

**On geometric quantisation of integrable systems  
with singularities**

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The 13th of September of 2013

# Acknowledgements

There is no way to overemphasise the importance of Eva Miranda in this work: she was not only an advisor, she was a coworker.

I had a set of discussions with some professors that were also important (although, probably, they are not aware of it): Francisco Presas (I am also in debt to him for his careful reading of this text), Henrique Bursztyn, Gert Heckman and Mark Hamilton.

Marcelo Alves is responsible for introducing me to Symplectic Geometry, and he deserves some credit, then. I am obliged to mention, once more, all of my professors at UFMG; Miguel Rodriguez Olmos; Chiara Esposito; and my officemates.

I want to thank Victor Guillermin and Francisco Presas for writing reports, and all the members of the jury: Pere Pascual, Francisco Presas, Carles Currás, Nguyen Tien Zung, Simone Gutt, Marcel Nicolau, and Jaume Amorós. Your work is really important to me and I am glad that you accepted it.

Partially supported by the taxpayers of the European Union and Brazil via:

- Erasmus Mundus External Cooperation Window EU-Brazil Startup 2009-2010 project;
- Contact And Symplectic Topology, Research Networking Programme of the European Science Foundation;
- Geometry, Mechanics and Control Theory network;
- DGICYT/FEDER project MTM2009-07594: Estructuras Geometricas: Deformaciones, Singularidades y Geometria Integral;
- MINECO project MTM2012-38122-C03-01: Geometria Algebraica, Simplectica, Aritmetica y Aplicaciones.

*“More seriously I took things  
The harder the rules became”*

Megadeth

*“ ‘Cause the truth about the world is that crime does pay”*

The Offspring

# Summary

This thesis shows an approach to geometric quantisation of integrable systems. It extends some results by Guillemin, Kostant, Rawnsley, Śniatycki and Sternberg in geometric quantisation, considering regular fibrations as real polarisations, to the singular setting: the real polarisations concerned here are given by integrable systems with nondegenerate singularities, and the definition of geometric quantisation used is the one suggested by Kostant (via higher cohomology groups). It also presents unifying proofs for results in geometric quantisation by exploring the existence of symplectic circle actions: the tools developed here highlight and unravel the role played by circle actions in known results in geometric quantisation.

The originality of this thesis relies on the following aspects. Firstly, the use of symplectic circle actions to obtain results in geometric quantisation, and secondly, the nonexistence of Poincaré lemmata for foliated cohomology when the foliation has singularities.

Previous results on circle actions, due to Rawnsley, could not be used when the circle action is not free, and it is not straightforward to adapt them to accommodate fixed points. After developing these techniques, the computation of geometric quantisation is performed in a series of situations, which includes: the cotangent bundle of the circle and products of it with any quantisable exact symplectic manifold, and neighbourhoods of nondegenerate singularities of integrable systems (hyperbolic singularities need special treatment, since there is no natural circle action).

These computations imply that the Kostant complex is a fine resolution (for the sheaf of sections of the prequantum line bundle which are flat along the polarisation) when the real polarisations are given by integrable systems with nondegenerate singularities. It is important to mention that the proofs are original, since, contrary to expectations, there is no Poincaré Lemma when singularities are allowed for the

foliated cohomology associated to foliations induced by integrable systems. This non-trivial result turns out to be interesting in its own right, but only the aspects related to geometric quantisation are presented in the thesis, e.g. the need for a new proof that the Kostant complex is a fine resolution for the sheaf of flat sections.

The thesis also provides a different proof of a theorem, firstly proved by Guillemin and Sternberg, that shows that the set of regular Bohr-Sommerfeld fibres is discrete—it not only bares the role played by circle actions, it also excludes the compactness assumption from the theorem. The exploitation of circle actions culminate in an alternative proof for the theorems of Śniatycki and Hamilton. It is an original and unifying proof: the argument works for both situations, Lagrangian fibre bundles and locally toric manifolds.

In addition, this approach casts some light on a conjecture about the contributions coming from focus-focus type of singularities. It actually proves that there is no contribution to geometric quantisation coming from focus-focus fibres for compact 4-dimensional almost toric manifolds in degree zero.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Organisation of the thesis . . . . .	3
<b>2</b>	<b>Preliminaries</b>	<b>6</b>
2.1	Symplectic manifolds . . . . .	6
2.1.1	Symplectic and Hamiltonian vector fields . . . . .	9
2.1.2	Poisson brackets . . . . .	10
2.1.3	Integrable systems . . . . .	11
2.1.4	Normal forms for nondegenerate singularities . . . . .	13
2.1.5	Symplectic actions and moment maps . . . . .	17
2.2	Hermitian line bundles . . . . .	20
2.2.1	Hermitian connections and curvature . . . . .	20
2.2.2	Parallel transport and holonomy . . . . .	24
2.3	Foliations and Lie pseudoalgebras . . . . .	25
2.3.1	Vector fields and some Lie subalgebras . . . . .	26
2.3.2	Representation theory . . . . .	27
<b>3</b>	<b>Prequantisation</b>	<b>31</b>
3.1	Prequantum line bundle . . . . .	31
3.2	Polarisation . . . . .	32
3.3	Bohr-Sommerfeld fibres . . . . .	33

<b>4</b>	<b>Geometric quantisation à la Kostant</b>	<b>37</b>
4.1	A cohomological definition . . . . .	37
<b>5</b>	<b>The Čech approach</b>	<b>40</b>
5.1	Sheaf cohomology . . . . .	40
5.1.1	Sheaf structure of $\mathcal{J}$ . . . . .	40
5.1.2	Construction of $\check{H}^k(M; \mathcal{J})$ . . . . .	41
5.1.3	The Mayer-Vietoris argument . . . . .	42
5.2	Known results for the singular case . . . . .	43
<b>6</b>	<b>The Kostant complex</b>	<b>46</b>
6.1	Line bundle valued polarised forms . . . . .	46
6.1.1	Interior product, Lie derivative and pullbacks . . . . .	49
<b>7</b>	<b>Circle actions and homotopy operators</b>	<b>55</b>
7.1	An almost homotopy operator . . . . .	55
<b>8</b>	<b>Poincaré lemma I: foliated cohomology</b>	<b>62</b>
8.1	Regular foliations . . . . .	63
8.2	Singular foliations . . . . .	63
<b>9</b>	<b>Poincaré lemma II: regular polarisations</b>	<b>67</b>
9.1	A fine resolution for $\mathcal{J}$ . . . . .	67
9.1.1	Symplectic vector spaces: linear polarisation . . . . .	69
9.2	Semilocal examples . . . . .	70
9.2.1	The cylinder: polarisation by circles . . . . .	70
9.2.2	Direct product type with a regular component . . . . .	71
<b>10</b>	<b>Poincaré lemma III: singular polarisations</b>	<b>74</b>
10.1	Elliptic singularities . . . . .	74
10.1.1	The complex plane: polarisation by circles . . . . .	75
10.1.2	Direct product type with an elliptic component . . . . .	76

10.2	Focus-focus singularities . . . . .	77
10.3	Hyperbolic singularities . . . . .	79
10.3.1	The hyperbolic-hyperbolic case . . . . .	85
<b>11</b>	<b>The resolution approach</b>	<b>89</b>
11.1	Nonsingular case . . . . .	89
11.1.1	Neighbourhood of a Liouville fibre . . . . .	89
11.1.2	Lagrangian fibre bundles . . . . .	90
11.2	Singular case . . . . .	92
11.2.1	Neighbourhood of an elliptic fibre . . . . .	92
11.2.2	Locally toric manifolds . . . . .	93
11.2.3	Focus-focus contribution to geometric quantisation . . . . .	94
	<b>References</b>	<b>96</b>



# Chapter 1

## Introduction

Classical Hamiltonian (conservative) systems can be described by Symplectic Geometry. One starts with a configuration space, which is a manifold representing the possible positions of a constrained system; states of the system are given by points of the cotangent bundle: any state can be characterised by its position and momentum. The dynamics of the system is given by a function, the so-called Hamiltonian function.

In the Schrödinger picture, the quantum phase space is a Hilbert space and the states are given by wave functions, which are, feckly, complex valued functions of the configuration space with finite norm. The dynamics of the wave functions are determined by the Schrödinger equation, in which the Hamiltonian is a selfadjoint operator on the Hilbert space.

The present thesis aims to describe a quantisation rule, geometric quantisation, that works at least for classical integrable Hamiltonian systems. This is a quote from Kirillov [14] to support this idea: “As we understand now, there is no canonical and universal correspondence: the quantum world is different from the classical one. Nonetheless, for many particular systems quantization rules which allow one to construct a quantum system from the classical one were formulated.”

Geometric quantisation tries to associate a Hilbert space to a symplectic manifold via a complex line bundle. Although it is possible to describe the canonical

quantisation using this language [17], most of the difficulties arise when one tries to mimic this procedure for symplectic manifolds which are not naturally cotangent bundles. Those appear in the context of reduction and are far from being artificial mathematical models.

The first difficulty is to isolate in a global way position and momentum, in order to define wave functions as sections of a complex line bundle over the symplectic manifold. This is done by introducing polarisations, which, roughly speaking, are Lagrangian foliations. The second issue, that will not be addressed here, is how to define a Hilbert structure; hence, the honest-to-goodness quantum phase space will not be constructed.

Usually, the quantum phase space is constructed using global sections of the line bundle which are flat along the polarisation. In case these global sections do not exist, Kostant suggested to associate wave functions to elements of higher cohomology groups, and to build the quantum phase space from these groups: by considering cohomology with coefficients in the sheaf of flat sections.

The main objective of the thesis is to compute these cohomology groups, for which at least two approaches can be used: Čech and de Rham. The results of Hamilton and Miranda [11, 12] are based on a Čech approach, this thesis takes the de Rham point of view, by finding a resolution for the sheaf. Following Kostant [27, 26], a resolution for the sheaf of sections can be obtained by twisting the sheaves relative to the foliated complex induced by the polarisation with the sheaf of flat sections.

This thesis follows closely Rawnsley's ideas [26] and explores the existence of circle actions to provide an alternative proof for the theorems of Śniatycki [27] and Hamilton [11]. The tools developed here highlight and unravel the role played by symplectic circle actions in known results in geometric quantisation. Not only that, this approach casts some light on a conjecture about the contributions coming from focus-focus type of singularities.

As a last remark, geometric quantisation might not have much to say about Physics, but it might say something about Mathematics. The orbit method [14]

alone justifies this: even if geometric quantisation is not able to give a satisfactory quantisation rule, it does not imply that it cannot be used to find irreducible unitary representations of Lie groups.

For completeness, some proofs of theorems that can be found on the literature are given. But instead of state them in their original or most generalised version, where possible, statements and proofs are adapted to the cases used in this thesis. The reason is to not deviate from the main tools and to keep the reader grounded. Inquiring readers are encouraged to go to the cited references.

Throughout this thesis and otherwise stated<sup>1</sup>, all the objects considered will be  $C^\infty$ ; manifolds are real, Hausdorff, paracompact and connected; and the units are such that  $\hbar = 1$ .

## 1.1 Organisation of the thesis

A collection of definitions and results about Symplectic Geometry, Hermitian line bundles and Lie pseudoalgebras is given in chapter 2.

Chapter 3 describes the notions of prequantum line bundle and polarisation required to define geometric quantisation; it also describes the important notion of Bohr-Sommerfeld fibres and provides a proof of a theorem (theorem 3.2 in this thesis) that shows that the set of regular, and compact, Bohr-Sommerfeld fibres is discrete [10].

The definition of geometric quantisation is provided in chapter 4, and the precise cohomological definitions are presented in chapter 5; as well as, the results of Hamilton and Miranda [11, 12].

In chapter 6 the line bundle polarised forms and Kostant complex are introduced. These are presented as an example of Lie pseudoalgebra representations, and it is this

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<sup>1</sup>One must reach to a compromise between being precise and being intelligible. Geometers tend to be sloppy, and as a physicist, I shall follow them. For instance, all figures must be seen from a topological point of view.

point of view that allows one to introduce singularities into the picture. Howbeit, the replacement of a subbundle of the tangent bundle by an integrable distribution offers no obstruction, and the propositions of chapter 6 are simple extensions of results contained in [26].

Chapter 7 further develops results from [26] and it contains the main tools of this thesis: it concerns the use of symplectic circle actions to obtain results in geometric quantisation (propositions 7.3 and 7.4). Rawnsley's results cannot be used when the circle action is not free, nor is it straightforward to adapt all the proofs to accommodate fixed points (e.g. lemma 7.2). In addition, this chapter contains a different proof of the theorem (for the alternative version, theorem 7.1 in this thesis) that shows that the set of regular Bohr-Sommerfeld fibres is discrete [10] —it not only bares the role played by circle actions, it also excludes the compactness assumption from the theorem.

Detouring from the motif of the thesis, chapter 8 justifies the need for a different proof that the Kostant complex is a fine resolution for the sheaf of flat sections. It summarises what is known about foliated cohomology —with special emphasis given to the existence of a Poincaré lemma when the foliation is singular. Contrary to expectations, there is no Poincaré Lemma when singularities are allowed (for the foliated cohomology associated to singular Lagrangian foliations induced by integrable systems). This nontrivial result (presented here as theorem 8.3) was obtained by Miranda and the author of this thesis [20], and it turns out to be interesting in its own right.

Chapters 9 and 10 prove that the Kostant complex is a fine resolution in a series of situations (theorem 9.2; propositions 9.1 and 9.2; corollary 9.2; theorems 10.1, 10.2, and 10.4). Chapter 9 deals with regular polarisations and, apart from the two proofs of the Poincaré lemma (both reducing to the Poincaré lemma for foliated cohomology), it computes the quantisation of the cotangent bundle of the circle (proposition 9.2) and gives a formula for products of the cotangent bundle of the circle with any quantisable exact symplectic manifold (corollary 9.2), as well. The computations involving the

cotangent bundle of the circle exploit the existence of a circle action.

The singularities are introduced in chapter 10, where Poincaré lemmata are proved when the Williamson type of the singularity has at least one elliptic (theorem 10.1) or focus-focus component (theorem 10.2), and when it has one or two hyperbolic components (theorem 10.4)—the rank of the singularity can be bigger than zero in all situations. It is important to mention that the proofs are original, since there is no Poincaré lemma for the foliated cohomology when these singularities are allowed. The circle action tools can be used when there are elliptic or focus-focus components, but hyperbolic singularities need special treatment: there is no natural circle action. The hyperbolic case has been done by Miranda and the author of this thesis [21], and a proof is included in chapter 10.

The alternative proof of Śniatycki's [27] and Hamilton's [11] theorem are presented in the last chapter of the thesis, chapter 11 (theorems 11.1 and 11.2). It is an original and unifying proof: the argument works for both situations, Lagrangian fibre bundles and locally toric manifolds. Action angle coordinates (more precisely, local normal forms) are still important, although not anymore the main characters in this new proof—a more prominent role is played by the circle actions. These circle actions actually prove that, in degree zero, there is no contribution to geometric quantisation coming from focus-focus fibres for compact 4-dimensional almost toric manifolds (proposition 11.2).

# Chapter 2

## Preliminaries

This is just a collection of definitions and results to fix notation.

### 2.1 Symplectic manifolds

Although this thesis is essentially on Symplectic Geometry, it would be a waste of time for readers to include an introduction to this field. For those who are not familiar with it, a good reference is [1].

**Definition 2.1.** *A pair  $(M, \omega)$  is a symplectic manifold, if  $\omega \in \Omega^2(M)$  is closed and nondegenerate (if this is the case,  $\omega$  is called a symplectic form); i.e.  $d\omega = 0$  and the map defined by  $X \mapsto \iota_X \omega$  gives an isomorphism between  $\mathfrak{X}(M)$  and  $\Omega^1(M)$ .*

**Remark 2.1.** The space of smooth vector fields will be denoted by  $\mathfrak{X}(M)$  when smooth vector fields are seen as derivations of the commutative algebra  $C^\infty(M)$  — which, by the way, can represent both the space of complex-valued or real-valued functions, depending on the context. When interpreted as smooth sections of the tangent bundle  $TM$ , their space will be denoted by  $\Gamma(TM)$ .

Nondegeneracy implies that all symplectic manifolds are even dimensional. Indeed, at every point of  $M$  the symplectic form  $\omega$  is an alternate bilinear form with an invertible and antisymmetric matrix  $A$  associated, and since the determinant is

invariant by transposition:

$$\det(A) = \det(A^\top) = \det(-A) = (-1)^{\dim M} \det(A) .$$

Symplectic manifolds are orientable as well; the wedge product of  $n$  copies of  $\omega$ , with  $2n$  being the dimension of  $M$ , is a volume form (this is also a characterisation of the nondegeneracy condition).

**Example 2.1.** Orientable surfaces with area form: the closedness comes from the top dimension of the de Rham cohomology, and the nondegeneracy from the orientability hypothesis.  $\diamond$

**Example 2.2.** Let  $Q$  be a manifold, its cotangent bundle,  $T^*Q$ , has a canonical symplectic structure: the pullback of the natural projection  $T^*Q \rightarrow Q$  is a 1-form on  $T^*Q$ , called action or tautological form, and the symplectic structure on  $T^*Q$  is obtained by differentiating the action form. Thus, the closedness is trivially satisfied and the nondegeneracy can be checked using trivialising coordinates on  $T^*Q$ .  $\diamond$

**Example 2.3.** Let  $G$  be a Lie group,  $(\mathfrak{g}, \text{ad})$  its Lie algebra and  $A : G \rightarrow \text{Diff}(G)$  the action by conjugation. For each  $g \in G$  the pullback of  $A_g$  at the identity,  $\text{Ad}_g^* := A_{g_e}^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ , defines an anti-representation of the Lie group in  $\mathfrak{g}^*$ . It is called the coadjoint representation and its orbits,  $\mathcal{O}_\xi = \{\text{Ad}_g^*(\xi)\}_{g \in G} \subset \mathfrak{g}^*$ , the coadjoint orbits. If the pushforward of the map  $g \mapsto \text{Ad}_g^*$  at the identity is denoted by  $\text{ad}^* : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}^*)$ , then the tangent space of the coadjoint orbit  $\mathcal{O}_\xi$  at a point  $\zeta = \text{Ad}_g^*(\xi)$  is given by  $\{\text{ad}_x^*(\zeta)\}_{x \in \mathfrak{g}}$ . The expression  $\omega|_\zeta(\text{ad}_x^*(\zeta), \text{ad}_y^*(\zeta)) := \zeta(\text{ad}_x(y))$  defines a<sup>1</sup> symplectic form on  $\mathcal{O}_\xi$ .  $\diamond$

All symplectic manifolds look alike locally, but before stating Darboux theorem, one needs to define a notion of equivalence.

---

<sup>1</sup>The expression  $\omega_\xi|_\zeta(\text{ad}_x^*(\zeta), \text{ad}_y^*(\zeta)) := \xi(\text{ad}_x(y))$  also defines a symplectic form on  $\mathcal{O}_\xi$ , which is compatible, in a suitable sense [24], with the form previously defined.

**Definition 2.2.** A diffeomorphism  $\varphi : M \rightarrow N$  between two symplectic manifolds  $(M, \omega)$  and  $(N, \varpi)$  is a symplectomorphism if  $\varphi^*(\varpi) = \omega$ . And the subgroup of symplectomorphisms of  $(M, \omega)$  is denoted by  $\text{Sympl}(M, \omega)$ .

**Theorem 2.1** (Darboux). For every symplectic manifold  $(M, \omega)$ , of dimension  $2n$ , there exists a local symplectomorphism  $\varphi : \mathbb{R}^{2n} \rightarrow M$ , such that in a neighbourhood of each point  $\varphi^*(\omega) = \sum_{j=1}^n dx_j \wedge dy_j$ , where  $(x_1, y_1, \dots, x_n, y_n)$  is a chart on  $\mathbb{R}^{2n}$ .

The charts on each point of  $M$ , given by the theorem, are called Darboux charts, and  $\omega$  is said to be in Darboux form under the symplectomorphism.

**Definition 2.3.** A submanifold  $N$  of a symplectic manifold  $(M, \omega)$  is isotropic if  $\omega$  restricted to  $TN$  vanishes,  $\omega|_{TN} = 0$ . The submanifold is called Lagrangian if also  $\dim N = \frac{1}{2} \dim M$ .

**Remark 2.2.** The notation  $\alpha|_A$  has a particular meaning depending on what  $A$  stands for. If  $A$  is a point, as  $\zeta$  is in exemple 2.3,  $\alpha|_A$  is the associated element of the dual of the tangent space of  $M$  at the point  $A$ , for a  $k$ -form  $\alpha \in \Omega^k(M)$ . If  $A$  is a set of points,  $\alpha|_A$  is the  $k$ -form restricted to the points of  $A$ : it makes sense to take the inner product of it with any vector field of  $M$  restricted to points of  $A$ . If  $A$  is a submodule of  $\mathfrak{X}(M)$ ,  $\alpha|_A$  is the restriction of the homomorphism  $\alpha \in \text{Hom}_{C^\infty(M)}(\wedge_{C^\infty(M)}^k \mathfrak{X}(M); C^\infty(M))$ , i.e.  $\alpha|_A \in \text{Hom}_{C^\infty(M)}(\wedge_{C^\infty(M)}^k A; C^\infty(M))$ ; in particular, if  $A$  is a submanifold of  $M$ ,  $\alpha|_{TA}$  stands for the associated element of  $\Omega^k(A)$ .

**Example 2.4.** Every 1-dimensional submanifold of a symplectic manifold is isotropic; the restriction of a 2-form always vanishes.  $\diamond$

Nontrivial examples of isotropic submanifolds appearing in integrable systems are discussed below.

**Remark 2.3.** The imaginary part of the standard Hermitian form on  $\mathbb{C}^n$  is a symplectic structure, equivalent to the Darboux structure on  $\mathbb{R}^{2n}$ . The restriction of



that symplectic structure over complex submanifolds of  $\mathbb{C}^n$  induces symplectic structures as well. Kähler manifolds and its complex submanifolds are also examples of symplectic manifolds. But none of them will be discussed in the thesis.

### 2.1.1 Symplectic and Hamiltonian vector fields

The nondegeneracy has yet another implication: it induces two particular Lie subalgebras of  $(\mathfrak{X}(M), [\cdot, \cdot])$ .

**Definition 2.4.** *A vector field  $X \in \mathfrak{X}(M)$  of a symplectic manifold  $(M, \omega)$  is a symplectic vector field if  $\iota_X \omega$  is closed. In the particular case where  $\iota_X \omega$  is exact,  $X$  is said to be a Hamiltonian vector field; and a function  $f$  satisfying  $\iota_X \omega = -df$  is called a Hamiltonian function of  $X$ .*

Each function  $f \in C^\infty(M)$  gives a unique Hamiltonian vector field,  $X_f$ , via the equation

$$\iota_{X_f} \omega = -df . \quad (2.1)$$

**Example 2.5.** For  $\mathbb{R}^{2n}$  with coordinates  $(x_1, y_1, \dots, x_n, y_n)$  and Darboux form  $\sum_{j=1}^n dx_j \wedge dy_j$ , the Hamiltonian vector field of a function  $f$  is given by  $\sum_{j=1}^n \left( \frac{\partial f}{\partial x_j} \frac{\partial}{\partial y_j} - \frac{\partial f}{\partial y_j} \frac{\partial}{\partial x_j} \right)$ . By Darboux theorem this is the local expression for Hamiltonian vector fields in any symplectic manifold.  $\diamond$

**Example 2.6.** The geodesic flow on a Riemannian manifold  $(Q, g)$  can be seen as the flow of a Hamiltonian vector field of  $T^*Q$  with the canonical form. The Hamiltonian function is given by  $(x, y) \mapsto \frac{1}{2}g_y(v, v)$ , where  $v \in T_y Q$  is defined by  $x(\cdot) = g_y(v, \cdot)$ .  $\diamond$

**Example 2.7.** For coadjoint orbits, the Hamiltonian vector field of a function  $f$  on a point  $\zeta$  on the orbit is given by  $\text{ad}_{df|_\zeta}^*(\zeta)$ . Where the derivative of  $f$  at the point  $\zeta$ ,  $df|_\zeta$ , is seen as an element of  $\mathfrak{g}$ ; since is a linear functional in  $\mathfrak{g}^*$ .  $\diamond$

The space of symplectic and Hamiltonian vector fields, respectively denoted by  $\mathfrak{X}^{\text{symp}}(M, \omega)$  and  $\mathfrak{X}^{\text{ham}}(M, \omega)$ , are Lie subalgebras of  $(\mathfrak{X}(M), [\cdot, \cdot])$  —as vector spaces they are, respectively, isomorphic to the space of closed and exact 1-forms on  $M$ .

The following inclusions respect the Lie algebra structure  $[\cdot, \cdot]$ ,

$$\mathfrak{X}^{\text{ham}}(M, \omega) \subset \mathfrak{X}^{\text{symp}}(M, \omega) \subset \mathfrak{X}(M) .$$

More can be said about these inclusions: the Lie bracket of two symplectic or Hamiltonian vector fields,  $X$  and  $Y$ , are Hamiltonian vector fields, and  $\omega(X, Y)$  is a Hamiltonian function of  $[X, Y]$ .

Symplectic vector fields preserve the symplectic structure.

**Proposition 2.1.** *Let  $X \in \mathfrak{X}(M)$  be a vector field and  $\phi_t \in \text{Diff}(M)$  its flow at time  $t$ . The following are equivalent:*

- $d(\iota_X \omega) = 0$ ;
- $\mathcal{L}_X(\omega) = 0$ ;
- $\phi_t^*(\omega) = \omega$  for all  $t$  (where the flow is defined).

### 2.1.2 Poisson brackets

On a symplectic manifold there exists a related structure, the so-called Poisson structure, endowing the space of smooth functions with a Lie algebra structure satisfying a Leibniz rule.

**Definition 2.5.** *The Poisson bracket of two functions  $f_1, f_2 \in C^\infty(M)$  on a symplectic manifold  $(M, \omega)$  is the function defined by  $\{f_1, f_2\}_\omega := \omega(X_{f_1}, X_{f_2})$ .*

The equality  $\{f, \cdot\}_\omega = X_f(\cdot)$  holds for each  $f \in C^\infty(M)$ , in other words,  $\{f, \cdot\}_\omega$  is the Hamiltonian vector field associated to  $f$ . Wherefore, the Poisson bracket satisfies the Leibniz rule,

$$\{f_1, f_2 f_3\}_\omega = \{f_1, f_2\}_\omega f_3 + f_2 \{f_1, f_3\}_\omega .$$

A consequence of the closedness of  $\omega$  is the Jacobi identity:

$$\{f_1, \{f_2, f_3\}_\omega\}_\omega + \{f_3, \{f_1, f_2\}_\omega\}_\omega + \{f_2, \{f_3, f_1\}_\omega\}_\omega = 0 .$$

As a result,  $X_{\{f_1, f_2\}_\omega} = [X_{f_1}, X_{f_2}]$ ; thus, the Poisson bracket endows  $C^\infty(M)$  with a Lie algebra structure homomorphic to the Lie algebra  $(\mathfrak{X}(M), [\cdot, \cdot])$ .

Another consequence of Cartan's magic formula is that integral curves of Hamiltonian vector fields are contained on the level sets of the Hamiltonian function — contrasting gradient vector fields of a Riemannian manifold, which are orthogonal to the level sets of the gradient function, Hamiltonian vector fields are tangent to the level sets of the Hamiltonian function. And a function  $f_2$  is a first integral of the Hamiltonian system generated by  $f_1$  if and only if  $\{f_1, f_2\}_\omega = 0$ .

**Example 2.8.** For functions  $f_1, f_2 \in C^\infty(\mathfrak{g}^*)$  the expression

$$\{f_1, f_2\}_{LP}(\xi) := \xi(\text{ad}_{df_1|_\xi}(df_2|_\xi)) \quad (2.2)$$

defines a Poisson structure on  $\mathfrak{g}^*$  called Lie-Poisson (again, the identification of  $\mathfrak{g}$  as linear functionals in  $\mathfrak{g}^*$  is been used). When restricted to the coadjoint orbits, this Poisson structure coincides with the one provided by the symplectic form<sup>2</sup> previously defined.  $\diamond$

### 2.1.3 Integrable systems

The classical Liouville theorem on the integrability of Hamiltonian systems is stated in this section (using modern language and results). This is one of the central theorems on which this thesis relies<sup>3</sup>. The book [5] is a good reference treating concrete examples of integrable systems.

**Definition 2.6.** *An integrable system on a  $2n$ -dimensional symplectic manifold  $(M, \omega)$  is a map  $F = (f_1, \dots, f_n) : M^{2n} \rightarrow \mathbb{R}^n$ , the so-called moment map, such that:*

---

<sup>2</sup>As before, one can define a different, though compatible, Poisson structure for each  $\zeta \in \mathfrak{g}^*$  by  $\{f_1, f_2\}_\zeta(\xi) := \zeta(\text{ad}_{df_1|_\xi}(df_2|_\xi))$

<sup>3</sup>And one of the most beautiful theorems of all times, in the humble author's opinion.

- *is a submersion on an open dense subset of  $M$ ;*
- *its components,  $f_j$ , Poisson commute among each other,  $\{f_j, f_k\}_\omega = 0$  for any  $j$  and  $k$ ;*
- *the Hamiltonian vector fields generated by its components are complete<sup>4</sup>.*

Examples (just to cite some) include Hamiltonian systems in dimension 2, the harmonic oscillator (in any dimension), the Kepler problem, the mathematical pendulum, the spherical pendulum, geodesic flows on surfaces of revolution, some left invariant geodesic flows on Lie groups [24] (the free rigid body is an example of this), and various tops.

**Theorem 2.2** (Liouville). *Let  $F = (f_1, \dots, f_n) : M \rightarrow \mathbb{R}^n$  be an integrable system on a symplectic manifold  $(M, \omega)$ .*

- *The Hamiltonian vector fields generated by its components define an integrable (in the Sussmann [28] sense) distribution of  $TM$  whose leaves are generically Lagrangian (with isotropic singular leaves).*
- *The connected components of the regular leaves are homogeneous  $\mathbb{R}^n$  spaces; they are diffeomorphic to  $\mathbb{R}^{n-k} \times \mathbb{T}^k$ .*
- *The foliation is a Lagrangian fibration in a neighbourhood of each regular leaf; it defines a fibre bundle with Lagrangian fibres.*
- *There exists a symplectomorphism on a local trivialisation of each Lagrangian leaf that puts  $\omega$  in a Darboux form and linearise the flows induced by  $f_j$ .*

The foliated structure immediately implies that the leaves are invariant with respect to the flows of the Hamiltonian vector fields. Also the preimages of critical points are a union of isotropic submanifolds, whilst the regular ones are a union of Lagrangian submanifolds.

---

<sup>4</sup>Some authors do not assume this condition, yet it holds in some cases, e.g. when the symplectic manifold is compact.

The regular leaves, when compact, are called Liouville tori and the Darboux chart that linearises the flows on a neighbourhood of each Liouville torus is called action-angle coordinates. The torus components are the angles and the action coordinates are first integrals of the system.

In other words, the Liouville theorem gives a description of integrable systems near the regular points of the map  $F$ . The next subsection deals with some known results about the structure near the critical points.

### 2.1.4 Normal forms for nondegenerate singularities

This is a summary of the known local models for integrable systems, the reader can find a detailed exposition in [3].

Let  $F = (f_1, \dots, f_n) : M \rightarrow \mathbb{R}^n$  be an integrable system and  $p \in M$  a critical point of rank zero, i.e.  $df_i|_p = 0$  for  $i = 1, \dots, n$ . One can linearise at the critical point  $p$  each Hamiltonian vector field associated to the components of the moment map, constructing in this way a linear operator  $T(f_i) : T_p M \rightarrow T_p M$  for each function  $f_i$ .

The linear operators  $T(f_i)$  belong to  $\mathfrak{sp}(T_p M, \omega|_p)$ , the Lie algebra of the group of symplectic transformations of  $(T_p M, \omega|_p)$ , with  $\omega|_p$  being the invertible and antisymmetric matrix associated to the symplectic form  $\omega$  at  $p$ . In fact, they form an Abelian Lie subalgebra: because  $\{f_i, f_j\}_\omega = 0$  implies  $[T(f_i), T(f_j)] = 0$ .

**Definition 2.7.** *A critical point  $p$  of rank zero is nondegenerate if  $\mathfrak{h}_p = \langle T(f_1), \dots, T(f_n) \rangle_{\mathbb{R}}$  is a Cartan subalgebra of  $\mathfrak{sp}(T_p M, \omega|_p)$ .*

The Lie algebra of homogeneous quadratic polynomials on  $(T_p M, \omega|_p)$ ,  $\mathfrak{Q}(2n, \mathbb{R})$ , is isomorphic to  $\mathfrak{sp}(T_p M, \omega|_p)$ , and Cartan subalgebras of  $\mathfrak{Q}(2n, \mathbb{R})$  were classified by Williamson [30].

**Theorem 2.3** (Williamson). *For any Cartan subalgebra  $\mathfrak{h}_p \subset \mathfrak{sp}(T_p M, \omega|_p)$ , there exist linear coordinates  $(x_1, y_1, \dots, x_n, y_n)$  in  $\mathbb{R}^{2n}$  such that,  $p = \mathbf{0}$ ,  $\omega|_p = \sum_{i=1}^n dx_i \wedge dy_i$ , and  $\mathfrak{h}_p$  has a basis  $\{h_1, \dots, h_n\}$  with each  $h_i$  of the following form (under the*

identification  $\mathfrak{sp}(T_p M, \omega|_p) \cong \mathfrak{Q}(2n, \mathbb{R})$ :

$$\begin{aligned}
h_i &= x_i^2 + y_i^2 && \text{for } 1 \leq i \leq k_e, && \text{(elliptic)} \\
h_i &= x_i y_i && \text{for } k_e + 1 \leq i \leq k_e + k_h, && \text{(hyperbolic)} \\
\begin{cases} h_i = x_i y_i + x_{i+1} y_{i+1}, \\ h_{i+1} = x_i y_{i+1} - x_{i+1} y_i \end{cases} &&& \text{for } i = k_e + k_h + 2j - 1, && \text{(focus-focus pair)} \\
&&& 1 \leq j \leq k_f &&
\end{aligned} \tag{2.3}$$

Therefore, the number of elliptic components  $k_e$ , hyperbolic components  $k_h$  and focus-focus components  $k_f$  is an invariant of  $\mathfrak{h}_p$ .

**Definition 2.8.** *The triple  $(k_e, k_h, k_f)$  is called the Williamson type of  $p$ .*

For rank zero singularities,  $n = k_e + k_h + 2k_f$ . Let  $\{h_1, \dots, h_n\}$  be a Williamson basis of this Cartan subalgebra, the Hamiltonian vector field of  $h_i$  with respect to the Darboux form will be denoted by  $X_i$ . A vector field  $X_i$  is said to be hyperbolic (resp. elliptic) if the corresponding function  $h_i$  is so, and a pair of vector fields  $X_i, X_{i+1}$  is a focus-focus pair if  $X_i$  and  $X_{i+1}$  are the Hamiltonian vector fields associated to functions  $h_i$  and  $h_{i+1}$  in a focus-focus pair.

In the local coordinates specified above, the vector fields  $X_i$  take the following form:

- $X_i$  is an elliptic vector field,

$$X_i = 2 \left( -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i} \right); \tag{2.4}$$

- $X_i$  is a hyperbolic vector field,

$$X_i = -x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}; \tag{2.5}$$

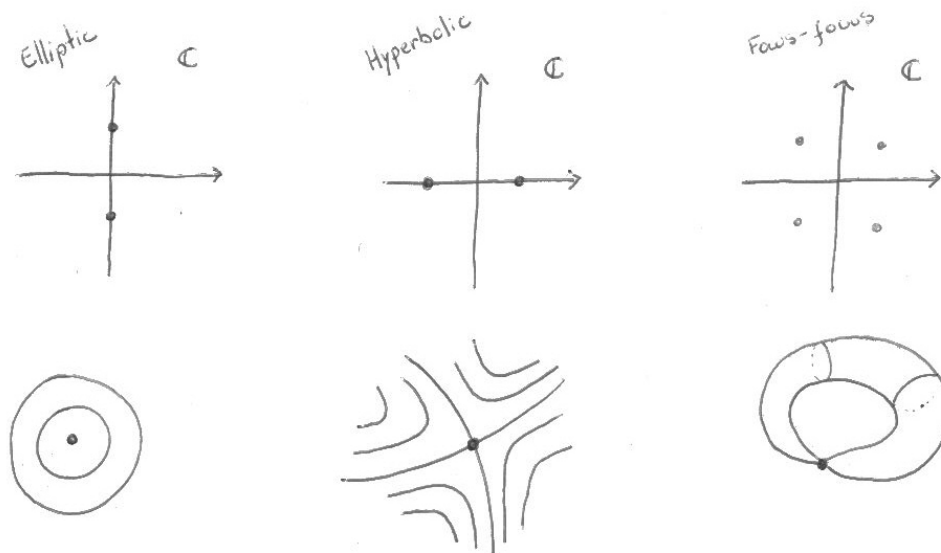
- $X_i, X_{i+1}$  is a focus-focus pair,

$$X_i = -x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}} + y_{i+1} \frac{\partial}{\partial y_{i+1}} \tag{2.6}$$

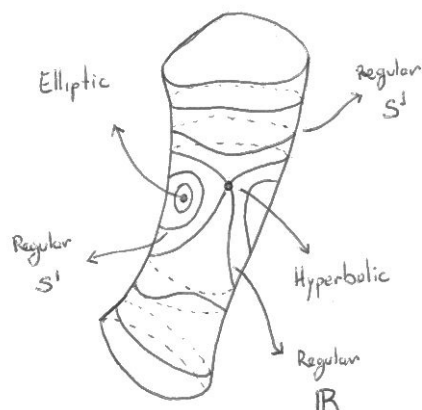
and

$$X_{i+1} = x_{i+1} \frac{\partial}{\partial x_i} + y_{i+1} \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial x_{i+1}} - y_i \frac{\partial}{\partial y_{i+1}}. \tag{2.7}$$

**Remark 2.4.** The nomenclature becomes clear when one looks at the eigenvalues of the derivative at a fixed point of the Hamiltonian vector fields (the linearised problem).



**Example 2.9.** For the simple pendulum the stable equilibrium point is an elliptic singularity, whilst the unstable one is a hyperbolic.



◇

**Example 2.10.** The spherical pendulum has a stable equilibrium point that is a purely elliptic singularity. The unstable equilibrium point is a focus-focus singularity.

◇

Let  $\mathcal{F}$  be a linear foliation on  $\mathbb{R}^{2n}$  with a rank zero singularity at the origin of Williamson type  $(k_e, k_h, k_f)$ ; the linear model for the foliation is, then, generated by the vector fields above. It turns out that these type of singularities are symplectically linearisable, and the local symplectic geometry of the foliation can be described by the algebraic data associated to the singularity (Williamson type). This is the content of the following symplectic linearisation result [8, 9, 19]:

**Theorem 2.4.** *Let  $\omega$  be a symplectic form defined in a neighbourhood  $V$  of the origin and  $\mathcal{F}$  a linear foliation with a rank zero singularity, of prescribed Williamson type, at the origin. Then, there exists a local diffeomorphism  $\phi : V \rightarrow \phi(V) \subset \mathbb{R}^{2n}$  such that,  $\phi$  preserves the foliation and  $\phi^*(\sum_{i=1}^n dx_i \wedge dy_i) = \omega$ , with  $(x_1, y_1, \dots, x_n, y_n)$  local coordinates on  $\phi(V)$ .*

Furthermore, if  $\mathcal{F}'$  is a foliation that has  $\mathcal{F}$  as a linear foliation model near a point, one can symplectically linearise  $\mathcal{F}'$  [19]. This is equivalent to Eliasson's theorem in the completely elliptic case [8, 9]: when the Williamson type of the singularity is  $(k_e, 0, 0)$ .

There are normal forms for higher rank ( $df_i|_p \neq 0$  for some  $i$ 's) and also in the case of singular nondegenerate compact leaves [19, 23]; the symplectic local normal form works not only in a neighbourhood of the singular point, it can be extended over a neighbourhood of a whole compact leaf.

When the rank of the singularity is greater than zero, a collection of regular vector fields is attached to it. Hence, one can reduce the  $k$ -rank case to the 0-rank case via a symplectic reduction associated to the natural Hamiltonian action given by the joint flow of the moment map.

**Definition 2.9.** *A moment map is said to be nondegenerate when all of its critical points are nondegenerate.*



### 2.1.5 Symplectic actions and moment maps

A Lie group action  $\rho : G \rightarrow \text{Diff}(M)$  induces a Lie anti-homomorphism between  $(\mathfrak{g}, \text{ad})$  and  $(\mathfrak{X}(M), [\cdot, \cdot])$ , via the exponential map  $\exp : \mathfrak{g} \rightarrow G$ . This Lie anti-homomorphism is given by the map  $\rho_{*e} : \mathfrak{g} \rightarrow \mathfrak{X}(M)$  that for each  $x \in \mathfrak{g}$  associates the vector field  $\rho_{*e}(x) \in \mathfrak{X}(M)$  whose flow at time  $t$  is  $\rho \circ \exp(tx) \in \text{Diff}(M)$ ; in particular,  $\rho_{*e}(x)$  is a complete vector field.

**Definition 2.10.** *A smooth Lie group action  $\rho : G \rightarrow \text{Diff}(M)$  on a symplectic manifold  $(M, \omega)$  is said to be symplectic when  $\rho(g)^*(\omega) = \omega$  for all  $g \in G$ . So the image of the action,  $\rho(G) \subset \text{Diff}(M)$ , is a subgroup of the group of symplectic diffeomorphisms of  $(M, \omega)$ . A symplectic action is called Hamiltonian when  $\rho_{*e}(\mathfrak{g}) \subset \mathfrak{X}^{\text{ham}}(M, \omega)$ .*

In other words, the notion of Hamiltonian action used in this thesis<sup>5</sup> is the following: a smooth Lie group action of Lie group on a symplectic manifold acting by symplectomorphisms, such that, the image of the associated infinitesimal action of the Lie algebra lies on the subspace of Hamiltonian vector fields of the symplectic manifold.

A Hamiltonian action is said to admit an equivariant comoment map (or equivalently, moment map) if there exists a Lie anti-homomorphism between the Lie algebra of the Lie Group and the Lie algebra of smooth functions on the symplectic manifold, satisfying that the image of any element of the Lie algebra is a Hamiltonian function for the Hamiltonian vector field corresponding to the infinitesimal action.

**Definition 2.11.** *Let  $\rho : G \rightarrow \text{Diff}(M)$  be a Hamiltonian action, a comoment map for this action is a linear map  $\mu^* : \mathfrak{g} \rightarrow C^\infty(M)$  such that  $\mu^*(x)$  is a Hamiltonian function of  $\rho_{*e}(x)$  for every  $x \in \mathfrak{g}$ . If in addition  $\mu^*$  is a Lie anti-homomorphism between  $(\mathfrak{g}, \text{ad})$  and  $(C^\infty(M), \{\cdot, \cdot\}_\omega)$ , then it is called an equivariant comoment map.*

---

<sup>5</sup>One can find (or easily deduce) all these results in Arnold's book [1]. He names things in a slightly different way: in [1], instead of saying Hamiltonian action admitting an equivariant moment map, Poisson action is used; the Lie bracket of vector fields is called a Poisson bracket there.

A comoment map for a Hamiltonian action is equivariant if and only if  $\mu^* \circ \text{Ad}_g(x) = \mu^*(x) \circ \rho(g)^{-1}$  for all  $g \in G$  and  $x \in \mathfrak{g}$ . For any Hamiltonian action there exists a comoment map. However, in general, there is no way to guarantee the Lie anti-homomorphism property: the Lie algebra cohomology of  $\mathfrak{g}$  has a say on this matter.

It is important to notice that the Lie homomorphism between the space of smooth functions (with the Poisson bracket associated to the symplectic structure) and the Lie subalgebra of Hamiltonian vector fields (with the Lie bracket of vector fields) given by  $f \mapsto X_f$ , although surjective, is not injective: its kernel being the space of constant functions. The existence of a right inverse of that map respecting the Lie structure is not a trivial problem, and it is what makes the notion of equivariance relevant.

If it was always true that there exists a right inverse respecting the Lie structure, then every Hamiltonian action would admit an equivariant comoment map: the composition of the infinitesimal action with the right inverse provides a Lie anti-homomorphism between the Lie algebra of the group  $G$  and the Lie algebra of smooth functions on  $M$ .

There is also a dual notion of a comoment map.

**Definition 2.12.** *A moment map for a Hamiltonian action  $\rho : G \rightarrow \text{Diff}(M)$  is a mapping  $\mu : M \rightarrow \mathfrak{g}^*$  such that, for each  $x \in \mathfrak{g}$  and  $p \in M$ , the function defined by  $f(x)(p) := \mu(p)(x)$  is a Hamiltonian function for  $\rho_{*e}(x)$ . And if  $\mu \circ \rho(g)^{-1} = \text{Ad}_g^* \circ \mu$  for all  $g \in G$ , then it is called an equivariant moment map.*

The existence of a moment map is equivalent to the existence of a comoment map, and the equivariance of  $\mu$  is equivalent to  $\mu^*$  be a Lie anti-homomorphism.

**Example 2.11.** The cotangent bundle of a Lie group  $G$  is trivial and can be naturally identified with  $G \times \mathfrak{g}^*$ . The left action of the group on itself can be lifted to the cotangent bundle, namely,  $\rho_g(h, \xi) = (gh, \xi)$ . The map  $\mu : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$  given by  $\mu(g, \xi) = \text{Ad}_{g^{-1}}^*(\xi)$  is a moment map for the canonical symplectic structure.  $\diamond$

**Example 2.12.** Integrable systems form a particular class of examples of Hamiltonian  $\mathbb{R}^n$ -actions admitting equivariant moment maps.

Supposing that  $\rho : G \rightarrow \text{Diff}(M)$  is an action of the additive Lie group  $G = \mathbb{R}^n$ , for each basis of its Lie algebra  $\{v_1, \dots, v_n\} \subset \mathfrak{g} = \mathbb{R}^n$  one can associate an integrable distribution  $\mathcal{P} := \langle X_1, \dots, X_n \rangle_{C^\infty(M)}$ , with  $X_j := \rho_{*e}(v_j)$ . The vector fields  $X_j$  are complete, and because  $\rho_{*e}$  is a Lie anti-homomorphism,  $[X_i, X_j] = 0$  for all  $i, j$ . The action is actually given by the joint flow of the vector fields  $X_j$ , since the exponential map is surjective.

In case this action is Hamiltonian, each  $X_j$  belongs to  $\mathfrak{X}^{ham}(M, \omega)$ , and a comoment map  $\mu^* : \mathbb{R}^n \rightarrow C^\infty(M)$  can be linearly defined by  $\mu^*(v_j) = f_j$ , with  $f_j \in C^\infty(M)$  an arbitrary Hamiltonian function for  $X_j$ . In order to have an equivariant comoment map from this construction, the choice of Hamiltonian functions must be such that  $\{f_i, f_j\}_\omega = 0$  for all  $i, j$  (their Poisson bracket is always a constant, but not necessarily zero). The Lie algebra is commutative,  $\text{ad}$  vanishes, and  $\mu^*$  is a Lie anti-homomorphism if and only if  $\{f_i, f_j\}_\omega = 0$  for all  $i, j$ .

It is clear now that the moment map associated to an equivariant  $\mu^*$ , if denoted by  $F : M \rightarrow \mathbb{R}^n$  (after the identification  $\mathfrak{g}^* \cong \mathbb{R}^n$ ), is an integrable system when the isotropy subgroups are discrete over an open dense subset of  $M$ .  $\diamond$

Another class of examples, which are also examples of integrable systems, are toric manifolds. They are compact  $2n$ -dimensional manifolds with an effective Hamiltonian  $n$ -torus action admitting an equivariant moment map. Those manifolds are classified [6] by their equivariant moment maps:

**Theorem 2.5** (Delzant). *Toric manifolds are classified by Delzant polytopes. More specifically, the bijective correspondence between these two sets is given by the equivariant moment map:*

$$\begin{aligned} \{\text{toric manifolds}\} &\longrightarrow \{\text{Delzant polytopes}\} \\ (M, \omega, \mathbb{T}^n, F) &\longrightarrow F(M) \end{aligned}$$

## 2.2 Hermitian line bundles

Under the language of geometric quantisation, wave functions are associated to sections of a complex line bundle; wherefore, some definitions and properties related to complex line bundles will be important. The assertions of this section can be found in [13].

A Hermitian line bundle  $L \xrightarrow{\mathbb{C}} M$  is a fibre bundle over  $M$  with fibres diffeomorphic to  $\mathbb{C}$ , together with a Hermitian structure  $\langle \cdot, \cdot \rangle$  defined on the space of sections  $\Gamma(L)$ . Any Hermitian line bundle is uniquely defined (up to isomorphism) by cocycle conditions lying in  $U(1)$ .

Sections of  $L$  can be represented by complex-valued functions over local trivialisations<sup>6</sup>. Let  $\mathcal{A} = \{A_j\}_{j \in I}$  be a contractible open cover of  $M$  such that each  $A_j$  is a local trivialisation of  $L$  with unitary section<sup>7</sup>  $s_j$  (this can always be obtained, e.g. using a convenient cover made of balls with respect to a Riemannian metric): each line bundle  $L|_{A_j} \xrightarrow{\mathbb{C}} A_j$  is bundle isomorphic to the trivial one,  $A_j \times \mathbb{C}$ , and under this bundle isomorphism, a section  $s \in \Gamma(L|_{A_j})$  is uniquely defined by a function  $f \in C^\infty(A_j; \mathbb{C})$ ;  $s(p) \simeq (p, f(p))$  for each point  $p \in A_j$ .

### 2.2.1 Hermitian connections and curvature

This subsection contains some properties of connections on complex line bundle.

**Remark 2.5.** From this point on, the distinction between the space of complex-valued functions,  $C^\infty(M; \mathbb{C})$ , and the real-valued ones,  $C^\infty(M; \mathbb{R})$ , will not always be explicitly made. The symbol  $C^\infty(V)$  will usually refer to smooth complex-valued functions over  $V$ .

---

<sup>6</sup>When there is an identification between a section  $s$  of the line bundle and a complex-valued function  $f$ , the bundle isomorphism will be omitted, for the sake of simplicity, and the equality  $s = f$  will be used.

<sup>7</sup>The existence of a trivialisation is equivalent to the existence of a unitary section. In particular, a complex line bundle is trivial if and only if it has global unitary sections.

**Definition 2.13.** A Hermitian connection is a linear map

$$\nabla : \Gamma(L) \rightarrow \Omega^1(M) \otimes_{C^\infty(M)} \Gamma(L) \quad (2.8)$$

satisfying:

$$\nabla(fs) = df \otimes s + f\nabla s , \quad (2.9)$$

with  $[\nabla s](X)$  denoted by  $\nabla_X s$ , and

$$X(\langle r, s \rangle) = \langle \nabla_X r, s \rangle + \langle r, \nabla_X s \rangle , \quad (2.10)$$

for any  $r, s \in \Gamma(L)$ ,  $f \in C^\infty(M)$  and  $X \in \mathfrak{X}(M)$ .

With respect to a unitary section  $s$ , defined over a submanifold<sup>8</sup>  $N \subset M$ , the connection  $\nabla$  can be represented by a potential 1-form  $\Theta_j \in \Omega^1(N)$ :

$$\nabla s = -i\Theta \otimes s . \quad (2.11)$$

Since  $s$  is unitary,  $\langle s, s \rangle = 1$ , the potential 1-form  $\Theta$  is real:

$$\begin{aligned} 0 = X(\langle s, s \rangle) &= \langle \nabla_X s, s \rangle + \langle s, \nabla_X s \rangle \\ &= \langle -i\Theta(X)s, s \rangle + \langle s, -i\Theta(X)s \rangle \\ &= -i\Theta(X)\langle s, s \rangle + i\overline{\Theta(X)}\langle s, s \rangle \\ &= -i\Theta(X) + i\overline{\Theta(X)} . \end{aligned}$$

**Lemma 2.1.** The potential 1-forms of  $\nabla$  for each unitary section defined in a submanifold of  $M$  are cohomologous. Conversely, if there is a unitary section over a submanifold of  $M$ , any 1-form which is cohomologous to the potential 1-form of  $\nabla$  is the potential 1-form of a unitary section.

*Proof:* Let  $s$  and  $r$  be two unitary sections defined over  $N \subset M$ , and  $\Theta$  and  $\vartheta$  the associated potential 1-forms. Since  $L$  is a Hermitian line bundle,  $s = e^{if}r$  for some real-valued  $f \in C^\infty(N)$ ; as a result,

$$\nabla s = -i\Theta \otimes s = -ie^{if}\Theta \otimes r ,$$

---

<sup>8</sup>This includes open neighbourhoods of local trivialisations.

but

$$\nabla s = \nabla(e^{if}r) = (de^{if} - ie^{if}\vartheta) \otimes r = ie^{if}(df - \vartheta) \otimes r$$

and, therefore,

$$\begin{aligned} -ie^{if}\Theta \otimes r &= ie^{if}(df - \vartheta) \otimes r \Rightarrow \\ \vartheta - \Theta &= df . \end{aligned} \tag{2.12}$$

Conversely, by the same computation, if  $s$  has  $\Theta = \vartheta - df$  as potential 1-form, then  $r = e^{-if}s$  is a unitary section having  $\vartheta$  as potential 1-form.  $\blacksquare$

**Definition 2.14.** *The curvature of the connection is the operator*

$$curv(\nabla) : \Gamma(L) \rightarrow \Omega^2(M) \otimes_{C^\infty(M)} \Gamma(L) \tag{2.13}$$

given by

$$[curv(\nabla)s](X, Y) = \nabla_X \circ \nabla_Y s - \nabla_Y \circ \nabla_X s - \nabla_{[X, Y]} s , \tag{2.14}$$

for any  $s \in \Gamma(L)$  and  $X, Y \in \mathfrak{X}(M)$ .

Again, over a submanifold  $N \subset M$  with unitary section  $s$ , the curvature of the connection  $\nabla$  can be represented by a closed 2-form  $\omega \in \Omega^2(N)$ :

$$curv(\nabla)s = -i\omega \otimes s . \tag{2.15}$$

The next proposition implies, in particular, that the curvature 2-form is closed and real-valued.

**Proposition 2.2.** *If  $\Theta$  is as potential 1-form for the connection  $\nabla$ , over a submanifold  $N \subset M$ , the curvature operator can be computed by  $curv(\nabla)|_{TN} = -id\Theta$ , and it is independent of the choice of trivialisation.*

*Proof:* Assuming that  $s$  is the unitary section associated with  $\Theta$ ,

$$\begin{aligned}
[\text{curv}(\nabla)s](X, Y) &= \nabla_X \circ \nabla_Y s - \nabla_Y \circ \nabla_X s - \nabla_{[X, Y]} s \\
&= \nabla_X(-i\Theta(Y)s) - \nabla_Y(-i\Theta(X)s) + i\Theta([X, Y])s \\
&= -i(d(\iota_Y \Theta)(X) - i\Theta(X)\Theta(Y) \\
&\quad - d(\iota_X \Theta)(Y) + i\Theta(Y)\Theta(X) - \Theta([X, Y]))s \\
&= -i(d(\iota_Y \Theta)(X) - d(\iota_X \Theta)(Y) - \Theta([X, Y]))s \\
&= -i(d\Theta)(X, Y)s .
\end{aligned}$$

Independence of the choice of trivialisation comes from the fact that the difference between potential 1-forms related to different unitary sections is exact (lemma 2.1).

■

The following lemma is the converse of proposition 2.2.

**Lemma 2.2.** *At a submanifold  $N \subset M$  where  $\text{curv}(\nabla)|_{TN} = -id\Theta$ , there exists a unitary section such that  $\Theta$  is its potential 1-form.*

*Proof:* Supposing  $\text{curv}(\nabla)|_{TN} = -id\Theta$ , let  $\mathcal{A} = \{A_j\}_{j \in I}$  be a contractible open cover of  $N$  such that each  $A_j$  is a local trivialisation of  $L$  with unitary section  $s_j$  (this can always be obtained, e.g. using a convenient cover made of balls with respect to a Riemannian metric). Each unitary section  $s_j$  has  $\Theta_j$  as a potential 1-form of  $\nabla$ , and since

$$\text{curv}(\nabla)|_{TA_j} = -id(\Theta|_{TA_j}) = -id\Theta_j , \quad (2.16)$$

there exists real-valued functions  $f_j \in C^\infty(A_j)$  such that  $\Theta|_{TA_j} = \Theta_j - df_j$ . By lemma 2.1, the unitary sections  $r_j = e^{-if_j}s_j$  have  $\Theta|_{TA_j}$  as potential 1-forms.

Any two sections  $r_j$  and  $r_k$  such that  $A_j \cap A_k \neq \emptyset$  share the same potential 1-form, and because of that, they differ by a nonzero constant function (lemma 2.1),  $r_j = c_{jk}r_k$  at  $A_j \cap A_k$ . Trivially,  $c_{jk}$  can be extended to the same constant over  $A_k$ , and  $c_{jk}r_k$  is a section defined over  $A_k$  such that its restriction to  $A_j \cap A_k$  is exactly  $r_j$ , and it still has  $\Theta|_{TA_k}$  as potential 1-form. Hence, they can be glued together, using

the glueing condition of sheaves, to a unitary section  $r$  defined over  $N$  and having  $\Theta$  as potential 1-form. ■

## 2.2.2 Parallel transport and holonomy

Let  $\gamma : N \hookrightarrow M$  be a regular curve (open,  $N = \mathbb{R}$ , or closed,  $N = S^1$ ) with tangent vector field  $\dot{\gamma} \in \mathfrak{X}(\gamma)$ , where, by simplicity, the submanifold  $\gamma(N)$  will be denoted by  $\gamma$ . Given a vector  $s_{\gamma(0)}$  belonging to the fibre  $L_{\gamma(0)}$  of  $L$  over the point  $\gamma(0)$ , there exists a unique section  $s$  of  $L$ , defined over  $\gamma$ , satisfying

$$\begin{cases} \nabla_{\dot{\gamma}} s = 0 \\ s(\gamma(0)) = s_{\gamma(0)} \end{cases} . \quad (2.17)$$

**Definition 2.15.** *The parallel transport  $\Pi_{\gamma(t)} : L_{\gamma(0)} \rightarrow L_{\gamma(t)}$ , from the point  $\gamma(0)$  to  $\gamma(t)$ , over the curve  $\gamma$  is the linear operator defined by the flow of the system 2.17:  $\Pi_{\gamma(t)}(z) = s(\gamma(t))$ , where the section  $s$  is the unique solution of the initial value problem  $s_{\gamma(0)} = z$ .*

The parallel transport is a bundle automorphism (respecting the Hermitian product), and one can write  $\Pi_{\phi_t}(s)$  to denote, for each  $p \in M$ , the parallel transport from  $p$  to  $\phi_t(p)$  of the section  $s \in \Gamma(L)$  through the integral curve of  $X \in \mathfrak{X}(M)$ : with  $\phi_t$  standing for its flow at time  $t$ . Moreover, by definition,

$$\nabla_X s = \frac{d}{dt} \Pi_{\phi_t}^{-1}(s \circ \phi_t) \Big|_{t=0} . \quad (2.18)$$

**Proposition 2.3.** *The parallel transport of a section  $r = fs \in \Gamma(L|_{\gamma})$ , where  $f \in C^\infty(\gamma)$  and  $s$  is a unitary section defined over  $\gamma$  with potential 1-form  $\Theta$ , is given by*

$$\Pi_{\gamma(t)}(r \circ \gamma(0)) = e^{i \int_0^t \Theta(\dot{\gamma}(t')) dt'} f \circ \gamma(0) s \circ \gamma(t) . \quad (2.19)$$

*Proof:* In fact,

$$\nabla_{\dot{\gamma}}(r) = \dot{\gamma}(f)s - if\Theta(\dot{\gamma})s \quad (2.20)$$

and  $\dot{\gamma}(f)|_{\gamma(t)} = \frac{d}{dt} f \circ \gamma(t)$ ; thus,

$$[\nabla_{\dot{\gamma}}(r)](\gamma(t)) = \left( \frac{d}{dt} f \circ \gamma(t) \right) s \circ \gamma(t) - if \circ \gamma(t) \Theta(\dot{\gamma}(t)) s \circ \gamma(t) \quad (2.21)$$



and the parallel transport equation (eq. 2.17) becomes

$$\begin{cases} \frac{d}{dt} f \circ \gamma(t) = i\Theta(\dot{\gamma}(t))f \circ \gamma(t) \\ r \circ \gamma(0) = f \circ \gamma(0)s \circ \gamma(0) \end{cases} . \quad (2.22)$$

■

Since both  $L_{\gamma(0)}$  and  $L_{\gamma(2\pi)}$  are isomorphic to  $\mathbb{C}$  and the connection is Hermitian, when the curve is closed,  $\gamma(0) = \gamma(2\pi)$ , there is an element of  $U(1)$  that realises the parallel transport.

**Definition 2.16.** *The holonomy of the loop,  $hol_{\nabla^\omega}(\gamma) \in U(1)$ , is the number given by  $\Pi_{\gamma(2\pi)}(z) = hol_{\nabla^\omega}(\gamma)z$ .*

The following is a corollary of proposition 2.3.

**Corollary 2.1.** *The holonomy is given by*

$$hol_{\nabla^\omega}(\gamma) = \exp\left(i \int_{\gamma} \Theta\right) , \quad (2.23)$$

*if there is a unitary section defined over the loop  $\gamma$  with potential 1-form  $\Theta$ .*

*Proof:* Putting  $t = 2\pi$  and taking  $z \in L_{\gamma(0)} \cong L_{\gamma(2\pi)}$  to be the value of the unitary section at  $\gamma(0)$ , equation 2.19 reads

$$\Pi_{\gamma(2\pi)}(z) = e^{i \int_0^{2\pi} \Theta(\dot{\gamma}(t')) dt'} z = \exp\left(i \int_{\gamma} \Theta\right) z . \quad (2.24)$$

■

## 2.3 Foliations and Lie pseudoalgebras

Relevant cohomology theories appearing in this thesis are going to be presented in a unified way —as Lie pseudoalgebra representations. This section introduces the basics of Lie pseudoalgebras and its relationship with foliated manifolds.

### 2.3.1 Vector fields and some Lie subalgebras

Considering a manifold  $M$ , the space  $\Gamma(TM)$  is both a  $C^\infty(M)$ -module and a Lie algebra (with the Lie bracket of vector fields  $[\cdot, \cdot]$  in mind), and these two structures are related by the Leibniz rule

$$[X, fY] = X(f)Y + f[X, Y] , \quad (2.25)$$

for any  $X, Y \in \Gamma(TM)$  and  $f \in C^\infty(M)$ . This Leibniz rule results from an ( $C^\infty(M)$ -module) identification<sup>9</sup> of  $\Gamma(TM)$  with the derivations of the commutative algebra  $C^\infty(M)$ ,  $\mathfrak{X}(M) := \text{Der}_{\mathbb{R}}(C^\infty(M))$ .

This is just an example of a Lie pseudoalgebra, and it motivates the following definition<sup>10</sup>.

**Definition 2.17.** *A Lie pseudoalgebra is a triplet  $(\mathfrak{g}, C^\infty(M; \mathbb{K}), T)$ , where:*

- $\mathbb{K}$  is either the field  $\mathbb{R}$  or  $\mathbb{C}$ ;
- $C^\infty(M; \mathbb{K})$  is the commutative  $\mathbb{K}$ -algebra of smooth functions over  $M$  with values in  $\mathbb{K}$ ;
- $\mathfrak{g}$  is both a  $C^\infty(M; \mathbb{K})$ -module and a  $\mathbb{K}$ -Lie algebra, with Lie bracket denoted by  $[\cdot, \cdot]$ ;
- $T : \mathfrak{g} \rightarrow \text{Der}_{\mathbb{K}}(C^\infty(M; \mathbb{K}))$  is a morphism of  $\mathbb{K}$ -Lie algebras and  $C^\infty(M; \mathbb{K})$ -modules, such that

$$[x, fy] = T(x)(f)y + f[x, y] , \quad (2.26)$$

for any  $x, y \in \mathfrak{g}$  and  $f \in C^\infty(M; \mathbb{K})$ .

When  $\mathfrak{g}$  is a  $\mathbb{K}$ -Lie subalgebra and  $C^\infty(M; \mathbb{K})$ -submodule of  $\text{Der}_{\mathbb{K}}(C^\infty(M; \mathbb{K}))$ ,  $T$  is going to be omitted if it is chosen to be the inclusion map.

The second example comes from foliated manifolds.

---

<sup>9</sup>This is true in the smooth category.

<sup>10</sup>The reader is invited to consult the survey [18] for a general and precise definition; as well for a nice account of the history and, various, names of this structure.

**Example 2.13.** Let  $(M, \mathcal{F})$  be a foliated  $m$ -dimensional manifold with  $n$ -dimensional leaves. The (regular) foliation can be thought as a subbundle of  $TM$ , which is often denoted by  $T\mathcal{F}$ . Therefore, the space of sections of  $T\mathcal{F}$  is a  $C^\infty(M)$ -module and, because of the integrability condition, the restriction of the Lie bracket of vector fields to  $\Gamma(T\mathcal{F})$  makes it a Lie subalgebra of  $(\Gamma(TM), [\cdot, \cdot])$ ; the Leibniz condition is obviously satisfied.  $\diamond$

The two aforementioned examples,  $\Gamma(TM)$  and  $\Gamma(T\mathcal{F})$ , are also Lie algebroids<sup>11</sup>; nevertheless, the next one is a genuine Lie pseudoalgebra.

**Example 2.14.** Integrable systems defined on  $(M, \omega)$  induce Lie subalgebras of  $(\Gamma(TM), [\cdot, \cdot])$ , namely,  $(\mathcal{P} := \langle X_1, \dots, X_n \rangle_{C^\infty(M)}, [\cdot, \cdot]_{\mathcal{P}})$ , where  $X_i$  is the Hamiltonian vector field of the  $i$ th component of a moment map  $F : M \rightarrow \mathbb{R}^n$ . These Lie subalgebras are  $C^\infty(M)$ -modules, as well, and the Leibniz rule applies. It is not an example of a Lie algebroid because the rank of the distribution  $\mathcal{P}$  may vary; thus, there is no vector bundle attached to  $\mathcal{P}$  in general.  $\diamond$

### 2.3.2 Representation theory

There is also a notion of representation for Lie pseudoalgebras [25].

**Definition 2.18.** A representation of a Lie pseudoalgebra  $(\mathfrak{g}, C^\infty(M; \mathbb{K}), T)$  on a  $C^\infty(M; \mathbb{K})$ -module  $E$  is a morphism of  $\mathbb{K}$ -Lie algebras and  $C^\infty(M; \mathbb{K})$ -modules  $\rho : \mathfrak{g} \rightarrow \text{Hom}_{\mathbb{K}}(E; E)$ , such that

$$\rho(x)(fs) = T(x)(f)s + f\rho(x)(s) , \quad (2.27)$$

for all  $x \in \mathfrak{g}$ ,  $s \in E$  and  $f \in C^\infty(M; \mathbb{K})$ .

In particular, the map  $\rho : \mathfrak{g} \rightarrow \text{Hom}_{\mathbb{K}}(E; E)$  is a representation of the  $\mathbb{K}$ -Lie algebra  $\mathfrak{g}$  on  $E$ .

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<sup>11</sup>The reader can easily check that this definition of Lie pseudoalgebras includes Lie algebroids as examples.

There exists a cohomology for each representation of a Lie pseudoalgebra, similar to the theory of Lie algebras. For each  $k \in \mathbb{Z}$ , one can define a  $C^\infty(M; \mathbb{K})$ -module

$$\Omega_{\mathfrak{g}}^k(M) := \text{Hom}_{C^\infty(M; \mathbb{K})}(\wedge_{C^\infty(M; \mathbb{K})}^k \mathfrak{g}; C^\infty(M; \mathbb{K})) , \quad (2.28)$$

if  $k < 0$  or  $k > \dim \mathfrak{g}$ ,  $\Omega_{\mathfrak{g}}^k(M) = \emptyset$ , and  $\Omega_{\mathfrak{g}}^0(M) = C^\infty(M; \mathbb{K})$ . Then, one can take the tensor product

$$S_{\mathfrak{g}}^k(E) := \Omega_{\mathfrak{g}}^k(M) \otimes_{C^\infty(M; \mathbb{K})} E \cong \text{Hom}_{C^\infty(M; \mathbb{K})}(\wedge_{C^\infty(M; \mathbb{K})}^k \mathfrak{g}; E) , \quad (2.29)$$

and define a  $\mathbb{K}$ -linear map  $d^\rho$  that takes  $\alpha \in S_{\mathfrak{g}}^k(E)$  to an element of  $S_{\mathfrak{g}}^{k+1}(E)$ , for any  $k \in \mathbb{Z}$ :

$$\begin{aligned} d^\rho \alpha(x_1, \dots, x_{k+1}) &:= \sum_{i=1}^{k+1} (-1)^{i+1} \rho(x_i) (\alpha(x_1, \dots, \hat{x}_i, \dots, x_{k+1})) \\ &\quad + \sum_{i < j} (-1)^{i+j} \alpha([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}) , \end{aligned} \quad (2.30)$$

with  $x_1, \dots, x_{k+1} \in \mathfrak{g}$ .

The fact that  $\rho$  is a representation implies that  $d^\rho$  is a coboundary operator,  $d^\rho \circ d^\rho = 0$ , and it is possible to construct a complex of  $\mathbb{K}$  vector spaces: the cochain spaces are the  $C^\infty(M; \mathbb{K})$ -modules  $S_{\mathfrak{g}}^k(E)$ , and the differential is given by  $d^\rho$ .

**Definition 2.19.** *The  $k$ th cohomology group associated to the complex*

$$0 \longrightarrow E_{\mathfrak{g}} \hookrightarrow E \xrightarrow{d^\rho} S_{\mathfrak{g}}^1(E) \xrightarrow{d^\rho} \dots \xrightarrow{d^\rho} S_{\mathfrak{g}}^{\dim \mathfrak{g}}(E) \xrightarrow{d^\rho} 0 , \quad (2.31)$$

with  $E_{\mathfrak{g}} := \ker(d^\rho : E \rightarrow S_{\mathfrak{g}}^1(E))$ , is denoted by  $H^k(S_{\mathfrak{g}}^\bullet(E))$  and called the Lie pseudoalgebra cohomology of  $\mathfrak{g}$  with respect to  $E$ .

In the particular case where  $E = C^\infty(M; \mathbb{K})$ , since  $\text{Der}_{\mathbb{K}}(C^\infty(M; \mathbb{K})) \subset \text{Hom}_{\mathbb{K}}(E; E)$ , one can use  $T$  in place of  $\rho$  and define a cohomology verbatim: the cochain spaces are the  $C^\infty(M; \mathbb{K})$ -modules  $\Omega_{\mathfrak{g}}^k(M)$ , the differential is denoted by  $d_{\mathfrak{g}}$ , and the cohomology is denoted by  $H_{\mathfrak{g}}^\bullet(M)$ .

**Example 2.15.** The de Rham cohomology. If  $E = C^\infty(M)$ ,  $\mathfrak{g} = \Gamma(TM)$ , endowed with the Lie bracket of vector fields, and  $\rho$  is the identification between sections of  $TM$  and derivations of  $C^\infty(M)$ :  $H_{\mathfrak{g}}^\bullet(M)$  is the de Rham cohomology of  $M$ ,  $H_{dR}^\bullet(M)$ .  $\diamond$

The previous example also holds if one considers Lie subalgebras of the Lie algebra of vector fields.

**Example 2.16.** Taking  $E = C^\infty(M)$ ,  $\mathfrak{g} = \Gamma(T\mathcal{F})$ , the Lie subalgebra of vector fields tangent to the foliation  $\mathcal{F}$ , and  $\rho$  the identification between sections of the subbundle  $T\mathcal{F}$  and derivations of  $C^\infty(M)$ , one has that  $\Omega_{\mathcal{F}}^k(M) \cong \Gamma(\wedge^k T\mathcal{F}^*)$  and  $H_{\mathfrak{g}}^\bullet(M)$  is the foliated cohomology,  $H_{\mathcal{F}}^\bullet(M)$ .  $\diamond$

The Lie pseudoalgebra  $(T\mathcal{F}, [\cdot, \cdot]|_{T\mathcal{F}})$  is being represented on  $C^\infty(M)$  as vector fields acting on smooth functions. The differential  $d_{\mathcal{F}}$  is the restriction of the exterior derivative,  $d$ , to  $T\mathcal{F}$ , and  $E_{\mathfrak{g}}$ , which is denoted by  $C_{\mathcal{F}}^\infty(M)$ , is the space of smooth functions which are constant along the leaves of the foliation (for the de Rham complex,  $E_{\mathfrak{g}} = \mathbb{R}$ ).

The Lie pseudoalgebra example of integrable systems, example 2.14, is of particular interest to this thesis. Its Lie pseudoalgebra cohomology with respect to  $C^\infty(M)$  is very similar to the one of the foliated cohomology:  $\mathcal{P}$  can be represented on  $C^\infty(M)$  as vector fields acting on smooth functions. Yet, it is worth to mention here one difference.

**Definition 2.20.** *The vanishing set of a vector field  $X_i$ , generating  $\mathcal{P}$ , is denoted by  $\Sigma_i := \{p \in M ; X_i(f)(p) = 0 \forall f \in C^\infty(M)\}$ .*

**Proposition 2.4.** *If  $\alpha \in \Omega_{\mathcal{P}}^k(M)$ , then  $\alpha(X_{j_1}, \dots, X_{j_k})|_{\Sigma_{j_1} \cup \dots \cup \Sigma_{j_k}} = 0$ .*

*Proof:* At every point  $p \in M$  the map  $\alpha \in \Omega_{\mathcal{P}}^k(M)$  reduces to an element of the dual of  $\wedge^k \mathcal{P}|_p$ , which is a finite dimensional vector space. Since  $X_i = 0$  at  $\Sigma_i$ , for any  $p \in \Sigma_i$  and vectors  $Y_1(p), \dots, Y_{k-1}(p) \in \mathcal{P}|_p$ , the following expression holds:

$$\alpha_p(X_i(p), Y_1(p), \dots, Y_{k-1}(p)) = 0 . \quad (2.32)$$

Therefore,  $\alpha(X_{j_1}, \dots, X_{j_k})|_{\Sigma_i} = 0$  for  $i = j_1, \dots, j_k$ . ■

# Chapter 3

## Prequantisation

This chapter deals with some concepts needed to define wave functions. The first attempt was to see them as sections of a complex line bundle over the symplectic manifold, the so-called prequantum line bundle. The other notion described here, polarisations, is a way to establish a global distinction between momentum and position.

### 3.1 Prequantum line bundle

Using a particular isomorphism between the Čech cohomology  $\check{H}^2(M; \mathbb{R})$  and de Rham cohomology  $H_{dR}^2(M; \mathbb{R})$ , a closed 2-form is integral if and only if it is in the image of the homomorphism between  $\check{H}^2(M; \mathbb{Z})$  and  $\check{H}^2(M; \mathbb{R})$ :

$$\begin{array}{ccc} \mathbb{R} & \check{H}^2(M; \mathbb{R}) & \longleftrightarrow & H_{dR}^2(M; \mathbb{R}) \\ \uparrow & & & \uparrow \\ \mathbb{Z} & \check{H}^2(M; \mathbb{Z}) & & \end{array}$$

the inclusion of  $\check{H}^2(M; \mathbb{Z})$  in  $\check{H}^2(M; \mathbb{R})$  is induced by a homomorphism between  $\mathbb{Z}$  and  $\mathbb{R}$ .

**Definition 3.1.** *A symplectic manifold  $(M, \omega)$  such that the de Rham class  $[\omega]$  is integral is called prequantisable. A prequantum line bundle of  $(M, \omega)$  is a Hermitian*

line bundle over  $M$  with connection, compatible with the Hermitian structure,  $(L, \nabla^\omega)$  that satisfies  $\text{curv}(\nabla^\omega) = -i\omega$ .

**Example 3.1.** Any exact symplectic manifold satisfies  $[\omega] = 0$ , in particular cotangent bundles with the canonical symplectic structure. The trivial line bundle is an example of a prequantum line bundle in this case.  $\diamond$

The following theorem<sup>1</sup> provides a relation between the above definitions:

**Theorem 3.1.** *A symplectic manifold  $(M, \omega)$  admits a prequantum line bundle  $(L, \nabla^\omega)$  if and only if it is prequantisable.*

*Proof:* Let  $\mathcal{A} = \{A_j\}_{j \in I}$  be a contractible open cover of  $M$  such that each  $A_j$  is a local trivialisation of  $L$  with unitary section  $s_j$  (this can always be obtained, e.g. using a convenient cover made of balls with respect to a Riemannian metric).

( $\text{curv}(\nabla^\omega) = -i\omega \Rightarrow [\omega]$  is integral) By proposition 2.2, on each  $A_j$  the formula  $\omega = d\Theta_j$  holds:  $\Theta_j$  is the potential 1-form with respect to  $s_j$ . As a consequence,  $d(\Theta_j - \Theta_k) = 0 \Rightarrow \Theta_j - \Theta_k = df_{jk}$  on  $A_j \cap A_k$ , where  $e^{if_{jk}} \in U(1)$  are the transition functions (equation 2.12). Now,  $d(f_{jk} + f_{kl} - f_{jl}) = 0 \Rightarrow f_{jk} + f_{kl} - f_{jl} = 2\pi a_{jkl}$  on  $A_j \cap A_k \cap A_l$ , with  $a_{jkl}$  a constant. The cocycle conditions of the line bundle,  $e^{if_{jk}} e^{if_{kl}} = e^{if_{jl}}$ , imply that  $e^{i2\pi a_{jkl}} = 1$ ; thus,  $a_{jkl} \in \mathbb{Z}$  and  $[\omega]$  is integral.

Conversely, given an integral  $[\omega]$  on each open set  $A_j$ , which is contractible, the closedness of the symplectic form implies exactness,  $\omega = d\theta_j$ . Again,  $\theta_j - \theta_k = df_{jk}$  on  $A_j \cap A_k$ ; therefore,  $e^{i2\pi f_{jk}}$  gives cocycle conditions (since  $[\omega]$  is integral) on the intersections of the open cover  $\mathcal{A}$ , defining (in a unique way, up to equivalence relations) a Hermitian line bundle.  $\blacksquare$

## 3.2 Polarisation

Classically, a real polarisation  $\mathcal{F}$  is an integrable subbundle of  $TM$  (the bundle  $T\mathcal{F}$ ) whose leaves are Lagrangian submanifolds: i.e.  $\mathcal{F}$  is a Lagrangian foliation. But

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<sup>1</sup>A proof can be found in [15].



due to the example below another definition is considered.

For an integrable system  $F : M \rightarrow \mathbb{R}^n$  on a symplectic manifold, the Liouville integrability condition implies that the distribution of the Hamiltonian vector fields of the components of the moment map generates a Lagrangian foliation (possibly) with singularities. This is an example of a generalised real polarisation —i.e. an integrable distribution on  $TM$  whose leaves are Lagrangian submanifolds, except for some singular leaves.

**Definition 3.2.** *A real polarisation  $\mathcal{P}$  is an integrable (in the Sussmann's [28] sense) distribution of  $TM$  whose leaves are generically Lagrangian. The complexification of  $\mathcal{P}$  is denoted by  $P$  and will be called polarisation.*

The most relevant real polarisation for this thesis is  $\mathcal{P} = \langle X_{f_1}, \dots, X_{f_n} \rangle_{C^\infty(M)}$ : the distribution of the Hamiltonian vector fields  $X_{f_i}$  of the components  $f_i$  of an integrable system  $F = (f_1, \dots, f_n) : M \rightarrow \mathbb{R}^n$ .

Here is an example of a real polarisation that do not come from an integrable system.

**Example 3.2.** The action of  $S^1$  on  $S^1 \times S^1$  given by  $(z, x, y) \mapsto (z \cdot x, y)$ , with  $z, x, y \in S^1$ , is symplectic (taking as symplectic form the area form of the torus). Because there are no fixed points, this action cannot be Hamiltonian —otherwise, one would have a function over a compact manifold without critical points.  $\diamond$

Henceforth,  $(L, \nabla^\omega)$  will be a prequantum line bundle and  $P$  the complexification of a real polarisation  $\mathcal{P}$  of  $(M, \omega)$ .

### 3.3 Bohr-Sommerfeld fibres

The following definition plays a very important role in the computation of the cohomology groups appearing in geometric quantisation:

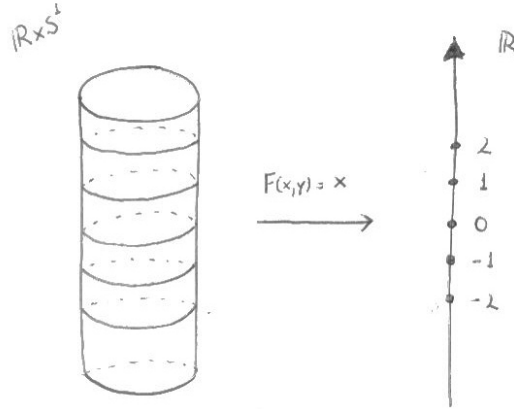
**Definition 3.3.** A leaf  $\ell$  of  $\mathcal{P}$  is a Bohr-Sommerfeld leaf if there exists a nonzero section  $s : \ell \rightarrow L$  such that  $\nabla_X^\omega s = 0$  for any vector field  $X$  of  $P$  restricted to  $\ell$ . Fibres which are a union of Bohr-Sommerfeld leaves are called Bohr-Sommerfeld fibres.

**Example 3.3.** Consider  $M = \mathbb{R} \times S^1$  with coordinates  $(x, y)$  and  $\omega = dx \wedge dy$ . Take as  $L$  the trivial complex line bundle with connection 1-form  $\Theta = xdy$ , with respect to the unitary section  $e^{ix}$ , and  $\mathcal{P} = \left\langle \frac{\partial}{\partial y} \right\rangle_{C^\infty(M)}$ .

Flat sections,  $s(x, y) = f(x, y)e^{ix}$ , satisfy

$$\left[ \nabla_{\frac{\partial}{\partial y}}^\omega s \right] (x, y) = \left( \frac{\partial f}{\partial y}(x, y) - ix f(x, y) \right) e^{ix} = 0 . \quad (3.1)$$

Thus,  $s(x, y) = g(x)e^{ixy}e^{ix}$ , for some function  $g$ , and it has period  $2\pi$  in  $y$  if and only if  $x \in \mathbb{Z}$ , for  $S^1$  the unity circle: flat sections are only well-defined for the set of points with  $x \in \mathbb{Z}$ ; wherefore, Bohr-Sommerfeld leaves are circles of integral height.



◇

**Proposition 3.1.** A leaf  $\ell$  of  $\mathcal{P}$  is a Bohr-Sommerfeld leaf if and only if the holonomy is trivial,  $hol_{\nabla^\omega}(\gamma) = 1$ , for any loop  $\gamma$  on a connected component of  $\ell$ .

*Proof:* In a Bohr-Sommerfeld leaf  $\ell$  the nonzero section  $s$  can be used to define a potential 1-form  $\Theta$  of the connection on the whole leaf, lemma 2.1. The potential 1-form vanishes on  $\ell$ , since  $0 = \nabla^\omega s|_{T\ell} = -i\Theta|_{T\ell} \otimes s$ . Thus, if  $\gamma$  is a loop on  $\ell$ , by corollary 2.1,  $hol_{\nabla^\omega}(\gamma) = e^{i \int_\gamma \Theta} = 1$ .

Now, supposing that  $hol_{\nabla^\omega}(\gamma) = 1$  for any loop on a connected component of a leaf  $\ell$  of  $\mathcal{P}$ , for any point  $p \in \ell$  and a nonzero  $s_p \in L_p$  (the fibre of  $L$  over  $p$ ) it is

possible to define a nonzero section  $s$  over  $\ell$  by parallel transport: i.e.  $s(q) = \Pi_{\gamma_1}(s_p)$ , where  $\gamma_1$  is any curve connecting  $p$  and  $q \in \ell$ . The section is well-defined because if  $\gamma_2$  is another curve connecting  $p$  and  $q$ , and  $\gamma$  the loop formed by composing  $\gamma_2$  and  $\gamma_1^{-1}$ ,

$$\begin{aligned} s(q) &= \text{hol}_{\nabla^\omega}(\gamma)s(q) = \Pi_\gamma(s(q)) = \Pi_{\gamma_2} \circ [\Pi_{\gamma_1}]^{-1}(s(q)) \\ &= \Pi_{\gamma_2} \circ [\Pi_{\gamma_1}]^{-1} \circ \Pi_{\gamma_1}(s_p) = \Pi_{\gamma_2}(s_p) . \end{aligned} \quad (3.2)$$

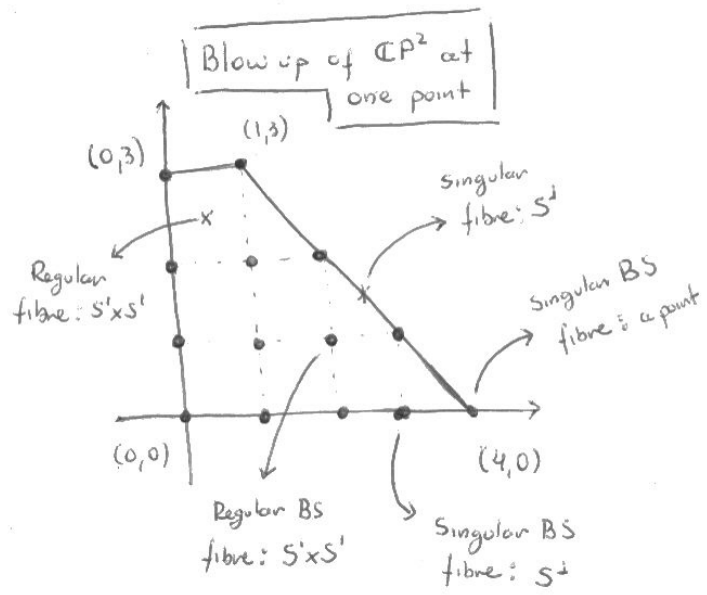
The parallel transport respects the Hermitian product, and this guarantees that the section defined in this way is nonzero. ■

There is a stronger characterisation for the Bohr-Sommerfeld leaves in the case of integrable systems.

On a neighbourhood  $N$  of a Liouville torus  $\ell$  the symplectic form is exact,  $\omega|_{TN} = d\theta$  (Liouville theorem 2.2), and lemma 2.2 provides a unitary section  $s \in \Gamma(L|_N)$  such that  $\nabla^\omega s = -i\theta \otimes s$ . If  $\ell$  is a Bohr-Sommerfeld leaf, there exist a flat unitary section  $r \in \Gamma(L|_\ell)$  and, because of lemma 2.1, a real-valued function  $f \in C^\infty(\ell)$  such that  $\theta|_{T\ell} - df$  is the potential 1-form associated to  $r$ . Since  $1 = \text{hol}_{\nabla^\omega}(\gamma) = e^{i \int_\gamma \theta - df}$  for any loop  $\gamma$  in  $\ell$  (proposition 3.1 and corollary 2.1), one has  $\int_\gamma \theta \in 2\pi\mathbb{Z}$ . Thus, using the action variables as a local chart on  $F(M)$ , the following theorem [10] has been proved:

**Theorem 3.2** (Guillemin and Sternberg). *Under the assumption that the zero fibre is Bohr-Sommerfeld, the image of the regular (Liouville tori) Bohr-Sommerfeld leaves by the moment map is contained in  $\mathbb{Z}^n$ .*

**Example 3.4.** For toric manifolds the Bohr-Sommerfeld leaves are the inverse image by the moment map of integer lattice points in the polytope, with regular ones inside the polytope.



◇

# Chapter 4

## Geometric quantisation à la Kostant

The original idea of geometric quantisation is to associate a Hilbert space to a symplectic manifold via a prequantum line bundle and a polarisation. Usually this is done using flat global sections of the line bundle; in case these global sections do not exist, one can define geometric quantisation via higher cohomology groups by considering cohomology with coefficients in the sheaf of flat sections.

### 4.1 A cohomological definition

The existence of global flat sections is a nontrivial matter. Actually, Rawnsley [26] (also proposition 7.3 in this thesis, under slightly different hypotheses) showed that the existence of a  $S^1$ -action may be an obstruction for the existence of nonzero global flat sections.

Example 3.3 and the complex plane, endowed with the canonical symplectic structure and a polarisation induced by a circle action, provide explicit examples of the nonexistence of nonzero global flat sections.

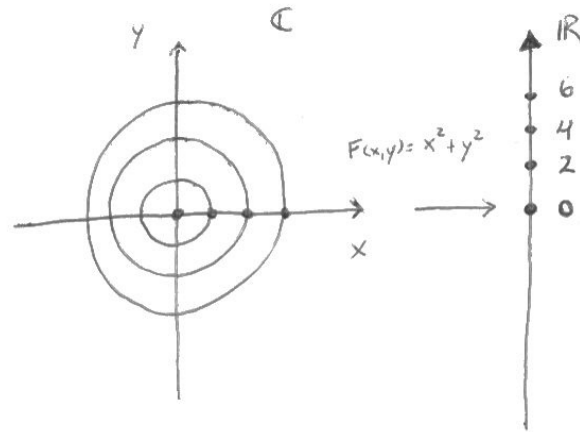
**Example 4.1.** Let  $M = \mathbb{C}$  with coordinates  $(x, y)$  and Darboux form  $\omega = dx \wedge dy$ ,  $L = \mathbb{C} \times \mathbb{C}$  the trivial bundle with connection 1-form  $\Theta = \frac{1}{2}(x dy - y dx)$ , with respect

to the unitary section  $e^{i(x^2+y^2)}$ , and  $\mathcal{P} = \left\langle -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right\rangle_{C^\infty(\mathbb{C})}$ .

Corollary 2.1 implies that the holonomy for a closed curve  $\gamma$  inside a leaf of  $\mathcal{P}$  is

$$[\text{hol}_{\nabla^\omega}(\gamma)](x, y) = \left[ \exp \left( i \int_\gamma \Theta \right) \right] (x, y) = e^{i2\pi \frac{(x^2+y^2)}{2}}. \quad (4.1)$$

Therefore, by proposition 3.1, flat sections are only well-defined for the set of points with  $\frac{(x^2+y^2)}{2} \in \mathbb{Z}$ .



◇

One is forced to work with delta functions with support over Bohr-Sommerfeld leaves in order to use flat sections as analogue for wave functions, or deal with sheaves and higher order cohomology groups. Both approaches can be found in the literature<sup>1</sup>, but here only the sheaf approach is treated: as suggested by Kostant.

**Definition 4.1.** *Let  $\mathcal{J}$  denotes the space of local sections  $s$  of a prequantum line bundle  $L$  such that  $\nabla_X^\omega s = 0$  for all vector fields  $X$  of a polarisation  $P$ . The space  $\mathcal{J}$  has the structure of a sheaf and it is called the sheaf of flat sections.*

Considering the triplet prequantisable symplectic manifold  $(M, \omega)$ , prequantum line bundle  $(L, \nabla^\omega)$ , and polarisation  $P$ :

**Definition 4.2.** *The quantisation of  $(M, \omega, L, \nabla^\omega, P)$  is given by*

$$\mathcal{Q}(M) = \bigoplus_{k \geq 0} \check{H}^k(M; \mathcal{J}), \quad (4.2)$$

<sup>1</sup>Rawnsley cites works of Simms, Śniatycki and Keller in [26].

where  $\check{H}^k(M; \mathcal{J})$  are Čech cohomology groups with values in the sheaf  $\mathcal{J}$ . In this case, one implicitly assumes the extra structures and calls  $M$  a quantisable manifold.

The detailed construction of  $\check{H}^k(M; \mathcal{J})$  and the sheaf structure of  $\mathcal{J}$  are described in chapter 5. The present thesis can be summarised as an approach to compute and understand the features of these cohomology groups.

**Remark 4.1.** Even though  $\mathcal{Q}(M)$  is just a vector space and a priori <sup>2</sup> has no Hilbert structure, it will be called quantisation. The true quantisation of the triplet  $(M, \omega, L, \nabla^\omega, P)$  shall be the completion of the vector space  $\mathcal{Q}(M)$ , after a Hilbert structure is given, together with a Lie algebra homomorphism (possibly defined over a smaller subset) between the Poisson algebra of  $C^\infty(M)$  and operators on the Hilbert space. In spite of the problems that may exist in order to define geometric quantisation using  $\mathcal{Q}(M)$ , the first step is to compute this vector space.

**Remark 4.2.** Flat sections behave in a different fashion for Kähler polarisations. This thesis does not deal with this case<sup>3</sup>; however, much can be found in the literature (e.g. [10, 11] and references therein). There is another aspect of the theory that will be left aside by this thesis<sup>4</sup>: metaplectic correction. To imbue  $\check{H}^0(M; \mathcal{J}) \cong \{s \in \Gamma(L) ; \nabla_X^\omega s = 0 \forall X \in P\}$  with a Hilbert structure, Kostant and Blattner [16, 2] introduced half-forms on geometric quantisation<sup>5</sup>. Besides inducing an inner product, half-forms also make a correction to the spectrum of the operators (Blattner, Rawnsley, Simms and Śniatycki are referred to for this in [17, 26, 27]), this correction does not always behaves as one would like, though (e.g. [7]).

---

<sup>2</sup>For the concrete cases presented in this thesis,  $\mathcal{Q}(M)$  does admit Hilbert structures.

<sup>3</sup>This is particularly due to the author's opinion on the famous Hadamard's quote: "The shortest path between two truths in the real domain passes through the complex domain".

<sup>4</sup>See remark 6.1 for an explanation.

<sup>5</sup>It is not clear who did what, but both Kostant and Blattner say that it has roots on a joint work of them with Sternberg.

# Chapter 5

## The Čech approach

The results of Hamilton and Miranda [11, 12] concerning geometric quantisation of integrable systems with elliptic and hyperbolic singularities are presented in this chapter. Part of their strategy relies on the existence of local normal forms near singularities of integrable systems.

### 5.1 Sheaf cohomology

Here is a review of the definition of sheaves and the construction of the Čech cohomology in the context of geometric quantisation. This is a humble account of a vast theory, and just the notions needed are introduced; for a more complete and general treatment see [4].

#### 5.1.1 Sheaf structure of $\mathcal{J}$

Let  $(L, \nabla^\omega)$  be a prequantum line bundle and  $P$  a polarisation for  $(M, \omega)$ . Take an open set  $V \subset M$  and denote by  $\mathcal{J}(V)$  the set of flat sections on  $V$ : i.e. sections  $s$  of  $L$  defined on  $V$  such that  $\nabla_X^\omega s = 0$  for all vector fields  $X$  of  $P$  defined over  $V$ . For each open set  $V$  this is an Abelian group (indeed, it is a module over the ring  $C_P^\infty(M)$ ).

Thus, for each open set of the manifold  $M$ , which is a topological space,  $\mathcal{J}(V)$



associates an Abelian group, and the restriction of sections to an open subset  $W \subset V$  induces restriction maps  $R_{W,V} : \mathcal{J}(V) \rightarrow \mathcal{J}(W)$ , i.e.  $R_{W,V}(s) := s|_W$  for any  $s \in \mathcal{J}(V)$ . The restriction maps respect the Abelian structure and satisfy  $R_{V,V} = \text{id}_{\mathcal{J}(V)}$  and  $R_{W,U} \circ R_{U,V} = R_{W,V}$ , whenever  $W \subset U \subset V$ . This is the definition of a presheaf of Abelian groups over a topological space.

If  $s, r \in \mathcal{J}(V \cup W)$  satisfy  $R_{W,V \cup W}(s) = R_{W,V \cup W}(r)$  and  $R_{V,V \cup W}(s) = R_{V,V \cup W}(r)$ , they are equal over  $V \cup W$ . If  $s \in \mathcal{J}(V)$  and  $t \in \mathcal{J}(W)$  are such that  $R_{V \cap W, V}(s) = R_{V \cap W, W}(t)$ , this implies that they are equal over  $V \cap W$ ; and one can define  $r \in \mathcal{J}(V \cup W)$  by the equations  $s = R_{V, V \cup W}(r)$  and  $t = R_{W, V \cup W}(r)$ . The first property is called local identity and the second one the glueing condition, a sheaf is a presheaf satisfying these extra conditions.

### 5.1.2 Construction of $\check{H}^k(M; \mathcal{J})$

Fix an open cover  $\mathcal{A} = \{A_\alpha\}$  of  $M$ . A section of  $\mathcal{J}$  is assigned to each  $(k+1)$ -fold intersection of elements from the cover,  $A_{\alpha_0 \dots \alpha_k} := A_{\alpha_0} \cap \dots \cap A_{\alpha_k}$ .

**Definition 5.1.** *A  $k$ -cochain is an assignment  $f_{\alpha_0 \dots \alpha_k} \in \mathcal{J}(A_{\alpha_0 \dots \alpha_k})$  for each  $(k+1)$ -fold intersection in the cover  $\mathcal{A}$ . The set of  $k$ -cochains is denote by  $C_{\mathcal{A}}^k(M; \mathcal{J})$ .*

A coboundary operator  $\delta$  is well-defined<sup>1</sup> by

$$(\delta^{k-1} f)_{\alpha_0 \dots \alpha_k} := \sum_{j=0}^k (-1)^j f_{\alpha_0 \dots \hat{\alpha}_j \dots \alpha_k} |_{A_{\alpha_0 \dots \alpha_k}}, \quad (5.1)$$

so one has a cochain complex for  $C_{\mathcal{A}}^k(M; \mathcal{J})$ .

**Definition 5.2.** *The sheaf cohomology with respect to the cover  $\mathcal{A}$  is the cohomology of this complex,*

$$\check{H}_{\mathcal{A}}^k(M; \mathcal{J}) := \frac{\ker \delta^k}{\text{im} \delta^{k-1}}. \quad (5.2)$$

---

<sup>1</sup>A two pages tedious computation shows that  $\delta \circ \delta = 0$ .

The sheaf cohomology of  $M$  will be given by a direct limit over open covers. Let  $\mathfrak{A}$  be a set of covers of  $M$  such that if  $\mathcal{A}$  is an open cover, then  $\mathcal{R} \in \mathfrak{A}$  is a refinement of  $\mathcal{A}$ . The order relation given by refinements makes  $\mathfrak{A}$  a direct set.

If  $\mathcal{S} \leq \mathcal{R}$ , then

$$\Phi_{\mathcal{R}\mathcal{S}}(f)_{\sigma_0 \dots \sigma_k} := f_{\rho_0 \dots \rho_k} \Big|_{\mathcal{S}_{\sigma_0 \dots \sigma_k}} \quad (5.3)$$

induces a homomorphism  $\Psi_{\mathcal{R}\mathcal{S}} : \check{H}_{\mathcal{R}}^k(M; \mathcal{J}) \rightarrow \check{H}_{\mathcal{S}}^k(M; \mathcal{J})$ . Thanks to the presheaf properties  $\{\check{H}_{\mathcal{R}}^k(M; \mathcal{J})\}_{\mathcal{R} \in \mathfrak{A}}$  and  $\{\varphi_{\mathcal{R}\mathcal{S}}\}_{\mathcal{R}, \mathcal{S} \in \mathfrak{A}}$  forms a direct system.

**Definition 5.3.** *The cohomology of  $M$  taking values on the sheaf  $\mathcal{J}$  is*

$$\lim_{\rightarrow} \check{H}_{\mathcal{R}}^k(M; \mathcal{J}) := \check{H}^k(M; \mathcal{J}) . \quad (5.4)$$

If  $\check{H}_{\mathcal{R}}^k(M; \mathcal{J}) \cong \check{H}_{\mathcal{S}}^k(M; \mathcal{J})$  for all  $\mathcal{R}, \mathcal{S} \in \mathfrak{A}$ , then  $\check{H}^k(M; \mathcal{J}) \cong \check{H}_{\mathcal{R}}^k(M; \mathcal{J})$ . The set  $\mathfrak{A}$  is called cofinal in this case, and any covering on it can be used to compute the cohomology.

### 5.1.3 The Mayer-Vietoris argument

Any open cover  $\mathcal{A} = \{A_\alpha\}$  of  $M$  induces a covering on open sets  $V, W \subset M$ , and also on their union and intersection. For simplicity they will be denoted by the same symbol. The goal is to show that the sequence

$$0 \rightarrow C_{\mathcal{A}}^k(V \cup W; \mathcal{J}) \rightarrow C_{\mathcal{A}}^k(V; \mathcal{J}) \oplus C_{\mathcal{A}}^k(W; \mathcal{J}) \rightarrow C_{\mathcal{A}}^k(V \cap W; \mathcal{J}) \rightarrow 0 \quad (5.5)$$

is an exact sequence and use it to glue together the cohomology of pieces of  $M$ .

An injective linear map  $q : C_{\mathcal{A}}^k(V \cup W; \mathcal{J}) \rightarrow C_{\mathcal{A}}^k(V; \mathcal{J}) \oplus C_{\mathcal{A}}^k(W; \mathcal{J})$  can be defined by

$$q(f_{\alpha_0 \dots \alpha_k}) = \left( f_{\alpha_0 \dots \alpha_k} \Big|_{V \cap A_{\alpha_0 \dots \alpha_k}} \right) \oplus \left( f_{\alpha_0 \dots \alpha_k} \Big|_{W \cap A_{\alpha_0 \dots \alpha_k}} \right) ; \quad (5.6)$$

if each component of  $q(f_{\alpha_0 \dots \alpha_k})$  is zero by the local identity property  $f_{\alpha_0 \dots \alpha_k}$  must be zero.

For the second part of the sequence, the linear map  $p : C_{\mathcal{A}}^k(V; \mathcal{J}) \oplus C_{\mathcal{A}}^k(W; \mathcal{J}) \rightarrow C_{\mathcal{A}}^k(V \cap W; \mathcal{J})$ , given by

$$p(f \oplus g)_{\alpha_0 \dots \alpha_k} := f_{\alpha_0 \dots \alpha_k} \Big|_{V \cap W \cap A_{\alpha_0 \dots \alpha_k}} - g_{\alpha_0 \dots \alpha_k} \Big|_{V \cap W \cap A_{\alpha_0 \dots \alpha_k}} , \quad (5.7)$$

is surjective and has its kernel in the image of  $q$ . It is clear, by definition, that  $p \circ q = 0$  and if  $p(f \oplus g)_{\alpha_0 \dots \alpha_k} = 0$  the glueing condition gives a  $h_{\alpha_0 \dots \alpha_k} \in \mathcal{J}(V \cup W)$  such that  $q(h_{\alpha_0 \dots \alpha_k}) = f_{\alpha_0 \dots \alpha_k} \oplus g_{\alpha_0 \dots \alpha_k}$ . The surjectivity comes from the existence of partitions of unity for  $\mathcal{J}$  (the sheaf is fine).

**Remark 5.1.** A proof that  $\mathcal{J}$  is fine depends on the properties of the polarisation, and the author of this thesis does not know any general proof of this when the polarisation is singular. Hamilton and Miranda [11, 12] proved this for their cases: with elliptic singularities in any dimension and with nondegenerate ones in dimension 2. Nevertheless, in this thesis, a fine resolution for the sheaf  $\mathcal{J}$  is obtained when the polarisation is nondegenerate. Thus, there is a Mayer-Vietoris sequence at the cohomology level.

As usual, the short exact sequence of cochains induces a long exact sequence of cohomology groups. There exists a lemma [4] asserting that direct limit of exact sequences is exact: they must be indexed by the same direct set and the diagram coming out of them needs to be commutative. Both conditions are satisfied using the restriction of the maps of the direct system; thus,

$$\cdots \rightarrow \check{H}^k(V \cup W; \mathcal{J}) \rightarrow \check{H}^k(V; \mathcal{J}) \oplus \check{H}^k(W; \mathcal{J}) \rightarrow \check{H}^k(V \cap W; \mathcal{J}) \rightarrow \cdots \quad (5.8)$$

is an exact sequence not depending on any particular covering.

When  $\check{H}^k(V \cap W; \mathcal{J}) = \{0\}$ , for all  $k$ , the cohomology  $\check{H}^k(V \cup W; \mathcal{J})$  will be isomorphic to  $\check{H}^k(V; \mathcal{J}) \oplus \check{H}^k(W; \mathcal{J})$ . In other words, if one finds a special covering of  $M$  lying inside a cofinal  $\mathfrak{A}$  and such that the cohomology groups on the intersections vanishes, the cohomology of the manifold can be computed piece by piece.

## 5.2 Known results for the singular case

Mark Hamilton found a very special covering for locally toric manifolds that not only permits the use of the Mayer-Vietoris argument and lies on a cofinal set, but allows one to use the local normal forms described in subsection 2.1.4 to compute

the cohomology groups of each open set of the covering. In [11] he has shown that Śniatycki's theorem [27] holds in this situation, and the elliptic singularities give no contribution to the quantisation (just the trivial vector space  $\{0\}$ ):

**Theorem 5.1** (Hamilton). *For  $M$  a  $2n$ -dimensional compact symplectic manifold equipped with a locally toric singular Lagrangian fibration:*

$$\mathcal{Q}(M) = \check{H}^n(M; \mathcal{J}) \cong \bigoplus_{p \in BS_r} \mathbb{C} , \quad (5.9)$$

$BS_r$  being the set of the regular Bohr-Sommerfeld fibres.

With Eva Miranda they found a covering sharing the same properties for compact orientable surfaces. The conclusion in [12] is that Śniatycki's theorem holds as well; elliptic singularities give no contribution and each hyperbolic singularity gives an infinite dimensional contribution.

**Theorem 5.2** (Hamilton and Miranda). *For an integrable system defined over a 2-dimensional compact symplectic manifold, whose moment map has only nondegenerate singularities,*

$$\mathcal{Q}(M) = \check{H}^1(M; \mathcal{J}) \cong \bigoplus_{p \in \mathcal{H}} (\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}) \oplus \bigoplus_{p \in BS_r} \mathbb{C} , \quad (5.10)$$

$BS_r$  being the set of the regular Bohr-Sommerfeld fibres and  $\mathcal{H}$  the set of hyperbolic singularities.

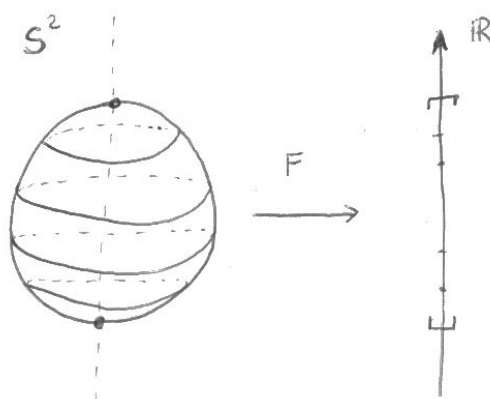
**Example 5.1.** The coadjoint orbits of  $\mathfrak{so}(3)^*$  are the origin and spheres centered at the origin (with symplectic form given by the area form), and the Euler equations are the free rigid body dynamics.

A choice of a momentum of inertia operator (a rigid body) is the same as a choice of a quadratic form on  $\mathfrak{so}(3)^*$ . The quadratic form gives the kinetic energy and its level sets are the concentric ellipsoids centered at the origin.

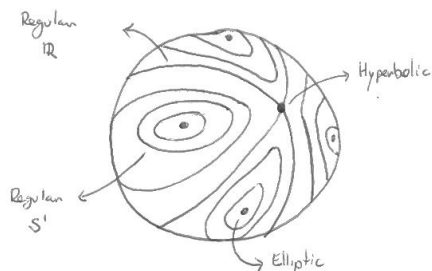
The moment map represents the angular momentum and its level sets are the coadjoint orbits. The prequantisable spheres are those with integral area, or equivalently, with radius  $\sqrt{k/4\pi}$ ,  $k \in \mathbb{N}$ .

Due to conservation of energy and angular momentum, the integral curves are contained in the intersection of the spheres and ellipsoids, and this gives the singular foliation associated to the integrable system of a free rigid body.

If the principal axis of inertia satisfies  $I_1 = I_2 \neq I_3$  the momentum of inertia operator has 2 distinct eigenvalues. It is an example of a toric manifold having two elliptic singularities and  $\mathcal{Q}(M)$  is finite dimensional.



A generic body satisfies  $I_1 < I_2 < I_3$  and it is an integrable system containing four elliptic and two hyperbolic singularities. It has an infinite dimensional  $\mathcal{Q}(M)$ .



◇

# Chapter 6

## The Kostant complex

Instead of computing directly the Čech cohomology groups  $\check{H}^k(M; \mathcal{J})$ , as done by Hamilton [11] and Hamilton and Miranda [12], the strategy of this thesis is to present a resolution for the sheaf  $\mathcal{J}$ . For regular polarisations this has been done by Kostant [27, 26]. In the singular case this can be achieved via Lie pseudoalgebra representations.

This chapter only recasts Geometric Quantisation under the language of Lie pseudoalgebras and its representations, the proof that the Kostant complex is a resolution for the sheaf is left to chapters 9 and 10.

### 6.1 Line bundle valued polarised forms

Recalling example 2.14, the polarisation induced by an integrable system provides a Lie pseudo algebra,  $(P, C^\infty(M), \mathbb{C})$ , and it can be represented on the space of sections of the prequantum line bundle,  $\Gamma(L)$ , via the Hermitian connection  $\nabla^\omega$ .

- The complex Lie algebra and  $C^\infty(M; \mathbb{C})$ -module  $\mathfrak{g}$  is going to be  $(P, [\cdot, \cdot]_P)$ ;
- the space of complex-valued functions  $C^\infty(M; \mathbb{C})$  will be denoted by  $C^\infty(M)$ ;
- the map  $T : P \rightarrow \mathfrak{X}(M)$  is going to be the inclusion map;
- the  $C^\infty(M)$ -module  $E$  is going to be  $\Gamma(L)$ ;

- the map  $\rho : P \rightarrow \text{Hom}_{\mathbb{C}}(\Gamma(L); \Gamma(L))$  is going to be the restriction of the connection  $\nabla^\omega$  to the polarisation,  $\nabla := \nabla^\omega|_P$ ;
- the cochain spaces  $\Omega_P^k(M) \otimes_{C^\infty(M)} \Gamma(L)$  will be denoted by  $S_P^k(L)$ .

**Proposition 6.1.** *The restriction of the connection  $\nabla^\omega$  to the polarisation,  $\nabla := \nabla^\omega|_P$ , defines a representation of the Lie pseudoalgebra  $(P, C^\infty(M), \mathbb{C})$  on  $\Gamma(L)$ .*

*Proof:* The space of sections of the prequantum line bundle  $L$  is clearly a  $C^\infty(M)$ -module, and

$$\nabla : \Gamma(L) \rightarrow \Omega_P^1(M) \otimes_{C^\infty(M)} \Gamma(L) \quad (6.1)$$

satisfies (by definition) the following property:

$$\nabla(fs) = d_P f \otimes s + f \nabla s, \quad (6.2)$$

for any  $f \in C^\infty(M)$  and  $s \in \Gamma(L)$ .

If  $X, Y \in P$ , thinking of  $\nabla$  as a linear map from  $P$  to endomorphisms of  $\Gamma(L)$ ,

$$\nabla_{[X, Y]} = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \text{curv}(\nabla)(X, Y). \quad (6.3)$$

But since  $\omega = i \text{curv}(\nabla^\omega)$  vanishes along  $P$ ,  $\text{curv}(\nabla)(X, Y) = 0$  and  $\nabla$  is a Lie algebra representation of  $(P, [\cdot, \cdot]|_P)$  on  $\Gamma(L)$  compatible with their  $C^\infty(M)$ -module structures. ■

**Definition 6.1.**  $\Omega_P^\bullet(M) := \bigoplus_{k \geq 0} \Omega_P^k(M)$  is the space of polarised forms, and the space of line bundle valued polarised forms is  $S_P^\bullet(L) := \bigoplus_{k \geq 0} S_P^k(L)$ .

Therefore,  $\nabla : S_P^0(L) \rightarrow S_P^1(L)$  and  $S_P^\bullet(L)$  has a module structure which enables an extension of  $\nabla$  to a derivation of degree +1 on the space of line bundle valued polarised forms, as follows: if  $\alpha \in \Omega_P^k(M)$  and  $\beta = \beta \otimes s \in S_P^l(L)$ ,

$$\alpha \wedge \beta = \alpha \wedge (\beta \otimes s) := (\alpha \wedge \beta) \otimes s \quad (6.4)$$

defines a left multiplication of the ring  $\Omega_P^\bullet(M)$  on  $S_P^\bullet(L)$ .

**Definition 6.2.** *The derivation on  $S_P^\bullet(L)$  is given by the degree +1 map  $d^\nabla : S_P^\bullet(L) \rightarrow S_P^\bullet(L)$ ,*

$$d^\nabla(\alpha \otimes s) := d_P \alpha \otimes s + (-1)^k \alpha \wedge \nabla s . \quad (6.5)$$

With respect to line bundle valued polarised forms, proposition 6.1 asserts that the degree +1 map  $d^\nabla$  is a coboundary (this map is the same as the one given by equation 2.30).

**Proposition 6.2.** *If  $\alpha \in \Omega_P^k(M)$  and  $\beta \in S_P^l(L)$ , then*

$$d^\nabla(\alpha \wedge \beta) = d_P \alpha \wedge \beta + (-1)^k \alpha \wedge d^\nabla \beta , \quad (6.6)$$

and

$$d^\nabla \circ d^\nabla \beta = \text{curv}(\nabla^\omega)|_P \wedge \beta . \quad (6.7)$$

*Proof:* By linearity, it suffices to prove it for elements of the form  $\beta = \beta \otimes s$ , with  $\beta \in \Omega_P^l(M)$  and  $s \in \Gamma(L)$ .

$$\begin{aligned} d^\nabla(\alpha \wedge \beta) &= d^\nabla[(\alpha \wedge \beta) \otimes s] = d_P(\alpha \wedge \beta) \otimes s + (-1)^{k+l}(\alpha \wedge \beta) \wedge \nabla s \\ &= [d_P \alpha \wedge \beta + (-1)^k \alpha \wedge d_P \beta] \otimes s + (-1)^k \alpha \wedge [(-1)^l \beta \wedge \nabla s] \\ &= d_P \alpha \wedge (\beta \otimes s) + (-1)^k \alpha \wedge [d_P \beta \otimes s + (-1)^l \beta \wedge \nabla s] \\ &= d_P \alpha \wedge \beta + (-1)^k \alpha \wedge d^\nabla \beta , \end{aligned}$$

which proves the first statement.

Equation 2.30 reads:

$$\begin{aligned} [d^\nabla(\nabla s)](X, Y) &= \nabla_X[(\nabla s)(Y)] - \nabla_Y[(\nabla s)(X)] - (\nabla s)([X, Y]) \\ &= \text{curv}(\nabla)(X, Y)s , \end{aligned}$$

for any  $X, Y \in P$ . As a result,  $(d^\nabla \circ d^\nabla)s = \text{curv}(\nabla^\omega)|_P \otimes s$  and

$$\begin{aligned} d^\nabla \circ d^\nabla \beta &= (d^\nabla \circ d^\nabla)(\beta \otimes s) = d^\nabla(d_P \beta \otimes s + (-1)^l \beta \wedge \nabla s) \\ &= d_P \circ d_P \beta \otimes s + (-1)^{l+1} d_P \beta \wedge \nabla s + (-1)^l [d_P \beta \wedge \nabla s + (-1)^l \beta \wedge d^\nabla(\nabla s)] \\ &= -(-1)^l d_P \beta \wedge \nabla s + (-1)^l d_P \beta \wedge \nabla s + \beta \wedge \text{curv}(\nabla^\omega)|_P \otimes s \\ &= \text{curv}(\nabla^\omega)|_P \wedge \beta \otimes s = \text{curv}(\nabla^\omega)|_P \wedge \beta . \end{aligned}$$



■

Since  $\omega = i \operatorname{curv}(\nabla^\omega)$  vanishes along  $P$ , one has  $d^\nabla \circ d^\nabla = 0$ .

**Corollary 6.1.**  $d^\nabla$  is a coboundary operator.

**Remark 6.1.** The only property of  $L$  being used here is the existence of flat connections along  $P$ ; any complex line bundle would do, not only a prequantum one —the results here work if metaplectic correction is included.

Thus, the associated Lie pseudoalgebra cohomology of this representation,  $H^\bullet(S_P^\bullet(L))$ , induces a complex (at the sheaf level). If  $\mathcal{S}_P^k(L)$  denotes the associated sheaf of  $S_P^k(L)$ , one can extend  $d^\nabla$  to a homomorphism of sheaves,  $d^\nabla : \mathcal{S}_P^k(L) \rightarrow \mathcal{S}_P^{k+1}(L)$ .  $\mathcal{S}_P^0(L) \cong \mathcal{S}$ , the sheaf of sections of the line bundle  $L$ , and  $\mathcal{J}$  is isomorphic to the kernel of  $\nabla : \mathcal{S} \rightarrow \mathcal{S}_P^1(L)$ .

**Definition 6.3.** The Kostant complex is

$$0 \longrightarrow \mathcal{J} \hookrightarrow \mathcal{S} \xrightarrow{\nabla} \mathcal{S}_P^1(L) \xrightarrow{d^\nabla} \cdots \xrightarrow{d^\nabla} \mathcal{S}_P^n(L) \xrightarrow{d^\nabla} 0. \quad (6.8)$$

### 6.1.1 Interior product, Lie derivative and pullbacks

As expected, the notions of interior product and Lie derivative are available for  $S_P^\bullet(L)$ . The Lie derivative can be seen as a derivation along a flow, but for that, a nontrivial notion of pullback is needed. The aim of this subsection is to describe all this.

**Definition 6.4.** The contraction between line bundle valued polarised forms and elements of  $P$  is given by a map  $\mathbf{i} : P \times S_P^\bullet(L) \rightarrow S_P^\bullet(L)$  that is a degree -1 map on  $S_P^\bullet(L)$ : i.e.

$$\mathbf{i}_X(\nabla s) := \nabla_X s \quad (6.9)$$

and

$$\mathbf{i}_X \boldsymbol{\beta} = \mathbf{i}_X(\beta \otimes s) := (\iota_X \beta) \otimes s \quad (6.10)$$

hold for each  $X \in P$  and  $\boldsymbol{\beta} = \beta \otimes s \in S_P^1(L)$ .

**Proposition 6.3.** *If  $X \in P$ ,  $\alpha \in \Omega_P^k(M)$  and  $\beta \in S_P^l(L)$ , then  $i_X \circ i_X = 0$  and*

$$i_X(\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^k \alpha \wedge i_X \beta . \quad (6.11)$$

*Proof:* Due to linearity, it suffices to prove it for elements  $\beta = \beta \otimes s$ .

$$i_X \circ i_X \beta = i_X[(i_X \beta) \otimes s] = (i_X \circ i_X \beta) \otimes s = 0 .$$

$$\begin{aligned} i_X(\alpha \wedge \beta) &= i_X([\alpha \wedge \beta] \otimes s) = i_X[\alpha \wedge \beta] \otimes s \\ &= [(i_X \alpha) \wedge \beta + (-1)^k \alpha \wedge i_X \beta] \otimes s = (i_X \alpha) \wedge \beta + (-1)^k \alpha \wedge i_X \beta . \end{aligned}$$

■

For  $\Omega_P^k(M)$  the pullback still makes sense, if one restricts to diffeomorphisms that preserve the polarisation  $P$ , but problems arise when one twists it with  $\Gamma(L)$ . A way to compare elements of  $L$  is by parallel transport, which in general is path dependent. When it does not depend on the path, the pullback on  $S_P^\bullet(L)$  is well-defined (for diffeomorphisms that preserve the polarisation); when it does, it is possible to make sense of pullbacks over paths.

**Definition 6.5.** *Let  $\phi : M \rightarrow M$  be a diffeomorphism that preserves  $P$  and  $\gamma : \mathbb{R} \hookrightarrow M$  a curve joining  $p = \gamma(0)$  to  $\phi(p) = \gamma(1)$  in  $M$ . The pullback  $(\phi, \gamma)^*$  of  $\alpha = \alpha \otimes s \in S_P^k(L)$  over the path  $\gamma$  at the point  $p$  is defined by*

$$[(\phi, \gamma)^* \alpha](X_1, \dots, X_k)(p) := \Pi_{\gamma(1)}^{-1}([\phi^* \alpha](X_1, \dots, X_k)(p) \cdot s \circ \phi(p)) , \quad (6.12)$$

for any  $X_1, \dots, X_k \in P$ , where  $\Pi_{\gamma(1)} : L_{\gamma(0)} \rightarrow L_{\gamma(1)}$  is the parallel transport.

The parallel transport is, indeed, a bundle automorphism, so it makes sense to write

$$[(\phi, \gamma)^*(\alpha \otimes s)] = (\phi^* \alpha) \otimes \Pi_{\gamma}^{-1}(s \circ \phi) . \quad (6.13)$$

Now, if  $X \in P$ , its flow  $\phi_t$  already encodes both a curve and a diffeomorphism.

**Definition 6.6.** The pullback  $\phi_t^*$  of  $\alpha \otimes s \in S_P^k(L)$  is defined by

$$\phi_t^*(\alpha \otimes s) := (\phi_t^*\alpha) \otimes \Pi_{\phi_t}^{-1}(s \circ \phi_t) ; \quad (6.14)$$

where, by the bundle automorphism property of the parallel transport,  $\Pi_{\phi_t}^{-1}(s \circ \phi_t)$  denotes the parallel transport between  $\phi_t(p)$  and  $p$  of  $s$  through the integral curve of the flow.

**Proposition 6.4.** Let  $X \in P$  with flow  $\phi_t$ ,  $\alpha \in \Omega_P^k(M)$  and  $\beta \in S_P^l(L)$ ; then,

$$\phi_t^*(\alpha \wedge \beta) = (\phi_t^*\alpha) \wedge \phi_t^*(\beta) , \quad (6.15)$$

and the pullback  $\phi_t^*$  commutes with  $d^\nabla$ .

*Proof:* Again, by linearity, it suffices to prove it for elements  $\beta = \beta \otimes s$ . The first assertion is just a simple computation,

$$\begin{aligned} \phi_t^*(\alpha \wedge \beta) &= \phi_t^*((\alpha \wedge \beta) \otimes s) = \phi_t^*(\alpha \wedge \beta) \otimes \Pi_{\phi_t}^{-1}(s \circ \phi_t) \\ &= (\phi_t^*\alpha) \wedge (\phi_t^*\beta \otimes \Pi_{\phi_t}^{-1}(s \circ \phi_t)) = (\phi_t^*\alpha) \wedge \phi_t^*(\beta) . \end{aligned}$$

For the commutation one has:

$$\begin{aligned} d^\nabla(\phi_t^*\beta) &= d^\nabla(\phi_t^*(\beta \otimes s)) = d^\nabla((\phi_t^*\beta) \otimes \Pi_{\phi_t}^{-1}(s \circ \phi_t)) \\ &= d_P(\phi_t^*\beta) \otimes \Pi_{\phi_t}^{-1}(s \circ \phi_t) + (-1)^l(\phi_t^*\beta) \wedge \nabla \Pi_{\phi_t}^{-1}(s \circ \phi_t) , \end{aligned} \quad (6.16)$$

and

$$\begin{aligned} \phi_t^*(d^\nabla\beta) &= \phi_t^*(d^\nabla(\beta \otimes s)) = \phi_t^*(d_P\beta \otimes s + (-1)^l\beta \wedge \nabla s) \\ &= \phi_t^*(d_P\beta) \otimes \Pi_{\phi_t}^{-1}(s \circ \phi_t) + (-1)^l(\phi_t^*\beta) \wedge \phi_t^*(\nabla s) . \end{aligned} \quad (6.17)$$

Since  $d_P$  commutes with  $\phi_t^*$ , the subtraction of these two expressions gives

$$(d^\nabla \circ \phi_t^* - \phi_t^* \circ d^\nabla)\beta = (-1)^l(\phi_t^*\beta) \wedge (\nabla \Pi_{\phi_t}^{-1}(s \circ \phi_t) - \phi_t^*(\nabla s)) . \quad (6.18)$$

Therefore, if  $\nabla \Pi_{\phi_t}^{-1}(s \circ \phi_t) = \phi_t^*(\nabla s)$ , then  $d^\nabla \circ \phi_t^* = \phi_t^* \circ d^\nabla$ : and one only needs to prove this locally.

Near any point, there exists a unitary section  $s$  of  $L$  such that,  $\nabla^\omega s = -i\Theta \otimes s$ . In case  $\dot{\gamma}(t)$  is the vector field  $X$  at the point  $\gamma(t)$ , with  $\gamma$  the integral curve of  $X$ , equation 2.19 reads

$$\Pi_{\phi_t}^{-1}(s \circ \phi_t) = \exp\left(-i \int_0^t \iota_X \circ \phi_{t'}^* \Theta \, dt'\right) s . \quad (6.19)$$

Defining  $f(t) = \int_0^t \iota_X \circ \phi_{t'}^* \Theta \, dt'$  and  $r(t) = e^{-if(t)}s$ , it is clear (lemma 2.1) that  $(df(t) + \Theta)$  is the potential 1-form relative to the unitary section  $r(t)$ .

Computing the differential of  $f(t)$ :

$$\begin{aligned} df(t) &= \int_0^t d \circ \iota_X \circ \phi_{t'}^* \Theta \, dt' = \int_0^t (\mathcal{L}_X - \iota_X \circ d) \circ \phi_{t'}^* \Theta \, dt' \\ &= \int_0^t \left( \frac{d}{d\tau} \phi_{\tau}^* \Theta \Big|_{t=t'} - \iota_X \circ \phi_{t'}^* \circ d\Theta \right) dt' \\ &= \phi_t^* \Theta - \Theta - \int_0^t \iota_X \circ \phi_{t'}^* \circ \omega \, dt' , \end{aligned} \quad (6.20)$$

where Cartan's magic formula and the commutation between the differential  $d$  and the pullback  $\phi_t^*$  were used.

Hence, noticing that  $\omega$  vanishes when restricted to the polarisation's directions,

$$\begin{aligned} \nabla \Pi_{\phi_t}^{-1}(s \circ \phi_t) &= \nabla^\omega r(t)|_P = -i \left( \phi_t^* \Theta|_P - \int_0^t \iota_X \circ \phi_{t'}^* \circ \omega|_P \, dt' \right) \otimes r(t) \\ &= \phi_t^*(-i\Theta|_P) \otimes \Pi_{\phi_t}^{-1}(s \circ \phi_t) = \phi_t^*(-i\Theta|_P \otimes s) \\ &= \phi_t^*(\nabla s) . \end{aligned} \quad (6.21)$$

■

**Definition 6.7.** The Lie derivative  $\mathcal{L}^\nabla : P \times S_P^\bullet(L) \rightarrow S_P^\bullet(L)$  is defined by:

$$\mathcal{L}_X^\nabla(\alpha) := \frac{d}{dt} \phi_t^* \alpha \Big|_{t=0} . \quad (6.22)$$

**Proposition 6.5.** The Lie derivative  $\mathcal{L}_X^\nabla$  commutes with the pullback  $\phi_t^*$ .

*Proof:*

$$\begin{aligned} \phi_t^* \circ \mathcal{L}_X^\nabla(\alpha) &= \phi_t^* \left( \frac{d}{d\tau} \phi_\tau^* \alpha \Big|_{\tau=0} \right) = \frac{d}{d\tau} \phi_t^*(\phi_\tau^* \alpha) \Big|_{\tau=0} \\ &= \frac{d}{d\tau} \phi_{\tau+t}^* \alpha \Big|_{\tau=0} = \mathcal{L}_X^\nabla(\phi_t^* \alpha) . \end{aligned}$$

■

Cartan's magic formula holds for the Lie derivative on  $S_P^\bullet(L)$ .

**Proposition 6.6.** *The Lie derivative  $\mathcal{L}^\nabla$  can be characterised by*

$$\mathcal{L}_X^\nabla(\alpha) = \mathbf{i}_X \circ d^\nabla \alpha + d^\nabla \circ \mathbf{i}_X \alpha . \quad (6.23)$$

*Proof:* By definition of the parallel transport, if  $\phi_t$  is the flow of  $X \in P$ , then  $\nabla_X s = \frac{d}{dt} \Pi_{\phi_t}^{-1}(s \circ \phi_t)|_{t=0}$  and similar for the Lie derivative:  $\mathcal{L}_X(\alpha) = \frac{d}{dt} \phi_t^* \alpha|_{t=0}$ . And once more, linearity implies that it suffices to prove the assertion for elements  $\alpha \otimes s \in S_P^k(L)$ .

On the one hand, by the Leibniz rule over tensor products,

$$\begin{aligned} \frac{d}{dt} \phi_t^*(\alpha \otimes s)|_{t=0} &= \frac{d}{dt} ((\phi_t^* \alpha) \otimes \Pi_{\phi_t}^{-1}(s \circ \phi_t))|_{t=0} \\ &= (\phi_0^* \alpha) \otimes \frac{d}{dt} \Pi_{\phi_t}^{-1}(s \circ \phi_t)|_{t=0} + \frac{d}{dt} \phi_t^* \alpha|_{t=0} \otimes \Pi_{\phi_0}^{-1}(s \circ \phi_0) \\ &= \alpha \otimes \nabla_X s + \mathcal{L}_X(\alpha) \otimes s . \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbf{i}_X \circ d^\nabla(\alpha \otimes s) + d^\nabla \circ \mathbf{i}_X(\alpha \otimes s) &= \mathbf{i}_X(d_P \alpha \otimes s + (-1)^k \alpha \wedge \nabla s) + d^\nabla(\iota_X \alpha \otimes s) \\ &= \iota_X(d_P \alpha) \otimes s + (-1)^k \mathbf{i}_X(\alpha \wedge \nabla s) \\ &\quad + d_P(\iota_X \alpha) \otimes s + (-1)^{k-1} (\iota_X \alpha) \wedge \nabla s \\ &= \mathcal{L}_X(\alpha) \otimes s + (-1)^k (\iota_X \alpha) \wedge \nabla s \\ &\quad + (-1)^{2k} \alpha \otimes \nabla_X s + (-1)^{k-1} (\iota_X \alpha) \wedge \nabla s \\ &= \mathcal{L}_X(\alpha) \otimes s + \alpha \otimes \nabla_X s . \end{aligned}$$

■

**Proposition 6.7.** *The Lie derivative  $\mathcal{L}^\nabla$  commutes with the derivation  $d^\nabla$ .*

*Proof:* It is just a simple application of propositions 6.2 and 6.6.

$$\begin{aligned} \mathcal{L}_X^\nabla \circ d^\nabla \alpha &= \mathbf{i}_X \circ d^\nabla \circ d^\nabla \alpha + d^\nabla \circ \mathbf{i}_X \circ d^\nabla \alpha \\ &= \mathbf{i}_X(\text{curv}(\nabla^\omega)|_P \wedge \alpha) + d^\nabla \circ \mathbf{i}_X \circ d^\nabla \alpha \\ &= d^\nabla \circ \mathbf{i}_X \circ d^\nabla \alpha + (\iota_X[\text{curv}(\nabla^\omega)|_P]) \wedge \alpha + \text{curv}(\nabla^\omega)|_P \wedge (\mathbf{i}_X \alpha) , \end{aligned}$$

$$d^\nabla \circ \mathcal{L}_X^\nabla \alpha = d^\nabla \circ \mathbf{i}_X \circ d^\nabla \alpha + d^\nabla \circ d^\nabla \circ \mathbf{i}_X \alpha = d^\nabla \circ \mathbf{i}_X \circ d^\nabla \alpha + \text{curv}(\nabla^\omega)|_P \wedge (\mathbf{i}_X \alpha),$$

and subtracting each other, noting that  $\text{curv}(\nabla^\omega)|_P = 0$ , one gets the result. ■

# Chapter 7

## Circle actions and homotopy operators

This chapter explains the construction of an almost homotopy operator for the Kostant complex when one has a symplectic  $S^1$ -action, and how this implies the vanishing of the stalks of points with nontrivial holonomy. Most results of this section were previously provided in [26] with slightly less general hypothesis; some proofs automatically hold (propositions 7.1, 7.2 and 7.3), but one (lemma 7.2) had to be adapted.

### 7.1 An almost homotopy operator

Let  $X \in P$  be a generator of a symplectic  $S^1$ -action. If  $\phi_t$  stands for the flow of  $X$  at time  $t$ , it is possible to define an induced action on  $S_P^k(L)$  via  $\phi_t^*$ . The holonomy of the loop generated by flowing during a time  $2\pi$  a point  $p \in M$  will be denoted by  $hol_{\nabla^\omega}(\gamma)(p)$ , and since  $\phi_{t+2\pi} = \phi_t$  for every  $t \in \mathbb{R}$ :

$$\phi_{2\pi}^*(\alpha) = \phi_{2\pi}^*(\alpha \otimes s) = \phi_{2\pi}^*\alpha \otimes \Pi_{\phi_{2\pi}}^{-1}(s \circ \phi_{2\pi}) = \alpha \otimes (hol_{\nabla^\omega}(\gamma)^{-1}s) = hol_{\nabla^\omega}(\gamma)^{-1}\alpha ,$$

and

$$\begin{aligned}
(hol_{\nabla\omega}(\gamma)^{-1} - 1)\alpha &= \phi_{2\pi}^* \alpha - \phi_0^* \alpha = \int_0^{2\pi} \frac{d}{dt}(\phi_t^* \alpha) dt \\
&= \int_0^{2\pi} \frac{d}{ds} \phi_{t+s}^* \alpha \Big|_{s=0} dt = \int_0^{2\pi} \frac{d}{ds} \phi_s^* (\phi_t^* \alpha) \Big|_{s=0} dt \\
&= \int_0^{2\pi} \mathcal{L}_X^\nabla(\phi_t^* \alpha) dt = \int_0^{2\pi} (\mathbf{i}_X \circ d^\nabla + d^\nabla \circ \mathbf{i}_X)(\phi_t^* \alpha) dt \\
&= \mathbf{i}_X \left( \int_0^{2\pi} d^\nabla(\phi_t^* \alpha) dt \right) + d^\nabla \circ \mathbf{i}_X \left( \int_0^{2\pi} \phi_t^* \alpha dt \right) .
\end{aligned}$$

Using that the pullback commutes with the derivative (proposition 6.4), one gets from the last equation

$$(hol_{\nabla\omega}(\gamma)^{-1} - 1)\alpha = \mathbf{i}_X \left( \int_0^{2\pi} \phi_t^* (d^\nabla \alpha) dt \right) + d^\nabla \circ \mathbf{i}_X \left( \int_0^{2\pi} \phi_t^* \alpha dt \right) , \quad (7.1)$$

which resembles the equation satisfied by a homotopy operator.

**Proposition 7.1.** *The expression  $\mathbf{J}_X(\alpha) = \mathbf{i}_X \left( \int_0^{2\pi} \phi_t^* \alpha dt \right)$  defines a degree  $-1$  derivation on  $S_P^\bullet(L)$ .*

*Proof:* Propositions 6.3 and 6.4 imply that  $\mathbf{J}_X$  is a derivation, and the degree comes from the fact that  $\mathbf{i}_X$  has degree  $-1$ . ■

The equation 7.1 implies that  $\mathbf{J}_X$  satisfies

$$(hol_{\nabla\omega}(\gamma)^{-1} - 1)\alpha = \mathbf{J}_X(d^\nabla \alpha) + d^\nabla \mathbf{J}_X(\alpha) , \quad (7.2)$$

for any  $\alpha \in S_P^k(L)$  if  $k \geq 1$ , whilst for  $k = 0$  it becomes

$$(hol_{\nabla\omega}(\gamma)^{-1} - 1)\alpha = \mathbf{J}_X(d^\nabla \alpha) , \quad (7.3)$$

since  $S_P^{-1}(L)$  is empty and  $\mathbf{J}_X$  has degree  $-1$ .

**Proposition 7.2.**  $d^\nabla((hol_{\nabla\omega}(\gamma)^{-1} - 1)\alpha) = (hol_{\nabla\omega}(\gamma)^{-1} - 1)d^\nabla \alpha$  for any  $\alpha \in S_P^k(L)$ ; hence,  $hol_{\nabla\omega}(\gamma)$  is constant along  $P$ .

*Proof:* It is a direct consequence of equation 7.2:

$$d^\nabla((hol_{\nabla\omega}(\gamma)^{-1} - 1)\alpha) = d^\nabla[\mathbf{J}_X(d^\nabla \alpha) + d^\nabla \mathbf{J}_X(\alpha)] = d^\nabla \mathbf{J}_X(d^\nabla \alpha) , \quad (7.4)$$



$$(hol_{\nabla^\omega}(\gamma))^{-1} - 1) d^\nabla \alpha = \mathbf{J}_X(d^\nabla \circ d^\nabla \alpha) + d^\nabla \mathbf{J}_X(d^\nabla \alpha) = d^\nabla \mathbf{J}_X(d^\nabla \alpha) . \quad (7.5)$$

■

**Lemma 7.1.** *Let  $X$  be the generator of a symplectic  $S^1$ -action; then,*

$$hol_{\nabla^\omega}(\gamma) = e^{i2\pi\theta(X)} ; \quad (7.6)$$

where  $\theta$  is a particular invariant potential 1-form for  $\omega$  in a neighbourhood of  $\gamma$ .

*Proof:* Weinstein's theorem for isotropic embeddings [29] asserts that in a neighbourhood  $N$  of an orbit the symplectic form is exact,  $\omega = d\theta$  —the potential 1-form can be chosen to be invariant by averaging it with respect to the flow of  $X$ . Let  $s \in \Gamma(L|_N)$  be the unitary section given by lemma 2.2 which has  $\theta$  as the potential 1-form for  $\nabla^\omega$ .

Cartan's magic formula and the invariance of  $\theta$  give:

$$0 = \mathcal{L}_X(\theta) = \iota_X d\theta + d(\iota_X \theta) \Rightarrow \iota_X \omega = -d\theta(X) ; \quad (7.7)$$

wherefore, near  $\gamma$ , the action is Hamiltonian, and  $\theta(X)$  is its Hamiltonian function. In particular, since  $\gamma$  is an integral curve of the Hamiltonian flow,  $\theta(\dot{\gamma}(t))$  is constant; thus, corollary 2.1 asserts that

$$hol_{\nabla^\omega}(\gamma) = e^{i \int_0^{2\pi} \theta(\dot{\gamma}(t')) dt'} = e^{i2\pi\theta(X)} . \quad (7.8)$$

■

**Proposition 7.3.** *Supposing that  $(M, \omega)$  admits a symplectic  $S^1$ -action preserving  $P$ , flat sections of  $L$  vanish if  $hol_{\nabla^\omega}(\gamma)$  is nontrivial over a dense set.*

*Proof:* Let  $s \in \Gamma(L)$  be a flat section,  $\nabla s = 0$ . By equation 7.3  $(hol_{\nabla^\omega}(\gamma))^{-1} - 1)s = 0$  and the flat section vanishes on the dense set where  $hol_{\nabla^\omega}(\gamma) \neq 1$ . Consequently, there are no nonzero flat sections. ■

Connected to this result, one can provide an alternative proof for the theorem of Guillemin and Sternberg 3.2 that holds not only for regular compact fibres; this is an application of lemma 7.1.

**Theorem 7.1.** *Under the assumption that the zero fibre is Bohr-Sommerfeld, the image of Bohr-Sommerfeld fibres by a moment map is contained in  $\mathbb{R}^{n-k} \times \mathbb{Z}^k$ ;  $k$  being the number of linearly independent Hamiltonian  $S^1$ -actions generated by the moment map.*

*Proof:* As already mentioned, this was proved by Guillemin and Sternberg in [10] when the fibres are Liouville tori —their proof holds for Lagrangian fibrations with compact connected fibres over simply connected basis. Lemma 7.1 and proposition 3.1 imply that over a Bohr-Sommerfeld fibre each component of the moment map generating a  $S^1$ -action takes an integral value, depending only on the fibre. ■

**Lemma 7.2.** *Under the hypothesis that  $\{hol_{\nabla\omega}(\gamma) \neq 1\}$  is dense, a form  $\alpha \in S_P^k(L)$  vanishes where  $hol_{\nabla\omega}(\gamma) = 1$  if and only if there exists a  $\beta \in S_P^k(L)$  such that  $\alpha = (hol_{\nabla\omega}(\gamma)^{-1} - 1)\beta$ .*

*Proof:* If  $\alpha = (hol_{\nabla\omega}(\gamma)^{-1} - 1)\beta$  it is obvious that  $\alpha$  vanishes where  $hol_{\nabla\omega}(\gamma) = 1$ . If the converse holds for functions on  $M$ , in any trivialising neighbourhood  $A$  with unitary section  $s$  and coordinates  $(z_1, \dots, z_{2n})$ , the form  $\alpha$  can be expressed by

$$\alpha = \left[ \sum_{j_1, \dots, j_k = 1}^{2n} \alpha_{j_1, \dots, j_k}(z_1, \dots, z_{2n}) dz_{j_1} \wedge \dots \wedge dz_{j_k} \right] \otimes s. \quad (7.9)$$

Furthermore,  $\alpha = 0$  at  $\{hol_{\nabla\omega}(\gamma) = 1\}$  if and only if all the functions  $\alpha_{j_1, \dots, j_k}$  vanish on  $A \cap \{hol_{\nabla\omega}(\gamma) = 1\}$ . Hence, there exist functions  $\beta_{j_1, \dots, j_k}$  such that  $\alpha_{j_1, \dots, j_k} = (hol_{\nabla\omega}(\gamma)^{-1} - 1)\beta_{j_1, \dots, j_k}$ . The manifold  $M$  can be covered by trivialising neighbourhoods, and the local functions  $\beta_{j_1, \dots, j_k}$  piece together to give the desired  $\beta \in S_P^k(L)$ .

Therefore, given  $f \in C^\infty(A)$  satisfying  $f|_{A \cap \{hol_{\nabla\omega}(\gamma)=1\}} = 0$  one must construct a  $g \in C^\infty(A)$  such that  $f = (hol_{\nabla\omega}(\gamma)^{-1} - 1)g$ .

For points where 1 is a regular value of  $hol_{\nabla\omega}(\gamma)$ , theorem 4 in [26] proves that this expression holds for functions. On the other hand, lemma 7.1 implies that critical points of  $hol_{\nabla\omega}(\gamma)$  are fixed points of the  $S^1$ -action, and that locally  $hol_{\nabla\omega}(\gamma) = e^{2\pi i h}$

for some function  $h$ . This means that the singularities of the set  $\{hol_{\nabla\omega}(\gamma) = 1\}$  form a closed submanifold —the set of points with trivial holonomy is a stratified submanifold, and its top dimensional strata have codimension 1.

Let  $A$  be a neighbourhood of a critical point of  $\{hol_{\nabla\omega}(\gamma) = 1\}$ . By shrinking  $A$ , and possibly changing  $h$  by a constant, one can assume that only 0, and no other integer, satisfies  $A \cap h^{-1}(\{0\}) \neq \emptyset$ . With the aid of the flow  $\varphi_t$  of the vector field  $-hZ$ , where in local coordinates around the critical point  $Z = h \sum_{j=1}^{2n} z_j \frac{\partial}{\partial z_j}$ , one can define a function  $g \in C^\infty(A)$ :

$$g = \frac{\int_0^\infty Z(f \circ \varphi_t) dt}{2\pi i \int_0^1 e^{-2\pi i t h} dt} . \quad (7.10)$$

In fact, for  $h = 0$

$$\int_0^1 e^{-2\pi i t h} dt = 1 , \quad (7.11)$$

and for  $h \neq 0$

$$\int_0^1 e^{-2\pi i t h} dt = \frac{hol_{\nabla\omega}(\gamma)^{-1} - 1}{-2\pi i h} . \quad (7.12)$$

Thus, the denominator in expression 7.10 never vanishes, whilst

$$\begin{aligned} g &= \frac{\int_0^\infty Z(f \circ \varphi_t) dt}{2\pi i (hol_{\nabla\omega}(\gamma)^{-1} - 1) / (-2\pi i h)} = \frac{-\int_0^\infty hZ(f \circ \varphi_t) dt}{hol_{\nabla\omega}(\gamma)^{-1} - 1} \\ &= \frac{\int_\infty^0 \frac{d}{dt} f \circ \varphi_t dt}{hol_{\nabla\omega}(\gamma)^{-1} - 1} = \frac{f - \lim_{t \rightarrow \infty} f \circ \varphi_t}{hol_{\nabla\omega}(\gamma)^{-1} - 1} . \end{aligned} \quad (7.13)$$

For any point  $p \in A$  the limit  $\lim_{t \rightarrow \infty} \varphi_t(p)$  is the critical point (which, in particular, has trivial holonomy) and  $f|_{A \cap \{hol_{\nabla\omega}(\gamma)=1\}} = 0$ ; consequently,  $f = (hol_{\nabla\omega}(\gamma)^{-1} - 1)g$  on  $A \cap \{hol_{\nabla\omega}(\gamma) \neq 1\}$ . By continuity of  $f, g$  and density of  $\{hol_{\nabla\omega}(\gamma) \neq 1\}$ , this must be true over all  $A$ . ■

The next proposition is a key tool to prove that the Kostant complex is a fine resolution when the (singular) polarisation comes from an almost or locally toric structure.

**Proposition 7.4.** *Let  $\alpha \in S_P^k(L)$  be closed,  $d^\nabla \alpha = 0$ , and  $k \neq 0$ .*

- *The form  $\alpha$  is exact everywhere  $hol_{\nabla\omega}(\gamma) \neq 1$ . It is also globally exact if  $\{hol_{\nabla\omega}(\gamma) \neq 1\}$  is dense and  $\mathbf{J}_X(\alpha) = 0$  where  $hol_{\nabla\omega}(\gamma) = 1$ .*
- *When  $\{hol_{\nabla\omega}(\gamma) = 1\}$  is a (not necessarily connected) submanifold,  $\alpha$  is exact on  $M$  if and only if  $\mathbf{J}_X(\alpha)|_{T\{hol_{\nabla\omega}(\gamma)=1\}}$  is exact.*

*Proof:* At points satisfying  $hol_{\nabla\omega}(\gamma) \neq 1$  a  $(k-1)$ -form  $\beta$  is well defined by

$$\beta = \frac{\mathbf{J}_X(\alpha)}{hol_{\nabla\omega}(\gamma)^{-1} - 1}. \quad (7.14)$$

Proposition 7.2 and equation 7.2, together with the hypothesis of  $\alpha$  being closed, imply that  $d^\nabla \beta = \alpha$ . In other words,  $\mathbf{J}_X/(hol_{\nabla\omega}(\gamma)^{-1} - 1)$  is a homotopy operator where  $hol_{\nabla\omega}(\gamma) \neq 1$ .

For  $\mathbf{J}_X(\alpha) = 0$  at  $\{hol_{\nabla\omega}(\gamma) = 1\}$ , lemma 7.2 gives a  $\sigma \in S_P^{k-1}(L)$  such that  $\mathbf{J}_X(\alpha) = (hol_{\nabla\omega}(\gamma)^{-1} - 1)\sigma$ ; therefore,  $\beta$  is well defined by the expression 7.14.

Assuming that  $\{hol_{\nabla\omega}(\gamma) = 1\}$  is a submanifold, one consequence of proposition 7.2 (as it was observed in [26]) is that the polarisation is tangent to it, and all definitions make sense with  $M$  replaced by  $\{hol_{\nabla\omega}(\gamma) = 1\}$ .

If  $\alpha = d^\nabla \beta$ , by applying equation 7.2,

$$\mathbf{J}_X(\alpha) = \mathbf{J}_X \circ d^\nabla \beta = (hol_{\nabla\omega}(\gamma)^{-1} - 1)d^\nabla \beta - d^\nabla \circ \mathbf{J}_X(\beta); \quad (7.15)$$

and  $\mathbf{J}_X(\alpha)|_{T\{hol_{\nabla\omega}(\gamma)=1\}}$  is exact.

Conversely, if  $\mathbf{J}_X(\alpha)|_{T\{hol_{\nabla\omega}(\gamma)=1\}} = d^\nabla|_{T\{hol_{\nabla\omega}(\gamma)=1\}} \zeta$ , taking an extension  $\eta \in S_P^{k-2}(L)$  of  $\zeta$ , the formula  $(\mathbf{J}_X(\alpha) - d^\nabla \eta)|_{T\{hol_{\nabla\omega}(\gamma)=1\}} = 0$  holds and lemma 7.2—the density of  $\{hol_{\nabla\omega}(\gamma) \neq 1\}$  is assumed—provides a  $\beta \in S_P^{k-1}(L)$  such that  $\mathbf{J}_X(\alpha) - d^\nabla \eta = (hol_{\nabla\omega}(\gamma)^{-1} - 1)\beta$ . Proposition 7.2 implies that

$$d^\nabla \circ \mathbf{J}_X(\alpha) = d^\nabla((hol_{\nabla\omega}(\gamma)^{-1} - 1)\beta) = (hol_{\nabla\omega}(\gamma)^{-1} - 1)d^\nabla \beta, \quad (7.16)$$

but equation 7.2 reads

$$d^\nabla \circ \mathbf{J}_X(\alpha) = (hol_{\nabla\omega}(\gamma)^{-1} - 1)\alpha; \quad (7.17)$$

thus,  $d^\nabla \beta = \alpha$  holds where  $hol_{\nabla^\omega}(\gamma) \neq 1$ . Since  $d^\nabla \beta$  is everywhere defined and  $\{hol_{\nabla^\omega}(\gamma) \neq 1\}$  is a dense set,  $\alpha$  must be exact. ■

# Chapter 8

## Poincaré lemma I: foliated cohomology

In [20], Miranda and the author of this thesis were able to compute the foliated cohomology for local models of integrable systems with singularities of nondegenerate type. They do explicit computations for the cohomology groups in some instances (in particular degree 1 and top degree for smooth systems, and in all the degrees for analytic ones), but the intention here is to justify the need for an alternative proof showing that the Kostant complex is a resolution for the sheaf of flat sections, when singularities are allowed in the polarisation.

This chapter revisits a Poincaré lemma for foliated forms with respect to a regular foliation, and computes the foliated cohomology for a local model of integrable systems with nondegenerate singularities. A key point in this computation is the use of some analytical tools; which are mainly a series of decomposition results for functions with respect to singular vector fields.

What makes the difference between the regular and singular case are the solutions of the equation  $X(f) = g$  for a given  $g$  and a given vector field  $X$ . When the vector field is regular, one can solve this equation by simple integration, no matter which function  $g$  is considered. If the vector field is singular, this is a nontrivial question: solutions may exist or not, depending on some properties of the function  $g$  and the

singularity of the vector field  $X$ .

## 8.1 Regular foliations

Let  $(M, \mathcal{F})$  be a foliated  $m$ -dimensional manifold and  $n$  the dimension of the leaves (for more details, the reader may consult section 2.3, in particular, examples 2.13 and 2.16). The foliated cohomology is the one associated to the following cochain complex:

$$0 \longrightarrow C_{\mathcal{F}}^{\infty}(M) \hookrightarrow C^{\infty}(M) \xrightarrow{d_{\mathcal{F}}} \Omega_{\mathcal{F}}^1(M) \xrightarrow{d_{\mathcal{F}}} \dots \xrightarrow{d_{\mathcal{F}}} \Omega_{\mathcal{F}}^n(M) \xrightarrow{d_{\mathcal{F}}} 0 . \quad (8.1)$$

**Theorem 8.1.** *The foliated cohomology groups  $H_{\mathcal{F}}^k(V)$  vanish for  $k \geq 1$  and  $V$  any contractible neighbourhood of a point of  $M$ .*

A sketch of the proof is provided in [20]; it is based on a parametric version of the homotopy formula used for proving a Poincaré lemma for the de Rham cohomology.

Just to illustrate what this theorem has to say, let  $\alpha$  be a closed foliated 1-form:  $\alpha \in \Omega_{\mathcal{F}}^1(V)$  and  $d_{\mathcal{F}}\alpha = 0$ . The foliation can be given locally by a set of vector fields,  $\mathcal{F}|_V = \langle X_1, \dots, X_n \rangle_{C^{\infty}(V)}$ , and the exactness of  $\alpha$  is equivalent to the existence of a function  $f \in C^{\infty}(V)$  satisfying

$$X_j(f) = \alpha(X_j) , \quad (8.2)$$

with  $j = 1, \dots, n$ . Hence, as a corollary, one can solve this equation provided that  $X_i(\alpha(X_j)) = X_j(\alpha(X_i))$ .

**Remark 8.1.** Whilst the de Rham complex is a fine resolution for the constant sheaf  $\mathbb{R}$ , the foliated cohomology is a fine resolution for the sheaf of smooth functions which are constant along the leaves of the foliation.

## 8.2 Singular foliations

The main objective of this section is to prove that one cannot assume, in general, the existence of a Poincaré lemma for foliated cohomology when singularities are al-

lowed. The nonexistence of solutions of equations of type  $X(f) = g$  can be interpreted as an obstruction for local solvability of the cohomological equation  $d_{\mathcal{F}}\beta = \alpha$ , for a given foliated closed  $k$ -form  $\alpha$ .

Before proving the nonexistence result, special decompositions for functions with respect to vector fields of a Williamson basis are presented.

**Theorem 8.2** (Miranda and Vu Ngoc). *Let  $g_1, \dots, g_k$ , be a set of smooth functions on  $\mathbb{R}^{2n}$  with  $k \leq n$  fulfilling the following commutation relations*

$$X_i(g_j) = X_j(g_i), \quad \forall i, j = 1, \dots, k, \quad (8.3)$$

where the  $X_i$ 's are the vector fields of a Williamson basis. Then, there exist a smooth function  $G$  and  $k$  smooth functions  $f_i$  such that,

$$X_j(f_i) = 0, \quad \forall i, j = 1, \dots, k \text{ and} \quad (8.4)$$

$$g_i = f_i + X_i(G), \quad \forall i = 1, \dots, k. \quad (8.5)$$

It is also included in [22] an interesting reinterpretation of this statement in terms of the deformation complex associated to an integrable system: an integrable system with nondegenerate singularities is  $C^\infty$ -infinitesimally stable at the singular point.

In order to fix notation, the next definition recalls what is meant by a Taylor flat function at a subset.

**Definition 8.1.** *For  $\mathbb{R}^m$  endowed with coordinates  $(p_1, \dots, p_m)$ , a smooth function  $g \in C^\infty(\mathbb{R}^m)$  is said to be Taylor flat at the subset  $\{p_1 = \dots = p_k = 0\}$  when*

$$\left. \frac{\partial^{j_1 + \dots + j_k} g}{\partial p_1^{j_1} \dots \partial p_k^{j_k}} \right|_{\{p_1 = \dots = p_k = 0\}} = 0, \quad (8.6)$$

for all  $j_1, \dots, j_k$  and some fixed  $k \leq m$ .

One can find special decompositions for smooth functions like  $f = f_i + X_i(F_i)$ . The following result is a summary of results contained in [8, 9, 19].



**Lemma 8.1** (Eliasson and Miranda). *Assuming that the origin is a singularity of Williamson type  $(k_e, k_h, 0)$ , for any  $f \in C^\infty(\mathbb{R}^{2n})$  there exist  $f_i, F_i \in C^\infty(\mathbb{R}^{2n})$  such that, for each vector field  $X_i$  in a Williamson basis,  $f = f_i + X_i(F_i)$ . Moreover,*

1.  $X_i(f_i) = 0$ ;
2.  $f_i$  is uniquely defined if  $X_i$  defines an  $S^1$ -action, otherwise  $f_i$  is uniquely defined up to Taylor flat functions at  $\Sigma_i$ ;
3. one can choose  $f_i$  and  $F_i$  such that  $X_j(f_i) = X_j(F_i) = 0$  whenever  $X_j(f) = 0$  for  $j \neq i$ ;
4. if  $f$  vanishes at the zero set of any vector of a Williamson basis, so does the function  $f_i$  and one can choose  $F_i$  vanishing at the zero set as well;
5.  $X_i(f) = 0$  implies that  $f$  depends on  $x_i$  and  $y_i$  via  $h_i$ :

$$f(x_1, y_1, \dots, x_n, y_n) = \tilde{f}(x_1, y_1, \dots, x_i^2 + y_i^2, \dots, x_n, y_n) ,$$

$$f(x_1, y_1, \dots, x_n, y_n)|_{Q_i^j} = \tilde{f}(x_1, y_1, \dots, x_i y_i, \dots, x_n, y_n) ,$$

where  $Q_i^1 = \{x_i > 0, y_i > 0\}$ ,  $Q_i^2 = \{x_i > 0, y_i < 0\}$ ,  $Q_i^3 = \{x_i < 0, y_i > 0\}$  and  $Q_i^4 = \{x_i < 0, y_i < 0\}$ .

The case when  $X_i$  is an elliptic vector field was proved in [8, 9, 19]; [19] also has a proof when  $X_i$  is a hyperbolic vector field.

Finally, the nonexistence result [20] can be presented.

**Theorem 8.3** (Miranda and Solha). *Considering  $(\mathbb{R}^{2n}, \sum_{i=1}^n dx_i \wedge dy_i)$  endowed with a distribution  $\mathcal{F}$  generated by a Williamson basis of type  $(k_e, k_h, 0)$ , the following decomposition holds:*

$$\ker(d_{\mathcal{F}} : \Omega_{\mathcal{F}}^1(\mathbb{R}^{2n}) \rightarrow \Omega_{\mathcal{F}}^2(\mathbb{R}^{2n})) = W_{\mathcal{F}}^1(\mathbb{R}^{2n}) \oplus d_{\mathcal{F}}(C^\infty(\mathbb{R}^{2n})) , \quad (8.7)$$

where  $W_{\mathcal{F}}^1(\mathbb{R}^{2n})$  is the set of 1-forms  $\beta \in \Omega_{\mathcal{F}}^1(\mathbb{R}^{2n})$  such that  $\mathcal{L}_{X_i}(\beta) = 0$  for all  $i$ , and if  $X_i$  is of hyperbolic type  $\beta(X_i)$  is not Taylor flat at  $\Sigma_i$  (when it is nonzero).

Thus, the foliated cohomology group in degree 1 is given by:

$$H_{\mathcal{F}}^1(\mathbb{R}^{2n}) \cong \bigoplus_{i=1}^{k_e} \{f \in C_{\mathcal{F}}^\infty(\mathbb{R}^{2n}) ; f|_{\Sigma_i} = 0\} \\ \bigoplus_{i=k_e+1}^n \{f \in C_{\mathcal{F}}^\infty(\mathbb{R}^{2n}) ; f = 0 \text{ or } f|_{\Sigma_i} = 0 \text{ and not Taylor flat at } \Sigma_i\}$$

*Proof:* For any  $\alpha \in \Omega_{\mathcal{F}}^1(\mathbb{R}^{2n})$  the condition  $d_{\mathcal{F}}\alpha = 0$  implies

$$d_{\mathcal{F}}\alpha(X_i, X_j) = X_i(\alpha(X_j)) - X_j(\alpha(X_i)) = 0, \quad (8.8)$$

and theorem 8.2 says that  $\alpha(X_i) = f_i + X_i(F)$ , where  $F \in C^\infty(\mathbb{R}^{2n})$  and  $f_i \in C_{\mathcal{F}}^\infty(\mathbb{R}^{2n})$ . Thus, any closed foliated 1-form  $\alpha$  is cohomologous to a foliated 1-form  $\beta$  satisfying  $\mathcal{L}_{X_i}(\beta) = 0$  for all  $i$  (proposition 2.4 and item 4 of lemma 8.1 guarantee that the forms are well defined); the condition  $\mathcal{L}_{X_i}(\beta) = 0$  automatically implies that  $\beta$  is closed.

There exists  $g \in C^\infty(\mathbb{R}^{2n})$  such that  $d_{\mathcal{F}}g = \beta$  if and only if  $\beta(X_i) = X_i(g)$ . Since  $\mathcal{L}_{X_i}(\beta) = 0$ , this implies  $X_i(\beta(X_i)) = 0$  and by uniqueness (up to Taylor flat functions, lemma 8.1)  $0 = \beta(X_i) + X_i(-g)$  has a solution if and only if  $\beta(X_i) = 0$  or  $\beta(X_i)$  is Taylor flat at  $\Sigma_i$  (for  $i = k_e + 1, \dots, n$ ). Wherefore,  $\beta$  is exact if and only if  $\beta = 0$  or, if  $\beta(X_i) \neq 0$  (for  $i = k_e + 1, \dots, n$ ),  $\beta(X_i)$  is Taylor flat at  $\Sigma_i$ .

The expression  $\ker = W_{\mathcal{F}}^1(\mathbb{R}^{2n}) \oplus d_{\mathcal{F}}(C^\infty(\mathbb{R}^{2n}))$  implies  $H_{\mathcal{F}}^1(\mathbb{R}^{2n}) = W_{\mathcal{F}}^1(\mathbb{R}^{2n})$ , by definition any  $\beta \in W_{\mathcal{F}}^1(\mathbb{R}^{2n})$  can be given by  $n$  functions vanishing at certain points (proposition 2.4) and satisfying some Taylor flat condition: e.g.  $\beta(X_n) = f \in C^\infty(\mathbb{R}^{2n})$ ,  $f|_{\Sigma_i} = 0$  and not Taylor flat at  $\Sigma_n$ , if it is nonzero. The Lie derivative condition yields  $f \in C_{\mathcal{F}}^\infty(\mathbb{R}^{2n})$ . ■

This theorem tells that, although in the regular case the foliated cohomology complex (8.1) is a fine resolution for  $\mathcal{C}_{\mathcal{F}}^\infty$  (the sheaf of functions which are constant along the leaves of the foliation), for singular foliations, in general, the foliated cohomology complex is not a resolution for the sheaf  $\mathcal{C}_{\mathcal{F}}^\infty$ .

# Chapter 9

## Poincaré lemma II: regular polarisations

It is included here a proof that the Kostant complex is a fine resolution for the sheaf of flat sections when the polarisation is regular.

### 9.1 A fine resolution for $\mathcal{J}$

The sheaves  $\mathcal{S}_P^k(L)$  are fine:  $\Gamma(L)$  and  $\Omega_P^k(M)$  are free modules over the ring of functions of  $M$ , and because of that, they admit partition of unity. Hence, via a Poincaré lemma, the abstract de Rham theorem [4] offers a proof for the following:

**Theorem 9.1.** *The Kostant complex is a fine resolution for  $\mathcal{J}$ . Therefore, each of its cohomology groups,  $H^k(S_P^\bullet(L))$ , is isomorphic to  $\check{H}^k(M; \mathcal{J})$ .*

**Remark 9.1.** There are particular situations in which a Poincaré lemma is available, and only in these cases theorem 9.1 holds. This is true<sup>1</sup> when  $\mathcal{P}$  is a subbundle of  $TM$ , and it can be extended to a more general setting; Chapter 10 of this thesis provides Poincaré lemmata when  $\mathcal{P}$  has nondegenerate singularities.

The following result uses the foliated Poincaré lemma for regular foliations, theorem 8.1.

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<sup>1</sup>Both Śniatycki and Rawnsley attribute this to Kostant, a proof is provided in [26].

**Lemma 9.1.** *There always exists a local unitary flat section on each point of  $M$  for a given regular polarisation  $P$ .*

*Proof:* The symplectic form is closed,  $d\omega = 0$ , and locally  $\omega = d\theta$ . Since  $P$  is Lagrangian,  $\omega$  vanishes in the directions tangent to the leaves of  $P$ , which implies  $\omega|_P = d_P\Theta = 0$ : where  $\Theta = \theta|_P$  is the restriction of  $\theta$  in the directions tangent to the leaves of the polarisation. By theorem 8.1, there exists a function  $f$  such that  $d_P f = \Theta$ ; therefore,  $\theta - df$  satisfies  $d(\theta - df) = \omega$  and  $\theta - df$  vanishes in the directions tangent to the leaves.

Lemma 2.2 offers a unitary section  $s$  satisfying  $\nabla^\omega s = -i(\theta - df) \otimes s$ , which happens to be a flat section,  $\nabla s = 0$ , because  $(\theta - df)|_P = 0$ . ■

As a consequence of the existence of unitary flat sections, elements of  $\mathcal{S}_P^k(L)$  which are closed can be interpreted as closed polarised  $k$ -forms taking values on the sheaf  $\mathcal{J}$ .

**Corollary 9.1.** *Let  $\Omega_P^k$  be the sheaf associated to  $\Omega_P^k(M)$ ; then,  $\mathcal{S}_P^k(L) \cong \Omega_P^k \otimes \mathcal{J}$  and  $\ker(d_\nabla) \cong \ker(d_P) \otimes \mathcal{J}$ .*

*Proof:* By lemma 9.1, for each point on  $M$  there exists a trivialising neighbourhood  $V \subset M$  of  $L$  with an unitary flat section  $s \in \Gamma(L|_V)$ . If  $\alpha \in \mathcal{S}_P^k(L)$ , it can be locally written as  $\alpha|_{TV} = \alpha \otimes s$ , where  $\alpha \in \Omega_P^k(V)$ . The condition  $d^\nabla(\alpha \otimes s) = 0$  is, then, equivalent to  $d_P \alpha = 0$ , because  $d^\nabla(\alpha \otimes s) = d_P \alpha \otimes s + (-1)^k \alpha \wedge \nabla s$ ,  $s \neq 0$  and  $\nabla s = 0$ . ■

The Kostant complex (6.8) is just the foliated complex twisted by the sheaf of sections  $\mathcal{S}$ , and the exactness of the foliated complex (8.1) (which is guaranteed by theorem 8.1) implies, by corollary 9.1, the exactness of the Kostant complex.

**Theorem 9.2.** *At a sufficiently small neighbourhood of any point of  $M$ , the cohomology groups  $H^k(S_P^\bullet(L))$  vanish for  $k \geq 1$  when  $\mathcal{P}$  is a subbundle of  $TM$ .*

Lemma 9.1 uses the existence of a Poincaré lemma for foliations in a crucial way. Since there is no hope for a Poincaré lemma when the foliation has singularities

(theorem 8.3), in order to prove a Poincaré lemma for the Kostant complex, different strategies need to be adopted when nondegenerate singularities are included in the picture. Chapter 10 provides Poincaré lemmata for this situation.

### 9.1.1 Symplectic vector spaces: linear polarisation

All symplectic vector spaces polarised by Lagrangian hyperplanes are equivalent to this particular case:  $(M = \mathbb{C}^n, \omega = \sum_{j=1}^n dx_j \wedge dy_j)$  and  $\mathcal{P} = \left\langle \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\rangle_{C^\infty(\mathbb{C}^n)}$ .

Since  $(M, \omega)$  is an exact symplectic manifold, the trivial line bundle is a prequantum line bundle for it:  $L = \mathbb{C} \times \mathbb{C}^n$  with connection 1-form  $\Theta = \sum_{j=1}^n x_j dy_j$ , with respect to the unitary section  $\exp\left(i \sum_{j=1}^n x_j y_j\right)$ .

The solutions of the flat equation,  $\nabla s = 0$ , are complex-valued functions of the type

$$s(x_1, y_1, \dots, x_n, y_n) = h(x_1, \dots, x_n) \exp\left(i \sum_{j=1}^n x_j y_j\right). \quad (9.1)$$

Therefore,

$$\check{H}^0(M; \mathcal{J}) = \{s \in \Gamma(L) ; \nabla s = 0\} \cong C^\infty(\mathbb{R}^n). \quad (9.2)$$

Using the unitary flat section  $r = \exp\left(i \sum_{j=1}^n x_j y_j\right)$  as basis, if  $\alpha \otimes r \in S_P^k(L)$  is closed:

$$0 = d^\nabla(\alpha \otimes r) = d_P \alpha \otimes r + (-1)^k \alpha \wedge \nabla r = d_P \alpha \otimes r. \quad (9.3)$$

Wherefore, Poincaré lemma for regular foliations (theorem 8.1) imply that  $H^k(S_P^\bullet(L)) = \{0\}$  for  $k \geq 1$ .

This provides an alternative proof of theorem 9.2; thus, theorem 9.1 concludes the proof of:

**Proposition 9.1.** *The quantisation of the cotangent bundle of  $\mathbb{R}^n$  with linear polarisation is  $C^\infty(\mathbb{R}^n)$ .*

## 9.2 Semilocal examples

Some examples, that are going to be important later on, are computed in this section. They also serve to illustrate how to apply the techniques from chapter 7.

### 9.2.1 The cylinder: polarisation by circles

Recalling example 3.3:  $(M = \mathbb{R} \times S^1, \omega = dx \wedge dy)$ ,  $L$  the trivial bundle with connection 1-form  $\Theta = xdy$ , with respect to the unitary section  $e^{ix}$ , and  $\mathcal{P} = \left\langle \frac{\partial}{\partial y} \right\rangle_{C^\infty(\mathbb{R} \times S^1)}$ .

The Hamiltonian vector field  $X = \frac{\partial}{\partial y}$  generates a  $S^1$ -action, and the holonomy of its orbits is given by  $hol_{\nabla^\omega}(\gamma) = e^{i2\pi x}$  (lemma 7.1, for  $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ ). Since Bohr-Sommerfeld leaves satisfy  $x \in \mathbb{Z}$ , the holonomy is nontrivial in a dense set of  $M$  and proposition 7.3 holds. Hence, applying theorem 9.1, one gets  $\check{H}^0(M; \mathcal{J}) = \{0\}$ . Furthermore, proposition 7.4 and theorem 9.1 can be applied, implying  $\check{H}^l(V; \mathcal{J}|_V) = \{0\}$ , for  $l \geq 1$ , for each neighbourhood  $V = (a, b) \times S^1$  that does not contain a Bohr-Sommerfeld leaf.

Let  $\ell_k$  be the inverse image by the height function of the point  $x = k \in \mathbb{Z}$ . Wherefore,  $\ell_k \cong S^1$  is a Bohr-Sommerfeld leaf and  $\{hol_{\nabla^\omega}(\gamma) = 1\} = \bigcup_{k \in \mathbb{Z}} \ell_k$ .

It is possible<sup>2</sup> to define a linear map  $\Psi : S_P^1(L) \rightarrow \bigoplus_{k \in \mathbb{Z}} \Gamma(L|_{\ell_k})$  by:

$$\Psi(\alpha) = \bigoplus_{k \in \mathbb{Z}} \mathbf{J}_X(\alpha) \Big|_{\ell_k}. \quad (9.4)$$

Because the dimension of  $M$  is 2,  $S_P^l(L) = \{0\}$  for  $l \geq 2$ , and, for any  $\alpha \in S_P^1(L)$ , equation 7.2 reads

$$\nabla \circ \mathbf{J}_X(\alpha) = (hol_{\nabla^\omega}(\gamma)^{-1} - 1)\alpha \Rightarrow \nabla_X \Psi(\alpha) = 0. \quad (9.5)$$

Thus, the image of  $\Psi$  is contained in the set of flat sections over Bohr-Sommerfeld leaves.

<sup>2</sup>This construction is due to Rawnsley [26].

Conversely, given  $\bigoplus_{k \in \mathbb{Z}} s_k \in \bigoplus_{k \in \mathbb{Z}} \Gamma(L|_{\ell_k})$ , where  $s_k$  are flat sections ( $\nabla_X s_k = 0$ ), there exists  $s \in \Gamma(L)$  such that  $s|_{\ell_k} = s_k$  for each  $k \in \mathbb{Z}$ : due the closedness of  $\bigcup_{k \in \mathbb{Z}} \ell_k$ . Lemma 7.2 implies the existence of an  $\alpha \in S_P^1(L)$  satisfying

$$\nabla s = (\text{hol}_{\nabla^\omega}(\gamma)^{-1} - 1)\alpha \Rightarrow (\text{hol}_{\nabla^\omega}(\gamma)^{-1} - 1)\mathbf{J}_X(\alpha) = \mathbf{J}_X(\nabla s) = (\text{hol}_{\nabla^\omega}(\gamma)^{-1} - 1)s. \quad (9.6)$$

By density and continuity,  $\mathbf{J}_X(\alpha) = s$ ; hence, the image of  $\Psi$  is the set of flat sections over Bohr-Sommerfeld leaves.

Proposition 7.4 asserts that  $\ker \Psi = \nabla(\Gamma(L))$ , and the first isomorphism theorem

$$\begin{array}{ccc} S_P^1(L) & \longrightarrow & \Psi(S_P^1(L)) \\ \downarrow & \swarrow & \\ S_P^1(L)/\ker \Psi & & \end{array} \quad (9.7)$$

implies that  $\check{H}^1(M; \mathcal{J}) \cong \bigoplus_{k \in \mathbb{Z}} \mathbb{C}$ : the ring of flat sections over  $\ell_k$  is isomorphic to  $\mathbb{C}$  (see example 3.3).

**Proposition 9.2.** *The quantisation of a cylinder polarised by circles is  $\mathbb{C}^{b_s}$ , where  $b_s$  is the number of Bohr-Sommerfeld leaves.*

## 9.2.2 Direct product type with a regular component

The following quantisation problem will be considered now:  $N = (-1, 1) \times S^1$  endowed with the same structures (symplectic form, polarisation and prequantum line bundle) of the model in subsection 9.2.1, and  $(M, \omega)$  a prequantisable symplectic manifold with real polarisation  $\mathcal{P}$  and prequantum line bundle  $(L, \nabla^\omega)$ . The product  $N \times M$  admits  $\left\langle \frac{\partial}{\partial y} \right\rangle_{C^\infty(N)} \oplus_{C^\infty(N \times M)} \mathcal{P}$  as a real polarisation for the symplectic form  $dx \wedge dy + \omega$  (its complexification will be denoted by  $\mathcal{P}$ ), and also a prequantum line bundle  $(\mathcal{L}, \bar{\nabla}^{\text{dx} \wedge \text{dy} + \omega})$ .

The vector field  $\frac{\partial}{\partial y}$  generates a Hamiltonian  $S^1$ -action with nontrivial holonomy over a dense set: the holonomy of its orbits is given by  $\text{hol}_{\nabla^\omega}(\gamma) = e^{i2\pi x}$  (lemma 7.1). Wherefore, proposition 7.3 can be used to show that  $H^0(S_{\mathcal{P}}^\bullet(\mathcal{L})) = \{0\}$ .

For the other groups one has:

**Theorem 9.3.** *Supposing that  $(M, \omega)$  is exact,  $\omega = d\theta$ , the map*

$$\Psi : H^k(S_{\mathcal{F}}^\bullet(\mathcal{L})) \rightarrow H^{k-1}(S_P^\bullet(L)) \quad (9.8)$$

defined by  $\Psi([\bar{\alpha}]) = \left[ \mathbf{J}_{\frac{\partial}{\partial y}}(\bar{\alpha}) \Big|_{\{hol_{\nabla\omega}(\gamma)=1\}} \right]$  is an isomorphism.

*Proof:* Since the product symplectic manifold,  $(N \times M, dx \wedge dy + \omega)$ , is exact, there exists a unitary section  $\bar{s} \in \Gamma(\mathcal{L})$  satisfying  $\bar{\nabla}\bar{s} = -i(xdy + \theta) \otimes \bar{s}$  (lemma 2.2). Let  $\bar{\alpha} = \bar{\alpha} \otimes \bar{s} \in S_{\mathcal{F}}^k(\mathcal{L})$  and  $\bar{\alpha} = dy \wedge \bar{\beta} + \bar{\sigma}$ , where  $\bar{\beta} = \iota_{\frac{\partial}{\partial y}} \bar{\alpha}$  and  $\bar{\sigma} = \bar{\alpha} - dy \wedge \bar{\beta}$ ; thus,  $\iota_{\frac{\partial}{\partial y}} \bar{\beta} = \iota_{\frac{\partial}{\partial y}} \bar{\sigma} = 0$ .

$$\mathbf{J}_{\frac{\partial}{\partial y}}(\bar{\alpha}) = \int_0^{2\pi} \phi_t^* \bar{\beta} \otimes e^{-itx} \bar{s} dt = \int_0^{2\pi} \phi_t^* \bar{\beta} e^{-itx} dt \otimes \bar{s} \Rightarrow \quad (9.9)$$

$$\mathbf{J}_{\frac{\partial}{\partial y}}(\bar{\alpha}) \Big|_{\{hol_{\nabla\omega}(\gamma)=1\}} = \eta \otimes \bar{s}|_{x=0}, \quad (9.10)$$

where  $\eta = \int_0^{2\pi} \phi_t^* \bar{\beta} e^{-itx} dt \Big|_{\{x=0\}}$ . The flow of  $\frac{\partial}{\partial y}$  preserves  $\eta$ ; therefore,  $\eta \in \Omega_P^{k-1}(M)$ .

For closed  $\bar{\alpha}$ ,

$$d^{\bar{\nabla}} \circ \mathbf{J}_{\frac{\partial}{\partial y}}(\bar{\alpha}) = (hol_{\nabla\omega}(\gamma)^{-1} - 1)\bar{\alpha} \Rightarrow d^{\bar{\nabla}} \circ \mathbf{J}_{\frac{\partial}{\partial y}}(\bar{\alpha}) \Big|_{\{hol_{\nabla\omega}(\gamma)=1\}} = 0. \quad (9.11)$$

By definition,  $\bar{\nabla}_{\frac{\partial}{\partial y}} \bar{s} = -ix\bar{s}$  and  $\bar{\nabla}_{\frac{\partial}{\partial y}} \bar{s}|_{x=0} = 0$ ; wherefore, for each point  $p \in M$ ,  $\bar{s}|_{x=0}$  is uniquely determined by its value at  $(0, 0, p) \in N \times M$  by parallel transport along integral curves of  $\frac{\partial}{\partial y}$ . This means that  $\bar{s}|_{x=0}$  identifies itself as a section of  $L$ : the restriction of  $\mathcal{L}$  to  $\{(0, 0)\} \times M$  is a line bundle over  $M$  with a connection such that its curvature is equal to  $\omega$ ; consequently, it must be isomorphic to  $L$ .

To summarise it, after some identifications,  $\mathbf{J}_{\frac{\partial}{\partial y}}(\cdot) \Big|_{\{hol_{\nabla\omega}(\gamma)=1\}}$  maps closed  $k$ -forms of  $S_{\mathcal{F}}^\bullet(\mathcal{L})$  to closed  $(k-1)$ -forms of  $S_P^\bullet(L)$ , and proposition 7.4 proves that  $\Psi$  is injective —the set  $\{hol_{\nabla\omega}(\gamma) = 1\}$  is equal to  $\{0\} \times S^1 \times M$ .

Now, given  $r \in \Gamma(L)$  a unitary section, let  $\bar{r} \in \Gamma(\mathcal{L})$  be an extension of the following section defined on  $\{x = 0\}$ : after identifying  $\mathcal{L}|_{\{(0,0)\} \times M}$  with  $L$ , for each point  $p \in M$ , the parallel transport of  $r(p)$  by the integral curve of  $\frac{\partial}{\partial y}$  passing through  $p$  defines a section of  $\mathcal{L}$  over the set  $\{x = 0\}$ .



Due to the inclusion  $\Omega_P^{k-1}(M) \subset \Omega_{\mathcal{P}}^{k-1}(N \times M)$ , the expression  $\bar{\zeta} = \zeta \otimes \bar{r}$  defines an element in  $S_{\mathcal{P}}^{k-1}(\mathcal{L})$  for any  $[\zeta \otimes r] \in H^{k-1}(S_P^\bullet(L))$ .

The form  $d^{\bar{\nabla}}\bar{\zeta}|_{\{hol_{\nabla\omega}(\gamma)=1\}}$  is completely determined by  $d^{\nabla}(\zeta \otimes r)$ , via parallel transport (which commutes with the derivation). And because  $d^{\nabla}(\zeta \otimes r) = 0$ , lemma 7.2 provides an  $\bar{\alpha} \in S_{\mathcal{P}}^k(\mathcal{L})$  such that  $d^{\bar{\nabla}}\bar{\zeta} = (hol_{\nabla\omega}(\gamma)^{-1} - 1)\bar{\alpha}$ . Hence,

$$0 = d^{\bar{\nabla}} \circ d^{\bar{\nabla}}\bar{\zeta} = (hol_{\nabla\omega}(\gamma)^{-1} - 1)d^{\bar{\nabla}}\bar{\alpha} , \quad (9.12)$$

implying that  $\bar{\alpha}$  is closed over the dense set  $\{x \neq 0\}$ , and, by continuity,  $\bar{\alpha}$  is closed.

As consequence of  $\iota_{\frac{\partial}{\partial y}}\zeta$  being zero,  $\mathbf{J}_{\frac{\partial}{\partial y}}(\bar{\zeta}) = 0$  and equation 7.2 reads

$$(hol_{\nabla\omega}(\gamma)^{-1} - 1)\bar{\zeta} = \mathbf{J}_{\frac{\partial}{\partial y}} \circ d^{\bar{\nabla}}\bar{\zeta} = \mathbf{J}_{\frac{\partial}{\partial y}}((hol_{\nabla\omega}(\gamma)^{-1} - 1)\bar{\alpha}) = (hol_{\nabla\omega}(\gamma)^{-1} - 1)\mathbf{J}_{\frac{\partial}{\partial y}}(\bar{\alpha}) , \quad (9.13)$$

which implies  $\mathbf{J}_{\frac{\partial}{\partial y}}(\bar{\alpha}) = \bar{\zeta}$  where  $x \neq 0$ ; and by density and continuity, it must hold true everywhere. This proves that  $\Psi$  is onto.  $\blacksquare$

The theorem still holds if  $N$  is replaced by  $(a, b) \times S^1$  with  $(a, b) \cap \mathbb{Z} = \{k\}$ . For  $(a, b) \cap \mathbb{Z} = \emptyset$ , propositions 7.3 and 7.4 assert that all cohomology groups  $H^l(S_{\mathcal{P}}^\bullet(\mathcal{L}))$  vanish: the quantisation of the product is trivial when there is no Bohr-Sommerfeld leaf.

By a Mayer-Vietoris argument<sup>3</sup>, similar to one that will be described below (subsection 10.1.1), one can compute the product quantisation for  $(a, b) \cap \mathbb{Z} = \{k_1, \dots, k_{b_s}\}$ . It suffices to take the cover  $\mathcal{A} = \{A_j\}_{j \in \{1, \dots, b_s\}}$ , where  $A_1 = (a, k_1 + 3/4) \times M$ ,  $A_{b_s} = (k_{b_s} - 3/4, b) \times M$  and  $A_j = (k_j - 3/4, k_j + 3/4) \times M$  (supposing  $k_1 \leq k_2 \leq \dots \leq k_{b_s}$ ).

**Corollary 9.2.** *Assuming that the Kostant complex is a fine resolution for the sheaf of flat sections of  $L$ , the quantisation of the product between a cylinder polarised by circles and an arbitrary quantisable exact symplectic manifold  $M$  is a direct sum of  $b_s$  copies of  $\mathcal{Q}(M)$ : where  $b_s$  is the number of Bohr-Sommerfeld leaves with respect to the quantisation of the cylinder.*

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<sup>3</sup>One might also use, instead, a global argument as in subsection 9.2.1.

# Chapter 10

## Poincaré lemma III: singular polarisations

Following Rawnsley [26], given a prequantisable symplectic manifold  $(M, \omega)$  with polarisation  $P$ , it is possible to construct a fine resolution for the sheaf of flat sections  $\mathcal{J}$ . Using the results of chapter 7 and [21], it is even possible to do it when  $P$  has nondegenerate singularities: this is the content of theorems 10.1, 10.2 and 10.4.

As it was said before, the proof of lemma 9.1 relies on the existence of a Poincaré lemma for foliations. When the foliation is not regular such theorem might not exist<sup>1</sup>, and the proof of lemma 9.1 is of no use; therefore, one needs a different method to prove that the Kostant complex is a fine resolution for the sheaf of flat sections.

### 10.1 Elliptic singularities

This section contains a Poincaré lemma for singularities of elliptic type.

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<sup>1</sup>This is exactly the situation for polarisations induced by nondegenerate integrable systems, as it was discussed in chapter 8, for which it has been proved that there is no Poincaré lemma for the foliated complex [20].

### 10.1.1 The complex plane: polarisation by circles

Let  $(M = \mathbb{C}, \omega = dx \wedge dy)$  and  $F : M \rightarrow \mathbb{R}$  be a nondegenerate integrable system of elliptic type, i.e.  $F(x, y) = x^2 + y^2$ . For this case, the real polarisation is  $\mathcal{P} = \left\langle -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right\rangle_{c^\infty(\mathbb{C})}$  and the Hamiltonian vector field  $X = -2y \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y}$  is the generator of a  $S^1$ -action —this is example 4.1, again.

As in the previous cases,  $(M, \omega)$  is an exact symplectic manifold and the trivial line bundle is a prequantum line bundle for it:  $L = \mathbb{C} \times \mathbb{C}$  with connection 1-form  $\Theta = \frac{1}{2}(xdy - ydx)$ , with respect to the unitary section  $e^{i(x^2+y^2)}$ .

**Proposition 10.1.** *At a sufficiently small neighbourhood of the origin of  $\mathbb{C}$ , the cohomology groups  $H^k(S_P^\bullet(L))$  vanish for  $k \geq 0$  when  $\mathcal{P}$  is generated by  $-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ .*

*Proof:* Using lemma 7.1 this time,  $[hol_{\nabla\omega}(\gamma)](x, y) = e^{i\pi(x^2+y^2)}$ , and it is clear that  $\{hol_{\nabla\omega}(\gamma) \neq 1\}$  is a dense set. As a result, proposition 7.3 can be applied to prove that  $H^0(S_P^\bullet(L)) = \{0\}$ .

The set  $\{hol_{\nabla\omega}(\gamma) = 1\}$  is the union of the origin and concentric circles with  $R^2/2$  integer ( $R$  being the radius), and since the origin is a fixed point, the operator  $\mathbf{J}_X$  is the null operator when restricted to the origin (proposition 2.4). Hence, proposition 7.4, applied for each contractible neighbourhood of the origin that does not contain any other Bohr-Sommerfeld leaf, implies that elliptic singularities give no contribution to quantisation<sup>2</sup>,  $H^k(S_P^\bullet(L)) = \{0\}$  for  $k \geq 1$ . ■

Refrasing the previous proposition, theorem 9.1 holds in this particular setting.

**Proposition 10.2.** *The quantisation of an open disk polarised by circles is  $\mathbb{C}^{b_s}$ , where  $b_s$  is the number of nonsingular Bohr-Sommerfeld leaves.*

*Proof:*  $M$  can be divided up<sup>3</sup> into an open disk  $V$  of radius  $b < 1$  centred at the origin, and an annulus  $W$  centred at the origin with small radius  $a \in (0, b)$  and an infinite big radius:  $M = V \cup W$  and  $V \cap W$  is an annulus with small radius  $a$  and big

<sup>2</sup>This was first proved in [11] using different techniques.

<sup>3</sup>The complex plane will be considered; for an arbitrary open disk the same argument works.

radius  $b$ . Proposition 10.1 implies that  $\check{H}^k(V; \mathcal{J}|_V) = \{0\}$  for all  $k$ , and proposition 9.2 gives  $\check{H}^k(V \cap W; \mathcal{J}|_{V \cap W}) = \{0\}$  for all  $k$  as well, since  $V \cap W \cong (a, b) \times S^1$  (polarised by circles). Thus, the Mayer-Vietoris argument works and  $\check{H}^k(M; \mathcal{J}) \cong \check{H}^k(W; \mathcal{J}|_W)$ . It happens that  $W \cong (a, \infty) \times S^1$  (polarised by circles), and proposition 9.2 concludes the proof.  $\blacksquare$

### 10.1.2 Direct product type with an elliptic component

The quantisation problem to be considered here is:  $N = \{(x, y) \in \mathbb{C} ; x^2 + y^2 < 1\}$  endowed with the same structures (symplectic form, polarisation and prequantum line bundle) of the model in subsection 10.1.1, and  $(M, \omega)$  a prequantisable symplectic manifold with real polarisation  $\mathcal{P}$  and prequantum line bundle  $(L, \nabla^\omega)$ . The product  $N \times M$  admits  $\left\langle -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right\rangle_{C^\infty(N)} \oplus_{C^\infty(N \times M)} \mathcal{P}$  as a real polarisation for the symplectic form  $dx \wedge dy + \omega$  (the complexification of it will be denoted by  $\mathcal{P}$ ), and also a prequantum line bundle  $(\mathcal{L}, \bar{\nabla}^{dx \wedge dy + \omega})$ .

**Lemma 10.1.** *The cohomology groups  $H^k(S_{\mathcal{P}}^\bullet(\mathcal{L}))$  vanish for  $k \geq 0$  in the particular case described in this subsection.*

*Proof:* The group  $H^0(S_{\mathcal{P}}^\bullet(\mathcal{L}))$  is trivial because  $X = -2y \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y}$  generates a Hamiltonian  $S^1$ -action with nontrivial holonomy over a dense set: lemma 7.1 gives  $hol_{\nabla^\omega}(\gamma) = e^{i\pi(x^2 + y^2)}$ ; wherefore, proposition 7.3 holds. Whilst for higher order groups, one needs to note that the set  $\{hol_{\nabla^\omega}(\gamma) = 1\}$  is equal to  $\{(0, 0)\} \times M$ , and that  $(0, 0, p)$  are fixed points for any  $p \in M$ ; thus, the operator  $\mathbf{J}_X$  is the null operator when restricted to  $\{(0, 0)\} \times M$  (proposition 2.4). Therefore, by applying proposition 7.4,  $H^k(S_{\mathcal{P}}^\bullet(\mathcal{L})) = \{0\}$  for  $k \geq 1$ .  $\blacksquare$

By a Mayer-Vietoris argument similar to the ones used in subsections 10.1.1 and 9.2.2, one has:

**Proposition 10.3.** *Assuming that the Kostant complex is a fine resolution for the sheaf of flat sections of a prequantum line bundle of  $(M, \omega)$ , the quantisation of the*

product between an open disk polarised by circles and an arbitrary quantisable manifold  $M$  is a direct sum of  $b_s$  copies of  $\mathcal{Q}(M)$ : where  $b_s$  is the number of nonsingular Bohr-Sommerfeld leaves with respect to the quantisation of the open disk.

The following theorem is the Poincaré lemma for singularities of elliptic type.

**Theorem 10.1.** *Assuming that  $p \in M$  is a nondegenerate critical point of Williamson type  $(k_e, k_h, k_f)$ , with  $k_e \geq 1$  and  $k_e + k_h + 2k_f \leq n$  (it does not need to be a rank zero critical point, but it has to have an elliptic component), for an integrable system  $F : M \rightarrow \mathbb{R}^n$  on a prequantisable symplectic manifold  $(M, \omega)$ , with polarisation induced by the moment map: the cohomology groups  $H^k(S_P^\bullet(L))$  vanish for  $k \geq 0$  in a sufficiently small neighbourhood of  $p$ .*

*Proof:* The normal form theorem (theorem 2.4 or its version for nonzero rank singularities [8, 9, 19]) says that the model near a critical point of that type is exactly the one of lemma 10.1: as long as  $k_e \geq 1$ , along one of the elliptic directions a neighbourhood of  $p$  splits into a product as the model of lemma 10.1.  $\blacksquare$

## 10.2 Focus-focus singularities

Let  $F = (f_1, f_2) : M \rightarrow \mathbb{R}^2$  be an integrable system on a prequantisable  $(M, \omega)$ , with a rank zero nondegenerate critical point of Williamson type  $(0, 0, k_f)$ . Near the singular point,  $f_2$  generates, via its Hamiltonian vector field flow, a Hamiltonian  $S^1$ -action —Zung [32] demonstrated that this action is defined semilocally, i.e. near a neighbourhood of a focus-focus singular fibre.

In a small enough neighbourhood  $W$  of a singular point of a focus-focus fibre,  $\mathbf{J}_X$  is the null operator over the points where  $\{hol_{\nabla\omega}(\gamma) = 1\}$ . Indeed, the symplectic local model is given by<sup>4</sup>,  $W \cong \mathbb{C}^2$  with coordinates  $(x_1, x_2, y_1, y_2)$ ,  $L|_W \cong \mathbb{C} \times \mathbb{C}^2$  with connection 1-form

$$\Theta = \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2) , \quad (10.1)$$

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<sup>4</sup>Zung [31, 32] cites Eliasson, Lerman and Umanskiy and Vey.

with respect to the unitary section  $s = e^{i(x_1y_1+x_2y_2+x_1y_2-x_2y_1)}$ .

The integrable system takes the form

$$F(x_1, x_2, y_1, y_2) = (x_1y_1 + x_2y_2, x_1y_2 - x_2y_1) , \quad (10.2)$$

and, therefore, the polarisation is generated by

$$X_1 = -x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \quad (10.3)$$

and

$$X_2 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2} . \quad (10.4)$$

The Hamiltonian vector field  $X_2$  is the generator of the  $S^1$ -action. Its periodic flow is given by:

$$\phi_t(x_1, x_2, y_1, y_2) = (x_1 \cos t + x_2 \sin t, x_2 \cos t - x_1 \sin t, y_1 \cos t + y_2 \sin t, y_2 \cos t - y_1 \sin t) . \quad (10.5)$$

By lemma 7.1, the holonomy of its orbits is

$$\text{hol}_{\nabla^\omega}(\gamma) = e^{i2\pi(x_1y_2-x_2y_1)} . \quad (10.6)$$

Now, given any  $\alpha \in S_{P|W}^1(L|W)$ , using the unitary section  $s$ , it can be written as

$$\alpha = \alpha \otimes s = (\alpha_1 dx_1 + \alpha_2 dx_2 + \alpha_3 dy_1 + \alpha_4 dy_2) \otimes s \quad (10.7)$$

and

$$\begin{aligned} \iota_{X_2} \phi_t^* \alpha|_{(x_1, x_2, y_1, y_2)} &= [\alpha(\dot{\gamma}(t))] \circ \phi_t(x_1, x_2, y_1, y_2) \\ &= \alpha_1 \circ \phi_t(x_1, x_2, y_1, y_2)(-x_1 \sin t + x_2 \cos t) \\ &\quad + \alpha_2 \circ \phi_t(x_1, x_2, y_1, y_2)(-x_2 \sin t - x_1 \cos t) \\ &\quad + \alpha_3 \circ \phi_t(x_1, x_2, y_1, y_2)(-y_1 \sin t + y_2 \cos t) \\ &\quad + \alpha_4 \circ \phi_t(x_1, x_2, y_1, y_2)(-y_2 \sin t - y_1 \cos t) . \end{aligned} \quad (10.8)$$

Therefore, using

$$A(t, p) = x_2 \alpha_1 \circ \phi_t(p) - x_1 \alpha_2 \circ \phi_t(p) + y_2 \alpha_3 \circ \phi_t(p) - y_1 \alpha_4 \circ \phi_t(p) \quad (10.9)$$

and

$$B(t, p) = x_1\alpha_1 \circ \phi_t(p) + x_2\alpha_2 \circ \phi_t(p) + y_1\alpha_3 \circ \phi_t(p) + y_2\alpha_4 \circ \phi_t(p) , \quad (10.10)$$

the expression  $\mathbf{J}_X(\boldsymbol{\alpha})$  at a point  $p = (x_1, x_2, y_1, y_2)$  near the singular point is:

$$\begin{aligned} [\mathbf{J}_X(\boldsymbol{\alpha})](p) &= \left( \int_0^{2\pi} A(t, p) e^{-it(x_1y_2 - x_2y_1)} \cos t \, dt \right. \\ &\quad \left. - \int_0^{2\pi} B(t, p) e^{-it(x_1y_2 - x_2y_1)} \sin t \, dt \right) s . \end{aligned} \quad (10.11)$$

The following upper bound proves that the expression above is zero over the set  $\{(x_1, x_2, y_1, y_2) \in \mathbb{C}^2; x_1y_2 - x_2y_1 = 0\}$  (the points where  $hol_{\nabla\omega}(\gamma) = 1$ ):

$$\begin{aligned} |\mathbf{J}_X(\boldsymbol{\alpha})| &\leq \left| \max_{t \in [0, 2\pi]} A(t, p) \right| \left| \int_0^{2\pi} e^{-it(x_1y_2 - x_2y_1)} \cos t \, dt \right| \\ &\quad + \left| \max_{t \in [0, 2\pi]} B(t, p) \right| \left| \int_0^{2\pi} e^{-it(x_1y_2 - x_2y_1)} \sin t \, dt \right| \\ &= \left| \max_{t \in [0, 2\pi]} A(t, p) \right| \left| \frac{(x_1y_2 - x_2y_1)(e^{-i2\pi(x_1y_2 - x_2y_1)} - 1)}{(x_1y_2 - x_2y_1)^2 - 1} \right| \\ &\quad + \left| \max_{t \in [0, 2\pi]} B(t, p) \right| \left| \frac{e^{-i2\pi(x_1y_2 - x_2y_1)} - 1}{(x_1y_2 - x_2y_1)^2 - 1} \right| . \end{aligned} \quad (10.12)$$

The proof of lemma 10.1 works verbatim if  $N$  is replaced by the local model  $W$  describe here. Hence, this can be interpreted as a proof of the Poincaré lemma needed for the proof of theorem 9.1 when the real distribution has focus-focus singularities.

**Theorem 10.2.** *If  $p \in M$  is a nondegenerate critical point of Williamson type  $(k_e, k_h, k_f)$ , with  $k_f \geq 1$  and  $k_e + k_h + 2k_f \leq n$  (it does not need to be a rank zero critical point, but it has to have a focus-focus component), for an integrable system  $F : M \rightarrow \mathbb{R}^n$  on a prequantisable symplectic manifold  $(M, \omega)$ , with polarisation induced by the moment map: the cohomology groups  $H^k(S_P^\bullet(L))$  vanish for  $k \geq 0$  in a sufficiently small neighbourhood of  $p$ .*

### 10.3 Hyperbolic singularities

The content of this section covers results of a joint work [21] between Miranda and the author of this thesis.

The proofs of theorems 10.1 and 10.2 are based on the existence of symplectic circle actions. Hyperbolic singularities do not share the same kind of symmetry as elliptic or focus-focus: i.e. there is no natural symplectic circle action near purely hyperbolic singularities. Wherefore, different techniques need to be employed to prove a Poincaré lemma for hyperbolic singularities.

Let  $(M = \mathbb{C}, \omega = dx \wedge dy)$  and  $h : M \rightarrow \mathbb{R}$  be a nondegenerate integrable system of hyperbolic type, i.e.  $h(x, y) = xy$ . For this case, the real polarisation is  $\mathcal{P} = \langle X \rangle_{C^\infty(\mathbb{C})}$ , with  $X$  the Hamiltonian vector field  $-x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ .

$(M, \omega)$  is an exact symplectic manifold and the trivial line bundle is a prequantum line bundle for it:  $L = \mathbb{C} \times \mathbb{C}$  with connection 1-form  $\Theta = \frac{1}{2}(x dy - y dx)$ , with respect to the unitary section  $e^{ih}$ .

Considering a section  $f e^{ih}$  of the prequantum line bundle, the flat section equation can be written as,

$$\nabla f e^{ih} = 0 \Leftrightarrow X(f) = ihf . \quad (10.13)$$

This equation has been studied in [12], in particular, proposition 3.5 of that paper says:

**Proposition 10.4** (Hamilton and Miranda). *Any flat section  $s$  can be written as a collection*

$$s_j = a_j(xy) e^{\frac{i}{2} xy \ln |\frac{x}{y}|} \quad j = 1, 2, 3, 4 ; \quad (10.14)$$

where  $a_j$  is a complex-valued smooth function of one variable, Taylor flat at 0, with domain such that  $a_j(xy)$  is defined on the  $j^{\text{th}}$  open quadrant of  $\mathbb{R}^2$ . Conversely, given four such  $a_j$ , they fit together to define a flat section  $s$  using the formula above.

Thus, (up to a different choice of sign) this implies that

$$f(x, y) e^{ixy} = \begin{cases} 0 & \text{if } x = 0, y = 0 \\ a_1(xy) e^{\frac{i}{2} xy \ln \frac{y}{x}} & \text{if } x > 0, y > 0 \\ a_2(xy) e^{\frac{i}{2} xy \ln \frac{-y}{x}} & \text{if } x > 0, y < 0 \\ a_3(xy) e^{\frac{i}{2} xy \ln \frac{-y}{-x}} & \text{if } x < 0, y > 0 \\ a_4(xy) e^{\frac{i}{2} xy \ln \frac{y}{-x}} & \text{if } x < 0, y < 0 , \end{cases} \quad (10.15)$$



where  $a_j$  is a smooth complex-valued function of one variable (defined for  $z \in [0, \infty)$  if  $j = 1, 4$  or  $z \in (-\infty, 0]$  if  $j = 2, 3$ ) and such that  $\frac{d^k a_j}{dz^k}(0) = 0$  for all  $j$  and  $k$ .

The converse of proposition 10.4 guarantees that  $H^0(S_P^\bullet(L))$  is not trivial and is given by quadruples of Taylor flat smooth complex-valued functions of one variable, as above.

For the computation of the first cohomology group the strategy is going to be close to the one used in [19, 22]: firstly, a formal solution is obtained and, thereafter, a closed formula is given for the case of Taylor flat functions.

A 1-form  $\alpha \otimes e^{ih} \in S_P^1(L)$  is exact if and only if there exists a  $g \in C^\infty(\mathbb{C})$  satisfying

$$\nabla g e^{ih} = \alpha \otimes e^{ih} \Leftrightarrow X(g) = ihg + \alpha(X) . \quad (10.16)$$

The Taylor series in  $(x, y)$  of  $g$  and  $\alpha(X)$  near the origin  $(0, 0) \in \mathbb{C}$  are denoted by,

$$\sum_{k, l = 0}^{\infty} g_{k, l} x^k y^l \quad (10.17)$$

and

$$\sum_{k, l = 0}^{\infty} f_{k, l} x^k y^l , \quad (10.18)$$

with  $f_{0, 0} = 0$ , by definition (proposition 2.4).

The cohomological equation in jets reads

$$\sum_{k, l = 0}^{\infty} (l - k) g_{k, l} x^k y^l = \sqrt{-1} \sum_{k, l = 0}^{\infty} g_{k, l} x^{k+1} y^{l+1} + \sum_{k, l = 0}^{\infty} f_{k, l} x^k y^l . \quad (10.19)$$

And the following recursive relations lead to a solution,

$$\begin{aligned}
g_{0,0} &= 0 ; \\
g_{k,k} &= \sqrt{-1} f_{k+1,k+1} , \quad k > 0 ; \\
g_{0,k} &= \frac{f_{0,k} + \sqrt{-1} g_{0,k-1}}{k} , \quad k > 0 ; \\
g_{k,0} &= \frac{-f_{k,0} - \sqrt{-1} g_{k-1,0}}{k} , \quad k > 0 ; \\
g_{k,l} &= \frac{f_{k,l} + \sqrt{-1} g_{k-1,l-1}}{l-k} , \quad k \neq l > 0 .
\end{aligned} \tag{10.20}$$

One can even write a closed expression for the jets,

$$\begin{aligned}
g_{0,0} &= 0 ; \\
g_{k,k} &= \sqrt{-1} f_{k+1,k+1} , \quad k > 0 ; \\
g_{0,k} &= \frac{1}{k!} \sum_{j=0}^{k-1} (-1)^{\frac{j}{2}} (k-j-1)! f_{0,k-j} , \quad k > 0 ; \\
g_{k,0} &= \frac{1}{k!} \sum_{j=0}^{k-1} (-1)^{\frac{j}{2}+1} (k-j-1)! f_{k-j,0} , \quad k > 0 ; \\
g_{k,l} &= \sum_{j=0}^{k-1} \frac{(-1)^{\frac{j}{2}}}{(l-k)^{j+1}} f_{k-j,l-j} + \sum_{j=0}^{l-k-1} \frac{(-1)^{\frac{k}{2}+\frac{j}{2}} (l-k-j-1)!}{(l-k)^k (l-k)!} f_{0,l-k-j} , \quad l > k > 0 ; \\
g_{k,l} &= \sum_{j=0}^{l-1} \frac{(-1)^{\frac{j}{2}}}{(l-k)^{j+1}} f_{k-j,l-j} + \sum_{j=0}^{k-l-1} \frac{(-1)^{\frac{l}{2}+\frac{j}{2}+1} (k-l-j-1)!}{(l-k)^l (k-l)!} f_{k-l-j,0} , \quad k > l > 0 .
\end{aligned} \tag{10.21}$$

This procedure solves the equation only formally. According to Borel's theorem, there exists, up to Taylor flat functions<sup>5</sup> at the origin, a unique smooth function with such Taylor series.

Hence, the following have been proved:

**Lemma 10.2.** *Any smooth function  $\tilde{g}$  whose Taylor series is defined by the previous recursive relations satisfies*

$$X(\tilde{g}) - ih\tilde{g} - \alpha(X) = F , \quad (10.22)$$

where  $F$  is a Taylor flat function at the origin.

Therefore, if it is possible to find a solution for

$$X(G) - ihG = F \quad (10.23)$$

such that,  $G$  is Taylor flat at the origin, the difference  $\tilde{g} - G$  defines a smooth solution for the cohomological equation.

One can solve this problem with the aid of the logarithmic function  $\ln \gamma : \{(x, y) \in \mathbb{C} ; xy \neq 0\} \rightarrow \mathbb{R}$ , where  $\ln \gamma(p)$  is the time that it takes for a point in the diagonal,  $\{(x, y) \in \mathbb{C} ; x = y\}$ , to reach  $p$  via the flow of  $X$  (the diagonal point and  $p$  lie over the same integral curve of  $X$ ). This function is well defined for  $xy \neq 0$ .

**Lemma 10.3.** *For a given Taylor flat function  $F$ , a solution to the equation  $X(G) - ihG = F$  is given by*

$$G = \int_{-\ln \gamma}^0 e^{-iht} F \circ \phi_t \, dt . \quad (10.24)$$

*This solution is actually well defined and smooth over all points of  $\mathbb{C}$ .*

**Remark 10.1.** The smoothness of this formula prevails if parameters are considered in the function  $F$ . This observation will be needed in the higher dimensional discussion.

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<sup>5</sup>Observe that, two smooth functions which have the same Taylor expand at a point differ by a smooth function which has vanishing jets at all order at that point.

*Proof:* The first thing, before proving that the expression for  $G$  is smooth and well defined, is to prove that  $G$  solves the equation by computing  $X(G)$ .

The composition of  $G$  with the flow of  $X$  at time  $s$  is:

$$G \circ \phi_s = \int_{-\ln \gamma \circ \phi_s}^0 e^{-ith \circ \phi_s} F \circ \phi_t \circ \phi_s \, dt = \int_{-\ln \gamma - s}^0 e^{-ith} F \circ \phi_{t+s} \, dt . \quad (10.25)$$

The logarithmic function satisfies  $\ln \gamma \circ \phi_s = \ln \gamma + s$  and  $h \circ \phi_s = h$ ; thus, by a change of coordinates  $\tau = t + s$ ,

$$G \circ \phi_s = \int_{-\ln \gamma - s + s}^s e^{-ih(\tau-s)} F \circ \phi_\tau \, d\tau = e^{ish} \int_{-\ln \gamma}^s e^{-ith} F \circ \phi_t \, dt . \quad (10.26)$$

Then, one differentiates  $G \circ \phi_s$  with respect to  $s$

$$\frac{d}{ds} G \circ \phi_s = ihe^{ish} \int_{-\ln \gamma}^s e^{-ith} F \circ \phi_t \, dt + F \circ \phi_s , \quad (10.27)$$

and, finally, evaluate it in  $s = 0$  to get

$$X(G) = \left. \frac{d}{ds} G \circ \phi_s \right|_{s=0} = ih \int_{-\ln \gamma}^0 e^{-ith} F \circ \phi_t \, dt + F = ihG + F . \quad (10.28)$$

It is clear that  $G$  is smooth and well defined over the points where the logarithmic function  $\ln \gamma$  is well defined (the set  $\{(x, y) \in \mathbb{C} ; xy \neq 0\}$ ). The idea now is to prove that it is continuous and well defined at the points where  $h = 0$ .

For each point of  $\{(x, y) \in \mathbb{C} ; xy \neq 0\}$ ,

$$|G| \leq \int_{-\ln \gamma}^0 |e^{-iht} F \circ \phi_t| \, dt = \int_{-\ln \gamma}^0 |F \circ \phi_t| \, dt \leq |\ln \gamma| \max_{t \in [-\ln \gamma, 0]} |F \circ \phi_t| . \quad (10.29)$$

When  $h$  approaches zero,  $\ln \gamma$  diverges in a logarithmic fashion. It is left to understand how  $\max_{t \in [-\ln \gamma, 0]} |F \circ \phi_t|$  behaves.

At a point  $p = (x, y) \in \mathbb{C}$ , the flow of the Hamiltonian vector field  $X = -x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$  is given by  $\phi_t(p) = (e^{-t}x, e^ty)$ . Let  $p_0 = (z, z)$  be a point of  $\mathbb{C}$  satisfying  $\phi_t(p_0) = p$ ; then,

$$\ln \gamma(p) = \begin{cases} \frac{1}{2} \ln \frac{y}{x} & \text{if } xy > 0 \\ \frac{1}{2} \ln \frac{-y}{x} & \text{if } xy < 0 \end{cases} , \quad (10.30)$$

since

$$e^{-t}z = x \Rightarrow t = \ln \frac{z}{x} \quad (10.31)$$

and

$$e^t z = y \Rightarrow t = \ln \frac{y}{z}. \quad (10.32)$$

Wherefore,

$$\phi_{-\ln \gamma(p)}(p) = (|h(p)|^{\frac{1}{2}}, |h(p)|^{\frac{1}{2}}), \quad (10.33)$$

which implies  $\lim_{|h| \rightarrow 0} F \circ \phi_{-\ln \gamma} = 0$  and it goes sufficiently fast to zero to guarantee that  $G$  is continuous and vanishes at  $h = 0$ , because the function  $F$  is Taylor flat at the origin.

One can see that  $G$  is actually smooth at  $h = 0$  by analysing its differential (it is clear that the argument that follows holds for the higher order partial derivatives):

$$dG = \int_{-\ln \gamma}^0 (e^{-iht} \phi_t^* \circ dF) dt - iGdh + e^{-ih \ln \gamma} F \circ \phi_{-\ln \gamma} d \ln \gamma. \quad (10.34)$$

The first term converges to zero, as  $h$  approaches to zero, by the same argument used above, the partial derivatives of a Taylor flat function are still Taylor flat by definition. The second term is continuous and well defined at  $h = 0$  because  $G$  is and  $h$  is smooth. It remains to analyse the term  $F \circ \phi_{-\ln \gamma} d \ln \gamma$ . By l'Hôpital's rule  $\lim_{h \rightarrow 0} e^{-ih \ln \gamma} = 1$  and the fact that  $F$  is Taylor flat guarantees that  $\lim_{h \rightarrow 0} F \circ \phi_{-\ln \gamma} d \ln \gamma = 0$ . ■

Since the dimension of the generic leaves is 1, the only cohomology group to check is the first cohomology group.

Lemmata 10.2 and 10.3 yield the following,

**Theorem 10.3.** *In a neighbourhood of a hyperbolic singularity, the first cohomology group  $H^1(S_P^\bullet(L))$  vanishes when the polarisation is given by an integrable system on a two-dimensional manifold.*

### 10.3.1 The hyperbolic-hyperbolic case

Let  $(M_1 \times M_2 = \mathbb{C}^2, \omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$  and  $H = (h_1, h_2) : M_1 \times M_2 \rightarrow \mathbb{R}$  be a nondegenerate integrable system of hyperbolic-hyperbolic type, i.e.  $H(x_1, y_1, x_2, y_2) =$

$(x_1y_1, x_2y_2)$ . For this case, the real polarisation is  $\mathcal{P} = \langle X_1, X_2 \rangle_{C^\infty(\mathbb{C}^2)}$ , with  $X_j$  the Hamiltonian vector field  $-x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}$ .

Again,  $(M_1 \times M_2, \omega)$  is an exact symplectic manifold and the trivial line bundle is a prequantum line bundle for it:  $L = \mathbb{C} \times \mathbb{C}^2$  with connection 1-form  $\Theta = \frac{1}{2}(x_1 dy_1 - y_1 dx_1) + \frac{1}{2}(x_2 dy_2 - y_2 dx_2)$ , with respect to the unitary section  $s = e^{i(h_1 + h_2)}$ .

To have a Poincaré lemma, one needs to prove that  $H^1(S_P^\bullet(L))$  and  $H^2(S_P^\bullet(L))$  are trivial<sup>6</sup>.

**Proposition 10.5.** *The second cohomology group  $H^2(S_P^\bullet(L))$  vanishes for hyperbolic-hyperbolic singularities.*

*Proof:* Any line bundle valued polarised 2-form,  $\alpha \otimes s$ , is automatically closed in dimension 4, and it is exact if and only if there exists a  $\beta$  such that,

$$d^\nabla(\beta \otimes s) = \alpha \otimes s . \quad (10.35)$$

Because  $\nabla s = -i\Theta \otimes s$ , the exactness of  $\alpha \otimes s$  is equivalent to

$$\alpha = d_P \beta - i\Theta \wedge \beta \Leftrightarrow \alpha(X_1, X_2) = X_1(\beta(X_2)) - ih_1\beta(X_2) - X_2(\beta(X_1)) + ih_2\beta(X_1) . \quad (10.36)$$

One can find a solution for this equation by taking  $\beta(X_1) = 0$  and solving for  $\beta(X_2)$ , using the parametric versions of lemmata 10.2 and 10.3. This ends the proof of the proposition. ■

In order to compute  $H^1(S_P^\bullet(L))$  one needs to prove parametric versions of lemmata 10.2 and 10.3 when the unknown functions possess a special property if the known functions have it. Concretely:

**Lemma 10.4.** *If  $X_1(f) = ih_1 f$ , there exists a smooth function  $\tilde{g}$  such that,  $X_1(\tilde{g}) = ih_1 \tilde{g}$  and  $X_2(\tilde{g}) - ih_2 \tilde{g} = f$ .*

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<sup>6</sup>The cohomology group  $H^0(S_P^\bullet(L))$  can also be computed by a parametric version of proposition 10.4. Since the aim here is to provide a Poincaré Lemma, this simple computation is left aside.

This can be proved by keeping track of the proofs of lemmata 10.2 and 10.3, since the condition on  $f$  is with respect to one set of variables,  $x_1$  and  $y_1$ , whilst the differential equation  $X_2(\tilde{g}) - ih_2\tilde{g} = f$  deals only with  $x_2$  and  $y_2$  and treats  $x_1$  and  $y_1$  merely as parameters.

**Proposition 10.6.** *The first cohomology group  $H^1(S_P^\bullet(L))$  vanishes for hyperbolic-hyperbolic singularities.*

*Proof:* A line bundle valued polarised 1-form,  $\alpha \otimes s$ , is closed if and only if

$$X_1(\alpha(X_2)) - ih_1\alpha(X_2) = X_2(\alpha(X_1)) - ih_2\alpha(X_1) , \quad (10.37)$$

and it is exact if and only if there exists a smooth function  $g$  such that

$$X_j(g) - ih_jg = \alpha(X_j) , \quad (10.38)$$

for  $j = 1, 2$ .

The first equation,

$$X_1(g) - ih_1g = \alpha(X_1) , \quad (10.39)$$

can be solved by using the parametric versions of lemmata 10.2 and 10.3. The closedness of  $\alpha \otimes s$  would, then, imply that

$$X_2(\alpha(X_1)) = X_2 \circ X_1(g) - iX_2(h_1g) = X_1(\alpha(X_2)) - ih_1\alpha(X_2) + ih_2\alpha(X_1) ; \quad (10.40)$$

and, because  $[X_1, X_2] = X_1(h_2) = X_2(h_1) = 0$ , one can write

$$X_1(X_2(g) - ih_2g - \alpha(X_2)) = ih_1(X_2(g) - ih_2g - \alpha(X_2)) . \quad (10.41)$$

Lemma 10.4 can be applied to prove that there exists a function  $\tilde{g}$  such that,

$$X_1(\tilde{g}) = ih_1\tilde{g} \quad (10.42)$$

and

$$X_2(\tilde{g}) - ih_2\tilde{g} = X_2(g) - ih_2g - \alpha(X_2) ; \quad (10.43)$$

As a result,

$$X_j(g - \tilde{g}) - ih_j(g - \tilde{g}) = \alpha(X_j) , \quad (10.44)$$

for  $j = 1, 2$ .

This proves that the system of equations above has a solution and finishes the proof of this proposition. ■

Wherefore, theorem 10.3, propositions 10.6 and 10.5, and corollary 9.2 entail the following:

**Theorem 10.4.** *If  $p \in M$  is a nondegenerate critical point of Williamson type  $(0, k_h, 0)$ , with  $k_h \leq 2 \leq n$  (it does not need to be a rank zero critical point), for an integrable system  $F : M \rightarrow \mathbb{R}^n$  (whose regular fibres are compact) on a prequantisable symplectic manifold  $(M, \omega)$ , with polarisation induced by the moment map: the cohomology groups  $H^k(S_P^\bullet(L))$  vanish for  $k \geq 0$  in a sufficiently small neighbourhood of  $p$ .*

Together with theorems 10.1 and 10.2, this theorem asserts that the Kostant complex computes geometric quantisation when the polarisation is given by nondegenerate integrable systems in dimension 2 or 4.



# Chapter 11

## The resolution approach

As opposed to the approach taken by Hamilton and Miranda [11, 12], this chapter deals with geometric quantisation à la de Rham. The results of chapters 9 and 10 provide a resolution for the sheaf of flat sections, and, by exploiting the existence of symplectic circle actions, this chapter presents alternative proofs for Śniatycki's [27] and Hamilton's theorems [11].

More than just an extension of Rawnsley's ideas [26], the tools developed in chapter 7 give a unifying view, and also add new information about the contributions coming from focus-focus type of singularities.

### 11.1 Nonsingular case

The aim of this section is to compute geometric quantisation for Lagrangian fibrations. The first section is devoted to compute geometric quantisation semilocally, near a Liouville fibre; the second section deals with the global computation.

#### 11.1.1 Neighbourhood of a Liouville fibre

The Liouville theorem for integrable systems provides a symplectic normal form for a neighbourhood of a regular fibre. What follows is the computation of the quantisation of that model.

Let  $M = \mathbb{R}^n \times (\mathbb{R}^{n-k} \times \mathbb{T}^k)$  and  $0 \leq k \leq n$ , where  $\mathbb{T}^k \cong \mathbb{R}^k / 2\pi\mathbb{Z}^k$ , with coordinates  $(x_1, \dots, x_n, y_1, \dots, y_{n-k}, \dots, y_n)$  and symplectic form  $\omega = \sum_{j=1}^n dx_j \wedge dy_j$ . It admits as a real polarisation  $\mathcal{P} = \left\langle \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\rangle_{C^\infty(M)}$ , and since  $(M, \omega)$  is an exact symplectic manifold, it also admits as a prequantum line bundle  $L = \mathbb{C} \times M$  with connection 1-form  $\Theta = \sum_{j=1}^n x_j dy_j$ , with respect to the unitary section  $\exp\left(i \sum_{j=1}^n x_j\right)$ .

The next lemma computes the contributions to geometric quantisation for each trivialising neighbourhood of a Lagrangian fibre bundle.

**Lemma 11.1.**  $H^{k+l}(S_P^\bullet(L)) = \{0\}$  for all  $l \neq 0$  and

$$H^k(S_P^\bullet(L)) \cong \begin{cases} \bigoplus_{m \in \mathbb{Z}^k} C^\infty(\mathbb{R}^{n-k}), & \text{if } k \neq n \\ \bigoplus_{m \in \mathbb{Z}^k} \mathbb{C}, & \text{if } k = n \end{cases}. \quad (11.1)$$

*Proof:* Supposing  $k \neq n$ , when  $M$  is written as  $(\mathbb{R} \times S^1)^k \times \mathbb{C}^{n-k}$  it becomes clear that the use of theorem 9.3 (more precisely, corollary 9.2)  $k$  times reduces the problem of computing the quantisation of  $M$  to the computation of the quantisation of  $\mathbb{C}^{n-k}$ : which by proposition 9.1 is just  $C^\infty(\mathbb{R}^{n-k})$ . If  $k = n$ , one just need to apply theorem 9.3  $n - 1$  times, and, then, proposition 9.2 to conclude.  $\blacksquare$

## 11.1.2 Lagrangian fibre bundles

In [27] Śniatycki studies the case when the polarisation is a Lagrangian fibration. He uses a resolution for the sheaf and proves the vanishing of the groups  $\check{H}^l(M; \mathcal{J})$ , for  $l \neq k$ :  $k$  being the rank of the fundamental group of a fibre.

**Theorem 11.1** (Śniatycki). *If the base space  $N$  is a manifold and the natural projection  $\mathcal{F} : M \rightarrow N$  is a Lagrangian fibration, then  $\mathcal{Q}(M) = \check{H}^k(M; \mathcal{J})$ , and  $\check{H}^k(M; \mathcal{J}) \cong \check{H}^0(\ell_{BS}; \mathcal{J}|_{\ell_{BS}})$ , where  $\ell_{BS} \subset M$  is the union of all Bohr-Sommerfeld fibres.*

A slightly different proof of his theorem is given here when  $k \neq 0$ . When  $k = 0$  there is no symplectic circle action and the techniques presented in chapter 7 are of

no use; wherefore, apart from the presentation, the proof is the same as the original one and is omitted.

Any atlas of the base space satisfies that the projection  $\mathcal{F} : M \rightarrow N$  on each open set  $V$  of the atlas is a moment map. Assuming  $\dim M = 2n$ , if  $\chi : V \rightarrow \mathbb{R}^n$  is a coordinate system over  $V$ ,  $F := \chi \circ \mathcal{F}|_{\mathcal{F}^{-1}(V)} : \mathcal{F}^{-1}(V) \rightarrow \mathbb{R}^n$  is an integrable system because each  $f_j := \text{pr}_j \circ F$  is constant along the fibres of  $\mathcal{F}$ ,  $df_j = 0$  along them. In other words,  $df_j$  annihilates vector fields tangent to the fibres and the Hamiltonian vector fields of the others  $f_j$ 's are, indeed, tangent to the fibres:  $\{f_i, f_j\}_\omega = 0$ .

The open sets  $\mathcal{F}^{-1}(V)$  are just the model in lemma 11.1 with a fixed number of Bohr-Sommerfeld fibres; thus, the quantisation of it is just a sum of copies of  $\mathbb{C}$ , or  $C^\infty(\mathbb{R}^{n-k})$ , depending on the value of  $k$ , for each Bohr-Sommerfeld fibre.

Assuming that  $k \neq 0$ , so that theorem 7.1 can be used, the atlas can —and it will— be chosen in such a way that no Bohr-Sommerfeld fibre is contained in more than one of the open sets  $\mathcal{F}^{-1}(V)$ . In particular, if  $V$  and  $W$  are two open sets of the atlas such that  $V \cap W \neq \emptyset$ , then  $\mathcal{F}^{-1}(V) \cap \mathcal{F}^{-1}(W)$  has no Bohr-Sommerfeld fibre. Proposition 3.1 implies that one of the periodic Hamiltonian vector fields of the components of  $F$  has orbits with nontrivial holonomy over  $\mathcal{F}^{-1}(V) \cap \mathcal{F}^{-1}(W)$ ; thus, by proposition 7.4, its quantisation is just the trivial vector space  $\{0\}$ . This means that a Mayer-Vietoris argument works for the cover  $\{\mathcal{F}^{-1}(V)\}$  of  $M$ , and this finishes the proof for  $k \neq 0$ .

**Remark 11.1.** Śniatycki works with the prequantum line bundle twisted by a bundle of half forms normal to the polarisation, and here the result is presented for the nontwisted prequantum line bundle. As it was mentioned before, the techniques used here apply to any complex line bundle admitting a flat connection along the polarisation: the only difference being that the Bohr-Sommerfeld fibres may not be the same.

**Example 11.1.** The Kodaira-Thurston manifold is an example of a Lagrangian bundle (there is a description of it in [11]). Moreover, it is a symplectic manifold which is

not Kähler that gives at least one reason for developing a truly symplectic geometric quantisation apparatus.  $\diamond$

## 11.2 Singular case

Śniatycki [27] computed the cohomology groups, but to achieve that he had to assume mild topological hypothesis for the base space. Unfortunately, it turns out that for a large class of examples —toric manifolds included— the base space is not a manifold, and what seems to be a mild assumption is, indeed, a strong restriction.

Hamilton and Miranda [11, 12] avoided this restriction by computing the cohomology groups over the manifold. Not only that, they do not use any resolution. So their approach is different from Rawnsley [26] and Śniatycki in two aspects.

For the singular case Śniatycki's result is expected to hold for integrable systems with nondegenerate singularities in any dimension.

**Conjecture 11.1** (Miranda and Hamilton). *For a  $2n$ -dimensional compact integrable system, whose moment map has only nondegenerate singularities,  $\mathcal{Q}(M) = \check{H}^n(M; \mathcal{J})$ . Moreover, the cohomology  $\check{H}^n(M; \mathcal{J})$  has contributions of the form  $\mathbb{C}$  for each regular Bohr-Sommerfeld leaf,  $\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}$  for the hyperbolic singularities and elliptic and focus-focus fibres give no contribution (just the trivial vector space  $\{0\}$ ).*

### 11.2.1 Neighbourhood of an elliptic fibre

Toric, or locally toric, manifolds also have a normal form for a neighbourhood of its fibres, even if they are singular: Zung [31] attributes this normal form to Dufour and Molino and Eliasson. The model and its quantisation are described below.

For  $0 \leq k \leq n$ , let  $(M = \mathbb{R}^{n+k} \times \mathbb{T}^{n-k}, \omega = \sum_{j=1}^n dx_j \wedge dy_j)$ , with coordinates  $(x_1, \dots, x_n, y_1, \dots, y_k, \dots, y_n)$ , and  $F : M \rightarrow \mathbb{R}^n$  be a nondegenerate integrable system of elliptic type, i.e.

$$F(x_1, \dots, x_n, y_1, \dots, y_n) = (x_1^2 + y_1^2, \dots, x_k^2 + y_k^2, x_{k+1}, \dots, x_n). \quad (11.2)$$

The real polarisation in this case is

$$\mathcal{P} = \left\langle -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1}, \dots, -y_k \frac{\partial}{\partial x_k} + x_k \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_{k+1}}, \dots, \frac{\partial}{\partial y_n} \right\rangle_{C^\infty(M)}, \quad (11.3)$$

and since  $(M, \omega)$  is an exact symplectic manifold, it also admits as a prequantum line bundle  $L = \mathbb{C} \times M$  with connection 1-form  $\Theta = \frac{1}{2} \sum_{j=1}^k (x_j dy_j - y_j dx_j) + \sum_{j=k+1}^n x_j dy_j$ ,

with respect to the unitary section  $\exp \left[ i \sum_{j=1}^k (x_j^2 + y_j^2) + i \sum_{j=k+1}^n x_j \right]$ .

**Proposition 11.1.**  $\mathcal{Q}(M) \cong \mathbb{C}^{b_s}$ , where  $b_s$  is the number of nonsingular Bohr-Sommerfeld fibres.

*Proof:* One can first use proposition 10.3  $k$  times, corollary 9.2  $n - k - 1$  times, and, finally, proposition 9.2. ■

It is important to notice that, if the case of  $b_s = 0$  was considered ( $x_1^2 + y_1^2, \dots, x_k^2 + y_k^2 < 1$ ), the previous proof would give that all cohomology groups vanish when  $k \neq 0$ . This implies, as a corollary, that Bohr-Sommerfeld fibres of elliptic type give no contribution to geometric quantisation.

### 11.2.2 Locally toric manifolds

Hamilton [11] has shown, via Čech approach, that Śniatycki's theorem holds for locally toric manifolds and that the elliptic fibres give no contribution to the quantisation.

**Theorem 11.2** (Hamilton). *For  $M$  a  $2n$ -dimensional compact symplectic manifold equipped with a locally toric singular Lagrangian fibration:*

$$\mathcal{Q}(M) = \check{H}^n(M; \mathcal{J}) \cong \bigoplus_{p \in BS_r} \mathbb{C}, \quad (11.4)$$

$BS_r$  being the set of the regular Bohr-Sommerfeld fibres.

**Remark 11.2.** Regarding metaplectic correction, contrary to Śniatycki's, Hamilton's result does not include a twisted prequantum line bundle. Using the framework

described in this thesis, it is straightforward to twist the prequantum line bundle by a bundle of half forms normal to the polarisation and achieve the same result —only noticing that the Bohr-Sommerfeld fibres may not be the same.

The previous reasoning used for the fibre bundle case works in this singular setting. This provides a proof for Hamilton’s theorem via a de Rham approach.

A locally toric singular Lagrangian fibration on a symplectic manifold  $(M, \omega)$  is a surjective map  $\mathcal{F} : M \rightarrow N$ , where  $N$  is a topological space such that for every point in  $N$  there exist an open neighbourhood  $V$  and a homeomorphism  $\chi : V \rightarrow U \subset \{z \in \mathbb{R}^k ; z \geq 0\} \times \mathbb{R}^{n-k}$  satisfying that  $(\mathcal{F}^{-1}(V), \omega|_{\mathcal{F}^{-1}(V)}, \chi \circ \mathcal{F}|_{\mathcal{F}^{-1}(V)})$  is an integrable system symplectomorphic to an open subset of the model of proposition 11.1.

Hence, by definition, the open sets  $\mathcal{F}^{-1}(V)$  are just the model in proposition 11.1 with a fixed number of Bohr-Sommerfeld fibres; thus, the quantisation of it is just a sum of copies of  $\mathbb{C}$ , or  $\{0\}$ , depending on the fibre dimension, for each Bohr-Sommerfeld fibre.

Choosing an open cover for  $N$  in such a way that no Bohr-Sommerfeld fibre is contained in more than one of the open sets  $\mathcal{F}^{-1}(V)$  (theorem 7.1 allows one to make this choice), if  $V$  and  $W$  are two open sets of the atlas such that  $V \cap W \neq \emptyset$ , then  $\mathcal{F}^{-1}(V) \cap \mathcal{F}^{-1}(W)$  has no Bohr-Sommerfeld fibre. Proposition 3.1 implies that one of the periodic Hamiltonian vector fields of the components of the integrable system has orbits with nontrivial holonomy over  $\mathcal{F}^{-1}(V) \cap \mathcal{F}^{-1}(W)$ ; wherefore, by proposition 7.4, its quantisation is just the trivial vector space  $\{0\}$ . This means that a Mayer-Vietoris argument works for the cover  $\{\mathcal{F}^{-1}(V)\}$  of  $M$ , and this finishes the proof.

### 11.2.3 Focus-focus contribution to geometric quantisation

Let  $F = (f_1, f_2) : M \rightarrow \mathbb{R}^2$  be an integrable system on a prequantisable  $(M, \omega)$ , with a nondegenerate focus-focus singular fibre  $\ell_{ff}$  which is Bohr-Sommerfeld. In [32] it is demonstrated the existence of a neighbourhood of  $\ell_{ff}$  over which  $f_2$  is a moment map for a Hamiltonian  $S^1$ -action.

**Proposition 11.2.** *In the neighbourhood of  $\ell_{ff}$  over which a Hamiltonian  $S^1$ -action is defined, there exists a neighbourhood  $V$  containing only  $\ell_{ff}$  as a Bohr-Sommerfeld fibre such that  $\check{H}^0(V; \mathcal{J}|_V) = \{0\}$ .*

*Proof:* Lemma 7.1 guarantees that the holonomy of the orbits of the Hamiltonian  $S^1$ -action is nontrivial over a dense set in  $V$ ; hence, proposition 7.3 asserts that there are no nonzero flat sections on  $V$ . ■

Concerning how focus-focus fibres behave under geometric quantisation, this partially answers conjecture 11.1.

Believing that the conjecture is true, one could try to use proposition 7.4 to prove it for the neighbourhood  $V$ . The first obstacle is that  $\{hol_{\nabla\omega}(\gamma) = 1\}$  is not a submanifold, and one needs to prove that  $\mathbf{J}_X$  is the null operator over the points where  $\{hol_{\nabla\omega}(\gamma) = 1\}$  (which, looking at the proof of theorem 10.2, seems to be the case). Another approach would be to prove only the exactness of  $\mathbf{J}_X$  and check out convergence over the singular points of  $\{hol_{\nabla\omega}(\gamma) = 1\}$ .

As it was seen from the quantisation of Lagrangian fibrations and locally toric manifolds, quantisation of neighbourhoods of Bohr-Sommerfeld fibres computes the quantisation of the whole manifold. Consequently, if one knows how to compute the higher cohomology groups for a neighbourhood of a focus-focus fibre, one is able to compute the quantisation for the almost toric case using the factorisation tools (corollary 9.2 and proposition 10.3) and proceeding like the Lagrangian bundle and locally toric cases.

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