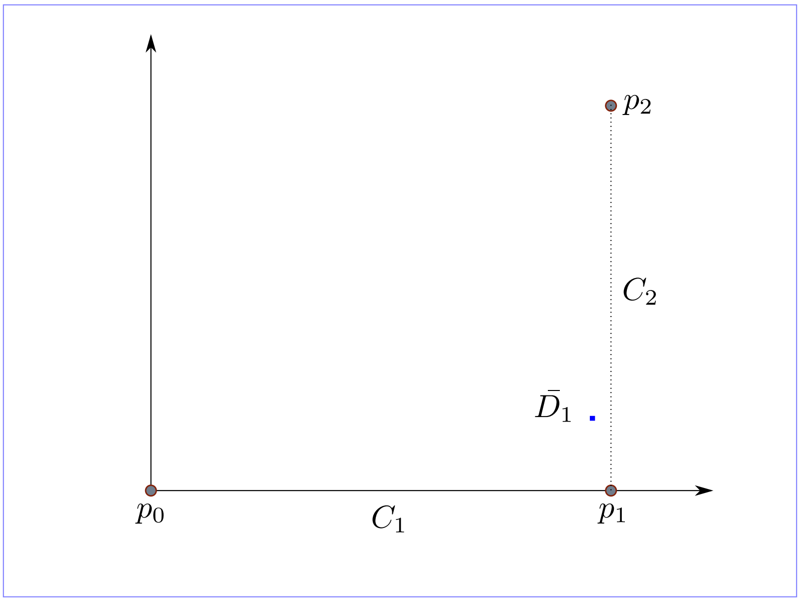
We can use, again, the linear prediction of the map, (2.13), to be sure that our domain is expanded in the x_2 direction.



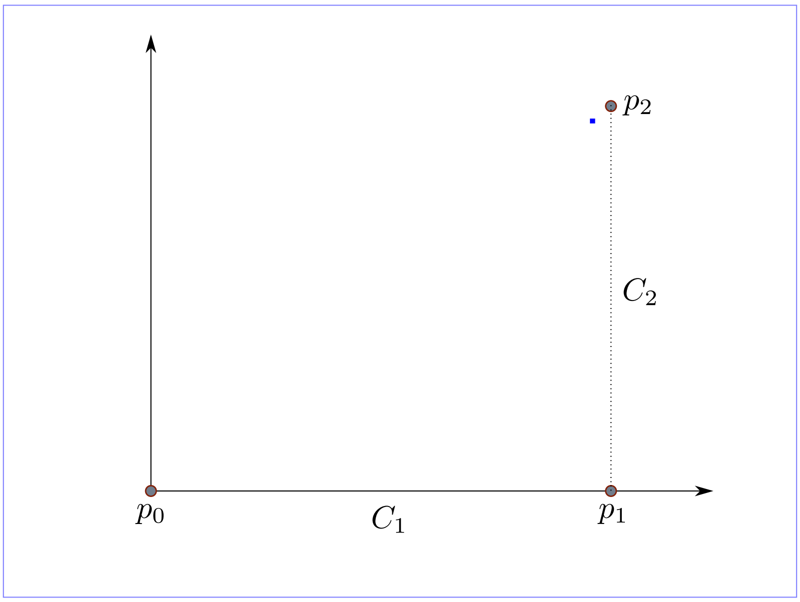
We use now the same argument. From all the possible directions that $f(D_1)$ covers, we want to escape through the one defined by the heteroclinic to p_2. So we put a section defined in the same spirit as before: $S_1 = \{x_2 = \sigma\}$.



We restrict now our domain in its intersection with the section S_1. The resulting domain $\bar{D}_1$ will have, then, one dimension less than D_1, that means that it will have dimension zero.



We have no more dimensions to lose, our initial domain become a single point. This point is close to the heteroclinic defined in C_2, so then, we are sure that after some iterates, it will approach the final fixed point p_2:



2.4.1.2 Proof of Theorem 5 using covering relations

We would like to prove now Theorem 5 in a rigorous way, that is, for any map f satisfying the previous hypothesis. However we are going to present only the proof when the map is linear close to the fixed points and an affine map represents the heteroclinic connection. To do so, we are going to use the language of the h-sets and covering relations. All the definitions and results concerning these tools can be found in Appendix A.

We first define the following notation. Let $z = (x_1, \dots, x_n)$. Define for $i = 0, \dots, n$, $z_i = (z_{i,p}, z_{i,\text{inc}}, z_{i,\text{out}}, z_{i,f})$ where

- $z_{i,p} = (x_1, \dots, x_{i-2})$ are the past coordinates
- $z_{i,\text{inc}} = x_i$ is the incoming coordinate
- $z_{i,\text{out}} = x_{i+1}$ is the outgoing coordinate
- $z_{i,f} = (x_{i+2}, \dots, x_n)$ are the future coordinates.

These are the local coordinates around each fixe point p_i.

As we have said, we are going to assume that we have a sequence of linear maps: f_i for $i = 0, \dots, n$ and $f_{i-1,i}$ for $i = 1, \dots, n$.

The maps f_i will correspond to the map close to the fixed point p_i and we will call them *local maps*. The maps $f_{i-1,i}$ will correspond to the maps that connect two consecutive fixed points, that we will call *transition maps*.

That means that we are proving the result when the map f is equal to the linear map f_i around the fixed point p_i and defined as a translation close to the heteroclinic connections.

Let $f_i(z_i) = (f_{i,p}(z_i), f_{i,\text{inc}}(z_i), f_{i,\text{out}}(z_i), f_{f,i}(z_i))$ the decomposition of the map f_i in terms of the previous splitting of the coordinates z_i. We are going to assume that:

$$
\begin{aligned}
f_{i,p}(z_i) &= A_{i,p} z_{i,p} \\
f_{i,\text{inc}}(z_i) &= \mu_i z_{i,\text{inc}} \\
f_{i,\text{out}}(z_i) &= \lambda_i z_{i,\text{out}} \\
f_{f,i}(z_i) &= A_{i,f} z_{i,f},
\end{aligned}
$$

where $A_{i,p}$ and $A_{i,f}$ are matrix of the corresponding dimension that satisfy

$$
|A_{i,p} z_{i,p}| \le \mu_{i,p} |z_{i,p}| \qquad |A_{i,f} z_{i,f}| \ge \lambda_{i,f} |z_{i,f}|,
$$

with $\mu_i \le \mu_{i,p} < 1$ and $1 < \lambda_{i,f} \le \lambda_i$. The last relations are not needed in our argument, but we include them to point out that the dominant directions are the ones defined by $z_{i,\text{inc}}$ and $z_{i,\text{out}}$. The norm that we are using here and for the rest of the proof is the maximum norm, $|.| = ||.||_\infty$.

We are assuming that the map is very uncoupled. We could also assume that the matrices $A_{i,p}$ and $A_{i,f}$ are diagonal with eigenvalues $\mu_{i,p}$ and $\lambda_{i,p}$ respectively.

For each $i = 0, \dots, n$, we want to define h-sets that will be centered in the following points $q_{i,\mathrm{inc}}$ and $q_{i,\mathrm{out}}$:

- $q_{i,\mathrm{inc}} = (0, \sigma, 0, 0)$
- $q_{i,\mathrm{out}} = (0, 0, \sigma, 0)$,

where $\sigma > 0$ is some small parameter that does not depend on the fixed point we are dealing with. Notice that $q_{i,\mathrm{inc}}$ is located close to the fixed point p_i in the direction of the incoming heteroclinic, defined by the segment C_i. The point $q_{i,\mathrm{out}}$ is also located close to the fixed point p_i but in the direction of the outgoing heteroclinic, defined by the segment C_{i+1}. Let $\delta > 0$ satisfying $\delta \leq \sigma$. Define the sets:

$$N_i^{\mathrm{inc}} = \{z_i \in \mathbb{R}^n, : |z_i - q_{i,\mathrm{inc}}| \leq \delta\} \tag{2.14}$$
$$N_i^{\mathrm{out}} = \{z_i \in \mathbb{R}^n, : |z_i - q_{i,\mathrm{out}}| \leq \delta\}. \tag{2.15}$$

Notice that they are boxes of size δ. We equip these sets with an h-set structure. We declare the directions $(z_{i,p}, z_{i,\mathrm{inc}})$ as directions nominally stable and $(z_{i,\mathrm{out}}, z_{i,f})$ as directions nominally unstable in both cases. Notice that, we can skip the word nominally since the nominally stable directions are stable directions close to the fixed point. The same thing happens for the nominally unstable ones.

It is time to relate the the h-sets (2.14) and (2.15) after some iterates of the map f_i.

Lemma 4. *There exists an integer k_i such that the following covering relation hold:*

$$N_i^{inc} \overset{f_i^{k_i}}{\Longrightarrow} N_i^{out},$$

for all $i = 0, \dots, n-1$.

Proof. Since the map is linear we only have to prove that the stable components of N_i^{inc} are mapped inside N_i^{out} and that the unstable directions of N_i^{inc} cover the the unstable components of N_i^{out}. This is, the boundary of the unstable directions of N_i^{inc} is mapped outside N_i^{out}.

Let us start with the past components. We have to show that $\left|f_{i,p}^{k_i}(z_i)\right| \leq \delta$ for $z_i \in N_i^{\mathrm{inc}}$. But

$$\left|f_{i,p}^{k_i}(z_i)\right| = \left|A_{i,p}^{k_i} z_{i,p}\right| \leq \mu_{i,p}^{k_i} |z_{i,p}| \leq \mu_{i,p}^{k_i}\delta,$$

and the requested inequality holds since $0 < \mu_{i,p} \leq 1$.

Consider the incoming component, $z_{i,\mathrm{inc}}$. We want k_i such that $f_{i,\mathrm{inc}}^{k_i}(z_i) \leq \delta$ for $z_i \in N_i^{\mathrm{inc}}$. But,

$$\left|f_{i,\mathrm{inc}}^{k_i}(z_i)\right| = \mu_i^{k_i} |z_{i,\mathrm{inc}}| \leq \mu_i^{k_i}(\sigma + \delta).$$

If we take

$$k_i \geq \frac{\ln \frac{\sigma+\delta}{\delta}}{\ln \mu_i^{-1}},$$

we obtain the desired inequality.

Now we want to study the unstable components. Take $z_i \in N_i^{\text{inc}}$ such that its outgoing component $z_{i,\text{out}}$ satisfies $|z_{i,\text{out}}| = \delta$. We want to see that

$$\left|f_{i,\text{out}}^{k_i}(z_i)\right| \geq \sigma + \delta.$$

Notice that we have:

$$\left|f_{i,\text{out}}^{k_i}(z_i)\right| = \lambda_i^{k_i} \left|z_{i,\text{out}}\right| = \lambda_i^{k_i}\delta.$$

If we take k_i such that

$$k_i \geq \frac{\ln \frac{\sigma+\delta}{\delta}}{\ln \lambda_i},$$

we obtain the desired inequality.

Finally, for the future components we proceed in the same way. Take $z_i \in N_i^{\text{inc}}$ such that its future component $z_{i,f}$ satisfies $|z_{i,f}| = \delta$. We want to see that

$$\left|f_{i,f}^{k_i}(z_i)\right| \geq \delta.$$

But,

$$\left|f_{i,f}^{k_i}(z_i)\right| = \left|A_{i,f}^{k_i} z_{i,f}\right| \geq \lambda_{i,f}^{k_i} \left|z_{i,f}\right| = \lambda_{i,f}^{k_i}\delta,$$

and the requested inequality holds since $\lambda_{i,f} > 1$.

To finish the proof we take

$$k_i = \max\left\{\frac{\ln \frac{\sigma+\delta}{\delta}}{\ln \mu_i^{-1}}, \frac{\ln \frac{\sigma+\delta}{\delta}}{\ln \lambda_i}\right\}.$$

□

Now we are going to equip N_i^{out} with another h-set structure, $\widetilde{N_i^{\text{out}}}$. We are going to put the outgoing coordinate $z_{i,\text{out}}$ in the set of stable directions. Notice that $\widetilde{N_i^{\text{out}}}$ is the same as N_i^{out} as sets. We are only changing the declaration of stable and unstable coordinates, that is, the h-set structure.

Notice that it is precisely at this moment where we have lost the outgoing direction. This argument is equivalent to the one in Section 2.4.1.1 where we intersect some domain with a section of co-dimension one located in the desired outgoing direction. Notice that $\widetilde{N_i^{\text{out}}}$ have the same number of stable (and unstable) components than N_{i+1}^{inc}.

As we have said, we are going to define the map close to the heteroclinic segment just as a translation, $f_{i,i+1}$. We are going to define the translation for points in $\widetilde{N_i^{\text{out}}}$, that is for points of the form $q_{i,\text{out}} + z_i$ with $|z_i| \leq \delta$. The map $f_{i,i+1}$ is defined as:

$$f_{i,i+1}(q_{i,\text{out}} + z_i) = q_{i+1,\text{inc}} + z_i.$$

Notice that, with the transition written in this way we do not have to perform a change of variables that would locate the fixed point p_{i+1} at the origin. The change is included in the transition.

Our goal is to prove that $\widetilde{N_i^{\text{out}}}$ covers N_{i+1}^{inc}. If we write the transition map in terms of z_i and z_{i+1} we have:

$$\begin{aligned}(z_{i,p}, z_{i,\text{inc}}) &= z_{i+1,p}\\ z_{i,\text{out}} &= z_{i+1,\text{inc}}\\ z_{i,f} &= (z_{i+1,\text{out}}, z_{i+1,f}).\end{aligned}$$

With this relation and the fact that we are using the maximum norm we can conclude that:

Lemma 5. *The following covering relation hold:*

$$\widetilde{N_i^{out}} \overset{f_{i,i+1}}{\Longrightarrow} N_{i+1}^{inc},$$

for all $i = 0, \ldots, n-1$.

So, we have obtained a sequence of covering relations. Now we can announce the shadowing Theorem that will prove Theorem 5 when the map can be taken as a sequence of linear maps:

Theorem 6. *Suppose that the following covering relations hold:*

$$N_i^{inc} \overset{f_i^{k_i}}{\Longrightarrow} N_i^{out}, \text{for all } i = 0, \ldots, n-1 \tag{2.16}$$

$$\widetilde{N_i^{out}} \overset{f_{i,i+1}}{\Longrightarrow} N_{i+1}^{inc}, \text{ for all } i = 0, \ldots, n-1. \tag{2.17}$$

Then, for all $\epsilon > 0$ *there exist a point* x_0 *and a sequence of integers* $0 = k_0 < k_1 < \cdots < k_n$ *such that:*

$$||f^{k_i}(x_0) - p_i|| < \epsilon \quad i = 0, \ldots, n.$$

When one has a sequence of h-sets related with covering relations one can apply, for instance, Collorary 11 from Appendix A. However, we are not dealing with a complete sequence of coverings since N_i^{out} and $\widetilde{N_i^{\text{out}}}$ are defined defined by the same set but with different h-set structure. Such modification is done in the proof of Theorem 7 so we will not include the proof of Theorem 6 since would be analogous.

Another important remark is that in Collorary 11 one can deal with an infinite sequence of h-sets and coverings. In our case we have proved the covering relations (2.16) and (2.17) without any assumptions on the relative sizes of the h-sets: the only dependence on the step i is on the number of iterates. This means that we can conjecture that the shadowing for an infinite sequence of fixed points is feasible if the stability of the fixed points is the one used above.

2.4.2 Generalizing Theorem 5

Our goal would be to produce a Theorem just like Theorem 5 for a wider class of system that would include the Toy Model System. However this is a very idealistic goal. What we are going to do is to prepare a general Theorem and show that its proof is equivalent to show a chain of covering relations between concrete h-sets. That is, we will define a a sequence of h-sets for the wider class of systems and we will prove a Theorem equivalent to Theorem 6 without proving that the covering relations hold.

2.4.2.1 Description of the class of systems

We are going to start generalizing the the class of systems for which we can find this kind of diffusion. We start decomposing our space $\mathbb{R}^n$ in the direct sum of L spaces. That is, let $n_i > 0$ for $i = 1, \ldots, L$ such $n_1 + n_2 + \cdots + n_L = n$.

For $i = 1, \ldots, L$ consider the subspaces V_i which are spanned by $e_{n_1+\cdots+n_{i-1}+j}$ for $j = 1, \ldots, n_i$. In this notation $\mathbb{R}^n = \bigoplus_{i=1}^{L} V_i$.

We will use the following notation: for $l \in \mathbb{N}$ $0^l, 1^l$ will denote the sequences of length l consisting from 0 or 1 respectively.

Assume that we have a diffeomorphism on $f : \mathbb{R}^n \to \mathbb{R}^n$ with the following properties:

- there exists a sequence of fixed points

$$\begin{aligned} p_1 &= (0^n), \\ p_2 &= (1, 0^{n_1-1}), \\ p_3 &= (1, 0^{n_1-1}1, 0^{n_2-1}), \\ &\ldots \\ p_k &= (1, 0^{n_1-1}, 1, 0^{n_2-1}, \ldots, 1, 0^{n_k-1}, 0^{n_k+\cdots+n_L}), \text{ for } 2 \leq k \leq L \\ p_{L+1} &= (1, 0^{n_1-1}, 1, 0^{n_2-1}, \ldots, 1, 0^{n_L-1}) \end{aligned}$$

- for any $i = 1, \ldots, L$, the interval connecting p_i and p_{i+1} denoted by C_i

$$C_i = \{z \mid (1, 0^{n_1-1}, 1, 0^{n_2-1}, \ldots, 1, 0^{n_{i-1}}, t, 0^{n_i+\cdots+n_L}) \quad t \in [0, 1]\}$$

  is invariant under f and for any $z \in C_i$

$$\lim_{k\to\infty} f^k(z) = p_{i+1}, \qquad \lim_{k\to-\infty} f^k(z) = p_i.$$

Notice that if we set $n_i = 1$ we recover the situation from Theorem 5. Now we are allowing that the transition takes place in a larger space. That means that when we approach a fixed point, we have many possible outgoing directions, defined in V_i, but only one will connect us to the following fixed point. This is precisely the situation when one considers the Toy Model System. There we had $n_i = 2$. So, in our argument we are going to get rid of n_i directions in the step i.

Once we have established the class of systems we define the tools that we are going to use.

2.4.2.2 Construction of h-sets

Using the decomposition above, in $\mathbb{R}^n$ we will represent points as $z = (z_1, \dots, z_L)$, where $z_i \in V_i$ for $i = 1, \dots, L$.

Definition 1. We define for $i = 1, \dots, L$ the h-sets N_i by

$$N_i = \{p_i\} + \Pi_{j=1}^{L} \overline{B}_{n_j}(0, r_{i,j}), \quad i = 1, \dots, L+1.$$

where we declare the directions in $V_i \oplus V_{i+1} \oplus \cdots \oplus V_L$ as the nominally unstable directions and the directions in $V_1 \oplus \cdots \oplus V_{i-1}$ as the nominally stable directions.

Notice that, each h-set is centered in p_i $i = 1, \dots, L+1$. Observe, also that N_{L+1} does not have any unstable directions.

In the spirit of the previous sections, we are going to define sections of codimension n_i in the vicinity of each point p_i in the direction defined by the outgoing heteroclinic we want to escape through.

Definition 2. Let $0 < \Delta \le r_{i,i}$. We define the *exit section* S_i as

$$S_i = \{z = (z_1, \dots, z_L) \in \mathbb{R}^n, : \; z_i = \delta_i = (\Delta_i, 0^{n_i - 1})\}.$$

Notice that, with the condition $\Delta \le r_{i,i}$, we ensure that $S_i \cap N_i \neq \emptyset$.

For $i \le L$ we will define also another set M_i, which is contained in N_i and is "centered" on the section S_i:

$$M_i = p_i + \delta_i + \Pi_{j=1}^{L} \overline{B}_{n_j}(0, t_{i,j}), \quad i = 1, \dots, L$$

where $t_{i,j}$ are positive real numbers, such that $M_i \subset N_i$.

We equip the set M_i with two different h-set structures. To define the first one, denoted by M_i for $i = 1, \dots, L$, we declare as the nominally unstable directions $V_i \oplus V_{i+1} \oplus \cdots \oplus V_L$ (i.e. spanned by $e_{n_1 + \cdots + n_{i-1} + j}$ for $j = 1, \dots, n_i + \cdots + n_L$). Notice that these are the same nominally unstable directions in N_i.

For the second one, that we will denote by $\widetilde{M}_i$, we declare as the unstable directions $V_{i+1} \oplus \cdots \oplus V_L$ (i.e. $e_{n_1+\cdots+n_i+j}$ for $j = 1, \ldots, n_{i+1} + \cdots + n_L$). Notice that if we compare it with M_i we can say that we have lost the subspace of nominally unstable directions generated by V_i ($e_{n_1+\cdots+n_{i-1}+j}$ for $j = 1, \ldots, n_i$).

Remark. Observe that $\widetilde{M}_L$ has no nominally unstable directions and M_L has n_L nominally unstable directions.

With this notation the nominally unstable directions of N_i and M_i could be identified with the subspace $V_i \oplus V_{i+1} \oplus \cdots \oplus V_L$ and in $\widetilde{M}_i$ the nominally unstable directions are $V_{i+1} \oplus \cdots \oplus V_L$.

2.4.2.3 Purely topological shadowing Theorem

We are going to present a result that is equivalent to a generalized version on Theorem 5. Using the definitions of sections above, we are going to suppose some covering relations between the h-sets. Then we could prove a shadowing Theorem.

Indeed, assume that the following covering relations are satisfied

$$\widetilde{M}_i \overset{f^{l_i}}{\Longrightarrow} M_{i+1} \quad i = 1, \ldots, L-1 \tag{2.18}$$
$$f^{l_L}(\widetilde{M}_L) \subset N_{L+1}, \tag{2.19}$$

where l_i for $i = 1, \ldots, L-1$ is some integer that represents the number of iterates that $\widetilde{M}_i$ needs to approach p_{i+1}.

Remark. From now on we will denote f^{l_i} by f_i.

Theorem 7. *Assume that covering relations (2.18,2.19) are satisfied.*

Then there exist z_1 and a sequence of integers $k_1 < k_2 < \cdots < k_L$, such that

$$\begin{aligned} z_1 &\in N_1, \\ f^{k_i}(z_1) &\in N_i \quad i = 1, \ldots, L+1. \end{aligned}$$

Proof. We will look for $q_1 \in S_1 \cap N_1$ such that, there exists a sequence $\{q_i\}_{i=2,3,\ldots,L+1}$ satisfying

$$q_{i+1} = f_i(q_i) \in S_{i+1} \cap M_{i+1} \subset N_{i+1}, \quad i = 1, \ldots, L-1 \tag{2.20}$$
$$q_{L+1} = f_L(q_L) \in N_{L+1} \tag{2.21}$$

This is a system of equations for q_i, $i = 1, 2, \ldots, L+1$. It is easy to see that we can drop equation (2.21), because from (2.20) and (2.19) it follows that $f_L(\widetilde{M}_L) \subset N_{L+1}$.

We define projections $z_i : \bigoplus_{j=1}^{L} V_j \to V_i$ by $z_i(z_1, \ldots, z_L) = z_i$.

We consider the following system of equations

$$z_i(q_i) - \delta_i = 0, \quad i = 1, \dots, L \tag{2.22}$$
$$f_i(q_i) - q_{i+1} = 0, \quad i = 1, \dots, L-1. \tag{2.23}$$

which we will be considered in the set

$$D = \Pi_{i=1}^{L} M_i.$$

Let us remind that the supports of M_i and $\widetilde{M}_i$ coincide, but $M_i^{\pm}$ and $\widetilde{M}_i^{\pm}$ differ as h-sets.

Observe that the number of equations in system (2.22–2.23) coincides with the number of variables in D. Indeed the equation count goes as follows:

- (2.22) consists of $n_1 + n_2 + \dots n_L = n$ equations
- (2.23) consists of $n(L-1)$ equations

which gives Ln equations in the system.

Let us denote by F the map given by the left hand side of system (2.22–2.23). We have for $q \in D$

$$F(q) = \begin{pmatrix} z_i(q_i) - \delta_i, & i = 1, \dots, L \\ f_i(q_i) - q_{i+1}, & i = 1, \dots, L-1. \end{pmatrix}$$

We will prove that system (2.22–2.23) has a solution in D, by using the homotopy argument to show that the local Brouwer degree $\deg(F, \operatorname{int} D, 0)$ is nonzero. For $\deg(F, \operatorname{int} D, 0)$ to be defined we need that $F(q) \neq 0$ for $q \in \partial D$. This fact will be established below for the one parameter family of maps in which F will be imbedded.

In the sequel coordinates in $\widetilde{M}_i$ and M_i are used (the coordinates from the h-set structure). In these coordinates $\delta_i = 0$

Let h_i for $i = 1, 2, \dots, L-1$ be the homotopies of the covering relation (2.18). Let A_i be a linear map which appear at the end of the homotopy h_i.

We imbed F into a one-parameter family of maps (a homotopy), H_t as follows

$$H_t(q) = \begin{pmatrix} z_i(q_i), & i = 1, \dots, L \\ h_{t,i}(q_i) - q_{i+1}, & i = 1, \dots, L-1 \end{pmatrix}$$

It is easy to see that $H_0(q) = F(q)$.

We show that if $q \in \partial D$ then for all $t \in [0,1]$ holds $H_t(q) \neq 0$. This will imply that $\deg(H_t, D, 0)$ is defined for all $t \in [0,1]$ and does not depend on t.

Let $q \in \partial D$. Then one of the following conditions holds for some $i = 1, \ldots, L$: $q_i \in \partial \widetilde{M}_i^-$ or $q_i \in \partial \widetilde{M}_i^+$.

Assume that $q_i \in \widetilde{M}_i^-$. Since $\widetilde{M}_L^- = \emptyset$ we must have $i < L$. Then from the covering relation (2.18) it follows that $h_{t,i}(q_i) \notin M_{i+1}$, hence the equation $h_{t,i}(q_i) - q_{i+1} = 0$ does not hold for any $q_{i+1} \in M_{i+1}$.

The case $q_i \in \widetilde{M}_i^+$ is more subtle. If $i > 1$, then we will use the covering relation $\widetilde{M}_{i-1} \overset{f_{i-1}}{\Longrightarrow} M_i$. Observe that $M_i^+ \subsetneq \widetilde{M}_i^+$. If $q_i \in M_i^+$, then $h_{t,i-1}(q_{i-1}) - q_i \neq 0$ for all $q_{i-1} \in M_{i-1}$. If $q_i \notin M_i^+$, then $z_i(q_i)$ is on the boundary in the i-th direction hence $z_i(q_i) \neq 0$. It remains to consider $q_1 \in \widetilde{M}_1^+$. In this situation either $z_1(q_1) \neq 0$.

We have proved that $\deg(H_t, \operatorname{int} D, 0)$ is defined. By the homotopy invariance we have

$$\deg(F, \operatorname{int} D, 0) = \deg(H_1, \operatorname{int} D, 0). \tag{2.24}$$

Observe that $H_1(q) = 0$ is the following system of linear equations

$$\begin{aligned}
z_{1,1} &= 0 \\
(0, A_1(z_{1,2}, \ldots, z_{1,L})) - (z_{2,1}, z_{2,2}, \ldots, z_{2,L}) &= 0 \\
z_{2,2} &= 0 \\
(0, 0, A_2(z_{2,3}, \ldots, z_{2,L})) - (z_{3,1}, z_{3,2}, \ldots, z_{3,L}) &= 0 \\
&\cdots \\
(0, \ldots, A_{L-1}(z_{L-1,L})) - (z_{L,1}, z_{L,2}, \ldots, z_{L,L}) &= 0 \\
z_{L,L} &= 0
\end{aligned}$$

It is not hard to see that $q = 0$ is the only solution of this system. To prove that $z_{i,j} = 0$ for $i, j = 1, \ldots, L$ we should start from two bottom equations to infer that $z_{L,i} = 0$ for $i = 1, \ldots, L$ and since A_{L-1} is an isomorphism then also $z_{L-1,L} = 0$. Now we consider $z_{L-1,i}$ from next two equations from the bottom and so on.

Therefore $\deg(H_1, \operatorname{int} D, 0) = \pm 1$.

This and (2.24) implies that

$$\deg(F, \operatorname{int} D, 0) = \pm 1$$

hence there exists a solution of equation $F(q) = 0$ in D. This finishes the proof. □

Notice that what we are proving is that if the covering relations (2.18,2.19) are satisfied, then we can shadow the heteroclinic chain. So, for a given problem it rests to prove such covering relations.

Also notice that the definition of the class of system is very vague and, however, we can prove Theorem 7. That means that the delicate study will come when one proves the covering relations. For instance, we are declaring nominally stable and nominally unstable directions in the h-sets. If the directions are not of the same type of stability, which is the case of the Toy Model System, then it will be hard to prove such coverings.

3 Applying the new scheme to the Toy Model System

3.1 Introduction

In this chapter our aim is to use the ideas and tools obtained in the previous one to prove Theorem 8. Then, we would have identified the diffusion mechanism that appears in the Toy Model System. However, instead of working with h-sets, we are going to present a different but equivalent argument with disks. We recall that the Theorem that we want to prove is:

Theorem 8. *Let $N > 1$, $\delta \ll 1$. There exists a time, $T_0^* = T_0^*(N, \delta)$ such that, for all time $T^* \geq T_0^*$, there exists an initial data $b(0) = (b_1(0), \dots, b_N(0)) \in \mathbb{C}^N$ for* (3.1) *such that*

$$\begin{aligned} |b_3(0)| \geq 1 - \delta, \quad |b_j(0)| \leq \delta, \quad j \neq 3 \\ |b_{N-2}(T^*)| \geq 1 - \delta, \quad |b_j(T^*)| \leq \delta, \quad j \neq N-2. \end{aligned}$$

In order to make this chapter more self-contained we will start recalling the structure and properties of the system in the following section. Later we will check if the shadowing argument that we propose is feasible when one consider only the linear part of the system around the objects of interest (periodic orbits and heteroclinic connections). Finally we prove Theorem 8 using the recurrent application of Proposition 1, once for each connection between two consecutive periodic orbits.

The more technical proofs can be found in Appendix C.

3.2 Dynamical structure of the Toy Model System

Let us begin with the definition of the system:

Definition 3 (The Toy Model System)**.** Let $N > 1$ and $b = (b_1, \dots, b_N) \in \mathbb{C}^N$. The *Toy Model System* is defined by the following equations:

$$\frac{\mathrm{d}b_j}{\mathrm{d}t} = -\mathrm{i}|b_j|^2 b_j + 2\mathrm{i}\overline{b_j}\left(b_{j-1}^2 + b_{j+1}^2\right), \quad j = 1, \dots, N \tag{3.1}$$

with the convention $b_0 = b_{N+1} = 0$.

We recall that the Toy Model System is a Hamiltonian system with Hamiltonian function

$$H_N(b_1, \dots, b_N) = \frac{1}{4}\sum_{j=1}^{N} |b_j|^4 - \frac{1}{2}\sum_{j=2}^{N-1}\left(b_j^2 \overline{b_{j-1}}^2 + \overline{b_j}^2 b_{j-1}^2\right).$$

It is also important to recall that the total mass

$$M_N(b) = \sum_{j=1}^{N} |b_j|^2$$

is constant along the solutions and we will restrict ourselves to the mass level $M_N = 1$, that is: we will consider system (3.1) on

$$\mathcal{S}_N = \{b \in \mathbb{C}^n \,:\, M_N(b) = 1\}.$$

The important invariant objects (or landmarks) of our discussion are the N periodic orbits:

$$\mathcal{T}_j = \{b \in \mathcal{S}_N \,:\, |b_j| = 1,\, b_k = 0 \,\forall\, k \neq j\}.$$

We recall that there exists an explicit family of heteroclinic connections between two consecutive periodic orbits:

$$\gamma^+_{j,j+1} : \{\mathcal{T}_j \to \mathcal{T}_{j+1}\} \qquad \gamma^-_{j+1,j} : \{\mathcal{T}_{j+1} \to \mathcal{T}_j\}.$$

So, in order to prove Theorem 8 we are going to look for solutions that shadow the forward heteroclinic connections, $\gamma_{j,j+1}$.

It is clear that our argument will have two distinct regimes: one when we properly connect two consecutive heteroclinics close to a periodic orbit $\mathcal{T}_j$ and the other in which we simply follow the heteroclinic $\gamma_{j,j+1}$ as a known solution of the system between two consecutive periodic orbits.

Then, it is reasonable to start analyzing system (3.1) close to the periodic orbit $\mathcal{T}_j$ by performing a local change of coordinates that refers the dynamics to the motion of the periodic orbit:

$$b_j = r e^{i\theta}, \qquad b_k = c_k e^{i\theta} \text{ for } k \neq j, \tag{3.2}$$

with $r \in \mathbb{R}$, $\theta \in \mathbb{T}$ and $c_k \in \mathbb{C}$ for all $k \neq j$. We see that, in these coordinates, the periodic orbit $\mathcal{T}_j$ is:

$$\mathcal{T}_j = \{r = 1,\, \theta \in \mathbb{T},\, c_k = 0,\, \forall\, k \neq j,\}.$$

Notice that we have not used yet that we work on $\mathcal{S}_N$ instead of $\mathbb{C}^N$. We can do so by substituting

$$r^2 = 1 - \sum_{k \neq j} |c_k|^2 \tag{3.3}$$

and forgetting about this radial coordinate in the system. Introducing the change

$$c_{j-1} = \omega^2 x_- + \omega y_-, \qquad c_{j+1} = \omega^2 x_+ + \omega y_+ \tag{3.4}$$

with $\omega = \mathrm{e}^{2\pi \mathrm{i}/3}$ we diagonalize the linear part of the system.

Lemma 6. *After the changes defined in* (3.2), (3.3) *and* (3.4), *the Toy Model System becomes:*

$$\begin{cases} \dot{x}_\pm &= \sqrt{3}x_\pm + R^{x_\pm}_{hyp}(x,y) + R^{x_\pm}_{mix}(x_\pm, y_\pm, c) \\ \dot{y}_\pm &= -\sqrt{3}y_\pm + R^{y_\pm}_{hyp}(x,y) + R^{y_\pm}_{mix}(x_\pm, y_\pm, c) \\ \dot{c}_k &= ic_k + R^{c_k}_{ell}(c) + R^{c_k}_{mix}(x,y,c) \\ \dot{\theta} &= -1 + 3x_-y_- + 3x_+y_+ + \sum_{l\in\mathcal{P}_j} |c_l|^2 \end{cases} \tag{3.5}$$

where $\theta \in \mathbb{T}$, $x = (x_-, x_+) \in \mathbb{R}^2$, $y = (y_-, y_+) \in \mathbb{R}^2$, $c = (c_1, \ldots, c_{j-2}, c_{j+2}, \ldots, c_N) \in \mathbb{C}^{N-3}$ *and* $\mathcal{P}_j = \{1 \le k \le N \,:\, k \ne j-1, j, j+1\}$.

The nonlinearities are given by:

$$R^{x_\pm}_{hyp}(x,y) = -x_\pm[f(x_\pm, y_\pm) + g(x_\mp, y_\mp)] - y_\pm h(x_\mp, y_\mp)$$
$$R^{y_\pm}_{hyp}(x,y) = x_\pm h(x_\mp, y_\mp) + y_\pm[f(y_\pm, x_\pm) + g(x_\mp, y_\mp)]$$

$$f(a,b) = \sqrt{3}(a^2 - 2ab + 3b^2)$$
$$g(a,b) = \frac{\sqrt{3}}{3}(4a^2 - 7ab + 4b^2)$$
$$h(a,b) = \frac{2\sqrt{3}}{3}(-a^2 + 4ab - b^2)$$

$$R^{x_\pm}_{mix}(x,y,c) = -x_\pm F(c) + 2x_\pm G_1(c_{j\pm2}) + y_\pm[-G_1(c_{j\pm2}) + G_2(c_{j\pm2})]$$
$$R^{y_\pm}_{mix}(x,y,c) = y_\pm F(c) - 2y_\pm G_1(c_{j\pm2}) + x_\pm[G_1(c_{j\pm2}) + G_2(c_{j\pm2})]$$

$$F(c) = \sqrt{3}\sum_{l\in\mathcal{P}_j} |c_l|^2$$
$$G_1(c_{j\pm2}) = \frac{2\sqrt{3}}{3}\left[Re^2(c_{j\pm2}) - Im^2(c_{j\pm2})\right]$$
$$G_2(c_{j\pm2}) = -4Re(c_{j\pm2})Im(c_{j\pm2})$$

If $k \ne j \pm 2$

$$R_{ell}^{c_k}(c) = -ic_k\left[|c_k|^2 + \sum_{l\in\mathcal{P}_j}|c_l|^2\right] + 2i\bar{c}_k\left[c_{k-1}^2 + c_{k+1}^2\right]$$
$$R_{mix}^{c_k}(x,y,c) = -ic_k\left[3x_-y_- + 3x_+y_+\right]$$

and, for $k = j \pm 2$

$$R_{ell}^{c_{j\pm2}}(c) = -ic_{j\pm2}\left[|c_{j\pm2}|^2 + \sum_{l\in\mathcal{P}_j}|c_l|^2\right] + 2i\bar{c}_{j\pm2}c_{j\pm3}^2$$
$$R_{mix}^{c_{j\pm2}}(x,y,c) = -ic_{j\pm2}\left[3x_-y_- + 3x_+y_+\right] + 2i\bar{c}_{j\pm2}\left(\omega^2x_\pm + \omega y_\pm\right)^2$$

Remark. (Symmetries)

- Notice that the angular variable θ does not appear in the equations of (x,y,c). This is due to the phase rotation symmetry that comes from the mass conservation. We can forget about this angle and work only with the other coordinates. In other words, we have taken the symplectic quotient of the phase space with respect to the rotation symmetry and reduced (the original Hamiltonian) in one degree of freedom, (r,θ). The elimination of these coordinates does not lead us only to a lower dimensional system but to a system in which the sizes of the coordinates are comparable.

- If we only take care of the equations for x, y and c, we have an equilibrium point at the origin that corresponds to $\{r = 1\}$, that is, the periodic orbit $\mathcal{T}_j$. We say that the periodic orbit has collapsed into an equilibrium point located at the origin of $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{C}^{N-3}$. We must recall that the original system is defined on the mass level $M_N = 1$. Since
$$|b_j| = r, \qquad |b_{j\pm1}|^2 = x_\pm^2 - x_\pm y_\pm + y_\pm^2, \qquad |b_k| = |c_k| \text{ for } k \in \mathcal{P}_j,$$
system (3.5) (when we remove the equation for θ) is defined in:
$$\mathcal{S}_N^j = \{(x,y,c) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{C}^{N-3} \,:\, M_N^j(x,y,c) \le 1\},$$
where
$$M_N^j(x,y,c) = x_-^2 - x_-y_- + y_-^2 + x_+^2 - x_+y_+ + y_+^2 + \sum_{l\in\mathcal{P}_j}|c_l|^2.$$

- One of the most remarkable properties of (3.1) is that if we set any mode b_k to zero it will remain at zero. This is what we call *the mode invariance*. With both changes we keep this symmetry, that is, the subspaces
$$L_k = \{c_k = 0\} \qquad L_\pm = \{x_\pm = y_\pm = 0\}$$
are invariant.

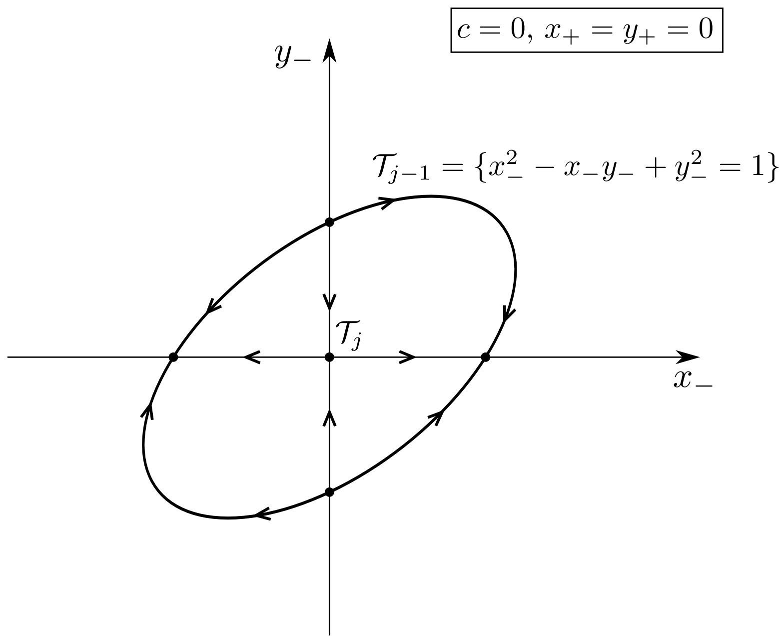
- Notice that the main structure of the system does not depend on j (same eigenvalues and same structure of nonlinearities). However, since these coordinates refer the dynamics around $\mathcal{T}_j$ we should keep some subscript for $x_\pm, y_\pm$ but there is no need to do so because during the main part of the discussion we will not combine coordinates that are referred to different periodic orbits. When we need to distinguish them we will use $(\tilde{x}_\pm, \tilde{y}_\pm)$ for the others.

If we look at the linear part of system (3.5) we see that we have four hyperbolic directions defined by x_-, y_-, x_+, y_+ and $N-3$ complex elliptic directions defined by c. We can say that the equilibrium point is saddle $\times$ saddle $\times$ center^{N-3}.

It is clear that these coordinates are local. So, if we want to analyze the system around the heteroclinic connection we could think that we should go back to the original ones. However, we can see that we can define the heteroclinics and work with these local coordinates for the whole transition from $\mathcal{T}_j$ to $\mathcal{T}_{j+1}$.

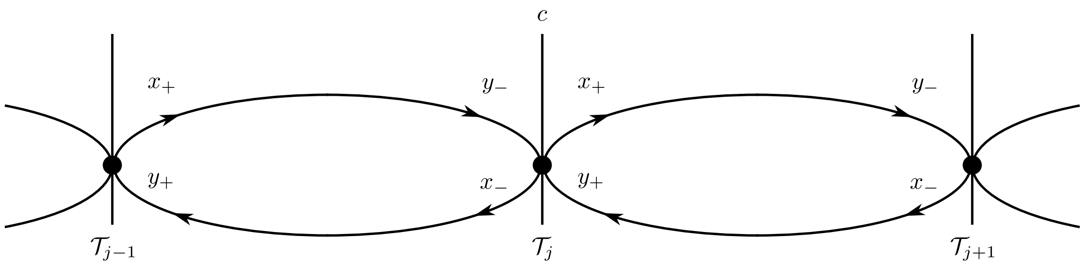
Indeed, we can easily check that the four hyperbolic positive semi-axis are invariant and the dynamics on them correspond to a heteroclinic motion between two equilibrium fixed points: the origin and a point at distance one.

We must point out that the previous change of coordinates is well defined everywhere except for the periodic orbits. That means that, for instance, the heteroclinics defined by the x_--axis and y_--axis do not end up (one backward and one forward in time) at the same equilibrium point because the periodic orbit $\mathcal{T}_{j-1}$ is defined by the ellipse $\{x_-^2 - x_- y_- + y_-^2 = 1\}$ with four equilibrium points:



Since we are in a rotating frame, these equilibrium points are precisely the points of $\mathcal{T}_{j-1}$ that are synchronized in a proper way with $\mathcal{T}_j$.

So, schematically we have the following picture:



The important heteroclinic for us will be the one that connects $\mathcal{T}_j$ with $\mathcal{T}_{j+1}$ forward in time and we will call it the outgoing heteroclinic. In this case it corresponds to x_+-axis and has the following expression:

$$x_+^h(t) = \frac{1}{\sqrt{1+\frac{1-x_+(0)^2}{x_+(0)^2}\mathrm{e}^{-2\sqrt{3}t}}},$$

and zero for the rest of the coordinates.

Now we can see that we can still work with the local coordinates from $\mathcal{T}_j$ in the study of the flow close to the heteroclinic. Once we have overcome it we will have to perform a change of coordinates between two consecutive local coordinates.

Lemma 7. *Using $(\tilde{x}, \tilde{y}, \tilde{c})$ for the coordinates referred to $\mathcal{T}_{j+1}$ the change of coordinates between two consecutive local coordinates is given by:*

$$\begin{aligned}
\tilde{y}_- &= \frac{r}{\tilde{r}}x_+ \\
\tilde{x}_- &= \frac{r}{\tilde{r}}y_+ \\
\tilde{y}_+ &= \frac{1}{\tilde{r}}\left[-\frac{2\sqrt{3}}{3}Im(c_{j+2})y_+ + \left(Re(c_{j+2}) + \frac{\sqrt{3}}{3}Im(c_{j+2})\right)x_+\right] \\
\tilde{x}_+ &= \frac{1}{\tilde{r}}\left[\left(Re(c_{j+2}) - \frac{\sqrt{3}}{3}Im(c_{j+2})\right)y_+ + \frac{2\sqrt{3}}{3}Im(c_{j+2})x_+\right] \\
\tilde{c}_k &= c_k\frac{\omega^2 y_+ + \omega x_+}{\tilde{r}} \quad \textit{for } k \in \{1, \ldots, j-2, j+3, \ldots, N\} \\
\tilde{c}_{j-1} &= \left(\omega y_- + \omega^2 x_-\right)\frac{\omega^2 y_+ + \omega x_+}{\tilde{r}},
\end{aligned}$$

where

$$\tilde{r}^2 = y_+^2 + x_+^2 - y_+x_+ \qquad r^2 = 1 - \tilde{r}^2 - \rho^2 = 1 - \tilde{r}^2 - (y_-^2 + x_-^2 - y_-x_-) - \sum_{l\in\mathcal{P}_j}|c_l|^2.$$

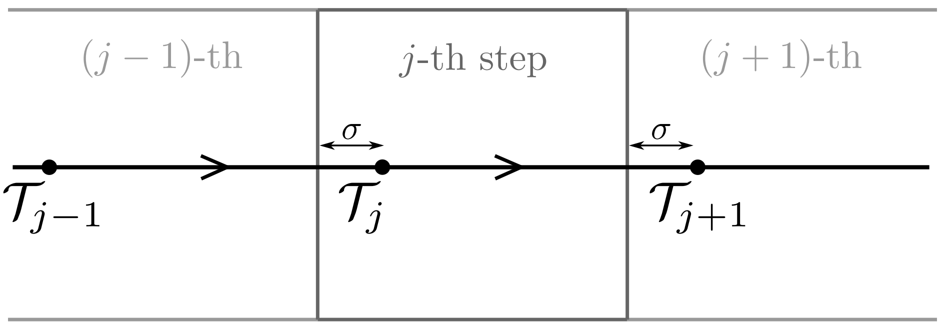
Remark. Notice that the outgoing heteroclinic defined by the x_+-axis becomes exactly the $\tilde{y}_-$-axis. This is why we call to the heteroclinic represented by the y_--axis the incoming heteroclinic.

After this introduction we are ready to propose an iterative scheme that will give us the proof of Theorem 8.

3.3 Inductive scheme of shadowing the heteroclinics

This is the main section of our work and this is where we define an inductive argument through which we shadow the heteroclinics. To do so we will define and let evolve some domains in the spirit of the previous chapter to find the diffusing orbit. Besides that, we will need to use some quantitative results that are independent of our argument but inherent to the structure of the dynamical system. These results will allow us to justify that our domains behave in the desired way. In the whole discussion we will combine both kind of arguments.

As all the previous works ([CKS+10] and [GK15]) we will split the shadowing argument in $N-5$ steps, one for each connection between two consecutive heteroclinics. So, for $j \in \{3, \dots, N-2\}$, we will have this schematic situation:



The junction point between steps is determined by some distance, that we call *macroscopic*, σ, to the periodic orbit. This is a small global parameter that does not depend on the step and its value will be determined later in terms of the parameter δ from Theorem 8.

Let us focus on the j-th step. As we have said we can work during the whole step with the local coordinates around $\mathcal{T}_j$. Recall that, in these coordinates, $\mathcal{T}_j$ has collapsed into an equilibrium point; the incoming heteroclinic from $\mathcal{T}_{j-1}$ to $\mathcal{T}_j$ is defined by the y_--axis and the outgoing from $\mathcal{T}_j$ to $\mathcal{T}_{j+1}$ by the x_+-axis. Then, the j-th step will begin at $y_- \sim \sigma$ and will end at $x_+ \sim \sqrt{1-\sigma^2}$ (that corresponds to $\tilde{y}_- \sim \sigma$ in the coordinates referred to $\mathcal{T}_{j+1}$). So the j-th step will consist on three parts: the study close to the equilibrium point, close to the heteroclinic channel and the change of coordinates that relates two consecutive local coordinates.

Now we are going to explain how we implement the technique of losing dimensions at each connection. We first need a concrete splitting of the coordinates.

Definition 4. Given $z \in \mathcal{S}_N^j$, consider the splitting $z = (p_j, f_j)$, where:

- p_j are the *past coordinates*
- f_j are the *future coordinates*

At the same time we define a new splitting for the past and future coordinates:

- $p_j = (cp_j, hp_j)$ where $cp_j = c_- = (c_1, \dots, c_{j-2})$ are the *center past coordinates* and $hp_j = (x_-, y_-)$ are the two *hyperbolic past coordinates.*
- $f_j = (hf_j, cf_j)$ where $hf_j = (x_+, y_+)$ are the two *hyperbolic future coordinates* and $cf_j = c_+ = (c_{j+2}, \dots, c_N)$ are the *center future coordinates.*

In an inductive argument we will assume that all the past coordinates are already fixed in previous steps while all the future coordinates are free. In the present step we will fix only the two hyperbolic future coordinates.

The way that we illustrate such a splitting of fixed and free coordinates is through what we call a generic disk.

Definition 5. Let $z = (p, f)$. A *generic disk* is:

$$D^{\text{gen}} = \{z = (p, f) \,:\, p = m(f),\, |f| \leq r_f\},$$

where m is a map from $B_{r_f}(0) \subset \mathbb{R}^{n_f}$ to $\mathbb{R}^{n_p}$ such that

$$||m|| = \sup_{f \in B_{r_f}(0)} ||m(f)|| \leq r_p.$$

We say that f is *free* and p is *fixed.*

With this definition we can understand better what we mean by fixed (past) or free (future) coordinate.

Our argument will consist on defining a sequence of disks and to prove that the flow acts on every disk including it in the following one. In this way we will guarantee that the last disk contains the evolution of the first.

This kind of argument is standard as used in Chapter 2. That is, to define a sequence of h-sets that are related consecutively through a covering relation. However, we will try to give a more intuitive argument working with disks instead of h-sets. The analog of the covering relations will be the following:

Lemma 8. *Let $F : U \subset \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, $z \mapsto \tilde{z}$ be a diffeomorphism. Consider the splitting:*

$$z = (p, f) \qquad \tilde{z} = (\tilde{p}, \tilde{f})$$

for which we can also split $F = \left(F_{\tilde{p}}, F_{\tilde{f}}\right)$. Assume that we have a generic disk, D^{gen} in U.

$$D^{gen} = \{z = (p, f) \,:\, p = m(f),\, |f| \le r_f\} \quad with \quad |m| \le r_p.$$

We define another disk in the image space as:

$$\tilde{D}^{gen} = \{\tilde{z} = (\tilde{p}, \tilde{f}) \,:\, \tilde{p} = \tilde{m}(\tilde{f}),\, |\tilde{f}| \le \tilde{r}_f\} \quad with \quad |\tilde{m}| \le \tilde{r}_p.$$

For this disk we do not give any concrete expression for the function $\tilde{m}$ that defines the disk. Assume that the map is close to a linear map. Then, the image of D^{gen} contains $\tilde{D}^{gen}$, $\tilde{D}^{gen} \subset F(D^{gen})$ for a concrete expression on the function $\tilde{m}$, if:

- *The map is consistent with the splitting:*
$$\frac{\partial F_{\tilde{f}}}{\partial f}(z) \tag{3.6}$$
is invertible.

- *We have upper bounds in the past coordinates:*
$$|F_{\tilde{p}}(m(f), f)| \le \tilde{r}_p. \tag{3.7}$$

- *We have lower bounds in the future coordinates: For f^* in the boundary of the domain, that is $|f^*| = r_f$, we have:*
$$\tilde{r}_f \le |F_{\tilde{f}}(m(f^*), f^*)|. \tag{3.8}$$

This result is equivalent to the definition of covering relation when one deals with maps close to linear or when the map has a strong uncoupling between fixed and free coordinates.

Coming back to our argument, we will start with some disk, D_j, located at a distance σ to $\mathcal{T}_j$ in the direction defined by the incoming heteroclinic. More precisely, we define what we call a generic incoming disk in the j-th step, D_j^{gen} that will be the starting disk at each step.

Definition 6. A *generic incoming disk* is defined as

$$D_j^{\mathrm{gen}} = \big\{ z \in \mathcal{S}_N^j \, : c_- = m_0^{c-}(x_+, y_+, c_+),\ x_- = m_0^{x-}(x_+, y_+, c_+),$$
$$y_- = \sigma + m_0^{y-}(x_+, y_+, c_+),$$
$$|x_+| \le r^0_{x+},\ |y_+| \le r^0_{y+},\ |c_{+,k}| \le r^0_{c_{+,k}} \, \big\},$$

where we assume

$$\left| m_0^{c_-,k}(x_+, y_+, c_+) \right| \le r^0_{c_-,k} \quad \left| m_0^{x-}(x_+, y_+, c_+) \right| \le r^0_{x_-} \quad \left| m_0^{y-}(x_+, y_+, c_+) \right| \le r^0_{y_-}.$$

Note that, on the one hand, the disk has the structure requested before: all the past coordinates fixed as a function of the free, future, coordinates. On the other hand we will assume that

$$r^0 = \max\{ r^0_{c_-}, r^0_{x_-}, r^0_{y_-}, r^0_{x_+}, r^0_{y_+}, r^0_{c_+} \}$$

is bounded by some *microscopic* quantity, much smaller than the macroscopic quantity. That means that at this point we have a distinguished macroscopic coordinate, y_-, the direction defined by the incoming heteroclinic, while the size of the other coordinates are, at most, microscopic. Hence our domain is truly localized in the incoming heteroclinic.

We are not interested in the evolution of the *whole* disk: we are only interested in the part of the disk that will flow close to the outgoing heteroclinic after the passage of the equilibrium point. This is why we have to restrict our disk or, equivalently, lose some dimensions. Since the outgoing heteroclinic is defined through the x_+-axis, the best restriction we could perform is to set all the future coordinates at zero except for x_+. However it is clear that, then, we could not continue with an inductive argument since we will not have more free coordinates to lose later on. The second best option is to fix only the two hyperbolic future coordinates since these are the ones that dominate the dynamics, in front of the central future coordinates. In such a way, we will keep enough freedom to continue with the inductive argument.

To obtain this we have two different options. The first one (explained in the previous chapter) consists on computing the evolution of the whole disk D_j^{gen} for a time τ (large enough) and then intersect the resulting disk with some section located in the desired outgoing direction. This section will be defined as

$$S_j = \{ x_+ = \sigma + x_+^*,\ y_+ = y_+^* \}. \tag{3.9}$$

Remark. The idea would be to set $x_+^* = y_+^* = 0$, but this is not exactly what we will obtain.

Then we would have a domain (the intersection of the evolved disk with that section) located at a macroscopic distance of $\mathcal{T}_j$ and in the right direction to escape.

However, we are going to use another strategy that saves us to compute the evolution of the whole disk. The idea is that there is no need to compute the evolution of the whole disk if we are going to get rid of two directions. The alternative is to use a Shilnikov scheme. That will give us the restriction of the original disk that ends in the desired section in an automatic way. In addition this technique will provide us good estimates of the solution (actually, good estimates on the deviation of the true solution with respect to the solution of the linearized problem).

Let us start defining the Shilnikov problem. These kind of problems were introduced in [Shi67].

Definition 7. Let F be a vector field in $\mathbb{R}^n$. Consider the following system of ordinary differential equations:

$$\dot{z} = F(z). \tag{3.10}$$

Split the coordinates in $z = (z_1, z_2)$, that is, consider a splitting $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, with $n_1 + n_2 = n$. Let $(\tau; z_1^0, z_2^1)$ be the the *Shilnikov data*, where $\tau \in \mathbb{R}^+$, $z_1 \in \mathbb{R}^{n_1}$ and $z_2 \in \mathbb{R}^{n_2}$.

We say that $z(t)$ is a solution for the *Shilnikov problem* with Shilnikov data $(\tau; z_1^0, z_2^1)$ if it solves equation (3.10) for $t \in [0, \tau]$ and

$$z_1(0) = z_1^0 \qquad z_2(\tau) = z_2^1.$$

Remark. Note that, for $\tau = 0$ we recover the initial value problem.

Remark. When one considers this kind of problems one has to prove that there exists a unique solution. This is not always true for all systems. A complete proof for saddle equilibrium points for a concrete splitting for the coordinates is given in [Den89].

In our case, we have already considered a splitting for the coordinates in terms of past and future coordinates. We are going to set all the past coordinates p_j for time $t = 0$. On the other hand, we want to fix precisely at this step the pair of hyperbolic future coordinates hf_j, and this is why we are going to fix them for time $t = \tau$ at the section, S_j, described above. For the rest of free coordinates, the center future cf_j, we want them to remain free after this step so we are going to consider a Shilnikov problem for each admissible value of cf_j at time $t = 0$.

That is, for $\tau > 0$ large enough, consider the Shilnikov problem defined by the following Shilnikov data:

$$\begin{aligned} c_-(0) &= m_0^{c-}(x_+^0, y_+^0, c_+^0) & c_+(0) &= c_+^0 \\ x_-(0) &= m_0^{x-}(x_+^0, y_+^0, c_+^0) & y_-(0) &= \sigma + m_0^{y-}(x_+^0, y_+^0, c_+^0) \\ x_+(\tau) &= \sigma + x_+^* & y_+(\tau) &= y_+^* \end{aligned} \tag{3.11}$$

for each $|c_+^0| \le r_+^0$.

Remark. Notice that we have introduced the parameters x_+^0, y_+^0 and c_+^0 that parameterize our starting disk.

Under some assumptions, we are going to see that these problems will have a unique solution that will depend on all the parameters:

$$\begin{aligned} c_-(t) &= c_-(t;\tau;x_+^0,y_+^0,c_+^0,x_+^*,y_+^*,c_+^0) \\ x_-(t) &= x_-(t;\tau;x_+^0,y_+^0,c_+^0,x_+^*,y_+^*,c_+^0) \\ y_-(t) &= y_-(t;\tau;x_+^0,y_+^0,c_+^0,x_+^*,y_+^*,c_+^0) \\ x_+(t) &= x_+(t;\tau;x_+^0,y_+^0,c_+^0,x_+^*,y_+^*,c_+^0) \\ y_+(t) &= y_+(t;\tau;x_+^0,y_+^0,c_+^0,x_+^*,y_+^*,c_+^0) \\ c_+(t) &= c_+(t;\tau;x_+^0,y_+^0,c_+^0,x_+^*,y_+^*,c_+^0) \end{aligned}$$

Next we will have to check that our solutions really start in D_j^{gen} for $t=0$. That means two things:

- The range where the future hyperbolic coordinates lie is large enough to contain the solutions:
$$|x_+(0)| \le r_{x+}^0, \qquad |y_+(0)| \le r_{y+}^0.$$
- The introduced parameters must match with the solution for $t=0$. For c_+^0 this is true by construction. For x_+^0 and y_+^0, they must be chosen as the ones that solve the following equation:
$$\begin{aligned} x_+^0 &= x_+(0) = x_+(0;\tau;x_+^0,y_+^0,c_+^0,x_+^*,y_+^*,c_+^0) \\ y_+^0 &= y_+(0) = y_+(0;\tau;x_+^0,y_+^0,c_+^0,x_+^*,y_+^*,c_+^0) \end{aligned}$$

Once we have checked these conditions, we only have to compute all the family of solutions at $t=\tau$. It will be a disk, that we will call the *outgoing disk*, in the right place (contained in S_j defined in (3.9)) and so, with two dimensions less.

Finally we will see that outgoing disk flows close to the outgoing heteroclinic until it reaches the proximity (distance σ) of $\mathcal{T}_{j+1}$. Then we will change the coordinates to the ones that refer the dynamics to the motion of the next periodic orbit, $\mathcal{T}_{j+1}$, and check that the transformed disk, D_{j+1}^{gen}, has the appropriate structure for a disk in the following step.

That means that we are looking to prove the following recurrent generic result: There exists a subdisk of the generic incoming disk, D_j^{gen}, such that, after the action of the flow for a time T^{gen} it contains the disk generic incoming disk for the next step, D_{j+1}^{gen}.

3.3.1 The prediction of the linear part

The first approach that we want to do is to check whether our argument is feasible, when we just consider the linear part of our system. This will be useful for two reasons. The first one is just because if our argument does not work with this simpler system we can neglect it as not feasible. The second one is because it will give us a quantitative prediction on many aspects, as the sizes of our coordinates.

Of course, we are not going to consider only the linear part of system (3.5) around the origin, because there are not heteroclinics in linear systems. The idea would be to combine linearization around different solutions: the fixed points and the heteroclinics. Since the coordinates we are working with are local, we will need to include changes between the local coordinates that refer the dynamics to the motion of one periodic orbit and the ones that refer the dynamics to the motion of the consecutive periodic orbit. For these changes we will consider only their linear part.

So, fix $j \in \{3, \ldots, N-2\}$ and consider the local coordinates around $\mathcal{T}_j$ and the generic incoming disk of Definition 6.

Up to now, we do not have a good idea of the size of $r^0_{c_-,k}$, $r^0_{x_-}$, $r^0_{x_-}$ and $r^0_{x_+}$, $r^0_{y_+}$, $r^0_{c_+,k}$ besides the fact that they are all smaller than σ in order to have a disk truly localized close to the incoming heteroclinic. As we have said, we will call these sizes *microscopic*. We plan to obtain a good guess for this size considering the linearized system.

We are now ready to study the flow of D_j^{gen} around an equilibrium point. As we have said, the tool that we are going to use will be a Shilnikov argument. Since we are close to the origin (the largest variable has size σ that is small), it is reasonable to linearize the system around it. That means to consider:

$$\begin{array}{rcl} \dot{x}_\pm & = & \sqrt{3}x_\pm \\ \dot{y}_\pm & = & -\sqrt{3}y_\pm \\ \dot{c}_k & = & \mathrm{i}c_k \end{array} \tag{3.12}$$

Once we have determined the system we must define the Shilnikov data. We are going to take the data defined in (3.11). A particularity of the linear system (3.12) is that the equations for the components are completely uncoupled. So, to consider a Shilnikov problem is equivalent to consider an initial value problem for each component, and thus, there exists always a unique solution. In our case, for each c_+^0 in its range, the solution of this Shilnikov problem (3.11) is given by:

$$\begin{aligned} c_-(t) &= \mathrm{e}^{\mathrm{i}t}c_-(0) = \mathrm{e}^{\mathrm{i}t}m_0^{c_-}\left(x_+^0, y_+^0, c_+^0\right) \\ x_-(t) &= \mathrm{e}^{\sqrt{3}t}x_-(0) = \mathrm{e}^{\sqrt{3}t}m_0^{x_-}\left(x_+^0, y_+^0, c_+^0\right) \\ y_-(t) &= \mathrm{e}^{-\sqrt{3}t}y_-(0) = \mathrm{e}^{-\sqrt{3}t}\left(\sigma + m_0^{y_-}\left(x_+^0, y_+^0, c_+^0\right)\right) \\ x_+(t) &= x_+(\tau)\mathrm{e}^{\sqrt{3}(t-\tau)} = \left(\sigma + x_+^*\right)\mathrm{e}^{\sqrt{3}(t-\tau)} \\ y_+(t) &= y_+(\tau)\mathrm{e}^{-\sqrt{3}(t-\tau)} = y_+^*\mathrm{e}^{-\sqrt{3}(t-\tau)} \\ c_+(t) &= \mathrm{e}^{\mathrm{i}t}c_+(0) = \mathrm{e}^{\mathrm{i}t}c_+^0, \end{aligned}$$

where the functions m_0^{c-}, m_0^{x-} and m_0^{y-} define the disk D_j^{gen} from Definition 6.

As we have said, the first thing we have to do is to determine the values of x_+^0 and y_+^0 in order to match them with the beginning of a solution. We recall the equations:

$$\begin{aligned} x_+^0 &= x_+(0) = x_+(0;\tau;x_+^0,y_+^0,c_+^0,x_+^*,y_+^*,c_+^0) \\ y_+^0 &= y_+(0) = y_+(0;\tau;x_+^0,y_+^0,c_+^0,x_+^*,y_+^*,c_+^0). \end{aligned}$$

Again, due to the uncoupling of the system, these equations become trivial in our linear situation:

$$\begin{aligned} x_+^0 &= x_+(0) = \left(\sigma + x_+^*\right)\mathrm{e}^{-\sqrt{3}\tau} \\ y_+^0 &= y_+(0) = y_+^*\mathrm{e}^{\sqrt{3}\tau}, \end{aligned}$$

and we can already write the expression of the solution if we take such values for the introduced parameters:

$$\begin{aligned} c_-(t) &= \mathrm{e}^{\mathrm{i}t} m_0^{c-}\left(\left(\sigma + x_+^*\right)\mathrm{e}^{-\sqrt{3}\tau}, y_+^*\mathrm{e}^{\sqrt{3}\tau}, c_+^0\right) \\ x_-(t) &= \mathrm{e}^{\sqrt{3}t} m_0^{x-}\left(\left(\sigma + x_+^*\right)\mathrm{e}^{-\sqrt{3}\tau}, y_+^*\mathrm{e}^{\sqrt{3}\tau}, c_+^0\right) \\ y_-(t) &= \mathrm{e}^{-\sqrt{3}t}\left(\sigma + m_0^{y-}\left(\left(\sigma + x_+^*\right)\mathrm{e}^{-\sqrt{3}\tau}, y_+^*\mathrm{e}^{\sqrt{3}\tau}, c_+^0\right)\right) \\ x_+(t) &= \left(\sigma + x_+^*\right)\mathrm{e}^{\sqrt{3}(t-\tau)} \\ y_+(t) &= y_+^*\mathrm{e}^{-\sqrt{3}(t-\tau)} \\ c_+(t) &= \mathrm{e}^{\mathrm{i}t} c_+^0. \end{aligned}$$

After that we can already think on sizes. First of all, we recall that we need to check another condition on the sizes of x_+ and y_+ for $t = 0$: the range in D_j^{gen} must contain these values. That means:

$$r_{x_+}^0 \geq x_+(0) = \left(\sigma + x_+^*\right)\mathrm{e}^{-\sqrt{3}\tau} \qquad r_{y_+}^0 \geq y_+(0) = y_+^*\mathrm{e}^{\sqrt{3}\tau}. \tag{3.13}$$

Assume for a moment that the inequalities are fulfilled. We will come back to that later.

For the central modes (both past and future) we see that the evolution does not change its size (it is a rotation). So, up to now, we only know that their sizes for $t = 0$ and $t = \tau$ should be microscopic.

From the expression for the largest components, y_- and x_+ we can give a first approximation of what we mean by microscopic size. Indeed, evaluate y_- at time $t = \tau$:

$$y_-(t) = \mathrm{e}^{-\sqrt{3}\tau}\left(\sigma + m_0^{y-}\left(\left(\sigma + x_+^*\right)\mathrm{e}^{-\sqrt{3}\tau}, y_+^*\mathrm{e}^{\sqrt{3}\tau}, c_+^0\right)\right).$$

Since we expect that the deviation of $y_-(0)$ with respect to σ will be small (microscopic) the dominant part of this expression is given by $\sigma\mathrm{e}^{-\sqrt{3}\tau}$. This size cannot be improved so, if we want all the components to be microscopic at $t = \tau$, we must define:

$$\text{microscopic} \sim \sigma \mathrm{e}^{-\sqrt{3}\tau}.$$

This is a first approximation of the size and a minimal value: we can allow (a little bit) larger microscopic sizes.

Once we have determined a bound for this microscopic value, we look at the small coordinates, x_- and y_+. We want them to be smaller than microscopic at any time during the transition. Consider, for instance, the component x_-. The maximum value that it will achieve is for time $t = \tau$:

$$x_-(\tau) = \mathrm{e}^{\sqrt{3}\tau} m_0^{x_-}\left(\left(\sigma + x_+^*\right)\mathrm{e}^{-\sqrt{3}\tau}, y_+^*\mathrm{e}^{\sqrt{3}\tau}, c_+^0\right).$$

The first guess is to set $x_-(0)$ at zero. When one only deals with the linear part this is possible, however this is not longer true when one considers the full system. So, if we want this component to be microscopic and we are not allowed to set it at zero we will need $x_-(0)$ to be smaller than microscopic, say *nanoscopic*, that will be defined as

$$\text{nanoscopic} \sim \sigma \mathrm{e}^{-2\sqrt{3}\tau}.$$

As a summary, we have introduced two different sizes, the microscopic and the nanoscopic. However we recall that the sizes for the future hyperbolic coordinates at the origin must obey (3.13). That means that we will need to allow larger sizes for the future coordinates.

After this discussion on sizes, we come back to our argument. Notice that we have captured the part of the incoming disk that ends in the desired location: the family of solutions really start inside D_j^{gen} and after a time $t = \tau$ is included in the section S_j. In addition, they form a disk over the central future coordinates:

$$\begin{aligned}
\hat{d}_j^{\text{gen}} = \{z \in \mathcal{S}_j \ : c_- &= \mathrm{e}^{\mathrm{i}\tau} m_0^{c_-}\left(\left(\sigma + x_+^*\right)\mathrm{e}^{-\sqrt{3}\tau}, y_+^*\mathrm{e}^{\sqrt{3}\tau}, c_+^0\right), \\
x_- &= \mathrm{e}^{\sqrt{3}\tau} m_0^{x_-}\left(\left(\sigma + x_+^*\right)\mathrm{e}^{-\sqrt{3}\tau}, y_+^*\mathrm{e}^{\sqrt{3}\tau}, c_+^0\right), \\
y_- &= \mathrm{e}^{-\sqrt{3}\tau}\left(\sigma + m_0^{y_-}\left(\left(\sigma + x_+^*\right)\mathrm{e}^{-\sqrt{3}\tau}, y_+^*\mathrm{e}^{\sqrt{3}\tau}, c_+^0\right)\right), \\
x_+ &= \sigma + x_+^* \\
y_+ &= y_+^*, \\
c_+ &= \mathrm{e}^{\mathrm{i}\tau} c_+^0 \\
|c_{+,k}^0| &\leq r_{c_{+,k}}^0 \text{ for } j+2 \leq k \leq N\Big\}.
\end{aligned}$$

Notice that the disk is parameterized by the central future coordinates at $t = 0$. We want it to be parameterized by the central future coordinates at time $t = \tau$. This is not a problem because we can invert these coordinates through

$$c_+^\tau = \mathrm{e}^{\mathrm{i}\tau} c_+^0,$$

and obtain the *generic outgoing disk*:

$$
\begin{aligned}
d_j^{\mathrm{gen}} = \{z \in \mathcal{S}_j \ : c_- &= m_\tau^{c_-}(c_+)\,, \\
x_- &= m_\tau^{x_-}(c_+)\,, \\
y_- &= m_\tau^{y_-}(c_+)\,, \\
x_+ &= \sigma + x_+^* \\
y_+ &= y_+^*, \\
|c_{+,k}| &\leq r_{c_{+,k}}^\tau \text{ for } j+2 \leq k \leq N\Big\}\,,
\end{aligned}
$$

with

$$
\begin{aligned}
m_\tau^{c_-}(c_+) &= \mathrm{e}^{\mathrm{i}\tau} m_0^{c_-}\left(\left(\sigma + x_+^*\right)\mathrm{e}^{-\sqrt{3}\tau}, y_+^*\mathrm{e}^{\sqrt{3}\tau}, \mathrm{e}^{-\mathrm{i}\tau}c_+\right)\,, \\
m_\tau^{x_-}(c_+) &= \mathrm{e}^{\sqrt{3}\tau} m_0^{x_-}\left(\left(\sigma + x_+^*\right)\mathrm{e}^{-\sqrt{3}\tau}, y_+^*\mathrm{e}^{\sqrt{3}\tau}, \mathrm{e}^{-\mathrm{i}\tau}c_+\right)\,, \\
m_\tau^{y_-}(c_+) &= \mathrm{e}^{-\sqrt{3}\tau}\left(\sigma + m_0^{y_-}\left(\left(\sigma + x_+^*\right)\mathrm{e}^{-\sqrt{3}\tau}, y_+^*\mathrm{e}^{\sqrt{3}\tau}, \mathrm{e}^{-\mathrm{i}\tau}c_+\right)\right)\,.
\end{aligned}
$$

We only need to take $r_{c_{+,k}}^\tau = r_{c_{+,k}}^0$.

Now is the time to flow close to the heteroclinic orbit. Following this argument we must linearize the system around the solution defined by the heteroclinic, that is:

$$c = 0,\ x_- = y_- = y_+ = 0,\ x_+ = x_+^h(t).$$

If we ask $x_+^h(0) = \sigma$ we have:

$$x_+^h(t) = \frac{1}{\sqrt{1 + \frac{1-\sigma^2}{\sigma^2}\mathrm{e}^{-2\sqrt{3}t}}},$$

and

$$x_+^h(T) = \sqrt{1-\sigma^2} \qquad \text{for } T = \frac{1}{\sqrt{3}}\ln\left(\frac{1-\sigma^2}{\sigma^2}\right).$$

So, we write $\alpha = x_+(t) - x_+^h(t)$ in system (3.5). Now the origin with respect to the variables $(x_-, y_-, \alpha, y_+, c)$ will correspond to the outgoing heteroclinic. If we linearize the system we obtain:

$$\begin{pmatrix} \dot{x}_- \\ \dot{y}_- \end{pmatrix} = \begin{pmatrix} \sqrt{3} - \frac{4\sqrt{3}}{3}x_+^h(t)^2 & \frac{2\sqrt{3}}{3}x_+^h(t)^2 \\ -\frac{2\sqrt{3}}{3}x_+^h(t)^2 & -\sqrt{3} + \frac{4\sqrt{3}}{3}x_+^h(t)^2 \end{pmatrix}\begin{pmatrix} x_- \\ y_- \end{pmatrix} \tag{3.14}$$

$$\begin{pmatrix} \dot{\alpha} \\ \dot{y}_+ \end{pmatrix} = \begin{pmatrix} \sqrt{3} - 3\sqrt{3}x_+^h(t)^2 & 2\sqrt{3}x_+^h(t)^2 \\ 0 & -\sqrt{3} + 3\sqrt{3}x_+^h(t)^2 \end{pmatrix}\begin{pmatrix} \alpha \\ y_+ \end{pmatrix} \tag{3.15}$$

$$\begin{pmatrix} \dot{c}_{j+2} \\ \dot{\bar{c}}_{j+2} \end{pmatrix} = \begin{pmatrix} \mathrm{i} & 2\mathrm{i}\omega x_+^h(t)^2 \\ -2\mathrm{i}\omega^2 x_+^h(t)^2 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} c_{j+2} \\ \bar{c}_{j+2} \end{pmatrix} \tag{3.16}$$

$$\begin{pmatrix} \dot{c}_k \\ \dot{\bar{c}}_k \end{pmatrix} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} c_k \\ \bar{c}_k \end{pmatrix} \text{ for } k \in \mathcal{P}_j, k \neq j+2 \tag{3.17}$$

Notice that the system is no longer diagonal. The main problem that we face now is that we cannot solve equations (3.14) and (3.16). So we have two options. The first one is to forget about this linearization and to consider the flow close to the heteroclinic just as a translation:

$$x_+(0) \mapsto x_+(T) = \sqrt{1-\sigma^2} + x_+(0) - \sigma,$$

and the identity in the rest of the coordinates. The second option is to solve whatever is solvable, use the solution as an approximation of the real system and to consider the identity as a solution for the equations that we cannot solve.

Since the second option will only introduce tedious computations we will take the first option. It is enough for the kind of argument we are using.

If we compute the image of our disk d_j^{gen} through this translation we obtain the *generic final disk:*

$$\begin{aligned} d_j^{\text{gen},T} = \{z \in \mathcal{S}_j \ : c_- &= m_\tau^{c_-}(c_+), \\ x_- &= m_\tau^{x_-}(c_+), \\ y_- &= m_\tau^{y_-}(c_+), \\ x_+ &= \sqrt{1-\sigma^2} + x_+^* \\ y_+ &= y_+^*, \\ |c_{+,k}| &\leq r_{c_{+,k}}^\tau \text{ for } j+2 \leq k \leq N\Big\}. \end{aligned}$$

Now we have approached the vicinity of $\mathcal{T}_{j+1}$. To continue with this argument we must change the coordinates into the ones that refer the dynamics to the motion of the $(j+1)$-th periodic orbit using Lemma 7. If we linearize that change of coordinates around the end point of the heteroclinic we obtain:

$$\tilde{c}_k = \omega c_k$$

$$\tilde{c}_{j-1} = x_- + \omega^2 y_-$$

$$\begin{pmatrix} \tilde{x}_- \\ \tilde{y}_- \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma \end{pmatrix} + D_\pm \begin{pmatrix} x_+ - \sqrt{1-\sigma^2} \\ y_+ \end{pmatrix} \text{ with } D_\pm = \begin{pmatrix} 0 & \frac{\sigma}{\sqrt{1-\sigma^2}} \\ -\frac{\sqrt{1-\sigma^2}}{\sigma} & \frac{1}{2\sigma\sqrt{1-\sigma^2}} \end{pmatrix}$$

$$\begin{pmatrix} \tilde{x}_+ \\ \tilde{y}_+ \end{pmatrix} = D_{\text{hyp}} \begin{pmatrix} c_{j+2} \\ \bar{c}_{j+2} \end{pmatrix} \text{ with } D_{\text{hyp}} = \begin{pmatrix} -\frac{\sqrt{3}}{3}\mathrm{i} & \frac{\sqrt{3}}{3}\mathrm{i} \\ \frac{1}{2} - \frac{\sqrt{3}}{6}\mathrm{i} & \frac{1}{2} + \frac{\sqrt{3}}{6}\mathrm{i} \end{pmatrix}$$

The first thing that we need to check is the transversality condition (3.6). Before the change we have some free coordinates, all the central future c_+. Now we will need free coordinates in the following step defined by $f_{j+1} = (\tilde{x}_+, \tilde{y}_+, cf_{j+1})$. That means that we will need a one-to-one relation between each point. Since the change of coordinates is not defined as the identity we have to check that some minor of the global matrix that defines the change of coordinates has determinant different from zero, in such a way that we can invert the map if we restrict it to the free coordinates. That is:

$$\left| \frac{\partial(\tilde{x}_+, \tilde{y_+}, \tilde{c}_+)}{\partial c_+} \right| \neq 0.$$

In our case, this condition is fulfilled since the minor is bidiagonal.

If we compute the image of $d^{\text{gen},T}$ through this change we obtain:

$$\begin{aligned} D^{\text{gen}}_{j+1} = \{ z \in \mathcal{S}_j \ : & \tilde{c}_- = \tilde{m}_0^{c-} (\tilde{x}_+, \tilde{y}_+, \tilde{c}_+), \\ & \tilde{c}_{j-1} = \tilde{m}_0^{c_{j-1}} (\tilde{x}_+, \tilde{y}_+, \tilde{c}_+), \\ & \tilde{x}_- = \tilde{m}_0^{x-} (\tilde{x}_+, \tilde{y}_+, \tilde{c}_+), \\ & \tilde{y}_- = \tilde{m}_0^{y-} (\tilde{x}_+, \tilde{y}_+, \tilde{c}_+), \\ & |\tilde{x}_+| \leq \tilde{r}^0_{y+} \\ & |\tilde{y}_+| \leq \tilde{r}^0_{y+}, \\ & |\tilde{c}_{+,k}| \leq \tilde{r}^0_{c_{+,k}} \text{ for } j+3 \leq k \leq N \Big\}, \end{aligned}$$

with

$$\begin{aligned} \tilde{m}_0^{c-} (\tilde{x}_+, \tilde{y}_+, \tilde{c}_+) &= \omega m_\tau^{c-}(c_+), \\ \tilde{m}_0^{c_{j-1}} (\tilde{x}_+, \tilde{y}_+, \tilde{c}_+) &= m_\tau^{x-} (c_+) + \omega^2 m_\tau^{y-} (c_+), \\ \tilde{m}_0^{x-} (\tilde{x}_+, \tilde{y}_+, \tilde{c}_+) &= \frac{\sigma}{\sqrt{1-\sigma^2}} y_+^*, \\ \tilde{m}_0^{y-} (\tilde{x}_+, \tilde{y}_+, \tilde{c}_+) &= \sigma - \frac{\sqrt{1-\sigma^2}}{\sigma} x_+^* + \frac{1}{2\sigma\sqrt{1-\sigma^2}} y_+^*. \end{aligned}$$

Notice that we are writing the functions that define our disk in terms of c_+ instead of $(\tilde{x}_+, \tilde{y}_+, \tilde{c}_+)$. However, due to the transversality condition we are sure that we can invert the transformation in these coordinates and, then, reparameterize the disk in terms of $(\tilde{x}_+, \tilde{y}_+, \tilde{c}_+)$.

Finally we only need to define the range where the new future coordinates lie. For the central future coordinates the change is just a rotation, so it does not change the size of the range and we can take $\tilde{r}^0_{c_{+,k}} = r^\tau_{c_{+,k}}$. For the new hyperbolic future coordinates we deal with another difficulty: a complex coordinate has split into two real coordinates. Since we want to define a domain by the product of intervals, that means that we are changing the norm, from $||.||_2$ to $||.||_\infty$. Since our new domain (a rectangle) must be strictly confined inside the image of a ball, we take

$c_{j+2} = r^T_{c_+,j+2} e^{i\theta_{j+2}}$ and compute the image of such a point (as we have explained in Lemma 8, condition (3.8)). To obtain this, we first need to invert the equation and get upper bounds:

$$\begin{aligned} c_{j+2} &= \omega\tilde{x}_+ + \tilde{y}_+, \\ |c_{j+2}| &= |\omega\tilde{x}_+ + \tilde{y}_+| = \sqrt{\tilde{x}_+^2 + \tilde{y}_+^2 - \tilde{x}_+\tilde{y}_+} \\ &\leq \sqrt{\frac{3}{2}}||(\tilde{x}_+, \tilde{y}_+)||_2 \leq \sqrt{3}||(\tilde{x}_+, \tilde{y}_+)||_\infty. \end{aligned}$$

Now, we see that if we select our domain defined by $||(\tilde{x}_+, \tilde{y}_+)||_\infty \leq \frac{\sqrt{3}}{3} r^T_{c_+,j+2}$ our domain will be defined inside the image of the original circle $|c_{j+2}| \leq r^T_{c_+,j+2}$. In other words, we are taking:

$$\tilde{r}^0_{x_+} = \frac{\sqrt{3}}{3} r^T_{c_+,j+2}, \qquad \tilde{r}^0_{y_+} = \frac{\sqrt{3}}{3} r^T_{c_+,j+2}.$$

Remark. Before dealing with the complete non-linear system we want to point out some facts:

- There is no geometric obstruction in following the sequences. Everything works for the linear part.
- In this trivial situation, we can obtain explicitly the expression of the map that defines a disk in terms of the previous ones. This situation will not longer be possible when we include the nonlinear part. We will only work with the bounds of these functions. Then, whenever we say that some disk is contained in another we will mean that the transformed range for the free coordinates will be contained in the range for the new free coordinates and that there exist functions that parameterize the image disk. We cannot give explicit expressions but precise estimates for these functions.
- This computation has allowed us to predict the sizes of our disk. However we will need to modify them for many reasons that we will explain when we define the proper incoming disk.

3.4 The implementation adding the non-linear terms

Once we have proved that the argument is feasible with an approximation of the system we are going to reproduce it for the complete one. Inspired with the estimates of the study of the linear part, the first thing that we do is to give the values for the sizes. However, as we have said the values will change slightly. More precisely we define the incoming disk as follows:

Definition 8. The *incoming disk* is defined as

$$D_j = \big\{ z \in \mathcal{S}_N^j \ : c_- = m_0^{c-}(x_+, y_+, c_+),\ x_- = m_0^{x-}(x_+, y_+, c_+), \\ y_- = \sigma + m_0^{y-}(x_+, y_+, c_+) \\ |x_+| \le r_{x_+}^0,\ |y_+| \le r_{y+}^0,\ |c_{+,k}| \le r_{c_{+,k}}^0 \ \big\},$$

where we assume for the past coordinates:

$$\begin{aligned} \left|m_0^{c_-,k}(x_+, y_+, c_+)\right| &\le r_{c_-,k}^0 = \frac{1}{N-3}\sigma\tau^{k_j}\mathrm{e}^{-\sqrt{3}\tau} \text{ for } 1 \le k \le j-1, \\ \left|m_0^{c_-,j-2}(x_+, y_+, c_+)\right| &\le r_{c_-,j-2}^0 = \frac{1}{N-3}\sigma\tau^{\frac{1}{2}k_j}\mathrm{e}^{-\sqrt{3}\tau}, \\ |m_0^{x-}(x_+, y_+, c_+)| &\le r_{x_-}^0 = \sigma\tau^{k_j}\mathrm{e}^{-2\sqrt{3}\tau}, \\ |m_0^{y-}(x_+, y_+, c_+)| &\le r_{y_-}^0 = \frac{1}{2}\sigma^2, \end{aligned}$$

and for the free future coordinates:

$$\begin{aligned} r_{x+}^0 &= 2\sigma\tau\mathrm{e}^{-\sqrt{3}\tau}, \\ r_{y+}^0 &= 2\sigma\tau\mathrm{e}^{-\sqrt{3}\tau}, \\ r_{c_{+,k}}^0 &= \frac{1}{N-3}\sigma\tau^{k_j}\mathrm{e}^{-\epsilon_j\sqrt{3}\tau} \text{ for } j+2 \le k \le N. \end{aligned}$$

We first notice that we have multiplied the established microscopic and nanoscopic sizes by a factor that depends on the time τ (the flight time close to the periodic orbit or equilibrium point). The reason is that our system (3.5) contains resonant terms of order three. That could generate a factor in the solution of polynomial type. Since the resonance is of order three, the largest coordinates that play a role in the transition, y_- and x_+, are not affected by these terms. However, the components that we need to be very small all the time (at least microscopic) are really affected by this resonance. Then, the second approach would be to define the microscopic and nanoscopic distances as:

$$\text{microscopic} \sim \sigma\tau\mathrm{e}^{-\sqrt{3}\tau} \qquad \text{nanoscopic} \sim \sigma\tau\mathrm{e}^{-2\sqrt{3}\tau}.$$

This modification is motivated by the study of the passage close to the equilibrium point. However, as we can see it does not correspond with the election in the previous definition. We need to include a factor in terms of a power of τ. The reason of choosing that is because, then, we will have different sizes depending on the step:

$$\text{microscopic}_j \sim \sigma\tau^{k_j}\mathrm{e}^{-\sqrt{3}\tau} \qquad \text{nanoscopic}_j \sim \sigma\tau^{k_j}\mathrm{e}^{-2\sqrt{3}\tau}.$$

Since we do not have a translation for the flow close to the heteroclinic and the change of coordinates between consecutive periodic orbits is not linear, we will lose accuracy, in such a way we could not guarantee that a component initially microscopic will remain microscopic for the next step if we have the same definition for that size for all the steps. To counteract this loss of accuracy we will take:

$$k_j \leq k_{j+1}.$$

Finally, we will need to include another parameter to counteract the same loss but in the other direction. For the future coordinates we are restricting at each step their range since we always want some disk to be contained in the image of the previous one. So we have a decreasing sequence of ranges. We will solve this issue including a sequence of small parameters ϵ for the free coordinates:

$$\epsilon_j \leq \epsilon_{j+1}.$$

After these considerations we can write the recurrent result in a concrete way that we are going to use to prove the Theorem 8.

Proposition 1. *Assume that* $1/2 < \epsilon_j \leq \epsilon_{j+1} < 1$*,* $k_j \leq k_{j+1}$*,* σ *is small enough and* τ *is large enough. More precisely,* σ*,* τ*,* k_j*,* k_{j+1}*,* ϵ_j *and* ϵ_{j+1} *satisfying* (S1)-(S14)*,* (Sh1*)-(Sh6*) *and* (R1)-(R6)*. Then, there exists a subdisk of the incoming disk* D_j *such that, after the action of the flow for a certain time* $\tau + T$*, it contains the incoming disk for the next step,* D_{j+1}*, for a concrete expression of the functions that define the disk.*

We have already explained how we are going to prove it. As we have pointed out we are going to split the argument in two parts. The first one concerns the study of the flow close to $\mathcal{T}_j$. It is in this step where we properly connect the incoming heteroclinic of $\mathcal{T}_j$ with the outgoing one. This is the *singular* part of the problem since we do not follow any known solution of the system. We will need to find a solution that contours the periodic orbit $\mathcal{T}_j$ through the Shilnikov Theorem. The second part corresponds to the *regular* part of the problem. We just need to flow close to a known solution of the system, the heteroclinic, in order to reach a proximity of the following periodic orbit $\mathcal{T}_{j+1}$. More precisely, we are going to prove Proposition 1 through Propositions 2 and 6. At the beginning of each part we are going to define the intermediate disk that will contain the evolution of the present one.

3.4.1 The singular problem

In this part of the proof we have to connect the incoming and the outgoing heteroclinic in a small but macroscopic neighborhood of the origin. Our starting point is the disk D_j from Definition 8. The final disk will be the *outgoing disk*, that will be located close to the outgoing heteroclinic and will have two dimensions less, since we fix x_+ and y_+:

Definition 9. The *outgoing disk* is defined as

$$d_j = \big\{ z \in \mathcal{S}_N^j : c_- = m_\tau^{c_-}(c_+),\ x_- = m_\tau^{x_-}(c_+),\ y_- = m_\tau^{y_-}(c_+) \\ x_+ = \sigma + m_\tau^{x_+}(c_+),\ y_+ = m_\tau^{y_+}(c_+),\ |c_{+,k}| \le r_{c_{+,k}}^\tau \big\},$$

where we assume for the past coordinates:

$$\begin{aligned} \left| m_\tau^{c_-,k}(c_+) \right| &\le r_{c_-,k}^\tau = \frac{1}{N-3}\sigma\tau^{k_j+1}\mathrm{e}^{-\sqrt{3}\tau} \\ |m_\tau^{x_-}(c_+)| &\le r_{x_-}^\tau = \sigma\tau^{k_j+2}\mathrm{e}^{-\sqrt{3}\tau} \\ |m_\tau^{y_-}(c_+)| &\le r_{y_-}^\tau = \sigma\tau^{k_j+2}\mathrm{e}^{-\sqrt{3}\tau} \\ |m_\tau^{x_+}(c_+)| &\le r_{x_+}^\tau = \sigma^2\mathrm{e}^{-\sqrt{3}\tau} \\ |m_\tau^{y_+}(c_+)| &\le r_{y_+}^\tau(c_{j+2}) = \sigma^2\tau^{2(k_j+1)}\mathrm{e}^{-2\sqrt{3}\tau} + |c_{j+2}|^2, \end{aligned}$$

and for the free future coordinates:

$$r_{c_{+,k}}^\tau = \frac{1}{N-3}\sigma(1-\sqrt{3}\sigma(1+\sigma))\tau^{k_j}\mathrm{e}^{-\epsilon_j\sqrt{3}\tau}.$$

Remark. Notice that now $r_{y_+}^\tau$ is a function. We do not want to give an absolute estimate for this coordinate and forget about its dependence. This is because in terms of the maximum size of c_{j+2}, the estimate for $r_{y_+}^\tau$ will be very large and will not allow us to complete the argument. Recall that y_+ is a component that is supposed to be nanoscopic and we need it nanoscopic after the transition close to the heteroclinic, since it will correspond to $\tilde{x}_-$ in the following step.

Proposition 2. *Assume σ is small enough, τ is large enough and $1/2 < \epsilon_j < 1$. More precisely, σ and τ satisfying inequalities* (S1)-(S14) *and* (Sh1*)-(Sh6*). *Then, there exists a subdisk of the incoming disk, D_j, such that after the action of the flow for the time τ it contains the outgoing disk, d_j, for a concrete expression of the functions that define the outgoing disk.*

As we have said we are going to use the Shilnikov scheme to prove such a result. However, in order to prove a Shilnikov theorem adapted to this problem it is necessary an additional ingredient. Since we have a partially hyperbolic equilibrium point we know that there exist some manifolds that, roughly speaking, determine the different dynamical behavior. To prove the existence of a solution and good estimates for it in the passage close to the equilibrium point, we will need these manifolds to be straightened.

3.4.1.1 Straightening the invariant manifolds

When one deals with a linear system, everything is easy for many reasons but the crucial for us is the uncoupling of the equations: the variables associated to the eigendirections depend on their initial/final condition so, the system becomes an initial value problem for these components. That means that, on the one hand, there will always exist a unique solution. On the other hand, this uncoupling leads us to distinguish very well the hyperbolic behavior of the directions. We say that we are working in the correct coordinates.

When one includes the nonlinear part of the problem generically one loses this uncoupling that allows us to distinguish the stable and the unstable directions in the current variables. However there exist invariant manifolds that reproduce the same behavior of the linearized system. Then, if we take adapted coordinates to these manifolds we will recover the uncoupling phenomenon.

Applying the theory of invariant manifolds (see [AR67] for a detailed explanation) to system (3.5) we can determine the existence of three analytic invariant manifolds that can be written, locally, as a graph of a function:

$$\mathcal{W}^{cs} = \{(x,y,c) \,:\, x_\pm = \phi_\pm(y,c)\} \text{ center-stable manifold}$$

$$\mathcal{W}^{cu} = \{(x,y,c) \,:\, y_\pm = \psi_\pm(x,c)\} \text{ center-unstable manifold}$$
$$\mathcal{W}^{c} = \{(x,y,c) \,:\, x_\pm = \chi^x_\pm(c),\, y_\pm = \chi^y_\pm(c)\} \text{ center manifold}$$

that satisfy

$$\mathcal{W}^c \subset \mathcal{W}^{cs} \text{ and } \mathcal{W}^c \subset \mathcal{W}^{cu}.$$

In [CKS+10] the authors do not need to do this step. Probably it is because the heteroclinics coincide with some axis and, hence, the manifolds are already straightened in the directions of their interest. However, if we did not know that the heteroclinics are already straightened we would need to straighten the invariant manifolds. That means, that we are producing an argument (the Shilnkov Theorem) that could be applied to more general systems, once we have straightened the invariant manifold, which is a thing that we always can do.

On the other hand in [GK15] the authors perform directly a change of coordinates to a resonant normal form (that includes, of course, this straightening). For our analysis we will stop in an intermediate step as we will see.

Lemma 9. *The change of coordinates that straightens the invariant manifolds and its inverse have the following form:*

$$\left\{\begin{array}{lcl} \xi_- &=& x_- - \phi_-(y,c) \\ \eta_- &=& y_- - \psi_-(x,c) \\ \xi_+ &=& x_+ - \phi_+(y,c) \\ \eta_+ &=& y_\pm - \psi_+(x,c) \end{array}\right. \qquad \Rightarrow \qquad \left\{\begin{array}{lcl} x_- &=& \xi_- + \Phi_-(\xi,\eta,c) \\ y_- &=& \eta_- + \Psi_-(\xi,\eta,c) \\ x_+ &=& \xi_+ + \Phi_+(\xi,\eta,c) \\ y_+ &=& \eta_+ + \Psi_+(\xi,\eta,c) \end{array}\right.$$

where the non-linearities are functions whose Taylor expansion around the origin *begin with terms of order three and can be written as:*

$$
\begin{aligned}
\psi_\pm(x,c) =& x_\pm \left[x_\mp^2 \psi_\pm^a(x,c) + |c_{j\pm 2}|^2 \psi_\pm^{b_1}(x_\pm,c)\right.\\
&\left. + \left(c_{j\pm 2}^2 + \overline{c_{j\pm 2}}^2\right)\psi_\pm^{b_2}(x_\pm,c) + i\left(c_{j\pm 2}^2 - \overline{c_{j\pm 2}}^2\right)\psi_\pm^{b_3}(x_\pm,c)\right]\\
\phi_\pm(y,c) =& y_\pm \left[y_\mp^2 \phi_\pm^a(y,c) + |c_{j\pm 2}|^2 \phi_\pm^{b_1}(y_\pm,c)\right.\\
&\left. + \left(c_{j\pm 2}^2 + \overline{c_{j\pm 2}}^2\right)\phi_\pm^{b_2}(y_\pm,c) + i\left(c_{j\pm 2}^2 - \overline{c_{j\pm 2}}^2\right)\phi_\pm^{b_3}(y_\pm,c)\right],
\end{aligned}
$$

with $\psi_\pm^{a,b}$, $\phi_\pm^{a,b}$ *functions of order one and*

$$
\begin{aligned}
\Psi_\pm(\xi,\eta,c) =& \xi_\pm \left[\xi_\mp^2 \Psi_\pm^1(\xi,\eta,c) + |c_{j\pm 2}|^2 \Psi_\pm^2(\xi,\eta,c)\right.\\
&+ \left(c_{j\pm 2}^2 + \overline{c_{j\pm 2}}^2\right)\Psi_\pm^3(\xi,\eta,c) + i\left(c_{j\pm 2}^2 - \overline{c_{j\pm 2}}^2\right)\Psi_\pm^4(\xi,\eta,c)\\
&\left. + \xi_\mp\eta_\mp \Psi_\pm^5(\xi,\eta,c) + \eta_\mp^2 \Psi_\pm^6(\xi,\eta,c)\right]\\
&+ \eta_\pm \left[\xi_\mp^2 \Psi_\pm^7(\xi,\eta,c) + |c_{j\pm 2}|^2 \Psi_\pm^8(\xi,\eta,c)\right.\\
&+ \left(c_{j\pm 2}^2 + \overline{c_{j\pm 2}}^2\right)\Psi_\pm^9(\xi,\eta,c) + i\left(c_{j\pm 2}^2 - \overline{c_{j\pm 2}}^2\right)\Psi_\pm^{10}(\xi,\eta,c)\\
&\left. + \xi_\mp\eta_\mp \Psi_\pm^{11}(\xi,\eta,c) + \eta_\mp^2 \Psi_\pm^{12}(\xi,\eta,c)\right]\\
\Phi_\pm(\xi,\eta,c) =& \eta_\pm \left[\eta_\mp^2 \Phi_\pm^1(\xi,\eta,c) + |c_{j\pm 2}|^2 \Phi_\pm^2(\xi,\eta,c)\right.\\
&+ \left(c_{j\pm 2}^2 + \overline{c_{j\pm 2}}^2\right)\Phi_\pm^3(\xi,\eta,c) + i\left(c_{j\pm 2}^2 - \overline{c_{j\pm 2}}^2\right)\Phi_\pm^4(\xi,\eta,c)\\
&\left. + \xi_\mp\eta_\mp \Phi_\pm^5(\xi,\eta,c) + \xi_\mp^2 \Phi_\pm^6(\xi,\eta,c)\right]\\
&+ \xi_\pm \left[\eta_\mp^2 \Phi_\pm^7(\xi,\eta,c) + |c_{j\pm 2}|^2 \Phi_\pm^8(\xi,\eta,c)\right.\\
&+ \left(c_{j\pm 2}^2 + \overline{c_{j\pm 2}}^2\right)\Phi_\pm^9(\xi,\eta,c) + i\left(c_{j\pm 2}^2 - \overline{c_{j\pm 2}}^2\right)\Phi_\pm^{10}(\xi,\eta,c)\\
&\left. + \xi_\mp\eta_\mp \Phi_\pm^{11}(\xi,\eta,c) + \xi_\mp^2 \Phi_\pm^{12}(\xi,\eta,c)\right],
\end{aligned}
$$

with $\Psi_\pm^{1,2,3,4}$ *and* $\Phi_\pm^{1,2,3,4}$ *functions of order one and*

$$
\begin{aligned}
\Psi_\pm^{5,6}, \Phi_\pm^{5,6} &= \mathcal{O}_2\left(\xi_\pm, \eta_\pm, c_{j\mp 2}\right)\\
\Psi_\pm^{7,8,9,10,11,12}, \Phi_\pm^{7,8,9,10,11,12} &= \mathcal{O}_2\left(\xi_\mp, \eta_\mp, c_{j\pm 2}\right).
\end{aligned}
$$

Remark. This change of coordinates keeps the invariant subspaces L_-, L_+ and L_k for all k.

Once we have performed this change of coordinates we realize that it is not enough. If we want to produce a scheme for the whole transition chain we need to kill some terms in the system. These terms are not resonant and can be eliminated with a step of quasi-normal form.

Lemma 10. *The quasi-normal form change is given by:*

$$
\begin{aligned}
\bar{\xi}_- &= \xi_- + \xi_+\eta_-\eta_+ \sum_{n=0} a_n \eta_-^{2n}\\
\bar{\eta}_+ &= \eta_+ + \xi_+\eta_-\xi_- \sum_{n=0} b_n \xi_+^{2n}.
\end{aligned}
$$

We are not going to present a proof for this result. However it is easy to check that the change will only contain the kind of terms that we want to eliminate from the equation.

Remark. We are going to abuse notation and write ξ_- and η_+ instead of $\bar{\xi}_-$ and $\bar{\eta}_+$.

With these coordinates we can identify easily the invariant manifolds while keeping the invariant subspaces remarked above. As a summary, we have the following invariant subspaces:

$$\mathcal{W}^{cs} = \{\xi_- = \xi_+ = 0\} \quad \mathcal{W}^{cu} = \{\eta_- = \eta_+ = 0\} \quad \mathcal{W}^{c} = \{\xi_- = \eta_- = \xi_+ = \eta_+ = 0\}$$
$$\Lambda_- = \{\xi_- = \eta_- = 0\} \qquad \Lambda_+ = \{\xi_+ = \eta_+ = 0\} \qquad \Lambda_k = \{c_k = 0\}.$$

We are going to define now the straightened version of our incoming disk:

Definition 10. The *incoming straightened disk* is defined as

$$\begin{aligned}\Delta_j = \big\{z \in \mathcal{S}_N^j : \; & c_- = \mu_0^{c-}(\xi_+, \eta_+, c_+),\ \xi_- = \mu_0^{\xi_-}(\xi_+, \eta_+, c_+),\\ & \eta_- = \sigma + \mu_0^{\eta_-}(\xi_+, \eta_+, c_+),\\ & |\xi_+| \le \rho^0_{\xi_+},\ |\eta_+| \le \rho^0_{\eta_+},\ |c_{+,k}| \le \rho^0_{c_{+,k}} \big\},\end{aligned}$$

where we assume for the past coordinates:

$$\begin{aligned}\left|\mu_0^{c_{-,k}}(\xi_+, \eta_+, c_+)\right| &\le \rho^0_{c_{-,k}} = \frac{1}{N-3}\sigma\tau^{k_j}\mathrm{e}^{-\sqrt{3}\tau}\\ \left|\mu_0^{\xi_-}(\xi_+, \eta_+, c_+)\right| &\le \rho^0_{\xi_-} = \sigma(1+\sigma)\tau^{k_j}\mathrm{e}^{-2\sqrt{3}\tau}\\ \left|\mu_0^{\eta_-}(\xi_+, \eta_+, c_+)\right| &\le \rho^0_{\eta_-} = \sigma^2,\end{aligned}$$

and for the free future coordinates:

$$\begin{aligned}\rho^0_{\xi_+} &= \sigma\tau\mathrm{e}^{-\sqrt{3}\tau}\\ \rho^0_{\eta_+} &= \sigma\tau\mathrm{e}^{-\sqrt{3}\tau}\\ \rho^0_{c_{+,k}} &= \frac{1}{N-3}\sigma\tau^{k_j}\mathrm{e}^{-\epsilon_j\sqrt{3}\tau}.\end{aligned}$$

Proposition 3. *Assume σ is small enough and k_j is large enough. More precisely, σ and k_j satisfying inequalities* (S1)-(S6). *Then, the image of the incoming disk, D_j, through the changes of coordinates defined in Lemmas 9 and 10 contains the straightened incoming disk, Δ_j, for some concrete expression for the function that defines the disk.*

Proof. Since we want to compose two changes we are going to introduce an intermediate disk. This disk will be exactly the same for all the components than Δ_j except for ξ_-, ξ_+ and η_+. On the one hand we recall that the changes are the identity over c. Then we can take the same sizes for those coordinates:

$$\rho^0_{c_{-,k}} = r^0_{c_{-,k}} \qquad \rho^0_{c_{+,k}} = r^0_{c_{+,k}}.$$

For c_{j-2} we are not taking the equality but something larger, but the inequality goes in the good direction.

On the other hand, from the expressions of Lemma 9 we can see that there exists $K > 0$ such that:

$$|\psi_\pm(x,c)| \leq K\,|x_\pm|\left(|x_\mp|^2 + |c_{j\pm2}|^2\right)$$
$$|\phi_\pm(y,c)| \leq K\,|y_\pm|\left(|y_\mp|^2 + |c_{j\pm2}|^2\right)$$

Then, for the hyperbolic future coordinates we have:

$$\begin{aligned}
|\xi_+ - x_+| \leq& K\,|y_+|\left(|y_-|^2 + |c_{j+2}|^2\right)\\
\leq& K r^0_{y+}\left(\sigma^2\left(1+\frac{1}{2}\sigma\right)^2 + \sigma^2\right)\\
\leq& 3K\sigma^2 r^0_{y+}\\
|\eta_+ - y_+| \leq& K\,|x_+|\left(|x_-|^2 + |c_{j+2}|^2\right)\\
\leq& K r^0_{x+}\left(\sigma^2 + \sigma^2\right)\\
\leq& 3K\sigma^2 r^0_{x+},
\end{aligned}$$

if

$$\left(1+\frac{1}{2}\sigma\right)^2 + 1 \leq 3. \tag{S1}$$

Now assume that

$$1 - 3K\sigma^2 \geq \frac{3}{4}. \tag{S2}$$

Then, for x_+ and y_+ in the boundary of their respective domains, we have:

$$\xi_+ \geq (1-3K\sigma^2)2\sigma\tau e^{-\sqrt{3}\tau} \geq \frac{3}{2}\sigma\tau e^{-\sqrt{3}\tau}$$
$$\eta_+ \geq (1-3K\sigma^2)2\sigma\tau e^{-\sqrt{3}\tau} \geq \frac{3}{2}\sigma\tau e^{-\sqrt{3}\tau}.$$

Then, it is enough to take:

$$\rho^*_{\xi+} = \frac{3}{2}\sigma\tau e^{-\sqrt{3}\tau}$$
$$\rho^*_{\eta+} = \frac{3}{2}\sigma\tau e^{-\sqrt{3}\tau}.$$

For the hyperbolic past coordinates, we have:

$$
\begin{aligned}
|\xi_-| \leq& |x_-| + K\,|y_-|\left(|y_+|^2 + |c_{j-2}|^2\right) \\
\leq& \sigma\tau^{k_j}\mathrm{e}^{-2\sqrt{3}\tau} + K\sigma\left(1+\frac{1}{2}\sigma\right)\left(4\sigma^2\tau^2\mathrm{e}^{-2\sqrt{3}\tau} + \frac{1}{(N-3)^2}\sigma^2\tau^{k_j}\mathrm{e}^{-2\sqrt{3}\tau}\right) \\
\leq& \sigma\tau^{k_j}\mathrm{e}^{-2\sqrt{3}\tau}\left(1 + K\sigma^2\left(1+\frac{1}{2}\sigma\right)\left(4+\frac{1}{(N-3)^2}\right)\right) \\
\leq& \sigma\tau^{k_j}\mathrm{e}^{-2\sqrt{3}\tau}\left(1+5K\sigma^2\left(1+\frac{1}{2}\sigma\right)\right) \\
\leq& \sigma\tau^{k_j}\mathrm{e}^{-2\sqrt{3}\tau}\left(1+\frac{1}{2}\sigma\right) \\
|\eta_- - \sigma| \leq& |y_- - \sigma| + |\eta_- - y_-| \\
\leq& \frac{1}{2}\sigma^2 + K\,|x_-|\left(|x_+|^2 + |c_{j-2}|^2\right) \\
\leq& \frac{1}{2}\sigma^2 + 2K\sigma^3 \\
\leq& \sigma^2,
\end{aligned}
$$

if σ and k_j satisfy:

$$10K\sigma\left(1+\frac{1}{2}\sigma\right) \leq 1, \tag{S3}$$

$$4\sigma K \leq 1, \tag{S4}$$

$$k_j > 2. \tag{S5}$$

To finish the proof we need to perform the second change of variables driven by a quasi-normal form. Increasing the value of K, if needed, we can assume that:

$$
\begin{aligned}
\left|\bar{\xi}_-\right| &\leq |\xi_-| + K\,|\xi_+|\,|\eta_-|\,|\eta_+| \\
|\bar{\eta}_+ - \eta_+| &\leq K\,|\xi_+|\,|\eta_-|\,|\xi_-|\,.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\left|\bar{\xi}_-\right| &\leq \sigma\tau^{k_j}\mathrm{e}^{-2\sqrt{3}\tau}\left(1+\frac{1}{2}\sigma\right) + K\frac{9}{4}\sigma\left(1+\sigma\right)\sigma^2\tau^2\mathrm{e}^{-2\sqrt{3}\tau} \\
&\leq \sigma(1+\sigma)\tau^{k_j}\mathrm{e}^{-2\sqrt{3}\tau},
\end{aligned}
$$

if σ satisfies:

$$\frac{9}{2}K\sigma(1+\sigma) \leq 1. \tag{S6}$$

For $\bar{\eta}_+$ we have:

$$|\bar{\eta}_+ - \eta_+| \leq 3K\sigma^2\frac{3}{2}\sigma\tau\mathrm{e}^{-\sqrt{3}\tau}$$

Taking that into account, we obtain

$$\bar{\eta}_+ \geq \left(1-3K\sigma^2\right)\frac{3}{2}\sigma\tau\mathrm{e}^{-\sqrt{3}\tau} \geq \frac{3}{4}\frac{3}{2}\sigma\tau\mathrm{e}^{-\sqrt{3}\tau} = \frac{9}{8}\sigma\tau\mathrm{e}^{-\sqrt{3}\tau} \geq \sigma\tau\mathrm{e}^{-\sqrt{3}\tau},$$

that is the desired bound for this coordinate. □

Next we compute the expression of system (3.5) in these new coordinates:

Lemma 11. *Denoting* $\xi = (\xi_-, \xi_+)$, $\eta = (\eta_-, \eta_+)$, $c = (c_1, \dots, c_{j-2}, c_{j+2}, \dots, c_N) \in \mathbb{C}^{N-3}$ *and* $z = (\xi, \eta, c)$, *the system* (3.5) *becomes:*

$$\left\{ \begin{array}{rcl} \dot{\xi}_\pm &=& \sqrt{3}\xi_\pm + R^{\xi_\pm}(\xi, \eta, c) \\ \dot{\eta}_\pm &=& -\sqrt{3}\eta_\pm + R^{\eta_\pm}(\xi, \eta, c) \\ \dot{c}_k &=& ic_k + R^{c_k}(\xi, \eta, c) \end{array} \right. \tag{3.18}$$

where the nonlinearities have the following expression:

$$\left\{ \begin{array}{rcl} R^{\xi_\pm}(\xi, \eta, c) &=& \xi_\pm R^{\xi_\pm}_{unc}(\xi, \eta, c) + \xi_\mp \eta_\pm R^{\xi_\pm}_{coup}(\xi_\mp, \eta, c) \\ R^{\eta_\pm}(\xi, \eta, c) &=& \eta_\pm R^{\eta_\pm}_{unc}(\xi, \eta, c) + \eta_\mp \xi_\pm R^{\eta_\pm}_{coup}(\xi, \eta_\mp, c) \\ R^{c_k}(\xi, \eta, c) &=& c_k R^{c_k}_1(\xi, \eta, c) + \bar{c}_k R^{c_k}_2(\xi, \eta, c) \end{array} \right. \tag{3.19}$$

with

$$R^{\xi_-}_{unc}(\xi, \eta, c) = \mathcal{O}_2(z) \qquad R^{\xi_-}_{coup}(\xi_+, \eta, c) = \mathcal{O}\left(\xi_+, \eta_+^2, c^2\right)$$

$$R^{\xi_+}_{unc}(\xi, \eta, c) = \mathcal{O}_2(z) \qquad R^{\xi_+}_{coup}(\xi_-, \eta, c) = \mathcal{O}(\xi_-, \eta, c)$$

$$R^{\eta_-}_{unc}(\xi, \eta, c) = \mathcal{O}_2(z) \qquad R^{\eta_-}_{coup}(\xi, \eta_+, c) = \mathcal{O}(\xi, \eta_+, c)$$

$$R^{\eta_+}_{unc}(\xi, \eta, c) = \mathcal{O}_2(z) \qquad R^{\eta_+}_{coup}(\xi, \eta_-, c) = \mathcal{O}\left(\xi_-^2, \eta_-, c^2\right)$$

$$R^{c_k}_1(\xi, \eta, c) = \mathcal{O}_2(z) \qquad R^{c_k}_2(\xi, \eta, c) = \mathcal{O}_2(z)$$

Proof. Notice that the changes defined in Lemmas 9 and 10 are defined as the identity plus some function of order three. Such changes keep the linear part of the problem, so we only have to check that the nonlinearities have the expression of (3.19). We split the nonlinearities in terms of the uncoupling and coupling terms. For instance in the equation for ξ_- we divide the nonlinearity in two parts: the one that depends on ξ_- and the one that does not. Recall that the system must show the invariance of the center-stable manifold $\mathcal{W}^{cs} = \{\xi_- = \xi_+ = 0\}$ and the mode invariance $\Lambda_- = \{\xi_- = \eta_- = 0\}$. This implies that the part that does not contain ξ_- must contain as a factor the product of $\xi_+\eta_-$ since the equation must vanish in these subspaces. This argument is analogous for the rest of hyperbolic coordinates and, taking into account that the nonlinearities are of order three, we have already proved that the equations for η_- and ξ_+ have the desired form.

For the equations of the central modes we have nothing to say: since in these variables the change is the identity we are only assuming that the nonlinearities are of order three and keep the mode invariance.

Finally we have to justify the expression of $R^{\xi_-}_{\text{coup}}$ and $R^{\eta_+}_{\text{coup}}$. Notice that its dependence in the central modes, c, in terms of c^2 is reasonable since the original system (3.5) has this kind of dependence. This property also can be checked for all the components. However, since we only need this assumption for these small coordinates, ξ_- and η_+, we will keep the more generic situation when proving the Shilnikov Theorem, in order to obtain Theorem that can be applied in a wider class of systems.

To prove the other assumptions we are going to restrict our argument in the ξ_- component. The argument for η_+ will be analogous. The first property to check is that $R^{\xi_-}_{\text{coup}}$ does not contain terms of the form η_-^n (ξ_+^n for the η_+ equation) for any n. This is a consequence of the form of the original system which does not contain the equivalent terms. Again, we could assume it for all the hyperbolic coordinates but, for the sake of generality, we will only assume it for the components that require it: ξ_- and η_+.

The only requirement for the Shilnikov Theorem that we cannot assume taking into account the expression of the original system is that $R^{\xi_-}_{\text{coup}}$ does not contain terms of the form $\eta_+\eta_-^n$ for any n. Indeed, these kind of terms appear in the original system. However they are non-resonant terms and are the kind of terms that will be killed through the quasi-normal form change from Lemma 10. □

Remark. As we have stated in the proof we want to recall that we are not breaking symmetries that originally appear in the Toy Model even if it seems so when we look at those expressions. We are only writing the minimal assumptions that we need for proving the Shilnikov Theorem.

3.4.1.2 The Shilnikov theorem

Once we have performed the changes of coordinates above, we are ready to connect the incoming heteroclinic with the outgoing one close to the equilibrium point. As always, we start defining the final disk of this part, the outgoing straightened disk:

Definition 11. The *outgoing straightened disk* is defined as

$$\begin{aligned}\delta_j = \Big\{ z \in \mathcal{S}_N^j \,:\, & c_- = \mu_\tau^{c_-}(c_+),\, \xi_- = \mu_\tau^{\xi_-}(c_+),\, \eta_- = \mu_\tau^{\eta_-}(c_+)\\ & \xi_+ = \sigma,\, \eta_+ = 0,\, |c_{+,k}| \leq \rho^\tau_{c_{+,k}} \Big\},\end{aligned}$$

where we assume for the past coordinates:

$$\begin{aligned}\left|\mu_\tau^{c_-,k}(c_+)\right| &\leq \rho^\tau_{c_{-,k}} = \frac{1}{N-3}\sigma\tau^{k_j+1}\mathrm{e}^{-\sqrt{3}\tau}\\ \left|\mu_\tau^{\xi_-}(c_+)\right| &\leq \rho^\tau_{\xi_-} = \sigma\tau^{k_j+1}\mathrm{e}^{-\sqrt{3}\tau}\\ \left|\mu_\tau^{\eta_-}(c_+)\right| &\leq \rho^\tau_{\eta_-} = \sigma\tau^{k_j+1}\mathrm{e}^{-\sqrt{3}\tau},\end{aligned}$$

and for the free future coordinates:

$$\rho^\tau_{c_{+,k}} = \frac{1}{N-3}\sigma(1-\sqrt{3}\sigma(1+\sigma))\tau^{k_j}\mathrm{e}^{-\epsilon_j\sqrt{3}\tau}.$$

As we have said, to connect both disks we are going to use a Shilnikov scheme. We notice that our Shilnikov problem is different from the standard one:

- We are generalizing the Shilnikov problem to a non hyperbolic equilibrium point, that is the case considered in [Den89] and [Shi67].
- We are not using the standard partition in initial and final conditions. When one considers a Shilnikov problem for a hyperbolic equilibrium point, the stable coordinates are fixed for time $t = 0$ while the unstable are fixed for $t = \tau$ as in [Den89] or [Shi67].
- We always keep the linear part of the solution as a reference. That means, for instance, that we are not going to get optimal transition times. To get them, we would need to use resonant normal forms as in [GK15].

Considering the previous modifications we can state the Shilnikov problem that guarantees the existence of a solution to a Shilnikov problem:

Theorem 9. *Let $\gamma_0 > 0$ and such that:*

$$\gamma_0 \leq \frac{2\sqrt{3}}{3K}, \tag{Sh1}$$

$$\frac{14K}{\sqrt{3}}(1+\gamma_0)^2\gamma_0^2 \leq \frac{1}{3}. \tag{Sh2}$$

Let $1/2 < \epsilon < 1$, $0 < \gamma \leq \gamma_0$, $k \geq 0$ and $\tau \in \mathbb{R}$ such that:

$$\tau \geq \frac{2\sqrt{3}}{3}, \tag{Sh3}$$

$$\frac{\sqrt{3}}{2}\tau^{k+1}e^{-\epsilon\sqrt{3}\tau} \leq 1, \tag{Sh4}$$

$$\frac{1}{18}\gamma\tau^{2k-1}e^{-\sqrt{3}\tau} \leq 1, \tag{Sh5}$$

$$\frac{1}{6}\gamma\tau^{2k-1}e^{(1-2\epsilon)\sqrt{3}\tau} \leq 1. \tag{Sh6}$$

Take $\xi_{-,0}, \eta_{-,0}, \xi_{+,1}, \eta_{+,1} \in \mathbb{R}$ and $\zeta = (\zeta_1, \dots, \zeta_{j-2}, \zeta_{j+2}, \dots, \zeta_N) \in \mathbb{C}^{N-3}$, such that:

$$\begin{array}{rclrcl} |\xi_{-,0}| &=& \gamma_{\xi_-} \leq \dfrac{1}{2\sqrt{3}}\gamma\tau^k e^{-2\sqrt{3}\tau} & |\eta_{-,0}| &=& \gamma_{\eta_-} \leq \dfrac{1}{2\sqrt{3}}\gamma \\ |\xi_{+,1}| &=& \gamma_{\xi_+} \leq \dfrac{1}{2\sqrt{3}}\gamma & |\eta_{+,1}| &=& \gamma_{\eta_+} \leq \dfrac{1}{2\sqrt{3}}\gamma\tau^k e^{-2\sqrt{3}\tau} \\ |\zeta_k| &=& \gamma_k \leq \dfrac{1}{N-3}\dfrac{1}{2\sqrt{3}}\gamma\tau^k e^{-\epsilon\sqrt{3}\tau} & & & \end{array}$$

Then there exists a unique solution of (3.18) *defined for $t \in [0,\tau]$ that satisfies the Shilnikov data, that is:*

$$\begin{array}{rclc} \xi_-(0) &=& \xi_{-,0} & \eta_-(0) = \eta_{-,0} \\ \xi_+(\tau) &=& \xi_{+,1} & \eta_+(\tau) = \eta_{+,1} \\ c_k(0) &=& \zeta_k & \end{array}$$

In addition, the solution deviates from the linear solution in the following way:

$$\left|\xi_-(t) - \xi_{-,0}e^{\sqrt{3}t}\right| \leq \frac{1}{2}\gamma\left(\gamma_{\xi_-} + \gamma_{\eta_-}\tau e^{-2\sqrt{3}\tau}\right)e^{\sqrt{3}t} \tag{3.20}$$

$$\left|\eta_-(t) - \eta_{-,0}e^{-\sqrt{3}t}\right| \leq \frac{1}{2}\gamma\left(\gamma_{\eta_-} + \frac{1}{2}\gamma_{\xi_-}e^{\sqrt{3}\tau}\right)e^{-\sqrt{3}t} \tag{3.21}$$

$$\left|\xi_+(t) - \xi_{+,1}e^{\sqrt{3}(t-\tau)}\right| \leq \frac{1}{2}\gamma\left(\gamma_{\xi_+} + \frac{1}{2}\gamma_{\eta_+}e^{\sqrt{3}\tau}\right)e^{\sqrt{3}(t-\tau)} \tag{3.22}$$

$$\left|\eta_+(t) - \eta_{+,1}e^{-\sqrt{3}(t-\tau)}\right| \leq \frac{1}{2}\gamma\left(\gamma_{\eta_+} + \gamma_{\xi_+}\tau e^{-2\sqrt{3}\tau}\right)e^{-\sqrt{3}(t-\tau)} \tag{3.23}$$

$$\left|c_k(t) - \zeta_k e^{it}\right| \leq \frac{1}{2}\gamma\gamma_k \tag{3.24}$$

We refer to the proof in Section C.3 in Appendix C.

Once we have announced the theoretical result, we are able to prove the following Proposition:

Proposition 4. *Assume σ is small enough, τ is large enough and $1/2 < \epsilon_j < 1$. More precisely, σ and τ satisfying inequalities* (S7)-(S10) *and* (Sh1*)-(Sh6*). *Then, there exists a subdisk of Δ_j such that after the action of the flow for the time τ it contains δ_j, for a concrete expression of the functions that define δ_j.*

Proof. We want all the solutions that start in our initial disk Δ_j and end, after a time τ in the section defined by $\{\xi_+ = \sigma, \eta_+ = 0\}$. That is, for each $c_+^0 \in \{c_+ : |c_+| \leq \rho_{c_+}^0\}$ we want to solve the Shilnikov Problem with data:

$$\begin{array}{rr} c_-(0) = \mu_0^{c_-}(\xi_+^0, \eta_+^0, c_+^0) & c_+(0) = c_+^0 \\ \xi_-(0) = \mu_0^{\xi_-}(\xi_+^0, \eta_+^0, c_+^0) & \eta_-(0) = \sigma + \mu_0^{\eta_-}(\xi_+^0, \eta_+^0, c_+^0) \\ \xi_+(\tau) = \sigma & \eta_+(\tau) = 0 \end{array}$$

To prove that there exists a solution for each considered Shilnikov problem we will apply Theorem 9. So we must, first, identify γ. Since the dominant coordinate of Δ_j is η_-, we are going to take $\gamma = 2\sqrt{3}\sigma(1+\sigma)$. It is clear that, for all the possible values of η_- in the disk, we have

$$|\eta_-| \leq \sigma + \rho_{\eta_-}^0 \leq \sigma(1+\sigma) = \frac{1}{2\sqrt{3}}\gamma.$$

The next inequality to check is

$$|\xi_-(0)| \leq \sigma(1+\sigma)\tau^k \mathrm{e}^{-2\sqrt{3}\tau}.$$

By the definition of Δ_j this condition is fulfilled by all the possible values of ξ_- in our disk if we take $k = k_j$. For the central modes we have the bounds:

$$|c_-| \leq \frac{1}{N-3}\sigma(1+\sigma)\tau^{k_j}\mathrm{e}^{-\sqrt{3}\tau} \qquad |c_+^0| \leq \frac{1}{N-3}\sigma(1+\sigma)\tau^{k_j}\mathrm{e}^{-\epsilon_j\sqrt{3}\tau}.$$

so the conditions for these coordinates in Theorem 9 are fulfilled if we select $\epsilon = \epsilon_j$.

The conditions for $\xi_+(0)$ and $\eta_+(\tau)$ are automatically fulfilled.

Recall that we need $\gamma \leq \gamma_0$ with γ_0 and τ satisfying conditions (Sh1)-(Sh6). In terms of σ, ϵ_j and k_j these inequalities become:

$$\sigma(1+\sigma) \leq \frac{1}{3K} \tag{Sh1*}$$

$$\frac{14K}{\sqrt{3}}\left(1+2\sqrt{3}\sigma(1+\sigma)\right)^2\left(2\sqrt{3}\sigma(1+\sigma)\right)^2 \leq \frac{1}{3} \tag{Sh2*}$$

$$\tau \geq \frac{2\sqrt{3}}{3} \tag{Sh3*}$$

$$\frac{\sqrt{3}}{2}\tau^{k_j+1}e^{-\epsilon_j\sqrt{3}\tau} \leq 1 \tag{Sh4*}$$

$$\frac{\sqrt{3}}{9}\sigma(1+\sigma)\tau^{2k_j-1}e^{-\sqrt{3}\tau} \leq 1 \tag{Sh5*}$$

$$\frac{\sqrt{3}}{3}\sigma(1+\sigma)\tau^{2k_j-1}e^{(1-2\epsilon_j)\sqrt{3}\tau} \leq 1. \tag{Sh6*}$$

So, we can apply Theorem 9 and we obtain a family of solutions parameterized by the initial value of $c_+(0)$ and estimates on the deviation of these solutions with respect to the solution of the linearized problem.

It is clear that these solutions end in the desired section. However we still have to check that all of them start at Δ_j for $t = 0$. On the one hand from (3.22) and (3.23) we have:

$$|\xi_+(0)| \leq \sigma\mathrm{e}^{-\sqrt{3}\tau}\left(1+\sqrt{3}\sigma(1+\sigma)\right) \leq \sigma\tau\mathrm{e}^{-\sqrt{3}\tau}$$
$$|\eta_+(0)| \leq \sigma\tau\mathrm{e}^{-\sqrt{3}\tau}\sqrt{3}\sigma(1+\sigma) \leq \sigma\tau\mathrm{e}^{-\sqrt{3}\tau},$$

if

$$1+\sqrt{3}\sigma(1+\sigma) \leq \tau, \tag{S7}$$

$$\sqrt{3}\sigma(1+\sigma) \leq 1. \tag{S8}$$

Notice that these maximum values are contained in the range that defines both coordinates in Δ_j.

On the other hand, we wonder which are the values that we have to give to the new introduced parameters: ξ_+^0 and η_+^0. In order to have the family of solutions starting in Δ_j we have to check that, for any admissible value of c_+^0 the following equation has a solution:

$$\begin{pmatrix} \xi_+^0 \\ \eta_+^0 \end{pmatrix} = \begin{pmatrix} \xi_+(0) \\ \eta_+(0) \end{pmatrix} = \begin{pmatrix} \xi_+\left(0;\tau,\mu_0^{(c_-,\xi_-,\eta_-)}(\xi_+^0,\eta_+^0,c_+^0),c_+^0\right) \\ \eta_+\left(0;\tau,\mu_0^{(c_-,\xi_-,\eta_-)}(\xi_+^0,\eta_+^0,c_+^0),c_+^0\right) \end{pmatrix}.$$

It is enough to check that

$$\left| \frac{\partial(\xi_+,\eta_+)}{\partial(c_-^0,\xi_-^0,\eta_-^0)}(0)\frac{\partial\mu_0^{(c_-,\xi_-,\eta_-)}}{\partial(\xi_+^0,\eta_+^0)} \right| \leq 1. \tag{3.25}$$

To prove above estimate we need to produce a result that gives estimates on the derivatives of the solution obtained by the Shilnikov Theorem with respect to the Shilnikov data. The proof for such a result is completely analogous to the proof of Theorem 9, and this is why we skip it. Notice that the smallness is only required in this factor of the product (3.25) since we will assume that the function that defines the straightened incoming disk, μ^0, and its derivative are bounded in the compact defined by the free coordinates of the disk.

Finally we have that our family of solutions begins at Δ_j forming a disk, parameterized by c_+^0. If we compute the evolution of this subdisk for the time τ we will obtain a disk in the desired section (by construction) parameterized again by c_+^0. We first compute the estimates for this new disk.

For the center past coordinates, using the estimate (3.24), we have:

$$\begin{aligned} |c_k(\tau)| &\leq \left(1+\sqrt{3}\sigma(1+\sigma)\right)|c_k^0| \leq \left(1+\sqrt{3}\sigma(1+\sigma)\right)\rho_{c_-,k}^0 \\ &= \frac{1}{N-3}\left(1+\sqrt{3}\sigma(1+\sigma)\right)\sigma\tau^{k_j}\mathrm{e}^{-\sqrt{3}\tau} \\ &\leq \frac{1}{N-3}\sigma\tau^{k_j+1}\mathrm{e}^{-\sqrt{3}\tau} = \rho_{c_-,k}^\tau. \end{aligned}$$

For the hyperbolic past coordinates, using estimates (3.20) and (3.21) we have:

$$\begin{aligned} |\xi_-(\tau)| &\leq \sigma(1+\sigma)\mathrm{e}^{-\sqrt{3}\tau}\left[\tau^{k_j}\left(1+\frac{1}{2}\gamma\right)+\frac{1}{2}\gamma\tau\right] \leq \sigma\tau^{k_j+1}\mathrm{e}^{-\sqrt{3}\tau} = \rho_{\xi_-}^\tau \\ |\eta_-(\tau)| &\leq \sigma(1+\sigma)\mathrm{e}^{-\sqrt{3}\tau}\left[1+\frac{1}{2}\gamma+\frac{1}{4}\gamma\tau^{k_j}\mathrm{e}^{-\sqrt{3}\tau}\right] \leq \sigma\tau^{k_j+1}\mathrm{e}^{-\sqrt{3}\tau} = \rho_{\eta_-}^\tau, \end{aligned}$$

if

$$(1+\sigma)\left(1+2\sqrt{3}\sigma(1+\sigma)\right) \leq 1, \tag{S9}$$

$$(1+\sigma)\left(1+\frac{3}{2}\sigma(1+\sigma)\right) \leq 1. \tag{S10}$$

Finally for the center future modes we cannot ensure the exact width that the disk will have. However, we are going to compute a range that, for sure, the disk will contain using the lower bounds from (3.24). Take a point in the boundary, that is $|c_{+,k}| = \rho^0_{c_{+,k}}$

$$\begin{aligned}|c_k(\tau)| &\geq \left(1-\sqrt{3}\sigma(1+\sigma)\right)|c^0_k| = \left(1-\sqrt{3}\sigma(1+\sigma)\right)\rho^0_{c_{+,k}}\\ &= \frac{1}{N-3}\left(1-\sqrt{3}\sigma(1+\sigma)\right)\tau^{k_j}\mathrm{e}^{-\epsilon_j\sqrt{3}\tau} = \rho^\tau_{c_{+,k}}.\end{aligned}$$

Notice that with these sizes we have already proved the Proposition. However, there is a small detail left. Our final disk δ_j should be parameterized by $c_+(\tau)$. We can solve this issue taking into account that the relation between $c_+(0)$ and $c_+(\tau)$ is invertible thanks to the bounds (3.24). On the other hand, we do not have to check the estimates for the rest of the coordinates in terms of this reparameterization because the bounds are independent of $c_{+,k}$. □

This is the main result that we will use to prove Proposition 2. Notice that we have already lost the future hyperbolic directions so our disk has two dimensions less. The only difference is that we have proved the result in different coordinates, the ones for which the invariant manifolds are straightened. It rests now only to translate this result in the original coordinates.

3.4.1.3 Returning to the original coordinates

We have now completed the most relevant result of our work. It is time now to perform the inverse changes of all the changes of coordinates: the step of normal form and the straightening of the invariant manifolds.

Lemma 12. *There exists K such that, for points in the outgoing straightened disk, the following estimates hold:*

$$\begin{aligned}|\xi_-| &\leq \bar{\xi}_-\left(1+K\xi_+^2\eta_-^2\right)\\ |\eta_+| &\leq K\bar{\xi}_-\xi_+\eta_-\end{aligned}$$

$$\begin{aligned}|\Psi_\pm(\xi,\eta,c)| \leq &K\left[|\xi_\pm|\left(|\xi_\mp|^2+|c_{j\pm2}|^2+|\eta_\mp|^2\left(|\xi_\pm|^2+|\eta_\pm|^2+|c_{j\mp2}|^2\right)\right)\right.\\ &\left.+|\eta_\pm|\left(|\xi_\mp|^2+|\eta_\mp|^2+|c_{j\pm2}|^2\right)\left(|\xi_\mp|^2+|\eta_\mp|^2+|c_{j\pm2}|^2\right)\right]\\ |\Phi_\pm(\xi,\eta,c)| \leq &K\left[|\eta_\pm|\left(|\eta_\mp|^2+|c_{j\pm2}|^2+|\xi_\mp|^2\left(|\xi_\pm|^2+|\eta_\pm|^2+|c_{j\mp2}|^2\right)\right)\right.\\ &\left.+|\xi_\pm|\left(|\eta_\mp|^2+|\xi_\mp|^2+|c_{j\pm2}|^2\right)\left(|\xi_\mp|^2+|\eta_\mp|^2+|c_{j\pm2}|^2\right)\right].\end{aligned}$$

Proof. We first perform the inverse change of the quasi-normal form change defined in (10):

$$\begin{aligned}\bar{\xi}_- &= \xi_- + \xi_+\eta_-\eta_+\sum_{n=0} a_n\eta_-^{2n}\\ \bar{\eta}_+ &= \eta_+ + \xi_+\eta_-\xi_-\sum_{n=0} b_n\xi_+^{2n}\end{aligned}$$

Notice that over our disk $\bar{\delta}_j$, we have $\bar{\eta}_+ = 0$. That means:

$$\eta_+ = -\xi_+\eta_-\xi_- \sum_{n=0} b_n\xi_+^{2n} = \xi_- f(\xi_+, \eta_-).$$

If we insert this fact in the equation for $\bar{\xi}_-$ we obtain:

$$\bar{\xi}_- = \xi_- + \xi_+\eta_-\eta_+ \sum_{n=0} a_n\eta_-^{2n} = \xi_-(1 + f(\xi_+, \eta_-)g(\xi_+, \eta_-)),$$

and then

$$\begin{aligned}
\xi_- &= \bar{\xi}_- \frac{1}{1 + f(\xi_+, \eta_+)g(\xi_+, \eta_-)} \\
\eta_+ &= \bar{\xi}_- \frac{1}{1 + f(\xi_+, \eta_+)g(\xi_+, \eta_-)} f(\xi_+, \eta_-).
\end{aligned}$$

Then, there exists K_1 such that

$$\begin{aligned}
|\xi_-| &\leq \bar{\xi}_- \left(1 + K_1\xi_+^2\eta_-^2\right) \\
|\eta_+| &\leq K_1\bar{\xi}_-\xi_+\eta_-.
\end{aligned}$$

For the straightening change, using the expressions form Lemma 9 we have that there exists K_2 such that:

$$\begin{aligned}
|\Psi_\pm(\xi, \eta, c)| \leq & K_2 \left[|\xi_\pm| \left(|\xi_\mp|^2 + |c_{j\pm2}|^2 + |\eta_\mp|^2 \left(|\xi_\pm|^2 + |\eta_\pm|^2 + |c_{j\mp2}|^2 \right) \right) \right. \\
& \left. + |\eta_\pm| \left(|\xi_\mp|^2 + |\eta_\mp|^2 + |c_{j\pm2}|^2 \right) \left(|\xi_\mp|^2 + |\eta_\mp|^2 + |c_{j\pm2}|^2 \right) \right] \\
|\Phi_\pm(\xi, \eta, c)| \leq & K_2 \left[|\eta_\pm| \left(|\eta_\mp|^2 + |c_{j\pm2}|^2 + |\xi_\mp|^2 \left(|\xi_\pm|^2 + |\eta_\pm|^2 + |c_{j\mp2}|^2 \right) \right) \right. \\
& \left. + |\xi_\pm| \left(|\eta_\mp|^2 + |\xi_\mp|^2 + |c_{j\pm2}|^2 \right) \left(|\xi_\mp|^2 + |\eta_\mp|^2 + |c_{j\pm2}|^2 \right) \right].
\end{aligned}$$

Let $K = \max\{K_1, K_2\}$. □

Proposition 5. *Assume σ is small enough and τ and k_j are large enough. More precisely, σ and τ satisfying* (S11)-(S14)*. Then the image of the disk δ_j contains the outgoing disk, d_j, after the change of coordinates for a concrete expression of the functions that define the disk d_j.*

Proof. Since the change in the central coordinates is the identity, we can take:

$$r^\tau_{c_{+,k}} = \rho^\tau_{c_{+,k}} \qquad r^\tau_{c_{-,k}} = \rho^\tau_{c_{-,k}}$$

Using the bounds for $\bar{\xi}_-$, η_- and ξ_+ in the definition of our disk $\bar{\delta}_j$ we obtain:

$$\begin{aligned}
|\xi_-| &\leq \sigma\tau^{k_j+1}\mathrm{e}^{-\sqrt{3}\tau}\left(1 + K\sigma^2\right) \\
|\eta_+| &\leq K\sigma^3\tau^{2(k_j+1)}\mathrm{e}^{-2\sqrt{3}\tau}.
\end{aligned}$$

Then,

$$\begin{aligned}|y_-| \leq& |\eta_-| + |\Psi_-(\xi,\eta,c)| \leq |\eta_-| + K\sigma^2 \left(|\xi_-| + |\eta_-|\right)\\ |x_-| \leq& |\xi_-| + |\Phi_-(\xi,\eta,c)| \leq |\xi_-| + K\sigma^2 \left(|\xi_-| + |\eta_-|\right)\end{aligned}$$

So, we have:

$$|\xi_-| + |\eta_-| \leq \sigma\tau^{k_j+1} \mathrm{e}^{-\sqrt{3}\tau} \left(2 + K\sigma^2\right),$$

and we can take:

$$\begin{aligned}r^\tau_{y_-} =& \sigma\tau^{k_j+2}\mathrm{e}^{-\sqrt{3}\tau}\\ r^\tau_{x_-} =& \sigma\tau^{k_j+2}\mathrm{e}^{-\sqrt{3}\tau},\end{aligned}$$

if

$$1 + K\sigma^2(2 + K\sigma^2) \leq \tau, \tag{S11}$$

$$1 + 3K\sigma^2 + K^2\sigma^4 \leq \tau. \tag{S12}$$

The macroscopic component, ξ_+ is transformed as:

$$|x_+ - \sigma| \leq |\Phi_+(\xi,\eta,c)| \leq K\sigma^3\mathrm{e}^{-\sqrt{3}\tau} \leq \sigma^2\mathrm{e}^{-\sqrt{3}\tau},$$

if

$$K\sigma \leq 1. \tag{S13}$$

Finally, the bounds for y_+ are the most delicate:

$$\begin{aligned}|y_+| \leq& |\eta_+| + |\Psi_+(\xi,\eta,c)| \leq |\eta_+| + K\sigma \left(|\xi_-|^2 + |\eta_-|^2 + |c_{j+2}|^2 + \sigma|\eta_+|\right)\\ =& |\eta_+|(1 + K\sigma^2) + K\sigma \left(|\xi_-|^2 + |\eta_-|^2 + |c_{j+2}|^2\right)\\ \leq& K\sigma^3\tau^{2(k_j+1)}\mathrm{e}^{-2\sqrt{3}\tau}(1 + K\sigma^2) + K\sigma^3\tau^{2(k_j+1)}\mathrm{e}^{-2\sqrt{3}\tau}\left(2 + K^2\sigma^4 + 2K\sigma^2\right)\\ &+ K\sigma|c_{j+2}|^2\\ \leq& \sigma^2\tau^{2(k_j+1)}\mathrm{e}^{-2\sqrt{3}\tau} + |c_{j+2}|^2,\end{aligned}$$

if

$$K\sigma\left(3 + 3K\sigma + K^2\sigma 4\right) \leq 1. \tag{S14}$$

□

After that we have proved Proposition 2 through Propositions 3, 4 and 5 just considering the smallness of σ and the largeness of τ, explicitly defined in inequalities (S1)-(S14) and (Sh1*)-(Sh6*).

3.4.2 The regular problem

The second part of the problem is, as we have said, the easy one. In that part we are going to follow a known solution of the system for a fixed time T: the outgoing heteroclinic. This is also the last part of our inductive argument and it will include the gluing with the next step, that is, the change of coordinates to the ones that refer the dynamics to the motion of the periodic orbit $\mathcal{T}_{j+1}$. We shall prove that the action of the flow for a fixed time T on the last disk of the singular part, the outgoing disk d_j, will contain the incoming disk in the $(j+1)$-th step, D_{j+1}. We recall the difference between two consecutive incoming disks is that the second one has two dimensions less. Since we have lost two dimensions in the Shilnikov problem, the dimension of d_j and D_{j+1} are the same.

So, the main goal of this regular part of the problem is to prove the following Proposition:

Proposition 6. *Assume that $\epsilon_j \leq \epsilon_{j+1}$, $k_j \leq k_{j+1}$, σ is small enough and τ is large enough. More precisely, σ, τ, k_j, k_{j+1}, ϵ_j and ϵ_{j+1} satisfying* (R1)-(R6). *Then, the action of the flow for a certain time T over the outgoing disk, d_j, contains the incoming disk for the $(j+1)$-th step, D_{j+1}, for a concrete expression of the functions that define the incoming disk.*

3.4.2.1 The heteroclinic channel

The first ingredient for the proof of Proposition 6 is the study of the flow close to the outgoing heteroclinic. Recall that this heteroclinic has an explicit expression:

$$x_+^h(t) = \frac{1}{\sqrt{1 + \frac{1-\sigma^2}{\sigma^2}\mathrm{e}^{-2\sqrt{3}t}}},$$

and zero all the other components.

This solution starts at the macroscopic distance σ for $t = 0$ and, after a time

$$T = \frac{1}{\sqrt{3}} \ln \left(\frac{1-\sigma^2}{\sigma^2} \right), \tag{3.26}$$

it ends up at a point $x_+^h(T) = \sqrt{1-\sigma^2}$, that is, at a macroscopic distance of the following equilibrium point.

The goal of this part, is to show our outgoing disk flows close to this heteroclinic solution for that concrete and finite time T. Since this time is finite a very detailed argument is not needed as in the previous singular case.

The target disk in this part is the so-called final disk, d^T, that up to a change of coordinates, will correspond to the incoming disk for the $(j+1)$-th step, D_{j+1}.

Definition 12. The *final disk* is defined as

$$\begin{aligned} d_j^T = \Big\{ z \in \mathcal{S}_N^j \, : \, & c_- = m_T^{c_-}(c_+),\ x_- = m_T^{x_-}(c_+),\ y_- = m_T^{y_-}(c_+) \\ & x_+ = \sqrt{1-\sigma^2} + m_T^{x_+}(c_+),\ y_+ = m_T^{y_+}(c_+),\ |c_{+,k}| \le r_{c_{+,k}}^T \Big\}, \end{aligned}$$

where we assume for the past coordinates:

$$\begin{aligned} \left| m_T^{c_-,k}(c_+) \right| \le r_{c_{-,k}}^T &= \frac{1}{N-3} \sigma \tau^{k_{j+1}} \mathrm{e}^{-\sqrt{3}\tau} \\ |m_T^{x_-}(c_+)| \le r_{x_-}^T &= \frac{\sqrt{2}}{2} \frac{1}{N-3} \sigma \tau^{k_{j+1}} \mathrm{e}^{-\sqrt{3}\tau} \\ |m_T^{y_-}(c_+)| \le r_{y_-}^T &= \frac{\sqrt{2}}{2} \frac{1}{N-3} \sigma \tau^{k_{j+1}} \mathrm{e}^{-\sqrt{3}\tau} \\ |m_T^{x_+}(c_+)| \le r_{x_+}^T &= \sigma^2 \tau^{k_{j+1}} \mathrm{e}^{-\sqrt{3}\tau} \\ |m_T^{y_+}(c_+)| \le r_{y_+}^T &= \sigma^2 \tau^{k_{j+1}} \mathrm{e}^{-2\sqrt{3}\tau}, \end{aligned}$$

and for the free future coordinates:

$$\begin{aligned} r_{c_{+,j+2}}^T &= 2\sqrt{3}\sigma\tau \mathrm{e}^{-\sqrt{3}\tau} \\ r_{c_{+,k}}^T &= \frac{1}{N-3} \sigma \tau^{k_{j+1}} \mathrm{e}^{-\epsilon_{j+1}\sqrt{3}\tau}. \end{aligned}$$

The main result that we will use to shadow the heteroclinic is based in crude Gronwall's estimates and is the following:

Proposition 7. *There exist K_k^-, K_k^+, K_-^-, $K_+^{y_+}$, $K_-^{y_+}$, $K_{j+2}^{y_+}$, $K_{x_+}^{x_+}$, $K_{y_+}^{x_+}$, $K_-^{x_+}$ and $K_{c_l}^{x_+}$ depending only on σ such that:*

$$K_k^- |c_k(0)| \le |c_k(T)| \le K_k^+ |c_k(0)| \qquad ||(x_-, y_-)(T)||_\infty \le K_-^- ||(x_-, y_-)(0)||_\infty$$

$$|y_+(T)| \le K_+^{y_+} |y_+(0)| + K_-^{y_+} ||(x_-, y_-)(0)||_\infty^2 + K_{j+2}^{y_+} |c_{j+2}(0)|^2$$

$$\begin{aligned} |x_+(T) - x_+^h(T)| \le & K_{x_+}^{x_+} |x_+(0) - x_+^h(0)| + K_{y_+}^{x_+} |y_+(0)| \\ & + K_-^{x_+} ||(x_-, y_-)(0)||_\infty^2 + \sum_{l \in \mathcal{P}_j} K_{c_l}^{x_+} |c_l(0)|^2. \end{aligned}$$

Since we are not happy enough with the maximum size of y_+ of d_j we are going to restrict it a little bit. We could not do this restriction before because it will depend on the parameters from the above Proposition.

Definition 13. Let K_{j+2}^- given in Proposition 7. The *intermediate outgoing disk* is defined as

$$\begin{aligned} d_j^* = \Big\{ z \in \mathcal{S}_N^j \, : \, & c_- = m_\tau^{c_-}(c_+),\ x_- = m_\tau^{x_-}(c_+),\ y_- = m_\tau^{y_-}(c_+) \\ & x_+ = \sigma + m_\tau^{x_+}(c_+),\ y_+ = m_\tau^{y_+}(c_+),\ |c_{+,k}| \le r_{c_{+,k}}^\tau \Big\}, \end{aligned}$$

where we assume for the past coordinates:

$$\begin{aligned}
\left|m_\tau^{c_-,k}(c_+)\right| &\leq r^\tau_{c_-,k} = \frac{1}{N-3}\sigma\tau^{k_j+1}\mathrm{e}^{-\sqrt{3}\tau}\\
\left|m_\tau^{x_-}(c_+)\right| &\leq r^\tau_{x_-} = \sigma\tau^{k_j+2}\mathrm{e}^{-\sqrt{3}\tau}\\
\left|m_\tau^{y_-}(c_+)\right| &\leq r^\tau_{y_-} = \sigma\tau^{k_j+2}\mathrm{e}^{-\sqrt{3}\tau}\\
\left|m_\tau^{x_+}(c_+)\right| &\leq r^\tau_{x_+} = \sigma^2\mathrm{e}^{-\sqrt{3}\tau}\\
\left|m_\tau^{y_+}(c_+)\right| &\leq r^\tau_{y_+} = \sigma^2\tau^{2(k_j+1)}\mathrm{e}^{-2\sqrt{3}\tau} + |c_{j+2}|^2,
\end{aligned}$$

and for the free future coordinates:

$$\begin{aligned}
r^{\tau,*}_{c_+,j+2} &= \frac{2\sqrt{3}}{K^-_{j+2}}\sigma\tau\mathrm{e}^{-\sqrt{3}\tau}\\
r^\tau_{c_+,k} &= \frac{1}{N-3}\sigma(1-\sqrt{3}\sigma(1+\sigma))\tau^{k_j}\mathrm{e}^{-\epsilon_j\sqrt{3}\tau}.
\end{aligned}$$

Lemma 13. *If τ is large enough, or more precisely, τ satisfies* (R1)*, then the intermediate outgoing disk, d_j^* defined as is contained in the outgoing disk.*

Proof. If τ is large enough to satisfy:

$$\frac{2\sqrt{3}}{K^-_{j+2}}\sigma\tau\mathrm{e}^{-\sqrt{3}\tau} \leq \frac{1}{N-3}\sigma(1-\sqrt{3}\sigma(1+\sigma))\tau^{k_j}\mathrm{e}^{-\epsilon_j\sqrt{3}\tau}, \tag{R1}$$

then we will have $r^{\tau,*}_{c_+,j+2} \leq r^\tau_{c_+,j+2}$. □

With this restriction we can give an uniform bound for y_+ in d_j^*:

$$\begin{aligned}
\left|m_\tau^{y_+}(c_+)\right| &\leq \sigma^2\tau^{2(k_j+1)}\mathrm{e}^{-2\sqrt{3}\tau} + \frac{12}{\left(K^-_{j+2}\right)^2}\sigma^2\tau^2\mathrm{e}^{-2\sqrt{3}\tau}\\
&\leq \sigma^2\mathrm{e}^{-2\sqrt{3}\tau}\left(\tau^{2(k_j+1)} + \frac{12}{\left(K^-_{j+2}\right)^2}\tau^2\right).
\end{aligned}$$

With this correction we are ready to announce and prove the main result of this part:

Proposition 8. *Assume that $\epsilon_j \leq \epsilon_{j+1}$, $k_j \leq k_{j+1}$, σ is small enough and τ is large enough. More precisely, σ, τ, k_j, k_{j+1}, ϵ_j and ϵ_{j+1} satisfying* (R2)-(R6)*. Then, the action of the flow for the time T defined in* (3.26) *over the outgoing intermediate disk, d_j^* contains the final disk, d_j^T, for a concrete expression of the functions that define the final disk.*

Proof. We are going to use the bounds of Proposition 7.

For the center past coordinates, if we take:

$$K_k^+\tau^{k_j+1} \leq \tau^{k_{j+1}}, \tag{R2}$$

we will have:

$$\begin{aligned}|c_{-,k}(T)| \leq K_k^+|c_{-,k}(0)| \leq K_k^+ r^0_{c_-,k} = \frac{1}{N-3}K_k^+\sigma\tau^{k_j+1}\mathrm{e}^{-\sqrt{3}\tau} \\ \leq \frac{1}{N-3}\sigma\tau^{k_{j+1}}\mathrm{e}^{-\sqrt{3}\tau} = r^T_{c_-,k}.\end{aligned}$$

For the past hyperbolic coordinates, (x_-, y_-), if we take

$$K_-^-\tau^{k_j+2} \leq \frac{\sqrt{2}}{2}\frac{1}{N-3}\tau^{k_{j+1}/2} \tag{R3}$$

we will have:

$$\begin{aligned}|x_-(T)| \leq ||(x_-,y_-)(T)||_\infty \leq K_-^-||(x_-,y_-)(0)||_\infty \leq K_-^-\sigma\tau^{k_j+2}\mathrm{e}^{-\sqrt{3}\tau} \\ \leq \frac{\sqrt{2}}{2}\frac{1}{N-3}\sigma\tau^{k_{j+1}/2}\mathrm{e}^{-\sqrt{3}\tau} = r^T_{x_-}.\end{aligned}$$

The case for $y_-(T)$ is analogous.

For the hyperbolic future but fixed coordinates, x_+, y_+ we have:

$$\begin{aligned}|y_+(T)| \leq & K_+^{y+}|y_+(0)| + K_-^{y+}|(x_-,y_-)(0)|^2 + K_{j+2}^{y+}|c_{j+2}(0)|^2 \\ \leq & K_+^{y+}\sigma^2\mathrm{e}^{-2\sqrt{3}\tau}\left(\tau^{2(k_j+1)} + \frac{12}{\left(K_{j+2}^-\right)^2}\tau^2\right) \\ & + K_-^{y+}\sigma^2\tau^{2(k_j+2)}\mathrm{e}^{-2\sqrt{3}\tau} + K_{j+2}^{y+}\frac{12}{\left(K_{j+2}^-\right)^2}\sigma^2\tau^2\mathrm{e}^{-2\sqrt{3}\tau} \\ \leq & \sigma^2\tau^{k_{j+1}}\mathrm{e}^{-2\sqrt{3}\tau},\end{aligned}$$

if we take k_{j+1} large enough to satisfy:

$$K_+^{y+}\left(\tau^{2(k_j+1)} + \frac{12}{\left(K_{j+2}^-\right)^2}\tau^2\right) + K_-^{y+}\tau^{2(k_j+2)} + K_{j+2}^{y+}\frac{12}{\left(K_{j+2}^-\right)^2}\tau^2 \leq \tau^{k_{j+1}}. \tag{R4}$$

$$\begin{aligned}
|x_+(T) - \sqrt{1-\sigma^2}| =& |x_+(T) - x_+^h(T)| \\
\leq& K_{x_+}^{x_+}|x_+(0) - x_+^h(0)| + K_{y_+}^{x_+}|y_+(0)| + K_-^{x_+}|(x_-,y_-)(0)|^2 \\
&+ \sum_{l\in\mathcal{P}_j} K_{c_l}^{x_+}|c_l(0)|^2 \\
\leq& K_{x_+}^{x_+}\sigma^2 e^{-\sqrt{3}\tau} \\
&+ K_{y_+}^{x_+}\sigma^2 e^{-2\sqrt{3}\tau}\left(\tau^{2(k_j+1)} + \frac{12}{\left(K_{j+2}^-\right)^2}\tau^2\right) \\
&+ K_-^{x_+}\sigma^2\tau^{2(k_j+2)}e^{-2\sqrt{3}\tau} \\
&+ \sum_{l\in\mathcal{P}_j} K_{c_l}^{x_+}\sigma^2 e^{-\sqrt{3}\tau} \\
\leq& \sigma^2\tau^{k_{j+1}}e^{-\sqrt{3}\tau},
\end{aligned}$$

if τ and k_{j+1} are large enough to satisfy:

$$K_{x_+}^{x_+} + K_{y_+}^{x_+}e^{-\sqrt{3}\tau}\left(\tau^{2(k_j+1)} + \frac{12}{\left(K_{j+2}^-\right)^2}\tau^2\right) + K_-^{x_+}\tau^{2(k_j+2)}e^{-\sqrt{3}\tau} + \sum_{l\in\mathcal{P}_j} K_{c_l}^{x_+} \leq \tau^{k_{j+1}}. \tag{R5}$$

For the center future coordinates, $c_{+,k}$ with $k > j+2$ we have to check that after the heteroclinic our domain covers the domain defined in the final disk. On the one hand, if we take $\epsilon_{j+1} > \epsilon_j$ we will have for τ large enough:

$$\tau^{k_{j+1}}e^{-\epsilon_{j+1}\sqrt{3}\tau} \leq K_k^-\left(1 - \sqrt{3}\sigma(1+\sigma)\right)\tau^{k_j}e^{-\epsilon_j\sqrt{3}\tau}. \tag{R6}$$

Then, for $|c_{+,k}(0)| = r_{c_+,k}^{\tau}$ we will have

$$\begin{aligned}
|c_{+,k}(T)| \geq K_k^-|c_{+,k}(0)| = K_k^- r_{c_+,k}^{\tau} &= \frac{1}{N-3}K_k^-\left(1 - \sqrt{3}\sigma(1+\sigma)\right)\sigma\tau^{k_j}e^{-\epsilon_j\sqrt{3}\tau} \\
&\geq \frac{1}{N-3}\sigma\tau^{k_{j+1}}e^{-\epsilon_{j+1}\sqrt{3}\tau} = r_{c_+,k}^{T}.
\end{aligned}$$

For $k = j+2$, we take $|c_{j+2}(0)| = r_{c_+,j+2}^{\tau,*}$ and then we have:

$$|c_{j+2}(T)| \geq K_{j+2}^-|c_+(0)| = K_{j+2}^- r_{c_+,j+2}^{\tau,*} = 2\sqrt{3}\sigma\tau e^{-\sqrt{3}\tau} = r_{c_+,j+2}^{T}.$$

□

3.4.2.2 From coordinates referred to $\mathcal{T}_j$ to coordinates referred to $\mathcal{T}_{j+1}$

This is the very last step to prove Proposition 6. As we have said we have already a disk located in the desired place, close to $\mathcal{T}_{j+1}$. The only difference is that is not expressed in the correct coordinates. The change of coordinates that we need to perform is defined in Lemma 7. However, we will only need the following estimates:

Lemma 14. *The change of coordinates defined in Lemma 7 satisfies:*

- $|\tilde{c}_k| = |c_k|$ *for* $k \in \{1,\dots,j-2, j+3,\dots,N\}$
- $|\tilde{c}_{j-1}| \leq \sqrt{2}||(x_-, y_-)||_\infty$
- $||(\tilde{x}_+, \tilde{y}_+)||_\infty \geq \frac{\sqrt{3}}{3}|c_{j+2}|$

Lemma 15. *Assume* $x_+ = \sqrt{1-\sigma^2} + m_x$ *and* $y_+ = m_y$*. Then:*

$$\tilde{r}^2 \geq 1 - \sigma^2 + \sqrt{1-\sigma^2}(2m_x - m_y) \qquad r^2 \leq \sigma^2 - \sqrt{1-\sigma^2}(2m_x - m_y)$$

and

$$|\tilde{x}_-| \leq \frac{\sqrt{\sigma^2 - \sqrt{1-\sigma^2}(2m_x - m_y)}}{\sqrt{1-\sigma^2 + \sqrt{1-\sigma^2}(2m_x - my)}}|m_y|$$

$$|\tilde{y}_- - \sigma| \leq \frac{\sqrt{\sigma^2 - \sqrt{1-\sigma^2}(2m_x - m_y)}}{\sqrt{1-\sigma^2 + \sqrt{1-\sigma^2}(2m_x - m_y)}}|m_x| \\ \frac{\sqrt{\sigma^2 - \sqrt{1-\sigma^2}(2m_x - m_y)}}{\sqrt{1-\sigma^2 + \sqrt{1-\sigma^2}(2m_x - m_y)}}\sqrt{1-\sigma^2} - \sigma.$$

Again the proof of these Lemmas can be found in Appendix C.

Proposition 9. *Assume that* σ *is small enough and* τ *is large enough. More precisely,* σ*,* τ*, satisfying* (R7) *and* (R8)*. The image of the final disk* d_j^T *through the change of coordinates defined in Lemma 7 contains the new incoming disk* D_{j+1}*, for some concrete expression for the functions that define the incoming disk.*

Proof. Notice that all the center coordinates c_k for $k \in \{1,\dots,j-2, j+3,\dots,N\}$ are transformed by a rotation. Then, we can take the same sizes before and after the change:

$$\begin{aligned} r^T_{c_-,k} &= \tilde{r}^0_{c_-,k} \text{ for } k \in \{1,\dots,j-2\} \\ r^T_{c_+,k} &= \tilde{r}^0_{c_+,k} \text{ for } k \in \{j+3,\dots,N\}, \end{aligned}$$

and this is what we have done.

After this change we obtain a new central past coordinate, $\tilde{c}_{j-1}$ that corresponds to the previous hyperbolic past mode: (x_-, y_-). Through the corresponding bound of Lemma 14, we have:

$$|\tilde{c}_{j-1}| \leq \sqrt{2}||(x_-, y_-)||_\infty \leq \sqrt{2}\frac{\sqrt{2}}{2}\frac{1}{N-3}\sigma\tau^{k_{j+1}}\mathrm{e}^{-\sqrt{3}\tau} = \frac{1}{N-3}\sigma\tau^{k_{j+1}}\mathrm{e}^{-\sqrt{3}\tau} = \tilde{r}^0_{c_-,j-1}.$$

From the future mode c_{j+2}, we obtain the new future hyperbolic coordinates, $\tilde{x}_+$ and $\tilde{y}_+$. Take $|c_{j+2}| = r^T_{c_+,j+2}$. Then, from Lemma 14, we have:

$$\begin{aligned} ||(\tilde{x}_+, \tilde{y}_+)||_\infty &\geq \frac{\sqrt{3}}{3}|c_{j+2}| = \frac{\sqrt{3}}{3} r^T_{c_+,j+2} \\ &= 2\sigma\tau\mathrm{e}^{-\sqrt{3}\tau} = \max\left(\tilde{r}^0_{x_+}, \tilde{r}^0_{y_+}\right) \end{aligned}$$

so, the new hyperbolic future coordinates are covered by the image of the final disk.

Taking now into account the values of $r^T_{x_+}$ and $r^T_{y_+}$ we can apply Lemma 15 knowing that $|m_x|, |m_y| \leq \sigma^2\tau^{k_{j+1}}\mathrm{e}^{-\sqrt{3}\tau}$. For the first equation, we can take:

$$|2m_x + my| \leq 3\sigma^2\tau^{k_{j+1}}\mathrm{e}^{-\sqrt{3}\tau} \leq \sigma^2,$$

and then

$$\begin{aligned} |\tilde{x}_-| &\leq \frac{\sqrt{\sigma^2 - \sqrt{1-\sigma^2}(2m_x - m_y)}}{\sqrt{1-\sigma^2 + \sqrt{1-\sigma^2}(2m_x - my)}} r^T_{y_+} \\ &\leq \frac{\sqrt{\sigma^2 + \sigma^2\sqrt{1-\sigma^2}}}{\sqrt{1-\sigma^2 - \sigma^2\sqrt{1-\sigma^2}}} r^T_{y_+} \\ &\leq \sigma\frac{\sqrt{1 + \sqrt{1-\sigma^2}}}{\sqrt{1-\sigma^2 - \sigma^2\sqrt{1-\sigma^2}}} r^T_{y_+} \\ &\leq 2\sigma r^T_{y_+}. \end{aligned}$$

So we obtain the desired bound if

$$2\sigma \leq 1. \tag{R7}$$

We have to be more careful with the expression for $\tilde{y}_-$. Using the previous bound we have:

$$\begin{aligned} |\tilde{y}_- - \sigma| \leq& 2\sigma r^T_{x_+} \\ & \frac{\sqrt{\sigma^2 - \sqrt{1-\sigma^2}(2m_x - m_y)}}{\sqrt{1-\sigma^2 + \sqrt{1-\sigma^2}(2m_x - m_y)}}\sqrt{1-\sigma^2} - \sigma. \end{aligned}$$

Let $v = 2m_x - m_y$. Then

$$\frac{\sqrt{\sigma^2 - \sqrt{1-\sigma^2}v}}{\sqrt{1-\sigma^2 + \sqrt{1-\sigma^2}v}}\sqrt{1-\sigma^2} - \sigma = \frac{\sigma\sqrt{1 - \frac{\sqrt{1-\sigma^2}}{\sigma^2}v}}{\sqrt{1-\sigma^2}\sqrt{1 + \frac{\sqrt{1-\sigma^2}}{1-\sigma^2}v}}\sqrt{1-\sigma^2} - \sigma.$$

Write now $\sigma^2 u = \sqrt{1-\sigma^1} v$. Then the previous expression becomes:

$$\frac{\sigma\sqrt{1-u}}{\sqrt{1+\frac{\sigma^2}{1-\sigma^2}u}} - \sigma = \frac{\sigma}{\sqrt{1+\frac{\sigma^2}{1-\sigma^2}u}}\left(\sqrt{1-u} - \sqrt{1+\frac{\sigma^2}{1-\sigma^2}u}\right)$$
$$\frac{\sigma}{\sqrt{1+\frac{\sigma^2}{1-\sigma^2}u}} \frac{1}{\sqrt{1-u}+\sqrt{1+\frac{\sigma^2}{1-\sigma^2}u}}\left(\frac{-1}{1-\sigma^2}u\right).$$

In our case, we can conclude, then, that:

$$\begin{aligned}|\tilde{y}_- - \sigma| \leq & 2\sigma r^T_{x+} + \frac{1}{\sigma\sqrt{1-\sigma^2}}\left(2r^T_{x+} + r^T_{y+}\right) \\ \leq & 2\sigma\sigma^2\tau^{k_{j+1}}\mathrm{e}^{-\sqrt{3}\tau} + \frac{1}{\sigma\sqrt{1-\sigma^2}} 3\sigma^2\tau^{k_{j+1}}\mathrm{e}^{-\sqrt{3}\tau} \\ \leq & \sigma^2,\end{aligned}$$

if τ is large enough to fulfill

$$\frac{3}{\sqrt{1-\sigma^2}}\tau^{k_{j+1}}\mathrm{e}^{-\sqrt{3}\tau} \leq \frac{1}{2}\sigma. \tag{R8}$$

□

Now we have finished the proof of Proposition 6 through Lemma 13, and Proposition 8 and 9 just assuming the smallness of σ, the largeness of τ and the correct relation between k_j and k_{j+1} and ϵ_j and ϵ_{j+1} given in (R1)-(R8).

3.4.3 Proof of Proposition 1 and Theorem 8

We have proved Proposition 1 as a consequence of applying Proposition 2 and Proposition 6. We still have to prove Theorem 8.

To do so, consider an increasing sequence of real numbers $\{k_j\}_j$ for $j = 3 \ldots N-2$. This is a sequence satisfying $k_j \leq k_{j+1}$. Consider also an increasing sequence of real numbers $\{\epsilon_j\}_j$ for $j = 3 \ldots N-2$ such that $1/2 < \epsilon_j \leq \epsilon_{j+1} < 1$. Let $\tau > 0$ be large enough and $\sigma > 0$ small enough such that conditions (S1)-(S14), (Sh1*)-(Sh6*) and (R1)-(R8) hold, for any tuple $(\sigma, \tau, k_j, \epsilon_j)$, for $j = 3 \ldots N-2$.

We can now apply Proposition 1 recursively. This means that the first incoming disk, D_3, contains a subdisk such that after the action of the flow for a time $(N-5)(\tau+T)$ contains the last incoming disk D_{N-2}.

Both the subdisk and the final disk have dimension 4, since we still have a free future in D_{N-2}, the hyperbolic part (that corresponds to the mode b_{N-1}) and the central part (that corresponds to the mode b_N). Since we will not need to continue with our argument, we can set them at zero in the whole discussion. That means, in particular, in the subdisk of D_3 and in D_{N-2}. After this reduction we have obtained two points: p_3 in D_3 and p_{N-2} in D_{N-2} connected by the flow for a time $T^* = (N-5)(\tau + T)$.

To end the proof we should compute the norm of these points in the original and global coordinates, check the statement of Theorem 8 and define δ in terms of σ. However, we are going to compute the norm in the original coordinates of any point in the disks D_3 and D_{N-2}.

Assume that p_3 and p_{N-2} can be written in coordinates as:

$$\begin{aligned} p_3 &= (c_-, x_-, y_-, x_+, y_+, c_+) \\ p_{N-2} &= (\tilde{c}_-, \tilde{x}_-, \tilde{y}_-, \tilde{x}_+, \tilde{y}_+, \tilde{c}_+). \end{aligned}$$

We know that the largest variable in both cases are y_- and $\tilde{y}_-$ and satisfy

$$\begin{aligned} |y_- - \sigma| &\leq \frac{1}{2}\sigma^2 \\ |\tilde{y}_- - \sigma| &\leq \frac{1}{2}\sigma^2. \end{aligned}$$

Due to condition (Sh4*) we know that the rest of the hyperbolic coordinates are bounded by $\frac{1}{2\sqrt{3}}\sigma$ while the center by $\frac{1}{N-3}\frac{1}{2\sqrt{3}}\sigma$ in both disks.

For D_3 we have:

$$\begin{aligned} |b_2|^2 &= x_-^2 - x_- y_- + y_-^2 \\ |b_4|^2 &= x_+^2 - x_+ y_+ + y_+^2 \\ |b_k|^2 &= |c_k| \text{ for } k \in \mathcal{P}_3 \\ |b_3|^2 &= 1 - (x_-^2 - x_- y_- + y_-^2) - (x_+^2 - x_+ y_+ + y_+^2) - \sum_{l \in \mathcal{P}_3} |c_l|^2, \end{aligned}$$

where we have used the conservation of the mass for the last equality. Taking into account the remark above, we have the estimates:

$$\begin{aligned} |b_2|^2 &\leq \sigma^2 \left(1 + \frac{5}{32} + \frac{1}{\sqrt{3}}\right) \leq \frac{7}{4}\sigma^2 \\ |b_4|^2 &\leq \frac{1}{4}\sigma^2 \\ |b_k|^2 &\leq \frac{1}{(N-3)^2}\frac{1}{12}\sigma^2 \text{ for } k \in \mathcal{P}_3 \\ |b_3|^2 &\geq 1 - \frac{7}{4}\sigma^2 - \frac{1}{4}\sigma^2 - \frac{1}{(N-3)}\frac{1}{12}\sigma^2 \geq 1 - 3\sigma^2. \end{aligned}$$

So, given δ, take $\sigma = \sqrt{3}\delta$. Then we will have:

$$
\begin{aligned}
|b_2| &\le \frac{\sqrt{7}}{2}\sigma = \frac{\sqrt{7}}{2\sqrt{3}}\delta \le \delta \\
|b_4| &\le \frac{1}{2}\sigma = \frac{1}{2\sqrt{3}}\delta \le \delta \\
|b_k| &\le \frac{1}{(N-3)}\frac{1}{2\sqrt{3}}\sigma = \frac{1}{(N-3)}\frac{1}{6}\delta \le \delta \text{ for } k \in \mathcal{P}_3 \\
|b_3| &\ge \sqrt{1-3\sigma^2} \ge 1 - \sqrt{3}\sigma = 1 - \delta.
\end{aligned}
$$

The argument for a point in D_{N-2} is analogous since we are considering that (Sh4*) holds for each step.

To finish the proof, notice that keeping σ and both sequences, $\{k_j\}_j$ and $\{\epsilon_j\}_j$, fixed and increasing τ we can still apply this result. That means that we have a minimal time that ensures the visit of $\mathcal{T}_{N-2}$ starting at $\mathcal{T}_3$. If we want to flow closer to the heteroclinic chain, we can increase the time τ and obtain a different pair on points p_3 and p_{N-2}, so Theorem 8 is proven.

Remark. Final comments, future work and possible open questions:

- Notice that all the conditions (S1)-(S14), (Sh1*)-(Sh6*) and (R1)-(R8) hold for any time τ large enough. This means that, as we have said, we can obtain a different pair of points connected through an orbit, improving, then, Theorem 2. However this unboundedness of τ forced us to obtain sharper bounds in the whole discussion and to perform a quasi-normal form change. We think that, if we allowed an upper bound for the time τ (that would depend on σ) we could possibly work without such precise tools.

- During the flow close to the heteroclinic we have used very crude estimates. If we improve them we will not need such a large diffusing time.

- The most immediate work that we plan to do is to rewrite the whole proofs using h-sets instead of disks.

- As we have said after the proof of Theorem 6, we could think wether the shadowing of an infinite sequence of periodic orbits is feasible. That does not mean unbounded growth of the Sobolev norm in (1.1) but it will have its own particular interest.

A h-sets, covering relations, cone conditions and....

A.1 h-sets, covering relations

The goal of this section is present the notions of h-sets and covering relations, and to state the theorem about the existence of point realizing the chain of covering relations.

A.1.1 h-sets and covering relations

Definition 14. [ZG04, Definition 1] An h-set, N, is a quadruple $(|N|, u(N), s(N), c_N)$ such that

- $|N|$ is a compact subset of $\mathbb{R}^n$
- $u(N), s(N) \in \{0, 1, 2, \dots\}$ are such that $u(N) + s(N) = n$
- $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ is a homeomorphism such that

$$c_N(|N|) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}.$$

We set

$$\begin{aligned}
\dim(N) &:= n, \\
N_c &:= \overline{B_{u(N)}} \times \overline{B_{s(N)}}, \\
N_c^- &:= \partial B_{u(N)} \times \overline{B_{s(N)}}, \\
N_c^+ &:= \overline{B_{u(N)}} \times \partial B_{s(N)}, \\
N^- &:= c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+).
\end{aligned}$$

Hence an h-set, N, is a product of two closed balls in some coordinate system. The numbers $u(N)$ and $s(N)$ are called the nominally unstable and nominally stable dimensions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. Observe that if $u(N)=0$, then $N^-=\emptyset$ and if $s(N)=0$, then $N^+=\emptyset$. In the sequel to make notation less cumbersome we will often drop the bars in the symbol $|N|$ and we will use N to denote both the h-sets and its support.

Sometimes we will call N^- *the exit set of* N and N^+ *the entry set of* N. These name are motivated by the Conley index theory and the role these sets will play in the context of covering relations.

Definition 15. [ZG04, Definition 3] Let N be a h-set. We define a h-set N^T as follows

- $|N^T|=|N|$
- $u(N^T)=s(N)$, $s(N^T)=u(N)$
- We define a homeomorphism $c_{N^T}:\mathbb{R}^n\to\mathbb{R}^n=\mathbb{R}^{u(N^T)}\times\mathbb{R}^{s(N^T)}$, by
$$c_{N^T}(x)=j(c_N(x)),$$
where $j:\mathbb{R}^{u(N)}\times\mathbb{R}^{s(N)}\to\mathbb{R}^{s(N)}\times\mathbb{R}^{u(N)}$ is given by $j(p,q)=(q,p)$.

□

Observe that $N^{T,+}=N^-$ and $N^{T,-}=N^+$. This operation is useful in the context of inverse maps.

Definition 16. [ZG04, Definition 6] Assume that N,M are h-sets, such that $u(N)=u(M)=u$ and $s(N)=s(M)=s$. Let $f:N\to\mathbb{R}^n$ be a continuous map. Let $f_c=c_M\circ f\circ c_N^{-1}:N_c\to\mathbb{R}^u\times\mathbb{R}^s$. Let w be a nonzero integer. We say that

$$N\overset{f,w}{\Longrightarrow}M$$

(N f-covers M with degree w) iff the following conditions are satisfied

1. there exists a continuous homotopy $h:[0,1]\times N_c\to\mathbb{R}^u\times\mathbb{R}^s$, such that the following conditions hold true

$$h_0 = f_c, \tag{A.1}$$
$$h([0,1],N_c^-)\cap M_c = \emptyset, \tag{A.2}$$
$$h([0,1],N_c)\cap M_c^+ = \emptyset. \tag{A.3}$$

2. If $u>0$, then there exists a map $A:\mathbb{R}^u\to\mathbb{R}^u$, such that

$$h_1(p,q) = (A(p),0), \text{ for } p\in\overline{B_u}(0,1) \text{ and } q\in\overline{B_s}(0,1), \tag{A.4}$$
$$A(\partial B_u(0,1)) \subset \mathbb{R}^u\setminus\overline{B_u}(0,1). \tag{A.5}$$

Moreover, we require that

$$\deg(A,\overline{B_u}(0,1),0)=w,$$

We will call condition (A.2) *the exit condition* and condition (A.3) will be called *the entry condition.*

Note that in the case $u = 0$, if $N \overset{f,w}{\Longrightarrow} M$, then $f(N) \subset \operatorname{int} M$ and $w = 1$.

In fact in the above definition $s(N)$ and $s(M)$ can be different, see [Wil06, Def. 2.2].

Remark. Observe, that since for any norm in $\mathbb{R}^n$ the closed unit ball is homeomorphic to $[-1,1]^n$, therefore for h-sets and covering relations we will use different norms in different contexts.

Remark. If the map A in condition 2 of Def. 16 is a linear map, then condition (A.5) implies, that

$$\deg(A, \overline{B_u}(0,1), 0) = \pm 1.$$

Hence condition (16) is in this situation automatically fulfilled with $w = \pm 1$.

In fact, this is the most common situation in the applications of covering relations.

Most of the time we will not interested in the value of w in the symbol $N \overset{f,w}{\Longrightarrow} M$ and we will often drop it and write $N \overset{f}{\Longrightarrow} M$, instead. Sometimes we may even drop the symbol f and write $N \Longrightarrow M$.

Definition 17. [ZG04, Definition 7] Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $g : \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$. Assume that $g^{-1} : |M| \to \mathbb{R}^n$ is well defined and continuous. We say that $N \overset{g}{\Longleftarrow} M$ (N g-backcovers M) iff $M^T \overset{g^{-1}}{\Longrightarrow} N^T$.

A.1.2 Main theorem about chains of covering relations

Theorem 10 (Thm. 9)**.** *[ZG04] Assume N_i, $i = 0, \ldots, k$, $N_k = N_0$ are h-sets and for each $i = 1, \ldots, k$ we have either*

$$N_{i-1} \overset{f_i, w_i}{\Longrightarrow} N_i$$

or

$$N_i \subset dom(f_i^{-1}) \quad and \quad N_{i-1} \overset{f_i, w_i}{\Longleftarrow} N_i.$$

Then there exists a point $x \in \operatorname{int} N_0$, such that

$$\begin{aligned} f_i \circ f_{i-1} \circ \cdots \circ f_1(x) &\in \operatorname{int} N_i, \quad i = 1, \ldots, k \\ f_k \circ f_{k-1} \circ \cdots \circ f_1(x) &= x \end{aligned}$$

We point the reader to [ZG04] for the proof. The basic idea of the proof of this theorem - the homotopy and the local Brouwer degree - appears in the proof Theorem 12.

The following corollary is an immediate consequence of Theorem 10.

Collorary 11. Let N_i, $i \in \mathbb{Z}_+$ be h-sets. Assume that for each $i \in \mathbb{Z}_+$ we have either

$$N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i$$

or

$$N_i \subset \operatorname{dom}(f_i^{-1}) \quad \text{and} \quad N_{i-1} \overset{f_i,w_i}{\Longleftarrow} N_i.$$

Then there exists a point $x \in \operatorname{int} N_0$, such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \quad \in \quad \operatorname{int} N_i, \quad i \in \mathbb{Z}_+.$$

Moreover, if $N_{i+k} = N_i$ for some $k > 0$ and all i, then the point x can be chosen so that

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x.$$

A.1.3 Natural structure of h-set

Observe that all the conditions appearing in the definition of the covering relation are expressed in 'internal' coordinates c_N and c_M. Also the homotopy is defined in terms of these coordinates. This sometimes makes the matter and the notation look a bit cumbersome. With this in mind we introduce the notion of a 'natural' structure on h-set.

Definition 18. We will say that $N = \{(x_0, y_0)\} + \overline{B}_u(0, r_1) \times \overline{B}_s(0, r_1) \subset \mathbb{R}^u \times \mathbb{R}^s$ is an *h-set* with a natural structure given by :
$u(N) = u$, $s(N) = s$, $c_N(x, y) = \left(\frac{x-x_0}{r_1}, \frac{y-y_0}{r_2}\right)$.

In context of $\mathbb{R}^2$ and $u = 1$, $s = 1$ we will sometimes write $N = z_0 + [-a, a] \times [-b, b]$. This in compatible with the above convention as a defines radius of ball $\overline{B}_u(0, a) = [-a, a]$ and b of $\overline{B}_s(0, b) = [-b, b]$.

A.1.4 Horizontal and vertical disks in an h-set

Definition 19. [WZ07, Definition 10] Let N be an h-set. Let $b : \overline{B_{u(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is *a horizontal disk in* N if there exists a homotopy $h : [0,1] \times \overline{B_{u(N)}} \to N_c$, such that

$$\begin{array}{rcll} h_0 & = & b_c & \\ h_1(x) & = & (x,0), & \text{for all } x \in \overline{B_{u(N)}} \\ h(t,x) & \in & N_c^-, & \text{for all } t \in [0,1] \text{ and } x \in \partial B_{u(N)} \end{array}$$

Definition 20. [WZ07, Definition 11] Let N be an h-set. Let $b : \overline{B_{s(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is *a vertical disk in* N if there exists a homotopy $h : [0,1] \times \overline{B_{s(N)}} \to N_c$, such that

$$\begin{array}{rcll} h_0 & = & b_c & \\ h_1(x) & = & (0,x), & \text{for all } x \in \overline{B_{s(N)}} \\ h(t,x) & \in & N_c^+, & \text{for all } t \in [0,1] \text{ and } x \in \partial B_{s(N)}. \end{array}$$

Definition 21. Let N be an h-set in $\mathbb{R}^n$ and b be a horizontal (vertical) disk in N. We will say that $x \in \mathbb{R}^n$ *belongs to* b, when $b(z) = x$ for some $z \in \operatorname{dom}(b)$.

By $|b|$ we will denote the image of b. Hence $z \in |b|$ iff z belongs to b.

A.1.5 Topological transversality theorem

Now we are ready to state the topological transversality theorem. A simplified version of this theorem was given in [Wil03] for the case of one unstable direction and covering relations chain without backcoverings. The argument in [Wil03], which was quite simple and was based on the connectivity only, cannot be carried over to a larger number of unstable directions or to the situation when both covering and backcovering relations are present.

Theorem 12. *[WZ07, Thm. 4] Let $k \geq 1$. Assume N_i, $i = 0, \ldots, k$, are h-sets and for each $i = 1, \ldots, k$ we have either*

$$N_{i-1} \overset{f_i, w_i}{\Longrightarrow} N_i$$

or

$$N_i \subset dom(f_i^{-1}) \quad and \quad N_{i-1} \overset{f_i, w_i}{\Longleftarrow} N_i.$$

Assume that b_0 is a horizontal disk in N_0 and b_e is a vertical disk in N_k.

Then there exists a point $x \in \operatorname{int} N_0$, such that

$$\begin{array}{rcll} x & = & b_0(t), & \textit{for some } t \in B_{u(N_0)}(0,1) \\ f_i \circ f_{i-1} \circ \cdots \circ f_1(x) & \in & \operatorname{int} N_i, & i = 1, \ldots, k \\ f_k \circ f_{k-1} \circ \cdots \circ f_1(x) & = & b_e(z), & \textit{for some } z \in B_{s(N_k)}(0,1) \end{array}$$

A.2 h-sets and cone conditions

The goal of this chapter is to introduce a method, which will allow to handle relatively easily the hyperbolic structure on h-sets. This material appeared first in [KWZ07, Zgl09].

Definition 22. Let $N \subset \mathbb{R}^n$ be an h-set and $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form

$$Q((x, y)) = \alpha(x) - \beta(y), \qquad (x, y) \in \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)},$$

where $\alpha : \mathbb{R}^{u(N)} \to \mathbb{R}$, and $\beta : \mathbb{R}^{s(N)} \to \mathbb{R}$ are positive definite quadratic forms.

The pair (N, Q) we be called an *h-set with cones.*

We will refer to the quadratic forms α and β as positive and negative parts of Q, respectively.

If (N, Q) is an h-set with cones, then we define a function $L_N : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$L_N(z_1, z_2) = Q(c_N(z_1) - c_N(z_2))$$

Quite often we will drop Q in the symbol (N, Q) and we will say that N is an h-set with cones.

A.2.1 Cone conditions for horizontal and vertical disks

Definition 23. Let (N, Q) be a h-set with cones.

Let $b : \overline{B_u} \to |N|$ be a horizontal disk.

We will say that b *satisfies the cone condition (with respect to Q)* iff for any $x_1, x_2 \in \overline{B_u}$, $x_1 \neq x_2$ holds

$$Q(b_c(x_1) - b_c(x_2)) > 0.$$

Definition 24. Let (N, Q) be a h-set with cones.

Let $b : \overline{B_s} \to |N|$ be a vertical disk.

We will say that b *satisfies the cone condition (with respect to Q)* iff for any $y_1, y_2 \in \overline{B_s}$, $y_1 \neq y_2$ holds

$$Q(b_c(y_1) - b_c(y_2)) < 0.$$

Lemma 16. *Let (N, Q) be a h-set with cones and let $b : \overline{B_u} \to |N|$ be a horizontal disk satisfying the cone condition.*

Then there exists a Lipschitz function $y : \overline{B_u} \to \overline{B_s}$ *such that*

$$b_c(x) = (x, y(x)).$$

Analogously, if $b : \overline{B_s} \to |N|$ *is a vertical disk satisfying the cone condition, then there exists a Lipschitz function* $x : \overline{B_s} \to \overline{B_u}$

$$b_c(y) = (x(y), y)).$$

Lemma 17. *Let* (N, Q) *be a h-set with cones and let* b_u *and* b_s *be horizontal and vertical disks satisfying cone conditions, respectively. Then* b_u *and* b_s *intersect in a single point.*

A.2.2 Cone conditions for maps

Definition 25. Assume that $(N, Q_N), (M, Q_M)$ are h-sets with cones, such that $u(N) = u(M) = u$ and let $f : N \to \mathbb{R}^{\dim(M)}$ be continuous. Assume that $N \overset{f}{\Longrightarrow} M$. We say that f satisfies the cone condition (with respect to the pair (N, M)) iff for any $x_1, x_2 \in N_c$, $x_1 \neq x_2$ holds

$$Q_M(f_c(x_1) - f_c(x_2)) > Q_N(x_1 - x_2).$$

Definition 26. Assume that $(N, Q_N), (M, Q_M)$ are h-sets with cones, such that $u(N) = u(M) = u$ and $s = s(N) = s(M)$ and let $f : N \to \mathbb{R}^{u+s}$ be continuous. Assume that $N \overset{f}{\Longleftarrow} M$. We say that f satisfies the cone condition (with respect to the pair $((N, Q_N), (M, Q_M))$) iff for any $y_1, y_2 \in M_c$, $y_1 \neq y_2$ holds

$$Q_M(y_1 - y_2) > Q_N(f_c^{-1}(y_1) - f^{-1}(y_2)).$$

Observe that Definition 26 is equivalent to Definition 25 applied to map f^{-1} with respect to pair $(M^T, -Q_M), (N^T, -Q_N))$.

The cone condition in Definition 25 is expressed in coordinates associated to h-sets, in the phase space it implies that

$$L_M(f(z_1), f(z_2)) > L_N(z_1, z_2), \qquad \text{for } z_1 \neq z_2,\ z_1, z_2 \in N. \tag{A.6}$$

Below we state two basic theorems relating covering relations and the cone conditions

Theorem 13. *Assume that for* $i = 0, \ldots, k-1$ *either*

$$N_i \overset{f_i}{\Longrightarrow} N_{i+1}$$

or

$$N_{i+1} \subset dom\,(f_i^{-1}) \quad and \quad N_i \overset{f_i}{\Longleftarrow} N_{i+1},$$

where all h-sets are h-sets with cones and f_i for $i = 0, \ldots, k-1$ satisfies the cone condition.

Assume that $b : \overline{B}_{s(N_k)} \to N_k$ is a vertical disk in N_k satisfying the cone condition.

Then the set of points $z \in N_0$ satisfying the following two conditions

$$\begin{aligned} f_{i-1} \circ f_{i-2} \circ \cdots \circ f_0(z) &\in N_i, \qquad for\ i = 1, \ldots, k \\ f_{k-1} \circ \cdots \circ f_0(z) &\in |b| \end{aligned}$$

is a vertical disk satisfying the cone condition.

Theorem 14. *Assume that for $i = 0, \ldots, k-1$ either*

$$N_i \overset{f_i}{\Longrightarrow} N_{i+1}$$

or

$$N_{i+1} \subset dom\,(f_i^{-1}) \quad and \quad N_i \overset{f_i}{\Longleftarrow} N_{i+1},$$

where all h-sets are h-sets with cones and f_i for $i = 0, \ldots, k-1$ satisfies the cone condition.

Assume that $b : \overline{B}_{n(N_0)} \to N_0$ is a horizontal disk in N_0 satisfying the cone condition.

Then exists a set $Z \subset |b|$, such that for all $z \in Z$ holds

$$f_{i-1} \circ f_{i-2} \circ \cdots \circ f_0(z) \in N_i, \qquad for\ i = 1, \ldots, k$$

and $f_{k-1} \circ f_{i-2} \circ \cdots \circ f_0(Z)$ a horizontal disk in N_k satisfying the cone condition.

B Proofs of results of Chapter 2

B.1 Proof of Lemma 1

Initial System:

$$H(b,\bar{b}) = \sum_{j=1}^{N}\left(\frac{1}{4}b_j^2\bar{b}_j^2 - \frac{1}{2}b_j^2\bar{b}_{j-1}^2 - \frac{1}{2}\bar{b}_j^2 b_{j-1}^2\right) \qquad \Omega = \frac{i}{2}\sum_{j=1}^{N} db_j \wedge d\bar{b}_j$$

Change of coordinates:

$$b_j = \sqrt{2I_j}e^{i\theta_j} \qquad \bar{b}_j = \sqrt{2I_j}e^{-i\theta_j}$$

$$\begin{aligned} db_j \wedge d\bar{b}_j &= d(\sqrt{2I_j}e^{i\theta_j}) \wedge d(\sqrt{2I_j}e^{-i\theta_j}) \\ &= \left(\frac{1}{\sqrt{2I_j}}e^{i\theta_j}dI_j + i\sqrt{2I_j}e^{i\theta_j}d\theta_j\right) \wedge \left(\frac{1}{\sqrt{2I_j}}e^{-i\theta_j}dI_j - i\sqrt{2I_j}e^{-i\theta_j}d\theta_j\right) \\ &= -idI_j \wedge d\theta_j + id\theta_j \wedge dI_j = -2idI_j \wedge d\theta_j \end{aligned}$$

$$\begin{aligned} H(I,\theta) &\equiv H(\sqrt{2I_j}e^{i\theta_j}, \sqrt{2I_j}e^{-i\theta_j}) \\ &= \sum_{j=1}^{N}\left(I_j^2 - 2I_jI_{j-1}e^{2i\theta_j - 2i\theta_{j-1}} - 2I_jI_{j-1}e^{-2i\theta_j + 2i\theta_{j-1}}\right) \\ &= \sum_{j=1}^{N}\left(I_j^2 - 4I_jI_{j-1}\cos 2(\theta_j - \theta_{j-1})\right) \end{aligned}$$

B.2 Proof of Lemma 2

Perform the symplectic change of coordinates:

$$\begin{cases} I_1 &= J_1 - \sum_{j=2}^{N} J_j \\ I_i &= J_i \text{ for } i \neq 1 \end{cases} \qquad \begin{cases} \theta_1 &= \varphi_1 \\ \theta_i &= \varphi_1 + \varphi_i \text{ for } i \neq 1 \end{cases}$$

$$\begin{cases} J_1 &= \sum_{j=1}^{N} I_j \\ J_i &= I_i \text{ for } i \neq 1 \end{cases} \qquad \begin{cases} \varphi_1 &= \theta_1 \\ \varphi_i &= \theta_i - \theta_1 \text{ for } i \neq 1 \end{cases}$$

$$\begin{aligned}
H(J,\varphi) &= \left(J_1 - \sum_{j=2}^{N} J_j\right)^2 + \sum_{j=2}^{N} J_j^2 - 4J_2\left(J_1 - \sum_{j=2}^{N} J_j\right)\cos 2\varphi_2 \\
&- 4\sum_{j=3}^{N} J_j J_{j-1}\cos 2(\varphi_j - \varphi_{j-1}) \\
&= J_1^2 + \left(\sum_{j=2}^{N} J_j\right)^2 - 2J_1\sum_{j=2}^{N} J_j + \sum_{j=2}^{N} J_j^2 \\
&- 4J_1 J_2\cos 2\varphi_2 + 4J_2\sum_{j=2}^{N} J_j\cos 2\varphi_2 - 4\sum_{j=3}^{N} J_j J_{j-1}\cos 2(\varphi_j - \varphi_{j-1}) \\
&= J_1^2 + 2\sum_{j=2}^{N} {J_j}^2 + 2\sum_{i=3}^{N}\sum_{j=2}^{i-1} J_i J_j - 2J_1\sum_{j=2}^{N} J_j \\
&- 4J_1 J_2\cos 2\varphi_2 + 4J_2\cos 2\varphi_2\sum_{j=2}^{N} J_j - 4\sum_{j=3}^{N} J_j J_{j-1}\cos 2(\varphi_j - \varphi_{j-1})
\end{aligned}$$

B.3 Proof of Lemma 3

First, note

$$\sum_{i=2}^{N} J_i = \sum_{i=2}^{N-1}(K_i - K_{i+1}) + K_N = K_2$$

$$\begin{aligned}
\sum_{i=2}^{N} J_i^2 = \sum_{i=2}^{N-1}(K_i - K_{i+1})^2 + K_N^2 &= \sum_{i=2}^{N-1}(K_i^2 + K_{i+1}^2 - 2K_iK_{i+1}) + K_N^2 \\
&= K_2^2 + 2\sum_{i=3}^{N} K_i^2 - 2\sum_{i=2}^{N-1} K_i K_{i+1}
\end{aligned}$$

$$\begin{aligned}
\sum_{i=3}^{N}\sum_{j=2}^{i-1} J_i J_j &= \sum_{i=3}^{N-1}\sum_{j=2}^{i-1}(K_i - K_{i+1})(K_j - K_{j-1}) + K_N\sum_{j=2}^{N-1} J_j \\
&= \sum_{i=3}^{N-1}\sum_{j=2}^{i-1}(K_iK_j - K_iK_{j+1} - K_{i+1}K_j + K_{i+1}K_{j+1}) \\
&+ K_N(K_2 - K_N)
\end{aligned}$$

$$\begin{aligned}\sum_{i=3}^{N-1}\sum_{j=2}^{i-1}(K_iK_j - K_iK_{j+1}) &= \sum_{i=3}^{N-1} K_i\left(\sum_{j=2}^{i-1}K_j - \sum_{j=3}^{i}K_j\right)\\ &= \sum_{i=3}^{N-1}K_i\left[K_2 + \sum_{j=3}^{i-1}K_j - \sum_{j=3}^{i-1}K_j - K_i\right]\\ &= K_2\sum_{i=3}^{N-1}K_i - \sum_{i=3}^{N-1}K_i^2\end{aligned}$$

$$\begin{aligned}\sum_{i=3}^{N-1}\sum_{j=2}^{i-1}(K_{i+1}K_{j+1} - K_{i+1}K_j) &= \sum_{i=4}^{N}\sum_{j=2}^{i-2}(K_iK_{j+1} - K_iK_j)\\ &= \sum_{i=4}^{N}K_i\left(\sum_{j=3}^{i-1}K_j - \sum_{j=2}^{i-2}K_j\right)\\ &= \sum_{i=4}^{N}K_i\left(K_{i-1} + \sum_{j=3}^{i-2}K_j - K_2 - \sum_{j=3}^{i-2}K_j\right)\\ &= \sum_{i=4}^{N}K_iK_{i-1} - K_2\sum_{i=4}^{N}K_i\end{aligned}$$

So,

$$\begin{aligned}\sum_{i=3}^{N}\sum_{j=2}^{i-1}J_iJ_j &= K_2\sum_{i=3}^{N-1}K_i - \sum_{i=3}^{N-1}K_i^2 + \sum_{i=4}^{N}K_iK_{i-1} - K_2\sum_{i=4}^{N}K_i + K_N(K_2 - K_N)\\ &= K_2K_3 - K_2K_N - \sum_{i=3}^{N-1}K_i^2 + \sum_{i=4}^{N}K_iK_{i-1} + K_N(K_2 - K_N)\\ &= -\sum_{i=3}^{N}K_i^2 + \sum_{i=3}^{N}K_iK_{i-1}\end{aligned}$$

Putting all together

$$
\begin{aligned}
H(K,\psi) =& \frac{1}{4} + 2\left(K_2^2 + 2\sum_{i=3}^{N} K_i^2 - 2\sum_{i=2}^{N-1} K_i K_{i+1}\right) \\
&+ 2\left(-\sum_{i=3}^{N} K_i^2 + \sum_{i=3}^{N} K_i K_{i-1}\right) - K_2 \\
&- 2(K_2 - K_3)\cos 2\psi_2 + 4(K_2 - K_3)K_2 \cos 2\psi_2 \\
&- 4\sum_{j=3}^{N-1} (K_j - K_{j+1})(K_{j-1} - K_j)\cos 2\psi_j \\
&- 4K_N(K_{N-1} - K_N)\cos 2\psi_N \\
=& \frac{1}{4} - K_2 + 2\sum_{i=2}^{N} K_i^2 - 2\sum_{i=2}^{N-1} K_i K_{i+1} \\
&- 2K_2 \cos 2\psi_2 + 2K_3 \cos 2\psi_2 + 4K_2^2 \cos 2\psi_2 - 4K_3 K_2 \cos 2\psi_2 \\
&- 4\sum_{j=3}^{N-1} K_j K_{j-1} \cos 2\psi_j + 4\sum_{j=3}^{N-1} K_j^2 \cos 2\psi_j \\
&+ 4\sum_{j=3}^{N-1} K_{j+1} K_{j-1} \cos 2\psi_j - 4\sum_{j=3}^{N-1} K_{j+1} K_j \cos 2\psi_j \\
&- 4K_N(K_{N-1} - K_N)\cos 2\psi_N \\
=& \frac{1}{4} - K_2 + 2\sum_{i=2}^{N} K_i^2 - 2\sum_{i=2}^{N-1} K_i K_{i+1} \\
&- 2K_2 \cos 2\psi_2 + 2K_3 \cos 2\psi_2 + 4\sum_{j=2}^{N} K_j^2 \cos 2\psi_j \\
&- 4\sum_{j=3}^{N} K_j K_{j-1} \cos 2\psi_j + 4\sum_{j=3}^{N-1} K_{j+1} K_{j-1} \cos 2\psi_j \\
&- 4\sum_{j=2}^{N-1} K_{j+1} K_j \cos 2\psi_j \\
=& \frac{1}{4} - K_2(1 + 2\cos 2\psi_2) + 2K_3 \cos 2\psi_2 \\
&+ 2\sum_{j=2}^{N} K_j^2 \left(1 + 2\cos 2\psi_j\right) \\
&- 2\sum_{j=3}^{N} K_j K_{j-1} \left(1 + 2\cos 2\psi_j + 2\cos 2\psi_{j-1}\right) \\
&+ 4\sum_{j=3}^{N-1} K_{j+1} K_{j-1} \cos 2\psi_j
\end{aligned}
$$

C Proofs of results of Chapter 3

C.1 Proof of Lemma 6

Let $b_j = r\mathrm{e}^{\mathrm{i}\theta}$ and $b_k = c_k\mathrm{e}^{\mathrm{i}\theta}$. Taking derivatives on both sides of the first equatilty:

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}\theta} + \mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= -\mathrm{i}|b_j|^2 b_j + 2\mathrm{i}\overline{b_j}\left(b_{j-1}^2 + b_{j+1}^2\right) \\ &= -\mathrm{i}r^3\mathrm{e}^{\mathrm{i}\theta} + 2\mathrm{i}r\mathrm{e}^{-\mathrm{i}\theta}\left(c_{j-1}^2\mathrm{e}^{2\mathrm{i}\theta} + c_{j+1}^2\mathrm{e}^{2\mathrm{i}\theta}\right).\end{aligned}$$

Multiplying both sides by $\mathrm{e}^{-\mathrm{i}\theta}$:

$$\frac{\mathrm{d}r}{\mathrm{d}t} + \mathrm{i}r\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\mathrm{i}r^3 + 2\mathrm{i}r\left(c_{j-1}^2 + c_{j+1}^2\right).$$

Equating now the real and imaginary part:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -2r\mathrm{Im}\left(c_{j-1}^2 + c_{j+1}^2\right) \tag{C.1}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -r^2 + 2\mathrm{Re}\left(c_{j-1}^2 + c_{j+1}^2\right). \tag{C.2}$$

Now we do the same for the first neighbors: c_{j-1} and c_{j+1}:

$$\begin{aligned}\frac{\mathrm{d}c_{j\pm1}}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}\theta} + \mathrm{i}c_{j\pm1}\mathrm{e}^{\mathrm{i}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= -\mathrm{i}|b_{j\pm1}|^2 b_{j\pm1} + 2\mathrm{i}\overline{b_{j\pm1}}\left(b_j^2 + b_{j\pm2}^2\right) \\ &= -\mathrm{i}|c_{j\pm1}|^2 c_{j\pm1}\mathrm{e}^{\mathrm{i}\theta} + 2\mathrm{i}\overline{c_{j\pm1}}\mathrm{e}^{-\mathrm{i}\theta}\left(r^2\mathrm{e}^{2\mathrm{i}\theta} + c_{j\pm2}^2\mathrm{e}^{2\mathrm{i}\theta}\right).\end{aligned}$$

Multiplying both sides by $\mathrm{e}^{-\mathrm{i}\theta}$:

$$\frac{\mathrm{d}c_{j\pm1}}{\mathrm{d}t} + \mathrm{i}c_{j\pm1}\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\mathrm{i}|c_{j\pm1}|^2 c_{j\pm1} + 2\mathrm{i}\overline{c_{j\pm1}}\left(r^2 + c_{j\pm2}^2\right). \tag{C.3}$$

Now for the far neighbors, that is $k \in \mathcal{P}_j = \{1 \le k \le N \,:\, k \ne j-1, j, j+1\}$:

$$\begin{aligned}\frac{\mathrm{d}c_k}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}\theta} + \mathrm{i}c_k\mathrm{e}^{\mathrm{i}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= -\mathrm{i}|b_k|^2 b_k + 2\mathrm{i}\overline{b_k}\left(b_{k-1}^2 + b_{k+1}^2\right) \\ &= -\mathrm{i}|c_k|^2 c_k\mathrm{e}^{\mathrm{i}\theta} + 2\mathrm{i}\overline{c_k}\mathrm{e}^{-\mathrm{i}\theta}\left(c_{k-1}^2\mathrm{e}^{2\mathrm{i}\theta} + c_{k+1}^2\mathrm{e}^{2\mathrm{i}\theta}\right).\end{aligned}$$

Multiplying both sides by $\mathrm{e}^{-\mathrm{i}\theta}$:

$$\frac{\mathrm{d}c_k}{\mathrm{d}t} + \mathrm{i}c_k\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\mathrm{i}|c_k|^2 c_k + 2\mathrm{i}\overline{c_k}\left(c_{k-1}^2 + c_{k+1}^2\right). \tag{C.4}$$

We must recall that the change of coordinates that diagonalizes the linear part is given by:

$$c_{j\pm1} = \omega^2 x_\pm + \omega y_\pm$$

with $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$, that is $\omega = -1/2 + \mathrm{i}\sqrt{3}/2$, and $x_\pm, y_\pm \in \mathbb{R}$. Notice that $|\omega| = 1$, $\omega^3 = 1$ and $\overline{\omega} = \omega^2$ and:

$$\begin{aligned} |c_{j\pm1}|^2 &= \left(\omega^2 x_\pm + \omega y_\pm\right)\left(\omega x_\pm + \omega^2 y_\pm\right) = x_\pm^2 + \left(\omega + \omega^2\right) x_\pm y_\pm + y_\pm^2 \\ &= x_\pm^2 - x_\pm y_\pm + y_\pm^2. \end{aligned}$$

$$\begin{aligned} c_{j\pm1}^2 &= \left(\omega^2 x_\pm + \omega y_\pm\right)^2 = \omega x_\pm^2 + 2x_\pm y_\pm + \omega^2 y_\pm^2 \\ \mathrm{Re}\left(c_{j\pm1}^2\right) &= -\frac{1}{2}x_\pm^2 + 2x_\pm y_\pm - \frac{1}{2}y_\pm^2. \end{aligned}$$

With that, we can already forget about the equation (C.1) using:

$$\begin{aligned} r^2 &= 1 - \sum_{k\neq j} |b_k|^2 = 1 - \sum_{k\neq j} |c_k|^2 \\ &= 1 - \left(x_-^2 - x_- y_- + y_-^2\right) - \left(x_+^2 - x_+ y_+ + y_+^2\right) - \sum_{l\in\mathcal{P}_j} |c_l|^2. \end{aligned}$$

Now we can express (C.2) in terms of the new variables without the radial coordinate:

$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}t} =& - r^2 + 2\mathrm{Re}\left(c_{j-1}^2 + c_{j+1}^2\right) \\ =& - 1 + \left(x_-^2 - x_- y_- + y_-^2\right) + \left(x_+^2 - x_+ y_+ + y_+^2\right) + \sum_{l\in\mathcal{P}_j} |c_l|^2 \\ &+ 2\left(-\frac{1}{2}x_-^2 + 2x_- y_- - \frac{1}{2}y_-^2 - \frac{1}{2}x_+^2 + 2x_+ y_+ - \frac{1}{2}y_+^2\right) \\ =& - 1 + 3x_- y_- + 3x_+ y_+ + \sum_{l\in\mathcal{P}_j} |c_l|^2 \end{aligned}$$

Now we can insert that values to equation (C.4) and obtain the final equation that satisfies c_k when $k \in \mathcal{P}_j \cap \{k \neq j-2, j+2\}$:

$$\begin{aligned} \frac{\mathrm{d}c_k}{\mathrm{d}t} =& - \mathrm{i}c_k \frac{\mathrm{d}\theta}{\mathrm{d}t} - \mathrm{i}|c_k|^2 c_k + 2\mathrm{i}\overline{c_k}\left(c_{k-1}^2 + c_{k+1}^2\right) \\ =& - \mathrm{i}c_k \left(-1 + 3x_- y_- + 3x_+ y_+ + \sum_{l\in\mathcal{P}_j} |c_l|^2\right) - \mathrm{i}|c_k|^2 c_k + 2\mathrm{i}\overline{c_k}\left(c_{k-1}^2 + c_{k+1}^2\right) \\ =& \mathrm{i}c_k \underbrace{-\mathrm{i}c_k\left(|c_k|^2 + \sum_{l\in\mathcal{P}_j} |c_l|^2\right) + 2\mathrm{i}\overline{c_k}\left(c_{k-1}^2 + c_{k+1}^2\right)}_{R_{\mathrm{ell}}^{c_k}(c)} \underbrace{-\mathrm{i}c_k\left(3x_- y_- + 3x_+ y_+\right)}_{R_{\mathrm{mix}}^{c_k}(x,y,c)} \end{aligned}$$

$$\frac{\mathrm{d}c_k}{\mathrm{d}t} = \mathrm{i}c_k + R^{c_k}_{\mathrm{ell}}(c) + R^{c_k}_{\mathrm{mix}}(x,y,c).$$

When one considers $k = j \pm 2$ the equation is slightly different because it contains a term that depends on $c_{j\pm1}$:

$$\begin{aligned}\frac{\mathrm{d}c_{j\pm2}}{\mathrm{d}t} =& -\mathrm{i}c_{j\pm2}\left(-1 + 3x_-y_- + 3x_+y_+ + \sum_{l\in\mathcal{P}_j}|c_l|^2\right)\\ &-\mathrm{i}|c_{j\pm2}|^2c_{j\pm2} + 2\mathrm{i}\overline{c_{j\pm2}}\left(c^2_{j\pm1} + c^2_{j\pm3}\right)\\ =&\mathrm{i}c_{j\pm2}\\ &\underbrace{-\mathrm{i}c_{j\pm2}\left(|c_{j\pm2}|^2 + \sum_{l\in\mathcal{P}_j}|c_l|^2\right) + 2\mathrm{i}\overline{c_{j\pm2}}c^2_{j\pm3}}_{R^{c_{j\pm2}}_{\mathrm{ell}}(c)}\\ &\underbrace{-\mathrm{i}c_{j\pm2}\left(3x_-y_- + 3x_+y_+\right) + 2\mathrm{i}\overline{c_{j\pm2}}\left(\omega^2x_\pm + \omega y_\pm\right)^2}_{R^{c_{j\pm2}}_{\mathrm{mix}}(x,y,c)}\end{aligned}$$

$$\frac{\mathrm{d}c_{j\pm2}}{\mathrm{d}t} = \mathrm{i}c_k + R^{c_{j\pm2}}_{\mathrm{ell}}(c) + R^{c_{j\pm2}}_{\mathrm{mix}}(x,y,c).$$

To deal with the hyperbolic modes we will work with $\{\omega,\omega^2\}$ instead of $\{1,\mathrm{i}\}$ as a base for $\mathbb{C}$ just using

$$\mathrm{i} = \frac{\sqrt{3}}{3}\omega - \frac{\sqrt{3}}{3}\omega^2 \qquad 1 = -\omega - \omega^2$$

From (C.3) we have

$$\begin{aligned}\frac{\mathrm{d}c_{j\pm1}}{\mathrm{d}t} =& -\mathrm{i}c_{j\pm1}\frac{\mathrm{d}\theta}{\mathrm{d}t} - \mathrm{i}|c_{j\pm1}|^2c_{j\pm1} + 2\mathrm{i}\overline{c_{j\pm1}}\left(r^2 + c^2_{j\pm2}\right)\\ =& -\mathrm{i}c_{j\pm1}\left(-1 + 3x_\pm y_\pm + 3x_\mp y_\mp + \sum_{l\in\mathcal{P}_j}|c_l|^2\right) - \mathrm{i}|c_{j\pm1}|^2c_{j\pm1}\\ &+ 2\mathrm{i}\overline{c_{j\pm1}}\left(1 - \left(x^2_\pm - x_\pm y_\pm + y^2_\pm\right) - \left(x^2_\mp - x_\mp y_\mp + y^2_\mp\right) - \sum_{l\in\mathcal{P}_j}|c_l|^2 + c^2_{j\pm2}\right)\\ =& -\mathrm{i}c_{j\pm1}\left(-1 + x^2_\pm + 2x_\pm y_\pm + y^2_\pm + 3x_\mp y_\mp + \sum_{l\in\mathcal{P}_j}|c_l|^2\right)\\ &+ 2\mathrm{i}\overline{c_{j\pm1}}\left(1 - \left(x^2_\pm - x_\pm y_\pm + y^2_\pm\right) - \left(x^2_\mp - x_\mp y_\mp + y^2_\mp\right) - \sum_{l\in\mathcal{P}_j}|c_l|^2 + c^2_{j\pm2}\right).\end{aligned}$$

Let

$$\begin{aligned}
R^{j\pm1}_{\text{hyp}}(x,y,c_{j\pm1}) =& \mathrm{i}c_{j\pm1} + 2\mathrm{i}\overline{c_{j\pm1}} \\
& + \left(-\mathrm{i}c_{j\pm1} - 2\mathrm{i}\overline{c_{j\pm1}}\right)\left(x_\pm^2 + y_\pm^2\right) \\
& + \left(-2\mathrm{i}c_{j\pm1} + 2\mathrm{i}\overline{c_{j\pm1}}\right)x_\pm y_\pm \\
& - 2\mathrm{i}\overline{c_{j\pm1}}\left(x_\mp^2 + y_\mp^2\right) \\
& + \left(-3\mathrm{i}c_{j\pm1} + 2\mathrm{i}\overline{c_{j\pm1}}\right)x_\mp y_\mp \\
R^{j\pm1}_{\text{mix}}(c_{j\pm1},c) =& \left(-\mathrm{i}c_{j\pm1} - 2\mathrm{i}\overline{c_{j\pm1}}\right)\sum_{l\in\mathcal{P}_j}|c_l|^2 + 2\mathrm{i}\overline{c_{j\pm1}}c_{j\pm2}^2.
\end{aligned}$$

Taking into account that $c_{j\pm1} = \omega^2 x_\pm + \omega y_\pm$ we obtain:

$$\omega^2\frac{\mathrm{d}x_\pm}{\mathrm{d}t} + \omega\frac{\mathrm{d}y_\pm}{\mathrm{d}t} = R^{j\pm1}_{\text{hyp}}(x,y,\omega^2x_\pm + \omega y_\pm) + R^{j\pm1}_{\text{mix}}(\omega^2 x_\pm + \omega y_\pm, c),$$

so, our aim is to decompose the right-hand side of the last equation in $\{\omega,\omega^2\}$.

For the first part we need:

$$\begin{aligned}
\mathrm{i}c_{j\pm1} + 2\mathrm{i}\overline{c_{j\pm1}} =& \sqrt{3}x_\pm\omega^2 - \sqrt{3}y_\pm\omega \\
-\mathrm{i}c_{j\pm1} - 2\mathrm{i}\overline{c_{j\pm1}} =& -\sqrt{3}x_\pm\omega^2 + \sqrt{3}y_\pm\omega \\
-2\mathrm{i}c_{j\pm1} + 2\mathrm{i}\overline{c_{j\pm1}} =& 2\sqrt{3}\left(x_\pm - y_\pm\right)\omega^2 + 2\sqrt{3}\left(x_\pm - y_\pm\right)\omega \\
-2\mathrm{i}\overline{c_{j\pm1}} =& \left(-\frac{4\sqrt{3}}{3}x_\pm + \frac{2\sqrt{3}}{3}y_\pm\right)\omega^2 + \left(-\frac{2\sqrt{3}}{3}x_\pm + \frac{4\sqrt{3}}{3}y_\pm\right)\omega \\
-3\mathrm{i}c_{j\pm1} + 2\mathrm{i}\overline{c_{j\pm1}} =& \left(\frac{7\sqrt{3}}{3}x_\pm - \frac{8\sqrt{3}}{3}y_\pm\right)\omega^2 + \left(\frac{8\sqrt{3}}{3}x_\pm - \frac{7\sqrt{3}}{3}y_\pm\right)\omega \\
2\mathrm{i}\overline{c_{j\pm1}}c_{j\pm2}^2 =& \left[2G_1(c_{j\pm2})x_\pm + \left(-G_1(c_{j\pm2}) + G_2(c_{j\pm2})\right)y_\pm\right]\omega^2 \\
& + \left[\left(G_1(c_{j\pm2}) + G_2(c_{j\pm2})\right)x_\pm - 2G_1(c_{j\pm2})y_\pm\right]\omega
\end{aligned}$$

where

$$\begin{aligned}
G_1(c_{j\pm2}) =& \frac{2\sqrt{3}}{3}\left(\mathrm{Re}^2(c_{j\pm2}) - \mathrm{Im}^2(c_{j\pm2})\right) \\
G_2(c_{j\pm2}) =& -4\mathrm{Re}(c_{j+2})\mathrm{Im}(c_{j\pm2}).
\end{aligned}$$

Now, we use that $\{\omega,\omega^2\}$ form a base and obtain:

$$\begin{aligned}
\frac{\mathrm{d}x_\pm}{\mathrm{d}t} =& \sqrt{3}x_\pm - \sqrt{3}x_\pm\left(x_\pm^2 + y_\pm^2\right) + 2\sqrt{3}\left(x_\pm - y_\pm\right)x_\pm y_\pm \\
& + \left(-\frac{4\sqrt{3}}{3}x_\pm + \frac{2\sqrt{3}}{3}y_\pm\right)\left(x_\mp^2 + y_\mp^2\right) + \left(\frac{7\sqrt{3}}{3}x_\pm - \frac{8\sqrt{3}}{3}y_\pm\right)x_\mp y_\mp \\
& - \sqrt{3}x_\pm\sum_{l\in\mathcal{P}_j}|c_l|^2 + 2G_1(c_{j\pm2})x_\pm + \left(-G_1(c_{j\pm2}) + G_2(c_{j\pm2})\right)y_\pm.
\end{aligned}$$

If we rearrange the terms we obtain the system announced in the Lemma. The equation for $y_\pm$ is equivalent.

C.2 Proof of Lemma 9

We know that there exist three invariant manifolds that can be written, locally, as a graph of a function:

$$\mathcal{W}^{cs} = \{(x, y, c) \,:\, x_{\pm} = \phi_{\pm}(y, c)\} \text{ center-stable manifold}$$

$$\mathcal{W}^{cu} = \{(x, y, c) \,:\, y_{\pm} = \psi_{\pm}(x, c)\} \text{ center-unstable manifold}$$

$$\mathcal{W}^{c} = \{(x, y, c) \,:\, x_{\pm} = \chi^{x}_{\pm}(c),\, y_{\pm} = \chi^{y}_{\pm}(c)\} \text{ center manifold}$$

that satisfy

$$\mathcal{W}^{c} \subset \mathcal{W}^{cs} \text{ and } \mathcal{W}^{c} \subset \mathcal{W}^{cu}$$

We want to know the expression of these functions without computing them.

First of all, we know that in the original coordinates the sets $\{b_j = 0\}$ are invariant for all j. So, then we have that $\{x_- = y_- = 0\}$, $\{x_+ = y_+ = 0\}$ and $\{c_k = 0\}$ for each $k \in \mathcal{P}_j$ are invariant subspaces.

For this reason, we can identify right now the center manifold. The set $\{x_- = y_- = x_+ = y_+ = 0\}$ is invariant and if we look at the restricted system we obtain a $2(N-3)$ dimensional system that consists in $2(N-3)$ elliptic directions. Then, this subset corresponds to the center manifold of the whole system:

$$\mathcal{W}^{c} = \{(x, y, c) \,:\, x_{\pm} = 0,\, y_{\pm} = 0\}$$

Then, since $\mathcal{W}^{c} \subset \mathcal{W}^{cs}$ and $\mathcal{W}^{c} \subset \mathcal{W}^{cu}$ we have

$$\phi_{\pm}(0, c) = 0 \tag{C.5}$$

$$\psi_{\pm}(0, c) = 0 \tag{C.6}$$

On the other hand, we know that this system possesses 4 heteroclinic connections to the previous and following mode that correspond to the hyperbolic axis. However, we could see that there are bigger invariant objects contained in the manifolds:

$$\{x_- = y_- = y_+ = 0,\, c_{j+2} = 0\} \subset \mathcal{W}^{cu} \tag{C.7}$$

$$\{x_- = y_- = x_+ = 0,\, c_{j+2} = 0\} \subset \mathcal{W}^{cs}$$

$$\{y_- = x_+ = y_+ = 0,\, c_{j-2} = 0\} \subset \mathcal{W}^{cu} \tag{C.8}$$

$$\{x_- = x_+ = y_+ = 0,\, c_{j-2} = 0\} \subset \mathcal{W}^{cs}$$

Let's impose these facts to obtain conditions on the parameterizations of the manifolds.

From (C.7) we obtain:

$$0 = \psi_-(x_- = 0, x_+, c_{j+2} = 0, c_{\neq j+2}) \Rightarrow \psi_-(x, c) = x_-\psi^1_-(x, c) + c_{j+2}\psi^2_-(x_+, c).$$

Now, impose equation (C.8):

$$x_-\psi_-^1(x_-, x_+ = 0, c_{j-2} = 0, c_{\neq j-2}) + c_{j+2}\psi_-^2(x_+ = 0, c_{j-2} = 0, c_{\neq j-2}) = 0.$$

Then, since the functions are analytic, they must vanish, so

$$\psi_-^1(x, c) = x_+\psi_-^a(x, c) + c_{j-2}\psi_-^b(x_-, c)$$
$$\psi_-^2(x_+, c) = x_+\psi_-^c(x_+, c) + c_{j-2}\psi_-^d(c).$$

Then

$$\psi_-(x, c) = x_- \left[x_+\psi_-^a(x, c) + c_{j-2}\psi_-^b(x_-, c)\right] + c_{j+2}\left[x_+\psi_-^c(x_+, c) + c_{j-2}\psi_-^d(c)\right].$$

Now we have to recall condition (C.5) that makes $\psi_-^d(c) = 0$.

Finally, if we impose the analogous conditions in the other functions, we obtain that, due to the symmetries, the parameterizations have the following form:

$$\psi_-(x, c) = x_- \left[x_+\psi_-^a(x, c) + c_{j-2}\psi_-^b(x_-, c)\right] + x_+c_{j+2}\psi_-^c(x_+, c)$$
$$\psi_+(x, c) = x_+ \left[x_-\psi_+^a(x, c) + c_{j+2}\psi_+^b(x_+, c)\right] + x_-c_{j-2}\psi_+^c(x_-, c)$$
$$\phi_-(y, c) = y_- \left[y_+\phi_-^a(y, c) + c_{j-2}\phi_-^b(y_-, c)\right] + y_+c_{j+2}\phi_-^c(y_+, c)$$
$$\phi_+(y, c) = y_+ \left[y_-\phi_+^a(y, c) + c_{j+2}\phi_+^b(y_+, c)\right] + y_-c_{j-2}\phi_+^c(y_-, c)$$

Or, summarized:

$$\psi_\pm(x, c) = x_\pm \left[x_\mp\psi_\pm^a(x, c) + c_{j\pm2}\psi_\pm^b(x_\pm, c)\right] + x_\mp c_{j\mp2}\psi_\pm^c(x_\mp, c)$$

$$\phi_\pm(y, c) = y_\pm \left[y_\mp\phi_\pm^a(y, c) + c_{j\pm2}\phi_\pm^b(y_\pm, c)\right] + y_\mp c_{j\mp2}\phi_\pm^c(y_\pm, c)$$

Now, we recall and use the fact that the subspaces $\{x_+ = y_+ = 0\}$ and $\{x_- = y_- = 0\}$ are invariant. Then the intersection of the first one (for instance) with the center-stable manifold should be invariant too. The equations of the intersection (Z^{cs}) are:

$$(Z^{cs})\left\{\begin{array}{rcl} 0 & = & \phi_+(y_-, y_+ = 0, c) = y_-c_{j-2}\phi_+^c(y_-, c) \\ x_- & = & \phi_-(y_-, y_+ = 0, c) = y_-c_{j-2}\phi_-^b(y_-, c) \end{array}\right.$$

On the other hand, consider the system restricted in $\{x_+ = y_+ = 0\}$ we still have a partially hyperbolic fixed point at the origin, so we can consider its center-stable manifold $(\tilde{\mathcal{W}}^{cs})$ that can be written as:

$$\tilde{\mathcal{W}}^{cs} = \{x_- = \tilde{\phi}_-(y_-, c)\}.$$

It is clear that $\tilde{\mathcal{W}}^{cs} \subset \mathcal{W}^{cs}$ but there is more, $\tilde{\mathcal{W}}^{cs} \subset Z^{cs}$. Actually is also clear the other inclusion, so the manifolds must coincide ($\tilde{\mathcal{W}}^{cs} = Z^{cs}$).

We know that the dimension of $\tilde{W}^{cs}$ is $2(N-3)+1$ (its free coordinates are y_- and c) and the dimension of Z^{cs} seams to be $2(N-3)$ since there is a equation that relates two coordinates. Since the dimensions must be equal, this relation must be a triviality, that is $\phi_+^c(y_-, c) = 0$ for all y_- and c.

Using the invariance of $\mathcal{W}^{cu}$ and then repeating the argument with $\{x_- = y_- = 0\}$ we can conclude that:

$$\phi_-^c(y_+, c) = \phi_+^c(y_-, c) = \psi_-^c(x_+, c) = \psi_+^c(x_-, c) = 0$$

and

$$\begin{aligned} \psi_\pm(x,c) &= x_\pm \left[x_\mp \psi_\pm^a(x,c) + c_{j\pm 2} \psi_\pm^b(x_\pm, c) \right] \\ \phi_\pm(y,c) &= y_\pm \left[y_\mp \phi_\pm^a(y,c) + c_{j\pm 2} \phi_\pm^b(y_\pm, c) \right]. \end{aligned}$$

However, we can say more. We can assume that

$$\begin{aligned} \psi_\pm(x,c) =& x_\pm \Big[x_\mp^2 \psi_\pm^a(x,c) + |c_{j\pm 2}|^2 \psi_\pm^{b_1}(x_\pm, c) \\ &+ \left(c_{j\pm 2}^2 + \overline{c_{j\pm 2}}^2\right) \psi_\pm^{b_2}(x_\pm, c) + \mathrm{i}\left(c_{j\pm 2}^2 - \overline{c_{j\pm 2}}^2\right) \psi_\pm^{b_3}(x_\pm, c) \Big] \\ \phi_\pm(y,c) =& y_\pm \Big[y_\mp^2 \phi_\pm^a(y,c) + |c_{j\pm 2}|^2 \phi_\pm^{b_1}(y_\pm, c) \\ &+ \left(c_{j\pm 2}^2 + \overline{c_{j\pm 2}}^2\right) \phi_\pm^{b_2}(y_\pm, c) + \mathrm{i}\left(c_{j\pm 2}^2 - \overline{c_{j\pm 2}}^2\right) \phi_\pm^{b_3}(y_\pm, c) \Big], \end{aligned}$$

We only sketch a justification. Indeed, consider for instance the equation for for $\phi_-(y,c)$. Notice that the terms in the equation of x_- that make the center-stable manifold not straightened are, precisely:

$$\frac{2\sqrt{3}}{3} y_- y_+^2 + y_- \left[-\frac{2\sqrt{3}}{3} \left(\mathrm{Re}^2(c_{j-2}) - \mathrm{Im}^2(c_{j-2}) \right) + 4\mathrm{Re}(c_{j-2}) \mathrm{Im}(c_{j-2}) \right].$$

So, it depends quadratically on y_+ and c_{j-2}. If, then, we look at the terms of order three in the parameterization of the center-stable manifold, we see that they also depend quadratically on these coordinates. One can see that such quadratic terms can be taken as a common factor of the higher order terms.

The expression of the inverse of the change can be obtained directly using the expression of the direct change.

C.3 Proof of Theorem 9

We first write the expression that the nonlinearities satisfy.

Lemma 18. *Denote* $\xi = (\xi_-, \xi_+)$, $\eta = (\eta_-, \eta_+)$, $c = (c_1, \dots, c_{j-2}, c_{j+2}, \dots, c_N) \in \mathbb{C}^{N-3}$, $z = (\xi, \eta, c)$, $z_+ = (\xi_+, \eta_+^2, c^2)$ *and* $z_- = (\xi_-^2, \eta_-, c^2)$.

There exists K *such that:*

$$\begin{aligned}
\left|R^{\xi_-}(z)\right| &\le K\left[|\xi_-|\,|z|^2 + |\xi_+||\eta_-|\,|z_+|\right]\\
\left|R^{\xi_+}(z)\right| &\le K\left[|\xi_+|\,|z|^2 + |\xi_-||\eta_+|\,|z|\right]\\
\left|R^{\eta_-}(z)\right| &\le K\left[|\eta_-|\,|z|^2 + |\eta_+||\xi_-|\,|z|\right]\\
\left|R^{\eta_+}(z)\right| &\le K\left[|\eta_+|\,|z|^2 + |\eta_-||\xi_+|\,|z_-|\right]\\
\left|R^{c_k}(z)\right| &\le K\,|c_k|\,|z|^2
\end{aligned}$$

$$\begin{aligned}
\left|R^{\xi_-}(z_1) - R^{\xi_-}(z_2)\right| \le K\Big\{&\left(|z_1|^2 + |z_2|^2\right)|\xi_{-,1} - \xi_{-,2}|\\
&+ \left[(|\xi_{-,1}| + |\xi_{-,2}|)(|z_1| + |z_2|)\right.\\
&\left.+ (|\eta_{-,1}| + |\eta_{-,2}|)(|z_{+,1}| + |z_{+,2}|)\right]|\xi_{+,1} - \xi_{+,2}|\\
&+ \left[(|\xi_{-,1}| + |\xi_{-,2}|)(|z_1| + |z_2|)\right.\\
&\left.+ (|\xi_{+,1}| + |\xi_{+,2}|)(|z_{+,1}| + |z_{+,2}|)\right]|\eta_{-,1} - \eta_{-,2}|\\
&+ \left[(|\xi_{-,1}| + |\xi_{-,2}|)(|z_1| + |z_2|)\right.\\
&\left.+ (|\xi_{+,1}| + |\xi_{+,2}|)(|\eta_{-,1}| + |\eta_{-,2}|)(|\eta_{+,1}| + |\eta_{+,2}|)\right]|\eta_{+,1} - \eta_{+,2}|\\
&+ \left[(|\xi_{-,1}| + |\xi_{-,2}|)(|z_1| + |z_2|)\right.\\
&\left.+ (|\xi_{+,1}| + |\xi_{+,2}|)(|\eta_{-,1}| + |\eta_{-,2}|)(|c_1| + |c_2|)\right]|c_1 - c_2|\Big\}
\end{aligned}$$

$$\begin{aligned}
\left|R^{\xi_+}(z_1) - R^{\xi_+}(z_2)\right| \le K\Big\{&\left(|z_1|^2 + |z_2|^2\right)|\xi_{+,1} - \xi_{+,2}|\\
&+ (|\xi_{+,1}| + |\xi_{+,2}| + |\eta_{+,1}| + |\eta_{+,2}|)(|z_1| + |z_2|)|\xi_{-,1} - \xi_{-,2}|\\
&+ (|\xi_{+,1}| + |\xi_{+,2}| + |\xi_{-,1}| + |\xi_{-,2}|)(|z_1| + |z_2|)|\eta_{+,1} - \eta_{+,2}|\\
&+ \left[(|\xi_{+,1}| + |\xi_{+,2}|)(|z_1| + |z_2|)\right.\\
&\left.+ (|\xi_{-,1}| + |\xi_{-,2}|)(|\eta_{+,1}| + |\eta_{+,2}|)\right]|\eta_{-,1} - \eta_{-,2}|\\
&+ \left[(|\xi_{+,1}| + |\xi_{+,2}|)(|z_1| + |z_2|)\right.\\
&\left.+ (|\xi_{-,1}| + |\xi_{-,2}|)(|\eta_{+,1}| + |\eta_{+,2}|)\right]|c_1 - c_2|\Big\}
\end{aligned}$$

$$\begin{aligned}
\left|R^{\eta_-}(z_1) - R^{\eta_-}(z_2)\right| \le K\Big\{&\left(|z_1|^2 + |z_2|^2\right)|\eta_{-,1} - \eta_{-,2}|\\
&+ (|\eta_{-,1}| + |\eta_{-,2}| + |\xi_{-,1}| + |\xi_{-,2}|)(|z_1| + |z_2|)|\eta_{+,1} - \eta_{+,2}|\\
&+ (|\eta_{-,1}| + |\eta_{-,2}| + |\eta_{+,1}| + |\eta_{+,2}|)(|z_1| + |z_2|)|\xi_{-,1} - \xi_{-,2}|\\
&+ \left[(|\eta_{-,1}| + |\eta_{-,2}|)(|z_1| + |z_2|)\right.\\
&\left.+ (|\eta_{+,1}| + |\eta_{+,2}|)(|\xi_{-,1}| + |\xi_{-,2}|)\right]|\xi_{+,1} - \xi_{+,2}|\\
&+ \left[(|\eta_{-,1}| + |\eta_{-,2}|)(|z_1| + |z_2|)\right.\\
&\left.+ (|\eta_{+,1}| + |\eta_{+,2}|)(|\xi_{-,1}| + |\xi_{-,2}|)\right]|c_1 - c_2|\Big\}
\end{aligned}$$

$$
\begin{aligned}
|R^{\eta_+}(z_1)\ -R^{\eta_+}(z_2)| \leq K \{ & \left(|z_1|^2 + |z_2|^2\right) |\eta_{+,1} - \eta_{+,2}| \\
& + [(|\eta_{+,1}| + |\eta_{+,2}|) (|z_1| + |z_2|) \\
& + (|\xi_{+,1}| + |\xi_{+,2}|) (|z_{-,1}| + |z_{-,2}|)] |\eta_{-,1} - \eta_{-,2}| \\
& + [(|\eta_{+,1}| + |\eta_{+,2}|) (|z_1| + |z_2|) \\
& + (|\eta_{-,1}| + |\eta_{-,2}|) (|z_{-,1}| + |z_{-,2}|)] |\xi_{+,1} - \xi_{+,2}| \\
& + [(|\eta_{+,1}| + |\eta_{+,2}|) (|z_1| + |z_2|) \\
& + (|\eta_{-,1}| + |\eta_{-,2}|) (|\xi_{+,1}| + |\xi_{+,2}|) (|\xi_{-,1}| + |\xi_{1,2}|)] |\xi_{-,1} - \xi_{-,2}| \\
& + [(|\eta_{+,1}| + |\eta_{+,2}|) (|z_1| + |z_2|) \\
& + (|\eta_{-,1}| + |\eta_{-,2}|) (|\xi_{+,1}| + |\xi_{+,2}|) (|c_1| + |c_2|)] |c_1 - c_2| \}
\end{aligned}
$$

$$
\begin{aligned}
|R^{c_k}(z_1) - R^{c_k}(z_1)| \leq K \Big\{ & \left(|z_1|^2 + |z_2|^2\right) |c_{k,1} - c_{k,2}| + (|c_{k,1}| + |c_{k,2}|) (|z_1| + |z_2|) \\
& (|\xi_{\pm,1} - \xi_{\pm,2}| + |\xi_{\mp,1} - \xi_{\mp,2}| + |\eta_{\pm,1} - \eta_{\pm,2}| \\
& + |\eta_{\mp,1} - \eta_{\mp,2}| + \sum_{l \neq k} |c_{l,1} - c_{l,2}| \Big) \Big\}
\end{aligned}
$$

To prove Theorem 9, first consider the sequence:

$$
z^{[0]}(t) = \begin{pmatrix} \xi_-^{[0]}(t) \\ \xi_+^{[0]}(t) \\ \eta_-^{[0]}(t) \\ \eta_+^{[0]}(t) \\ c^{[0]}(t) \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\sqrt{3}t} \xi_{-,0} \\ \mathrm{e}^{\sqrt{3}(t-\tau)} \xi_{+,1} \\ \mathrm{e}^{-\sqrt{3}t} \eta_{-,0} \\ \mathrm{e}^{-\sqrt{3}(t-\tau)} \eta_{+,1} \\ \mathrm{e}^{it} \zeta \end{pmatrix}
$$

and $z^{[n+1]}(t)$ defined as the unique solution of

$$
\begin{cases}
\dot{\xi}_\pm^{[n+1]} &= \sqrt{3} \xi_\pm^{[n+1]} + R^{\xi_\pm}\left(z^{[n]}(t)\right) \\
\dot{\eta}_\pm^{[n+1]} &= -\sqrt{3} \eta_\pm^{[n+1]} + R^{\eta_\pm}\left(z^{[n]}(t)\right) \\
\dot{c}_k^{[n+1]} &= i c_k^{[n+1]} + R^{c_k}\left(z^{[n]}(t)\right)
\end{cases}
$$

that satisfies the Shilnikov conditions:

$$
\begin{aligned}
\xi_-^{[n+1]}(0) &= \xi_{-,0} \qquad \eta_-^{[n+1]}(0) = \eta_{-,0} \\
\xi_+^{[n+1]}(\tau) &= \xi_{+,1} \qquad \eta_+^{[n+1]}(\tau) = \eta_{+,1} \\
c_k^{[n+1]}(0) &= \zeta_k
\end{aligned}
$$

Let's see that $\{z^{[n]}(t)\}$ is a Cauchy sequence. We first rewrite the system:

$$\begin{cases} \xi_-^{[n+1]}(t) &=& \mathrm{e}^{\sqrt{3}t}\xi_{-,0} + \displaystyle\int_0^t \mathrm{e}^{\sqrt{3}(t-s)} R^{\xi_-}\left(z^{[n]}(s)\right) \mathrm{d}s \\ \xi_+^{[n+1]}(t) &=& \mathrm{e}^{\sqrt{3}(t-\tau)}\xi_{+,1} + \displaystyle\int_\tau^t \mathrm{e}^{\sqrt{3}(t-s)} R^{\xi_+}\left(z^{[n]}(s)\right) \mathrm{d}s \\ \eta_-^{[n+1]}(t) &=& \mathrm{e}^{-\sqrt{3}t}\eta_{-,0} + \displaystyle\int_0^t \mathrm{e}^{-\sqrt{3}(t-s)} R^{\eta_-}\left(z^{[n]}(s)\right) \mathrm{d}s \\ \eta_+^{[n+1]}(t) &=& \mathrm{e}^{-\sqrt{3}(t-\tau)}\eta_{+,1} + \displaystyle\int_\tau^t \mathrm{e}^{-\sqrt{3}(t-s)} R^{\eta_+}\left(z^{[n]}(s)\right) \mathrm{d}s \\ c_k^{[n+1]}(t) &=& \mathrm{e}^{\sqrt{3}(t-\tau)}\zeta_k + \displaystyle\int_0^t \mathrm{e}^{i(t-s)} R^{c_k}\left(z^{[n]}(s)\right) \mathrm{d}s \end{cases}$$

Notice that (using $\tau^k \mathrm{e}^{-\sqrt{3}\tau} \leq 1$)

$$\begin{aligned} \left|\xi_-^{[0]}(t)\right| &= \mathrm{e}^{\sqrt{3}t}|\xi_{-,0}| = \gamma_{\xi_-}\mathrm{e}^{\sqrt{3}t} \leq \frac{\gamma}{2\sqrt{3}}\tau^k \mathrm{e}^{-2\sqrt{3}\tau}\mathrm{e}^{\sqrt{3}t} =: \left|\xi_-^{[0]}(t)^*\right| \\ \left|\xi_+^{[0]}(t)\right| &= \mathrm{e}^{\sqrt{3}(t-\tau)}|\xi_{+,1}| = \gamma_{\xi_+}\mathrm{e}^{\sqrt{3}(t-\tau)} \leq \frac{\gamma}{2\sqrt{3}}\mathrm{e}^{\sqrt{3}(t-\tau)} =: \left|\xi_+^{[0]}(t)^*\right| \end{aligned}$$

$$\begin{aligned} \left|\xi_-^{[0]}(t)^*\right| &\leq \frac{\gamma}{2\sqrt{3}}\mathrm{e}^{\sqrt{3}(t-\tau)} \\ \left|\xi^{[0]}(t)\right| &\leq \frac{\gamma}{\sqrt{3}}\mathrm{e}^{\sqrt{3}(t-\tau)} =: \left|\xi^{[0]}(t)^*\right| \end{aligned}$$

$$\begin{aligned} \left|\eta_-^{[0]}(t)\right| &= \mathrm{e}^{-\sqrt{3}t}|\eta_{-,0}| = \gamma_{\eta_-}\mathrm{e}^{-\sqrt{3}t} \leq \frac{\gamma}{2\sqrt{3}}\mathrm{e}^{-\sqrt{3}t} =: \left|\eta_-^{[0]}(t)^*\right| \\ \left|\eta_+^{[0]}(t)\right| &= \mathrm{e}^{-\sqrt{3}(t-\tau)}|\eta_{+,1}| = \gamma_{\eta_+}\mathrm{e}^{-\sqrt{3}(t-\tau)} \leq \frac{\gamma}{2\sqrt{3}}\tau^k\mathrm{e}^{-2\sqrt{3}\tau}\mathrm{e}^{-\sqrt{3}(t-\tau)} =: \left|\eta_+^{[0]}(t)^*\right| \end{aligned}$$

$$\begin{aligned} \left|\eta_+^{[0]}(t)^*\right| &\leq \frac{\gamma}{2\sqrt{3}}\mathrm{e}^{-\sqrt{3}t} \\ \left|\eta^{[0]}(t)\right| &\leq \frac{\gamma}{\sqrt{3}}\mathrm{e}^{-\sqrt{3}t} =: \left|\eta^{[0]}(t)^*\right| \end{aligned}$$

$$\begin{aligned} \left|c_k^{[0]}(t)\right| &= |\zeta_k| = \gamma_k \leq \frac{\gamma}{2\sqrt{3}(N-3)}\tau^k\mathrm{e}^{-\epsilon\sqrt{3}\tau} \\ \left|c^{[0]}(t)\right| &< \frac{\gamma}{2\sqrt{3}}\tau^k\mathrm{e}^{-\epsilon\sqrt{3}\tau} =: \left|c^{[0]}(t)^*\right| \end{aligned}$$

then

$$\begin{aligned}
\left|R^{\xi_-}(z^{[0]}(s))\right| &\leq K\left[\left|\xi_-^{[0]}(s)\right|\left|z^{[0]}(s)\right|^2+\left|\xi_+^{[0]}(s)\right|\left|\eta_-^{[0]}(s)\right|\left|z_+^{[0]}(s)\right|\right]\\
&\leq K\left[\gamma_{\xi_-}\mathrm{e}^{\sqrt{3}s}\left|z^{[0]}(s)\right|^2+\frac{\gamma}{2\sqrt{3}}\mathrm{e}^{\sqrt{3}(s-\tau)}\gamma_{\eta_-}\mathrm{e}^{-\sqrt{3}s}\left|z_+^{[0]}(s)\right|\right]\\
&\leq K\mathrm{e}^{\sqrt{3}s}\left[\gamma_{\xi_-}\left|z^{[0]}(s)\right|^2+\frac{1}{2\sqrt{3}}\gamma_{\eta_-}\gamma\mathrm{e}^{-\sqrt{3}s}\mathrm{e}^{-\sqrt{3}\tau}\left|z_+^{[0]}(s)\right|\right]\\
\left|R^{\xi_+}(z^{[0]}(s))\right| &\leq K\left[\left|\xi_+^{[0]}(s)\right|\left|z^{[0]}(s)\right|^2+\left|\xi_-^{[0]}(s)\right|\left|\eta_+^{[0]}(s)\right|\left|z^{[0]}(s)\right|\right]\\
&\leq K\left[\gamma_{\xi_+}\mathrm{e}^{\sqrt{3}(s-\tau)}\left|z^{[0]}(s)\right|^2+\frac{\gamma}{2\sqrt{3}}\mathrm{e}^{\sqrt{3}(s-\tau)}\gamma_{\eta_+}\mathrm{e}^{-\sqrt{3}(s-\tau)}\left|z^{[0]}(s)\right|\right]\\
&\leq K\mathrm{e}^{\sqrt{3}(s-\tau)}\left[\gamma_{\xi_+}\left|z^{[0]}(s)\right|^2+\frac{1}{2\sqrt{3}}\gamma_{\eta_+}\gamma\mathrm{e}^{-\sqrt{3}s}\mathrm{e}^{\sqrt{3}\tau}\left|z^{[0]}(s)\right|\right]
\end{aligned}$$

Lemma 19. *If*

$$\begin{aligned}
\tau^k e^{-\epsilon\sqrt{3}\tau} &\leq 1,\\
\frac{2\sqrt{3}}{4}\tau^{2k+1}e^{-2\epsilon\sqrt{3}\tau} &\leq 1,\\
\tau e^{-\sqrt{3}\tau} &\leq \frac{1}{2\sqrt{3}},\\
\frac{1}{18}\gamma\tau^{2k-1}e^{-\sqrt{3}\tau} &\leq 1,\\
\frac{1}{6}\gamma\tau^{2k-1}e^{(1-2\epsilon)\sqrt{3}\tau} &\leq 1\\
\frac{\sqrt{3}}{2}\tau^{k+1}e^{-\epsilon\sqrt{3}\tau}
\end{aligned}$$

then:

$$I_1=\int_0^\tau\left|z^{[0]}(s)^*\right|^2\,ds\leq\frac{\gamma^2}{2\sqrt{3}}$$

$$I_2=\int_0^\tau e^{-\sqrt{3}s}\left|z^{[0]}(s)^*\right|\,ds\leq\frac{\gamma}{2}\quad I_2^+=\int_0^\tau e^{-\sqrt{3}s}\left|z_+^{[0]}(s)^*\right|\,ds\leq\gamma\tau e^{-\sqrt{3}\tau}$$

$$I_3=\int_0^\tau e^{\sqrt{3}(s-\tau)}\left|z^{[0]}(s)^*\right|\,ds\leq\frac{\gamma}{2}\quad I_3^-=\int_0^\tau e^{\sqrt{3}(s-\tau)}\left|z_-^{[0]}(s)^*\right|\,ds\leq\gamma\tau e^{-\sqrt{3}\tau}$$

Proof.

$$\begin{aligned}
I_1 &= \int_0^\tau \left|z^{[0]}(s)\right|^2 \mathrm{d}s \le \int_0^\tau \left[\left|\xi^{[0]}(s)^*\right|^2 + \left|\eta^{[0]}(s)^*\right|^2 + \left|c^{[0]}(s)^*\right|^2\right] \mathrm{d}s \\
&\le \int_0^\tau \left[\frac{\gamma^2}{3}\mathrm{e}^{2\sqrt{3}(s-\tau)} + \frac{\gamma^2}{3}\mathrm{e}^{-2\sqrt{3}s} + \frac{\gamma^2}{12}\tau^{2k}\mathrm{e}^{-2\epsilon\sqrt{3}\tau}\right] \mathrm{d}s \\
&= \frac{\gamma^2}{3}\left[\frac{1}{2\sqrt{3}}\left(1-\mathrm{e}^{-2\sqrt{3}\tau}\right) + \frac{1}{2\sqrt{3}}\left(1-\mathrm{e}^{-2\sqrt{3}\tau}\right) + \frac{1}{4}\tau^{2k+1}\mathrm{e}^{-2\epsilon\sqrt{3}\tau}\right] \\
&\le \frac{\gamma^2}{2\sqrt{3}}\frac{1}{3}\left(2 + \underbrace{\frac{2\sqrt{3}}{4}\tau^{2k+1}\mathrm{e}^{-2\epsilon\sqrt{3}\tau}}_{\le 1}\right) \le \frac{\gamma^2}{2\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_0^\tau \mathrm{e}^{-\sqrt{3}s}\left|z^{[0]}(s)^*\right| \mathrm{d}s \le \int_0^\tau \mathrm{e}^{-\sqrt{3}s}\left[\left|\xi^{[0]}(s)^*\right| + \left|\eta^{[0]}(s)^*\right| + \left|c^{[0]}(s)^*\right|\right] \mathrm{d}s \\
&\le \int_0^\tau \mathrm{e}^{-\sqrt{3}s}\left[\frac{\gamma}{\sqrt{3}}\mathrm{e}^{\sqrt{3}(s-\tau)} + \frac{\gamma}{\sqrt{3}}\mathrm{e}^{-\sqrt{3}s} + \frac{\gamma}{2\sqrt{3}}\tau^k\mathrm{e}^{-\epsilon\sqrt{3}\tau}\right] \mathrm{d}s \\
&= \frac{\gamma}{\sqrt{3}}\left[\tau\mathrm{e}^{-\sqrt{3}\tau} + \frac{1}{2\sqrt{3}}\left(1-\mathrm{e}^{-2\sqrt{3}\tau}\right) + \frac{1}{2}\tau^k\mathrm{e}^{-\epsilon\sqrt{3}\tau}\frac{1}{\sqrt{3}}\left(1-\mathrm{e}^{-\sqrt{3}\tau}\right)\right] \\
&\le \frac{\gamma}{\sqrt{3}}\left(\underbrace{\tau\mathrm{e}^{-\sqrt{3}\tau}}_{\le 1/2\sqrt{3}} + \frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{3}}\underbrace{\tau^k\mathrm{e}^{-\epsilon\sqrt{3}\tau}}_{\le 1}\right) \le \frac{\gamma}{2}
\end{aligned}$$

$$\begin{aligned}
I_2^+ &= \int_0^\tau \mathrm{e}^{-\sqrt{3}s}\left|z_+^{[0]}(s)^*\right| \mathrm{d}s \le \int_0^\tau \mathrm{e}^{-\sqrt{3}s}\left[\left|\xi_+^{[0]}(s)^*\right| + \left|\eta_+^{[0]}(s)^*\right|^2 + \left|c^{[0]}(s)^*\right|^2\right] \mathrm{d}s \\
&\le \int_0^\tau \mathrm{e}^{-\sqrt{3}s}\left[\frac{\gamma}{2\sqrt{3}}\mathrm{e}^{\sqrt{3}(s-\tau)} + \left(\frac{\gamma}{2\sqrt{3}}\tau^k\mathrm{e}^{-2\sqrt{3}\tau}\mathrm{e}^{-\sqrt{3}(s-\tau)}\right)^2\right] \mathrm{d}s \\
&\quad + \int_0^\tau \mathrm{e}^{-\sqrt{3}s}\left[\left(\frac{\gamma}{2\sqrt{3}}\tau^k\mathrm{e}^{-\epsilon\sqrt{3}\tau}\right)^2\right] \mathrm{d}s \\
&\le \frac{\gamma}{2\sqrt{3}}\left[\tau\mathrm{e}^{-\sqrt{3}\tau} + \frac{1}{2\sqrt{3}}\gamma\tau^{2k}\mathrm{e}^{-2\sqrt{3}\tau}\frac{1}{3\sqrt{3}}\left(1-\mathrm{e}^{-3\sqrt{3}\tau}\right)\right. \\
&\quad \left. + \frac{1}{2\sqrt{3}}\gamma\tau^{2k}\mathrm{e}^{-2\epsilon\sqrt{3}\tau}\frac{1}{\sqrt{3}}\left(1-\mathrm{e}^{-\sqrt{3}\tau}\right)\right] \\
&\le \frac{\gamma}{2\sqrt{3}}\tau\mathrm{e}^{-\sqrt{3}\tau}\left[1 + \frac{1}{18}\gamma\tau^{2k-1}\mathrm{e}^{-\sqrt{3}\tau} + \frac{1}{6}\gamma\tau^{2k-1}\mathrm{e}^{(1-2\epsilon)\sqrt{3}\tau}\right] \\
&\le \frac{3\gamma}{2\sqrt{3}}\tau\mathrm{e}^{-\sqrt{3}\tau} \le \gamma\tau\mathrm{e}^{-\sqrt{3}\tau}
\end{aligned}$$

$$
\begin{aligned}
I_3 &= \int_0^\tau e^{\sqrt{3}(s-\tau)} \left|z^{[0]}(s)^*\right| \,\mathrm{d}s \le \int_0^\tau e^{\sqrt{3}(s-\tau)} \left[\left|\xi^{[0]}(s)^*\right| + \left|\eta^{[0]}(s)^*\right| + \left|c^{[0]}(s)^*\right|\right] \,\mathrm{d}s \\
&\le \int_0^\tau e^{\sqrt{3}(s-\tau)} \left[\frac{\gamma}{\sqrt{3}} e^{\sqrt{3}(s-\tau)} + \frac{\gamma}{\sqrt{3}} e^{-\sqrt{3}s} + \frac{\gamma}{2\sqrt{3}} \tau^k e^{-\epsilon\sqrt{3}\tau}\right] \mathrm{d}s \\
&= \frac{\gamma}{\sqrt{3}} \left[\frac{1}{2\sqrt{3}}\left(1 - e^{-2\sqrt{3}\tau}\right) + \tau e^{-\sqrt{3}\tau} + \frac{1}{2}\tau^k e^{-\epsilon\sqrt{3}\tau} \frac{1}{\sqrt{3}}\left(1 - e^{-\sqrt{3}\tau}\right)\right] \\
&\le \frac{\gamma}{2}
\end{aligned}
$$

$$
\begin{aligned}
I_3^- &= \int_0^\tau e^{\sqrt{3}(s-\tau)} \left|z_-^{[0]}(s)^*\right| \,\mathrm{d}s \le \int_0^\tau e^{\sqrt{3}(s-\tau)} \left[\left|\xi_-^{[0]}(s)^*\right|^2 + \left|\eta_-^{[0]}(s)^*\right| + \left|c^{[0]}(s)^*\right|^2\right] \mathrm{d}s \\
&\le \int_0^\tau e^{\sqrt{3}(s-\tau)} \left[\left(\frac{\gamma}{2\sqrt{3}} \tau^k e^{-2\sqrt{3}\tau} e^{\sqrt{3}s}\right)^2 + \frac{\gamma}{2\sqrt{3}} e^{-\sqrt{3}s} + \left(\frac{\gamma}{2\sqrt{3}} \tau^k e^{-\epsilon\sqrt{3}\tau}\right)^2\right] \mathrm{d}s \\
&\le \frac{\gamma}{2\sqrt{3}} \left[\frac{1}{2\sqrt{3}} \gamma \tau^{2k e^{-2\sqrt{3}\tau}} \frac{1}{3\sqrt{3}} \left(1 - e^{-3\sqrt{3}\tau}\right) + \tau e^{-\sqrt{3}\tau}\right. \\
&\quad \left. + \frac{1}{2\sqrt{3}} \gamma \tau^{2k} e^{-2\epsilon\sqrt{3}\tau} \frac{1}{\sqrt{3}} \left(1 - e^{-\sqrt{3}\tau}\right)\right] \\
&\le \frac{3\gamma}{2\sqrt{3}} \tau e^{-\sqrt{3}\tau} \le \gamma \tau e^{-\sqrt{3}\tau}
\end{aligned}
$$

□

$$
\begin{aligned}
\left|\xi_-^{[1]}(t) - \xi_-^{[0]}(t)\right| &= \left|\int_0^t e^{\sqrt{3}(t-s)} R^{\xi_-}\left(z^{[0]}(s)\right) \mathrm{d}s\right| \le \int_0^t e^{\sqrt{3}(t-s)} \left|R^{\xi_-}\left(z^{[0]}(s)\right)\right| \mathrm{d}s \\
&\le \int_0^t e^{\sqrt{3}(t-s)} K e^{\sqrt{3}s} \left[\gamma_{\xi_-} \left|z^{[0]}(s)\right|^2 + \frac{1}{2\sqrt{3}} \gamma_{\eta_-} \gamma e^{-\sqrt{3}s} e^{-\sqrt{3}\tau} \left|z_+^{[0]}(s)\right|\right] \mathrm{d}s \\
&\le K e^{\sqrt{3}t} \left[\gamma_{\xi_-} \int_0^t \left|z^{[0]}(s)^*\right|^2 \mathrm{d}s + \frac{1}{2\sqrt{3}} \gamma_{\eta_-} \gamma e^{-\sqrt{3}\tau} \int_0^t e^{-\sqrt{3}s} \left|z_+^{[0]}(s)^*\right| \mathrm{d}s\right] \\
&\le K e^{\sqrt{3}t} \left[\gamma_{\xi_-} I_1 + \frac{1}{2\sqrt{3}} \gamma_{\eta_-} \gamma e^{-\sqrt{3}\tau} I_2^+\right] \\
&\le \frac{\gamma^2 K}{2\sqrt{3}} e^{\sqrt{3}t} \left[\gamma_{\xi_-} + \gamma_{\eta_-} \tau e^{-2\sqrt{3}\tau}\right]
\end{aligned}
$$

$$\begin{aligned}\left|\xi_+^{[1]}(t) - \xi_+^{[0]}(t)\right| &= \left|\int_\tau^t \mathrm{e}^{\sqrt{3}(t-s)} R^{\xi_+}\left(z^{[0]}(s)\right)\,\mathrm{d}s\right| \leq \int_t^\tau \mathrm{e}^{\sqrt{3}(t-s)}\left|R^{\xi_+}\left(z^{[0]}(s)\right)\right|\,\mathrm{d}s\\
&\leq \int_t^\tau \mathrm{e}^{\sqrt{3}(t-s)} K\mathrm{e}^{\sqrt{3}(s-\tau)}\left[\gamma_{\xi_+}\left|z^{[0]}(s)\right|^2 + \frac{1}{2\sqrt{3}}\gamma_{\eta_+}\gamma\mathrm{e}^{-\sqrt{3}s}\mathrm{e}^{\sqrt{3}\tau}\left|z^{[0]}(s)\right|\right]\mathrm{d}s\\
&\leq K\mathrm{e}^{\sqrt{3}(t-\tau)}\left[\gamma_{\xi_+}\int_t^\tau \left|z^{[0]}(s)^*\right|^2\,\mathrm{d}s + \frac{1}{2\sqrt{3}}\gamma_{\eta_+}\gamma\mathrm{e}^{\sqrt{3}\tau}\int_t^\tau \mathrm{e}^{-\sqrt{3}s}\left|z^{[0]}(s)^*\right|\,\mathrm{d}s\right]\\
&\leq K\mathrm{e}^{\sqrt{3}(t-\tau)}\left[\gamma_{\xi_+}I_1 + \frac{1}{2\sqrt{3}}\gamma_{\eta_+}\gamma\mathrm{e}^{\sqrt{3}\tau}I_2\right]\\
&\leq \frac{\gamma^2 K}{2\sqrt{3}}\mathrm{e}^{\sqrt{3}(t-\tau)}\left[\gamma_{\xi_+} + \frac{\gamma_{\eta_+}}{2}\mathrm{e}^{\sqrt{3}\tau}\right]\end{aligned}$$

Analogously

$$\left|\eta_-^{[1]}(t) - \eta_-^{[0]}(t)\right| \leq \frac{\gamma^2 K}{2\sqrt{3}}\mathrm{e}^{-\sqrt{3}t}\left[\gamma_{\eta_-} + \frac{\gamma_{\xi_-}}{2}\mathrm{e}^{\sqrt{3}\tau}\right]$$

$$\left|\eta_+^{[1]}(t) - \eta_+^{[0]}(t)\right| \leq \frac{\gamma^2 K}{2\sqrt{3}}\mathrm{e}^{-\sqrt{3}(t-\tau)}\left[\gamma_{\eta_+} + \gamma_{\xi_+}\tau\mathrm{e}^{-2\sqrt{3}\tau}\right]$$

Similarly

$$\left|R^{c_k}\left(z^{[0]}(s)\right)\right| \leq K\left|c_k^{[0]}(s)\right|\left|z^{[0]}(s)\right|^2 \leq K\left|\zeta_k\right|\left|z^{[0]}(s)\right|^2 \leq K\gamma_k\left|z^{[0]}(s)\right|^2,$$

so then

$$\begin{aligned}\left|c_k^{[1]}(t) - c_k^{[0]}(t)\right| &\leq \left|\int_0^t \mathrm{e}^{i(t-s)} R^{c_k}\left(z^{[0]}(s)\right)\,\mathrm{d}s\right| \leq \int_0^t \left|\mathrm{e}^{i(t-s)}\right|\left|R^{c_k}\left(z^{[0]}(s)\right)\right|\,\mathrm{d}s\\
&\leq K\gamma_k\int_0^t \left|z^{[0]}(s)^*\right|^2\,\mathrm{d}s \leq K\gamma_k I_1 \leq \frac{\gamma^2 K}{2\sqrt{3}}\gamma_k\end{aligned}$$

We write now the recursive Lemma that we are going to use the prove that we are dealing with a Cauchy sequence:

Lemma 20. *Assume* (Sh1) -(Sh6). *Then, the following bounds hold:*

$$\begin{aligned}\left|\xi_-^{[n+1]}(t) - \xi_-^{[n]}(t)\right| &\leq \frac{1}{3^{n+1}}\gamma e^{\sqrt{3}t}\left[\gamma_{\xi_-} + \gamma_{\eta_-}\tau e^{-2\sqrt{3}\tau}\right]\\
\left|\xi_+^{[n+1]}(t) - \xi_+^{[n]}(t)\right| &\leq \frac{1}{3^{n+1}}\gamma e^{\sqrt{3}(t-\tau)}\left[\gamma_{\xi_+} + \frac{\gamma_{\eta_+}}{2}e^{\sqrt{3}\tau}\right]\\
\left|\eta_-^{[n+1]}(t) - \eta_-^{[n]}(t)\right| &\leq \frac{1}{3^{n+1}}\gamma e^{-\sqrt{3}t}\left[\gamma_{\eta_-} + \frac{\gamma_{\xi_-}}{2}e^{\sqrt{3}\tau}\right]\\
\left|\eta_+^{[n+1]}(t) - \eta_+^{[n]}(t)\right| &\leq \frac{1}{3^{n+1}}\gamma e^{-\sqrt{3}(t-\tau)}\left[\gamma_{\eta_+} + \gamma_{\xi_+}\tau e^{-2\sqrt{3}\tau}\right]\\
\left|c_k^{[n+1]}(t) - c_k^{[n]}(t)\right| &\leq \frac{1}{3^{n+1}}\gamma\gamma_k\end{aligned}$$

In particular,

$$\left|\xi_-^{[n+1]}(t)\right| \leq \left[\gamma_{\xi_-} + \frac{1}{2}\gamma\left(\gamma_{\xi_-} + \gamma_{\eta_-}\tau e^{-2\sqrt{3}\tau}\right)\right] e^{\sqrt{3}t}$$
$$\left|\xi_+^{[n+1]}(t)\right| \leq \left[\gamma_{\xi_+} + \frac{1}{2}\gamma\left(\gamma_{\xi_+} + \frac{\gamma_{\eta_+}}{2} e^{\sqrt{3}\tau}\right)\right] e^{\sqrt{3}(t-\tau)}$$
$$\left|\eta_-^{[n+1]}(t)\right| \leq \left[\gamma_{\eta_-} + \frac{1}{2}\gamma\left(\gamma_{\eta_-} + \frac{\gamma_{\xi_-}}{2} e^{\sqrt{3}\tau}\right)\right] e^{-\sqrt{3}t}$$
$$\left|\eta_+^{[n+1]}(t)\right| \leq \left[\gamma_{\eta_+} + \frac{1}{2}\gamma\left(\gamma_{\eta_+} + \gamma_{\xi_+}\tau e^{-2\sqrt{3}\tau}\right)\right] e^{-\sqrt{3}(t-\tau)}$$
$$\left|c_k^{[n+1]}(t)\right| \leq \gamma_k + \frac{1}{2}\gamma\gamma_k$$

Proof. First of all we will prove the implication written in the lemma.

$$\begin{aligned}
\left|\xi_+^{[n+1]}(t)\right| &\leq \left|\xi_+^{[0]}(t)\right| + \sum_{i=0}^{n}\left|\xi_+^{[i+1]}(t) - \xi_+^{[i]}(t)\right| \\
&\leq \gamma_{\xi_+}\mathrm{e}^{\sqrt{3}(t-\tau)} + \sum_{i=0}^{n}\frac{1}{3^{i+1}}\gamma\mathrm{e}^{\sqrt{3}(t-\tau)}\left[\gamma_{\xi_+} + \frac{\gamma_{\eta_+}}{2}\mathrm{e}^{\sqrt{3}\tau}\right] \\
&\leq \left[\gamma_{\xi_+} + \gamma\left(\gamma_{\xi_+} + \frac{\gamma_{\eta_+}}{2}\mathrm{e}^{\sqrt{3}\tau}\right)\sum_{i=0}^{\infty}\frac{1}{3^{i+1}}\right]\mathrm{e}^{\sqrt{3}(t-\tau)} \\
&= \left[\gamma_{\xi_+} + \frac{1}{2}\gamma\left(\gamma_{\xi_+} + \frac{\gamma_{\eta_+}}{2}\mathrm{e}^{\sqrt{3}\tau}\right)\right]\mathrm{e}^{\sqrt{3}(t-\tau)}
\end{aligned}$$

This argument is analogous for the other coordinates. Now, as we did in the case of $n = 0$ we get the estimates:

$$\left|\xi_-^{[n+1]}(t)\right| \leq \frac{\gamma}{2\sqrt{3}}\alpha\tau^k\mathrm{e}^{-2\sqrt{3}\tau}\mathrm{e}^{\sqrt{3}t} = \alpha\left|\xi_-^{[0]}(t)^*\right|$$
$$\left|\xi_+^{[n+1]}(t)\right| \leq \frac{\gamma}{2\sqrt{3}}\alpha\mathrm{e}^{\sqrt{3}(t-\tau)} = \alpha\left|\xi_+^{[0]}(t)^*\right|$$

$$\left|\xi^{[n+1]}(t)\right| \leq \frac{\gamma}{\sqrt{3}}\alpha\mathrm{e}^{\sqrt{3}(t-\tau)} = \alpha\left|\xi^{[0]}(t)^*\right|$$

$$\left|\eta_-^{[n+1]}(t)\right| \leq \frac{\gamma}{2\sqrt{3}}\alpha\mathrm{e}^{-\sqrt{3}t} = \alpha\left|\eta_-^{[0]}(t)^*\right|$$
$$\left|\eta_+^{[n+1]}(t)\right| \leq \frac{\gamma}{2\sqrt{3}}\alpha\tau^k\mathrm{e}^{-2\sqrt{3}\tau}\mathrm{e}^{-\sqrt{3}(t-\tau)} = \alpha\left|\eta_+^{[0]}(t)^*\right|$$

$$\left|\eta^{[n+1]}(t)\right| \leq \frac{\gamma}{\sqrt{3}}\alpha\mathrm{e}^{-\sqrt{3}t} = \alpha\left|\eta^{[0]}(t)^*\right|$$

$$\left|c_k^{[n+1]}(t)\right| \leq \frac{\gamma}{2\sqrt{3}(N-3)}\alpha\tau^k \mathrm{e}^{-\epsilon\sqrt{3}\tau}$$
$$\left|c^{[n+1]}(t)\right| < \frac{\gamma}{2\sqrt{3}}\alpha\tau^k \mathrm{e}^{-\epsilon\sqrt{3}\tau} = \alpha\left|c^{[0]}(t)^*\right|$$

with $\alpha = 1+\gamma$.

Now we proceed by induction. We have already seen the case for $n=0$ if we take $\gamma_0 \leq \frac{2\sqrt{3}}{3K}$. Let us evaluate the expressions of Lemma 18 with $z_1 = z^{[n]}(s)$ and $z_2 = z^{[n-1]}(s)$, since, for instance:

$$\begin{aligned}\left|\xi_-^{[n+1]}(t) - \xi_-^{[n]}(t)\right| &= \left|\int_0^t \mathrm{e}^{\sqrt{3}(t-s)}\left[R^{\xi_-}\left(z^{[n]}(s)\right) - R^{\xi_-}\left(z^{[n-1]}(s)\right)\right]\mathrm{d}s\right| \\ &\leq \int_0^t \mathrm{e}^{\sqrt{3}(t-s)}\left|R^{\xi_-}\left(z^{[n]}(s)\right) - R^{\xi_-}\left(z^{[n-1]}(s)\right)\right|\mathrm{d}s.\end{aligned}$$

$$\begin{aligned}\left|\xi_+^{[n+1]}(t) - \xi_+^{[n]}(t)\right| &= \left|\int_\tau^t \mathrm{e}^{\sqrt{3}(t-s)}\left[R^{\xi_+}\left(z^{[n]}(s)\right) - R^{\xi_+}\left(z^{[n-1]}(s)\right)\right]\mathrm{d}s\right| \\ &\leq \int_t^\tau \mathrm{e}^{\sqrt{3}(t-s)}\left|R^{\xi_+}\left(z^{[n]}(s)\right) - R^{\xi_+}\left(z^{[n-1]}(s)\right)\right|\mathrm{d}s.\end{aligned}$$

$$\begin{aligned}\left|R^{\xi_\pm}\left(z^{[n]}(s)\right) - R^{\xi_\pm}\left(z^{[n-1]}(s)\right)\right| \leq K\Big\{&\rho_{\xi_\pm}^{\xi_\pm}\left(z^{[n]}(s), z^{[n-1]}(s)\right)\left|\xi_\pm^{[n]}(s) - \xi_\pm^{[n-1]}(s)\right| \\ &+ \rho_{\xi_\mp}^{\xi_\pm}\left(z^{[n]}(s), z^{[n-1]}(s)\right)\left|\xi_\mp^{[n]}(s) - \xi_\mp^{[n-1]}(s)\right| \\ &+ \rho_{\eta_\pm}^{\xi_\pm}\left(z^{[n]}(s), z^{[n-1]}(s)\right)\left|\eta_\pm^{[n]}(s) - \eta_\pm^{[n-1]}(s)\right| \\ &+ \rho_{\eta_\mp}^{\xi_\pm}\left(z^{[n]}(s), z^{[n-1]}(s)\right)\left|\eta_\mp^{[n]}(s) - \eta_\mp^{[n-1]}(s)\right| \\ &+ \rho_c^{\xi_\pm}\left(z^{[n]}(s), z^{[n-1]}(s)\right)\left|c^{[n]}(s) - c^{[n-1]}(s)\right|\Big\}\end{aligned}$$

with

$$\rho_{\xi_\pm}^{\xi_\pm}\left(z^{[n]}(s), z^{[n-1]}(s)\right) = \left|z^{[n]}(s)\right|^2 + \left|z^{[n-1]}(s)\right|^2$$

$$\begin{aligned}\rho_{\xi_+}^{\xi_-}\left(z^{[n]}(s), z^{[n-1]}(s)\right) &= \left(\left|\xi_-^{[n]}(s)\right| + \left|\xi_-^{[n-1]}(s)\right|\right)\left(\left|z^{[n]}(s)\right| + \left|z^{[n-1]}(s)\right|\right) \\ &+ \left(\left|\eta_-^{[n]}(s)\right| + \left|\eta_-^{[n-1]}(s)\right|\right)\left(\left|z_+^{[n]}(s)\right| + \left|z_+^{[n-1]}(s)\right|\right) \\ \rho_{\xi_-}^{\xi_+}\left(z^{[n]}(s), z^{[n-1]}(s)\right) &= \left(\left|\xi_+^{[n]}(s)\right| + \left|\xi_+^{[n-1]}(s)\right|\right)\left(\left|z^{[n]}(s)\right| + \left|z^{[n-1]}(s)\right|\right) \\ &+ \left(\left|\eta_+^{[n]}(s)\right| + \left|\eta_+^{[n-1]}(s)\right|\right)\left(\left|z^{[n]}(s)\right| + \left|z^{[n-1]}(s)\right|\right)\end{aligned}$$

$$\begin{aligned}\rho_{\eta_-}^{\xi_-}\left(z^{[n]}(s),z^{[n-1]}(s)\right) &= \left(\left|\xi_-^{[n]}(s)\right|+\left|\xi_-^{[n-1]}(s)\right|\right)\left(\left|z^{[n]}(s)\right|+\left|z^{[n-1]}(s)\right|\right)\\ &\quad+\left(\left|\xi_+^{[n]}(s)\right|+\left|\xi_+^{[n-1]}(s)\right|\right)\left(\left|z_+^{[n]}(s)\right|+\left|z_+^{[n-1]}(s)\right|\right)\\ \rho_{\eta_+}^{\xi_+}\left(z^{[n]}(s),z^{[n-1]}(s)\right) &= \left(\left|\xi_+^{[n]}(s)\right|+\left|\xi_+^{[n-1]}(s)\right|\right)\left(\left|z^{[n]}(s)\right|+\left|z^{[n-1]}(s)\right|\right)\\ &\quad+\left(\left|\xi_-^{[n]}(s)\right|+\left|\xi_-^{[n-1]}(s)\right|\right)\left(\left|z^{[n]}(s)\right|+\left|z^{[n-1]}(s)\right|\right)\end{aligned}$$

$$\begin{aligned}\rho_{\eta_+}^{\xi_-}\left(z^{[n]}(s)\ ,z^{[n-1]}(s)\right) &= \left(\left|\xi_-^{[n]}(s)\right|+\left|\xi_-^{[n-1]}(s)\right|\right)\left(\left|z^{[n]}(s)\right|+\left|z^{[n-1]}(s)\right|\right)\\ &\quad+\left(\left|\xi_+^{[n]}(s)\right|+\left|\xi_+^{[n-1]}(s)\right|\right)\left(\left|\eta_-^{[n]}(s)\right|+\left|\eta_-^{[n-1]}(s)\right|\right)\left(\left|\eta_+^{[n]}(s)\right|+\left|\eta_+^{[n-1]}(s)\right|\right)\\ \rho_{\eta_-}^{\xi_+}\left(z^{[n]}(s)\ ,z^{[n-1]}(s)\right) &= \left(\left|\xi_+^{[n]}(s)\right|+\left|\xi_+^{[n-1]}(s)\right|\right)\left(\left|z^{[n]}(s)\right|+\left|z^{[n-1]}(s)\right|\right)\\ &\quad+\left(\left|\xi_-^{[n]}(s)\right|+\left|\xi_-^{[n-1]}(s)\right|\right)\left(\left|\eta_+^{[n]}(s)\right|+\left|\eta_+^{[n-1]}(s)\right|\right)\end{aligned}$$

$$\begin{aligned}\rho_c^{\xi_-}\left(z^{[n]}(s)\ ,z^{[n-1]}(s)\right) &= \left(\left|\xi_-^{[n]}(s)\right|+\left|\xi_-^{[n-1]}(s)\right|\right)\left(\left|z^{[n]}(s)\right|+\left|z^{[n-1]}(s)\right|\right)\\ &\quad+\left(\left|\xi_+^{[n]}(s)\right|+\left|\xi_+^{[n-1]}(s)\right|\right)\left(\left|\eta_-^{[n]}(s)\right|+\left|\eta_-^{[n-1]}(s)\right|\right)\left(\left|c^{[n]}(s)\right|+\left|c^{[n-1]}(s)\right|\right)\\ \rho_c^{\xi_+}\left(z^{[n]}(s)\ ,z^{[n-1]}(s)\right) &= \left(\left|\xi_+^{[n]}(s)\right|+\left|\xi_+^{[n-1]}(s)\right|\right)\left(\left|z^{[n]}(s)\right|+\left|z^{[n-1]}(s)\right|\right)\\ &\quad+\left(\left|\xi_-^{[n]}(s)\right|+\left|\xi_-^{[n-1]}(s)\right|\right)\left(\left|\eta_+^{[n]}(s)\right|+\left|\eta_+^{[n-1]}(s)\right|\right)\end{aligned}$$

and

$$\begin{aligned}\left|\xi_-^{[n]}(s)-\xi_-^{[n-1]}(s)\right| &\le \frac{1}{3^n}\gamma e^{\sqrt{3}s}\left[\gamma_{\xi_-}+\gamma_{\eta_-}\tau^k e^{-2\sqrt{3}\tau}\right]\le\frac{1}{3^n}\frac{1}{\sqrt{3}}\gamma^2\tau e^{-\sqrt{3}\tau}e^{\sqrt{3}(s-\tau)}\\ \left|\xi_+^{[n]}(s)-\xi_+^{[n-1]}(s)\right| &\le \frac{1}{3^n}\gamma e^{\sqrt{3}(s-\tau)}\left[\gamma_{\xi_+}+\frac{\gamma_{\eta_+}}{2}e^{\sqrt{3}\tau}\right]\le\frac{1}{3^n}\frac{3}{4\sqrt{3}}\gamma^2 e^{\sqrt{3}(s-\tau)}\\ \left|\eta_-^{[n]}(s)-\eta_-^{[n-1]}(s)\right| &\le \frac{1}{3^n}\gamma e^{-\sqrt{3}s}\left[\gamma_{\eta_-}+\frac{\gamma_{\xi_-}}{2}e^{\sqrt{3}\tau}\right]\le\frac{1}{3^n}\frac{3}{4\sqrt{3}}\gamma^2 e^{-\sqrt{3}s}\\ \left|\eta_+^{[n]}(s)-\eta_+^{[n-1]}(s)\right| &\le \frac{1}{3^n}\gamma e^{-\sqrt{3}(s-\tau)}\left[\gamma_{\eta_+}+\gamma_{\xi_+}\tau e^{-2\sqrt{3}\tau}\right]\le\frac{1}{3^n}\frac{1}{\sqrt{3}}\gamma^2\tau^k e^{-\sqrt{3}\tau}e^{-\sqrt{3}s}\\ \left|c^{[n]}(s)-c^{[n-1]}(s)\right| &\le \frac{1}{3^n}\gamma\gamma_c\end{aligned}$$

where $\gamma_c=\sum\gamma_k$.

$$\begin{aligned}&\rho_{\xi_-}^{\xi_-}\left(z^{[n]}(s),z^{[n-1]}(s)\right)\left|\xi_-^{[n]}(s)-\xi_-^{[n-1]}(s)\right|\\ &\qquad\le\frac{1}{3^n}\gamma e^{\sqrt{3}s}\left[\gamma_{\xi_-}+\tau e^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right]\left(\left|z^{[n]}(s)\right|^2+\left|z^{[n-1]}(s)\right|^2\right)\end{aligned}$$

$$\rho^{\xi_-}_{\xi_+}\left(z^{[n]}(s),z^{[n-1]}(s)\right)\left|\xi^{[n]}_+(s)-\xi^{[n-1]}_+(s)\right|\leq$$
$$\frac{1}{3^n}\gamma e^{\sqrt{3}s}\left\{\frac{3}{2\sqrt{3}}\gamma e^{\sqrt{3}(s-\tau)}\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\xi_-}+\frac{1}{2}\gamma\tau e^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right]\left(\left|z^{[n]}(s)\right|+\left|z^{[n-1]}(s)\right|\right)\right.$$
$$\left.+\frac{3}{2\sqrt{3}}\gamma e^{-\sqrt{3}\tau}e^{-\sqrt{3}s}\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\eta_-}+\frac{1}{4}\gamma e^{\sqrt{3}\tau}\gamma_{\xi_-}\right]\left(\left|z^{[n]}_+(s)\right|+\left|z^{[n-1]}_+(s)\right|\right)\right\}$$

$$\rho^{\xi_-}_{\eta_-}\left(z^{[n]}(s),z^{[n-1]}(s)\right)\left|\eta^{[n]}_-(s)-\eta^{[n-1]}_-(s)\right|\leq$$
$$\frac{1}{3^n}\gamma e^{\sqrt{3}s}\left\{\frac{3}{2\sqrt{3}}\gamma e^{-\sqrt{3}s}\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\xi_-}+\frac{1}{2}\gamma\tau e^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right]\left(\left|z^{[n]}(s)\right|+\left|z^{[n-1]}(s)\right|\right)\right.$$
$$\left.+\frac{2}{\sqrt{3}}\alpha\gamma e^{-\sqrt{3}\tau}e^{-\sqrt{3}s}\left[\gamma_{\eta_-}+\frac{1}{2}e^{\sqrt{3}\tau}\gamma_{\xi_-}\right]\left(\left|z^{[n]}_+(s)\right|+\left|z^{[n-1]}_+(s)\right|\right)\right\}$$

$$\rho^{\xi_-}_{\eta_+}\left(z^{[n]}(s),z^{[n-1]}(s)\right)\left|\eta^{[n]}_+(s)-\eta^{[n-1]}_+(s)\right|\leq$$
$$\frac{1}{3^n}\gamma e^{\sqrt{3}s}\left\{\frac{2\gamma\tau}{\sqrt{3}}e^{-\sqrt{3}\tau}e^{-\sqrt{3}s}\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\xi_-}+\frac{1}{2}\gamma\tau e^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right]\left(\left|z^{[n]}(s)\right|+\left|z^{[n-1]}(s)\right|\right)\right.$$
$$\left.+\frac{4}{\sqrt{3}}\alpha^2\gamma e^{-\sqrt{3}\tau}e^{-\sqrt{3}s}\left|\eta^{[0]}_+(s)^*\right|^2\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\eta_-}+\frac{1}{4}\gamma e^{\sqrt{3}\tau}\gamma_{\xi_-}\right]\right\}$$

$$\rho^{\xi_-}_{c}\left(z^{[n]}(s)\ ,z^{[n-1]}(s)\right)\left|c^{[n]}(s)-c^{[n-1]}(s)\right|\leq$$
$$\frac{1}{3^n}\gamma e^{\sqrt{3}s}\left\{2\gamma_c\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\xi_-}+\frac{1}{2}\gamma\tau e^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right]\left(\left|z^{[n]}(s)\right|+\left|z^{[n-1]}(s)\right|\right)\right.$$
$$\left.+\frac{4}{\sqrt{3}}\alpha^2\gamma\gamma_c^2e^{-\sqrt{3}\tau}e^{-\sqrt{3}s}\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\eta_-}+\frac{1}{4}\gamma e^{\sqrt{3}\tau}\gamma_{\xi_-}\right]\right\}$$

Remark. Notice that we have $e^{\sqrt{3}s}$ as a common factor. It will cancel the $e^{-\sqrt{3}s}$ in the integral.

Lemma 21.

$$I_4=\int_0^\tau\left[\left|z^{[n]}(s)\right|^2+\left|z^{[n-1]}(s)\right|^2\right]ds\leq\frac{1}{\sqrt{3}}\alpha^2\gamma^2$$

$$I_5 = \int_0^\tau e^{\sqrt{3}(s-\tau)} \left[\left| z^{[n]}(s) \right| + \left| z^{[n-1]}(s) \right| \right] \, ds \leq \alpha\gamma$$

$$I_6 = \int_0^\tau e^{-\sqrt{3}s} \left[\left| z^{[n]}(s) \right| + \left| z^{[n-1]}(s) \right| \right] \, ds \leq \alpha\gamma$$

$$I_7 = \gamma_c \int_0^\tau \left[\left| z^{[n]}(s) \right| + \left| z^{[n-1]}(s) \right| \right] \, ds \leq \frac{1}{\sqrt{3}} \alpha\gamma^2$$

$$I_5^- = \int_0^\tau e^{\sqrt{3}(s-\tau)} \left[\left| z_-^{[n]}(s) \right| + \left| z_-^{[n-1]}(s) \right| \right] \, ds \leq 2\alpha\gamma\tau e^{-\sqrt{3}\tau}$$

$$I_6^+ = \int_0^\tau e^{-\sqrt{3}s} \left[\left| z_+^{[n]}(s) \right| + \left| z_+^{[n-1]}(s) \right| \right] \, ds \leq 2\alpha\gamma\tau e^{-\sqrt{3}\tau}$$

Proof. It is very easy to see that $I_4 = 2\alpha I_1$. As the rest of integrals. On the other hand,

$$\begin{aligned}
I_7 &= \gamma_c \int_0^\tau \left[\left| z^{[n]}(s) \right| + \left| z^{[n-1]}(s) \right| \right] \mathrm{d}s \\
&\leq 2\gamma_c \int_0^\tau \left[\frac{1}{\sqrt{3}} \alpha\gamma \mathrm{e}^{\sqrt{3}(s-\tau)} + \frac{1}{\sqrt{3}} \alpha\gamma \mathrm{e}^{-\sqrt{3}s} + \frac{1}{2\sqrt{3}} \alpha\gamma\tau^k \mathrm{e}^{-\epsilon\sqrt{3}\tau} \right] \mathrm{d}s \\
&\leq \frac{2}{\sqrt{3}} \alpha\gamma\gamma_c \left[\frac{1}{\sqrt{3}} \left(1 - \mathrm{e}^{-\sqrt{3}\tau} \right) + \frac{1}{\sqrt{3}} \left(1 - \mathrm{e}^{-\sqrt{3}\tau} \right) + \frac{1}{2} \tau^{k+1} \mathrm{e}^{-\epsilon\sqrt{3}\tau} \right] \\
&\leq \frac{2}{\sqrt{3}} \alpha\gamma\gamma_c \left[\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right] \\
&\leq 2\alpha\gamma\gamma_c \leq 2\alpha\gamma \frac{1}{2\sqrt{3}} \gamma\tau^k \mathrm{e}^{-\epsilon\sqrt{3}\tau} \leq \frac{1}{\sqrt{3}} \alpha\gamma^2 \tau^k \mathrm{e}^{-\epsilon\sqrt{3}\tau} \leq \frac{1}{\sqrt{3}} \alpha\gamma^2.
\end{aligned}$$

providing that

$$\frac{\sqrt{3}}{2} \tau^{k+1} \mathrm{e}^{-\epsilon\sqrt{3}\tau}.$$

□

After computing the integrals we get:

$$
\begin{aligned}
\left|\xi_-^{[n+1]}(t) - \xi_-^{[n]}(t)\right| \leq & \frac{1}{3^n}\gamma K \mathrm{e}^{\sqrt{3}t}\left\{\left[\gamma_{\xi_-} + \tau \mathrm{e}^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right] I_4\right. \\
& + \frac{3}{2\sqrt{3}}\gamma\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\xi_-} + \frac{1}{2}\gamma\tau \mathrm{e}^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right] I_5 \\
& + \frac{3}{2\sqrt{3}}\gamma \mathrm{e}^{-\sqrt{3}\tau}\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\eta_-} + \frac{1}{4}\gamma \mathrm{e}^{\sqrt{3}\tau}\gamma_{\xi_-}\right] I_6^+ \\
& + \frac{3}{2\sqrt{3}}\gamma\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\xi_-} + \frac{1}{2}\gamma\tau \mathrm{e}^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right] I_6 \\
& + \frac{2}{\sqrt{3}}\alpha\gamma \mathrm{e}^{-\sqrt{3}\tau}\left[\gamma_{\eta_-} + \frac{1}{2}\mathrm{e}^{\sqrt{3}\tau}\gamma_{\xi_-}\right] I_6^+ \\
& + \frac{2}{\sqrt{3}}\gamma\tau \mathrm{e}^{-\sqrt{3}\tau}\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\xi_-} + \frac{1}{2}\gamma\tau \mathrm{e}^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right] I_6 \\
& + \frac{4}{\sqrt{3}}\alpha^2\gamma \mathrm{e}^{-\sqrt{3}\tau}\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\eta_-} + \frac{1}{4}\gamma \mathrm{e}^{\sqrt{3}\tau}\gamma_{\xi_-}\right]\int_0^\tau \mathrm{e}^{-\sqrt{3}s}\left|\eta_+^{[0]}(s)^*\right|^2 \mathrm{d}s \\
& + 2\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\xi_-} + \frac{1}{2}\gamma\tau \mathrm{e}^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right] I_7 \\
& \left. + \frac{4}{\sqrt{3}}\alpha^2\gamma \mathrm{e}^{-\sqrt{3}\tau}\left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\eta_-} + \frac{1}{4}\gamma \mathrm{e}^{\sqrt{3}\tau}\gamma_{\xi_-}\right]\gamma_c^2\int_0^\tau \mathrm{e}^{-\sqrt{3}s}\right\} \\
\leq & \frac{1}{3^n}\gamma K \mathrm{e}^{\sqrt{3}t}\left\{\left[\gamma_{\xi_-} + \tau \mathrm{e}^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right] I_4\right. \\
& + \left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\xi_-} + \frac{1}{2}\gamma\tau \mathrm{e}^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right]\left(\frac{3}{2\sqrt{3}}\gamma I_5 + \frac{3}{2\sqrt{3}}\gamma I_6\right) \\
& + \left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\xi_-} + \frac{1}{2}\gamma\tau \mathrm{e}^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right]\left(\frac{2}{\sqrt{3}}\gamma\tau \mathrm{e}^{-\sqrt{3}\tau} I_6 + 2I_7\right) \\
& + \left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\eta_-} + \frac{1}{4}\gamma \mathrm{e}^{\sqrt{3}\tau}\gamma_{\xi_-}\right]\left(\frac{3}{2\sqrt{3}}\gamma \mathrm{e}^{-\sqrt{3}\tau} I_6^+ + \frac{2}{3}\alpha^2\gamma^2\tau \mathrm{e}^{-2\sqrt{3}\tau}\right) \\
& + \left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\eta_-} + \frac{1}{4}\gamma \mathrm{e}^{\sqrt{3}\tau}\gamma_{\xi_-}\right]\left(\frac{4}{3}\alpha^2\gamma \mathrm{e}^{-\sqrt{3}\tau}\gamma_c^2\right) \\
& \left. + \left[\gamma_{\eta_-} + \frac{1}{2}\mathrm{e}^{\sqrt{3}\tau}\gamma_{\xi_-}\right]\frac{2}{\sqrt{3}}\alpha\gamma \mathrm{e}^{-\sqrt{3}\tau} I_6^+\right\} \\
\leq & \frac{1}{3^n}\gamma K \mathrm{e}^{\sqrt{3}t}\left\{\left[\gamma_{\xi_-} + \tau \mathrm{e}^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right]\frac{1}{\sqrt{3}}\alpha^2\gamma^2\right. \\
& + \left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\xi_-} + \frac{1}{2}\gamma\tau \mathrm{e}^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right]\frac{6}{\sqrt{3}}\alpha\gamma^2 \\
& + \left[\left(1+\frac{1}{2}\gamma\right)\gamma_{\eta_-} + \frac{1}{4}\gamma \mathrm{e}^{\sqrt{3}\tau}\gamma_{\xi_-}\right]\frac{3}{2}\alpha\gamma^2\tau \mathrm{e}^{-2\sqrt{3}\tau}(1+\alpha) \\
& \left. + \left[\gamma_{\eta_-} + \frac{1}{2}\mathrm{e}^{\sqrt{3}\tau}\gamma_{\xi_-}\right]\frac{4}{\sqrt{3}}\alpha^2\gamma^2\tau \mathrm{e}^{-2\sqrt{3}\tau}\right\} \\
\leq & \frac{1}{3^n}\gamma K \mathrm{e}^{\sqrt{3}t}\frac{1}{\sqrt{3}}\alpha\gamma^2\left[\gamma_{\xi_-} + \tau \mathrm{e}^{-2\sqrt{3}\tau}\gamma_{\eta_-}\right] 14(1+\gamma) \leq \frac{1}{3^{n+1}}\gamma \mathrm{e}^{\sqrt{3}t}
\end{aligned}
$$

if

$$\frac{14K}{\sqrt{3}}\gamma^2(1+\gamma)^2 \le \frac{1}{3}.$$

Since

$$K \le \frac{2\sqrt{3}}{3\gamma},$$

we can ask

$$28\gamma(1+\gamma)^2 \le 1.$$

The situation for ξ_+, η_- and η_+ is completely equivalent. The only discussion left is the case for c.

$$\begin{aligned}
|R^{c_k}\ \left(z^{[n]}(s)\right) - R^{c_k}\left(z^{[n-1]}(s)\right)| \le K &\left\{ \left(\left|z^{[n]}(s)\right|^2 + \left|z^{[n-1]}(s)\right|^2 \right) \left|c_k^{[n]}(s) - c_k^{[n-1]}(s)\right| \right.\\
&+ \left(\left|c_k^{[n]}(s)\right| + \left|c_k^{[n-1]}(s)\right| \right) \left(\left|z^{[n]}(s)\right| + \left|z^{[n-1]}(s)\right| \right)\\
&\left(\left|\xi_-^{[n]}(s) - \xi_-^{[n-1]}(s)\right| + \left|\xi_+^{[n]}(s) - \xi_+^{[n-1]}(s)\right| \right.\\
&\left.\left. + \left|\eta_-^{[n]}(s) - \eta_-^{[n-1]}(s)\right| + \left|\eta_+^{[n]}(s) - \eta_+^{[n-1]}(s)\right| + \sum_{l\neq k} \left|c_l^{[n]}(s) - c_l^{[n-1]}(s)\right| \right) \right\}\\
\le K &\left\{ \left(\left|z^{[n]}(s)\right|^2 + \left|z^{[n-1]}(s)\right|^2 \right) \frac{1}{3^n}\gamma\gamma_k \right.\\
&+ 2\gamma_k \left(1 + \frac{1}{2}\gamma\right) \left(\left|z^{[n]}(s)\right| + \left|z^{[n-1]}(s)\right| \right)\\
&\left(\frac{1}{3^n}\frac{1}{\sqrt{3}}\gamma^2\tau \mathrm{e}^{-\sqrt{3}\tau}\mathrm{e}^{\sqrt{3}(s-\tau)} + \frac{1}{3^n}\frac{3}{4\sqrt{3}}\gamma^2\mathrm{e}^{\sqrt{3}(s-\tau)} \right.\\
&\left.\left. + \frac{1}{3^n}\frac{3}{4\sqrt{3}}\gamma^2\mathrm{e}^{-\sqrt{3}s} + \frac{1}{3^n}\frac{1}{\sqrt{3}}\gamma^2\tau\mathrm{e}^{-\sqrt{3}\tau}\mathrm{e}^{-\sqrt{3}s} + \sum_{l\neq k}\frac{1}{3^n}\gamma\gamma_l \right) \right\}\\
\le \frac{1}{3^n}\gamma\gamma_k K &\left\{ \left(\left|z^{[n]}(s)\right|^2 + \left|z^{[n-1]}(s)\right|^2 \right) \right.\\
&+ 2\left(1 + \frac{1}{2}\gamma\right) \left(\left|z^{[n]}(s)\right| + \left|z^{[n-1]}(s)\right| \right)\\
&\left.\left(\frac{3}{2\sqrt{3}}\gamma \mathrm{e}^{\sqrt{3}(s-\tau)} + \frac{3}{2\sqrt{3}}\gamma\mathrm{e}^{-\sqrt{3}s} + \gamma_c \right) \right\}
\end{aligned}$$

$$\begin{aligned}
\left|c_k^{[n+1]}(s) - c_k^{[n]}(s)\right| &\le \frac{1}{3^n}\gamma\gamma_k K \left\{ I_4 + 2\left(1+\frac{1}{2}\gamma\right)\left(\frac{3}{2\sqrt{3}}\gamma I_5 + \frac{3}{2\sqrt{3}}\gamma I_6 + I_7\right) \right\}\\
&\le \frac{1}{3^n}\gamma\gamma_k K \left\{ \frac{1}{\sqrt{3}}\alpha^2\gamma^2 + 2\left(1+\frac{1}{2}\gamma\right)\frac{4}{\sqrt{3}}\alpha\gamma^2 \right\}\\
&\le \frac{1}{3^n}\gamma\gamma_k K\alpha^2\gamma^2\frac{9}{\sqrt{3}} \le \frac{1}{3^{n+1}}\gamma\gamma_k
\end{aligned}$$

□

Once we have proved the lemma, we see that we have a Cauchy sequence, so then is convergent to a limit $(\xi(t), \eta(t), c(t))$ that corresponds to the (unique) solution of the Shilnikov problem. In addition, this limit must satisfy the same bounds, that is:

$$\begin{aligned}
|\xi_-(t)| &\leq \left[\gamma_{\xi_-} + \frac{1}{2}\gamma\left(\gamma_{\xi_-} + \gamma_{\eta_-}\tau e^{-2\sqrt{3}\tau}\right)\right] e^{\sqrt{3}t} \\
|\xi_+(t)| &\leq \left[\gamma_{\xi_+} + \frac{1}{2}\gamma\left(\gamma_{\xi_+} + \frac{\gamma_{\eta_+}}{2} e^{\sqrt{3}\tau}\right)\right] e^{\sqrt{3}(t-\tau)} \\
|\eta_-(t)| &\leq \left[\gamma_{\eta_-} + \frac{1}{2}\gamma\left(\gamma_{\eta_-} + \frac{\gamma_{\xi_-}}{2} e^{\sqrt{3}\tau}\right)\right] e^{-\sqrt{3}t} \\
|\eta_+(t)| &\leq \left[\gamma_{\eta_+} + \frac{1}{2}\gamma\left(\gamma_{\eta_+} + \gamma_{\xi_+}\tau e^{-2\sqrt{3}\tau}\right)\right] e^{-\sqrt{3}(t-\tau)} \\
|c_k(t)| &\leq \gamma_k + \frac{1}{2}\gamma\gamma_k
\end{aligned}$$

However, we can also say more, we can give estimates on the deviation respect to the linear flow:

$$\begin{aligned}
\left|\xi_-(t) - \xi_{-,0} e^{\sqrt{3}t}\right| &\leq \frac{1}{2}\gamma\left(\gamma_{\xi_-} + \gamma_{\eta_-}\tau e^{-2\sqrt{3}\tau}\right) e^{\sqrt{3}t} \\
\left|\eta_-(t) - \eta_{-,0} e^{-\sqrt{3}t}\right| &\leq \frac{1}{2}\gamma\left(\gamma_{\eta_-} + \frac{1}{2}\gamma_{\xi_-} e^{\sqrt{3}\tau}\right) e^{-\sqrt{3}t} \\
\left|\xi_+(t) - \xi_{+,1} e^{\sqrt{3}(t-\tau)}\right| &\leq \frac{1}{2}\gamma\left(\gamma_{\xi_+} + \frac{1}{2}\gamma_{\eta_+} e^{\sqrt{3}\tau}\right) e^{\sqrt{3}(t-\tau)} \\
\left|\eta_+(t) - \eta_{+,1} e^{-\sqrt{3}(t-\tau)}\right| &\leq \frac{1}{2}\gamma\left(\gamma_{\eta_+} + \gamma_{\xi_+}\tau e^{-2\sqrt{3}\tau}\right) e^{-\sqrt{3}(t-\tau)} \\
\left|c_k(t) - \zeta_k e^{it}\right| &\leq \frac{1}{2}\gamma\gamma_k
\end{aligned}$$

C.4 Proof of Proposition 7

For our argument we need lower bounds of the solution for the free components and upper bounds for the fixed ones.

We are going to use strongly the fact that if we take some mode equal to zero at the beginning it will be always zero and that since we are working on a compact set, the functions that define our system are globally bounded.

We first consider the center modes c, and get crude estimates that are not particularized for solutions close to the heteroclinic because it is enough.

For $k \neq j \pm 2$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|c_k|^2 = \frac{\mathrm{d}}{\mathrm{d}t}[c_k\bar{c}_k] &= \frac{\mathrm{d}c_k}{\mathrm{d}t}\bar{c}_k + c_k\frac{\mathrm{d}\bar{c}_k}{\mathrm{d}t} = \frac{\mathrm{d}c_k}{\mathrm{d}t}\bar{c}_k + c_k\overline{\frac{\mathrm{d}c_k}{\mathrm{d}t}} = 2\mathrm{Re}\left\{\frac{\mathrm{d}c_k}{\mathrm{d}t}\bar{c}_k\right\}\\ &= 2\mathrm{Re}\left\{i|c_k|^2 + \bar{c}_k R_{ell}^{c_k}(c) + \bar{c}_k R_{mix}^{c_k}(x,y,c)\right\}\\ &= 2\mathrm{Re}\left\{\bar{c}_k\left(-ic_k\left[|c_k|^2 + \sum_{l\in\mathcal{P}_j}|c_l|^2\right] + 2i\bar{c}_k\left[c_{k-1}^2 + c_{k+1}^2\right]\right) + \bar{c}_k(-ic_k)w(x,y)\right\}\\ &= 2\mathrm{Re}\left\{2i\bar{c}_k^2\left(c_{k-1}^2 + c_{k+1}^2\right)\right\} = -4\mathrm{Im}\left\{\bar{c}_k^2\left(c_{k-1}^2 + c_{k+1}^2\right)\right\}\end{aligned}$$

where we have used that $w(x,y)$ is a real function.

Then

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}|c_k|^2\right| = 4\left|\mathrm{Im}\left\{\bar{c}_k^2\left(c_{k-1}^2 + c_{k+1}^2\right)\right\}\right| < 4|c_k|^2.$$

Then,

$$\mathrm{e}^{-4t}|c_k(0)|^2 \leq |c_k(t)|^2 \leq \mathrm{e}^{4t}|c_k(0)|^2.$$

The argument is analogous for $c_{j\pm 2}$, so we obtain

$$\mathrm{e}^{-2t}|c_k(0)| \leq |c_k(t)| \leq \mathrm{e}^{2t}|c_k(0)| \qquad \text{for all } k \in \{1,\ldots,j-1,j+1,\ldots,N\}.$$

For the past hyperbolic coordinates we use a similar argument. Write the system as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_- \\ y_- \end{pmatrix} = A(t)\begin{pmatrix} x_- \\ y_- \end{pmatrix}.$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}||(x_-,y_-)||_\infty \leq ||A(t)||_\infty||(x_-,y_-)||_\infty.$$

As we have said using that we are working in a compact set we can determine $\tilde{K}_-^- > 0$ such that

$$||A(t)||_\infty \leq \tilde{K}_-^-,$$

so

$$||(x_-,y_-)(t)||_\infty \leq \mathrm{e}^{\tilde{K}_-^- t}||(x_-,y_-)(0)||_\infty.$$

Finally we want to express a similar bound for the pair defined by $(x_+ - x_+^h(t), y_+)$. However, it is not true that the subspace $\{x_+ = x_+^h(t), y_+ = 0\}$ is invariant. For that pair of coordinates we are going to use the Gronwall's inequality:

Lemma 22 (Gronwall's inequality). *Let η be a nonnegative, absolutely continuous function on $[0,T]$ which satisfies for a.e. t the differential inequality*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$, $\psi(t)$ are nonnegative on $[0,T]$. Then

$$\eta(t) \leq e^{\int_0^t \phi(s)\,ds}\left[\eta(0) + \int_0^t \psi(s)ds\right]$$

for all $0 \leq t \leq T$.

The equation for y_+ can be written as

$$\dot{y}_+ = a_1(t)y_+ + a_2(t),$$

$$|a_2(t)| \leq |a_{21}(t)| ||(x_-, y_-)(t)||_\infty^2 + |a_{22}(t)||c_{j+2}(t)|^2.$$

Then, using the previous bounds and the fact that all the functions $a_1(t)$, $a_{21}(t)$ and $a_{22}(t)$ are bounded because we are working on a compact set, we obtain, using Gronwall's Lemma:

$$|y_+(T)| \leq K_+^{y_+}|y_+(0)| + K_-^{y_+}||(x_-, y_-)(0)||_\infty^2 + K_{j+2}^{y_+}|c_{j+2}(0)|^2$$

for some constants that only depend on σ.

Finally, for the component $x_+ - x_+^h(t)$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_+ - x_+^h(t)) = a_3(t)(x_+ - x_+^h(t)) + a_4(t),$$

where

$$|a_4(t)| \leq |a_{41}(t)||y_+(t)| + |a_{42}(t)| ||(x_-, y_-)(t)||_\infty^2 + \sum_{l \in \mathcal{P}_j} |a_{4,l}(t)||c_l(t)|^2.$$

Using an analogous argument through the Gronwall's inequality, we obtain the desired bound.

C.5 Proof of Lemmas 7 and 14

We denote $(x, y, c...)$ to the coordinates referred to the j mode and $(\tilde{x}, \tilde{y}, \tilde{c}...)$ to the ones referred to the $j+1$ mode. Then, we have

$$b_j = r\mathrm{e}^{i\theta} \quad b_k = c_k\mathrm{e}^{i\theta} \qquad b_{j+1} = \tilde{r}\mathrm{e}^{i\tilde{\theta}} \quad b_k = \tilde{c}_k\mathrm{e}^{i\tilde{\theta}}$$

If $c = \omega y + \omega^2 x$, then $\bar{c} = \omega^2 y + \omega x$ and inverting these relations we obtain:

$$y = -\frac{\omega^2}{\omega^2 - 1}c + \frac{1}{\omega^2 - 1}\bar{c} \qquad x = \frac{1}{\omega^2 - 1}c - \frac{\omega^2}{\omega^2 - 1}\bar{c}. \tag{C.9}$$

Through these changes the norms can be computed as:

$$|c| = x^2 - xy + y^2 = ||(x, y)||_*$$

This notation defines a norm in $\mathbb{R}^2$ equivalent to $||.||_2$ since:

$$-\frac{1}{2}\left(x^2 + y^2\right) \leq xy \leq \frac{1}{2}\left(x^2 + y^2\right),$$

we have:

$$\frac{1}{2}||(x,y)||_2^2 = x^2+y^2-\frac{1}{2}\left(x^2+y^2\right) \leq ||(x,y)||_* \leq x^2+y^2+\frac{1}{2}\left(x^2+y^2\right) = \frac{3}{2}||(x,y)||_2.$$

We will also use:

$$\frac{\sqrt{2}}{2}||(x,y)||_2 \leq ||(x,y)||_\infty \leq ||(x,y)||_2.$$

Now we have

$$c_{j+2} = b_{j+2}\mathrm{e}^{-i\theta} = \tilde{c}_{j+2}\mathrm{e}^{i(\theta-\tilde{\theta})} = \mathrm{e}^{i(\theta-\tilde{\theta})}\left(\omega\tilde{y}_+ + \omega^2\tilde{x}_+\right),$$

That means

$$||c_{j+2}|| = ||\tilde{c}_{j+2}|| = ||(\tilde{x}_+,\tilde{y}_+)||_*,$$

$$|c_{j+2}| \leq \sqrt{\frac{3}{2}}||(\tilde{x}_+,\tilde{y}_+)||_2 \leq \sqrt{3}||(\tilde{x}_+,\tilde{y}_+)||_\infty$$

Now we have an estimate but we want to compute the exact change. Taking (C.9) we have:

$$\begin{aligned}
\tilde{y}_+ &= -\frac{\omega^2}{\omega^2-1}\tilde{c}_{j+2} + \frac{1}{\omega^2-1}\bar{\tilde{c}}_{j+2} \\
&= -\frac{\omega^2}{\omega^2-1}b_{j+2}\mathrm{e}^{-i\tilde{\theta}} + \frac{1}{\omega^2-1}\bar{b}_{j+2}\mathrm{e}^{i\tilde{\theta}} \\
&= -\frac{\omega^2}{\omega^2-1}c_{j+2}\mathrm{e}^{i(\theta-\tilde{\theta})} + \frac{1}{\omega^2-1}\bar{c}_{j+2}\mathrm{e}^{-i(\theta-\tilde{\theta})}.
\end{aligned}$$

Analogously,

$$\tilde{x}_+ = \frac{1}{\omega^2-1}c_{j+2}\mathrm{e}^{i(\theta-\tilde{\theta})} - \frac{\omega^2}{\omega^2-1}\bar{c}_{j+2}\mathrm{e}^{-i(\theta-\tilde{\theta})}.$$

On the other hand,

$$\begin{aligned}
\tilde{y}_- &= -\frac{\omega^2}{\omega^2-1}\tilde{c}_j + \frac{1}{\omega^2-1}\bar{\tilde{c}}_j \\
&= -\frac{\omega^2}{\omega^2-1}b_j\mathrm{e}^{-i\tilde{\theta}} + \frac{1}{\omega^2-1}\bar{b}_j\mathrm{e}^{i\tilde{\theta}} \\
&= -\frac{\omega^2}{\omega^2-1}r\mathrm{e}^{i(\theta-\tilde{\theta})} + \frac{1}{\omega^2-1}r\mathrm{e}^{-i(\theta-\tilde{\theta})}.
\end{aligned}$$

Analogously,

$$\tilde{x}_- = \frac{1}{\omega^2-1}r\mathrm{e}^{i(\theta-\tilde{\theta})} - \frac{\omega^2}{\omega^2-1}r\mathrm{e}^{-i(\theta-\tilde{\theta})}.$$

Notice that

$$\tilde{r}\mathrm{e}^{i\tilde{\theta}} = b_{j+1} = c_{j+1}\mathrm{e}^{i\theta} = (\omega y_+ + \omega^2 x_+)\mathrm{e}^{i\theta},$$

so

$$\mathrm{e}^{i(\tilde{\theta}-\theta)} = \frac{\omega y_+ + \omega^2 x_+}{\tilde{r}} \qquad \mathrm{e}^{i(\theta-\tilde{\theta})} = \frac{\omega^2 y_+ + \omega x_+}{\tilde{r}}.$$

Then,

$$\begin{aligned}
\tilde{y}_- &= -\frac{\omega^2}{\omega^2-1} r\mathrm{e}^{i(\theta-\tilde{\theta})} + \frac{1}{\omega^2-1} r\mathrm{e}^{-i(\theta-\tilde{\theta})} \\
&= \frac{r}{\tilde{r}}\left(-\frac{\omega^2}{\omega^2-1}(\omega^2 y_+ + \omega x_+) + \frac{1}{\omega^2-1}(\omega y_+ + \omega^2 x_+)\right) \\
&= \frac{r}{\tilde{r}} x_+,
\end{aligned}$$

and, analogously,

$$\tilde{x}_- = \frac{r}{\tilde{r}} y_+.$$

On the other hand,

$$\begin{aligned}
\tilde{y}_+ &= -\frac{\omega^2}{\omega^2-1} c_{j+2}\mathrm{e}^{i(\theta-\tilde{\theta})} + \frac{1}{\omega^2-1}\bar{c}_{j+2}\mathrm{e}^{-i(\theta-\tilde{\theta})} \\
&= -\frac{\omega^2}{\omega^2-1} c_{j+2}\frac{\omega^2 y_+ + \omega x_+}{\tilde{r}} + \frac{1}{\omega^2-1}\bar{c}_{j+2}\frac{\omega y_+ + \omega^2 x_+}{\tilde{r}} \\
&= \frac{1}{\tilde{r}}\left(\frac{y_+}{\omega^2-1}\left(-\omega c_{j+2} + \omega \bar{c}_{j+2}\right) + \frac{x_+}{\omega^2-1}\left(-c_{j+2} + \omega^2 \bar{c}_{j+2}\right)\right) \\
&= \frac{1}{\tilde{r}}\left(-\frac{2\sqrt{3}}{3}\mathrm{Im}(c_{j+2}) y_+ + \left(\mathrm{Re}(c_{j+2}) + \frac{\sqrt{3}}{3}\mathrm{Im}(c_{j+2})\right) x_+\right)
\end{aligned}$$

$$\begin{aligned}
\tilde{x}_+ &= \frac{1}{\omega^2-1} c_{j+2}\mathrm{e}^{i(\theta-\tilde{\theta})} - \frac{\omega^2}{\omega^2-1}\bar{c}_{j+2}\mathrm{e}^{-i(\theta-\tilde{\theta})} \\
&= \frac{1}{\omega^2-1} c_{j+2}\frac{\omega^2 y_+ + \omega x_+}{\tilde{r}} - \frac{\omega^2}{\omega^2-1}\bar{c}_{j+2}\frac{\omega y_+ + \omega^2 x_+}{\tilde{r}} \\
&= \frac{1}{\tilde{r}}\left(\frac{y_+}{\omega^2-1}\left(\omega^2 c_{j+2} - \bar{c}_{j+2}\right) + \frac{x_+}{\omega^2-1}\left(\omega c_{j+2} - \omega \bar{c}_{j+2}\right)\right) \\
&= \frac{1}{\tilde{r}}\left(\left(\mathrm{Re}(c_{j+2}) - \frac{\sqrt{3}}{3}\mathrm{Im}(c_{j+2})\right) y_+ + \frac{2\sqrt{3}}{3}\mathrm{Im}(c_{j+2}) x_+\right).
\end{aligned}$$

For $k \neq j-1, j+1, j+2$,

$$\tilde{c}_k = b_k \mathrm{e}^{-i\tilde{\theta}} = c_k \mathrm{e}^{i(\theta-\tilde{\theta})} = \frac{\omega^2 y_+ + \omega x_+}{\tilde{r}} c_k.$$

Notice that

$$|\tilde{c}_k| = \left|c_k \mathrm{e}^{i(\theta-\tilde{\theta})}\right| = |c_k|.$$

Finally

$$\tilde{c}_{j-1} = b_{j-1}\mathrm{e}^{-i\tilde{\theta}} = c_{j-1}\mathrm{e}^{i(\theta-\tilde{\theta})} = \frac{\omega^2 y_+ + \omega x_+}{\tilde{r}} \left(\omega y_- + \omega^2 x_-\right).$$

Now we can compute:

$$|\tilde{c}_{j-1}| = |c_{j-1}| = ||(x_-, y_-)||_*,$$

and obtain the same bound that in $\tilde{c}_{j+2}$ but in terms of x_- and y_-.

C.6 Proof of Lemma 15

$$\begin{aligned}
\tilde{r}^2 &= x_+^2 - x_+ y_+ + y_+^2 \\
&= (\sqrt{1-\sigma^2} + m_x)^2 - (\sqrt{1-\sigma^2} + m_x)m_y + m_y^2 \\
&= 1 - \sigma^2 + 2m_x\sqrt{1-\sigma^2} + m_x^2 - m_y\sqrt{1-\sigma^2} - m_x m_y + m_y^2 \\
&= 1 - \sigma^2 + \sqrt{1-\sigma^2}(2m_x - m_y) + ||(m_x, m_y)||_*^2 \\
&\geq 1 - \sigma^2 + \sqrt{1-\sigma^2}(2m_x - m_y)
\end{aligned}$$

$$\begin{aligned}
r^2 &= 1 - ||(x_-, y_-)||_*^2 - \tilde{r}^2 - \sum_{l \in \mathcal{P}_j} |c_l|^2 \\
&\leq 1 - \tilde{r}^2 \leq \sigma^2 - \sqrt{1-\sigma^2}(2m_x - m_y)
\end{aligned}$$

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