Theorem 7.8. *If G is a (p,q)-graph with degree sequence $2k_1, 2k_2, \ldots, 2k_p$, then there exists $\vec{G}$ an orientation of G such that the graph $H_{\vec{G}}$ is a bipartite graph with bipartite sets V_1 and V_2 then the degree sequence of the vertices of V_1 is equal to the degree sequence of the vertices of V_2 and equal to $k_1, k_2, \ldots, k_p$.*

Proof.

Since all the degrees of the vertices of G are even, then G is decomposable into cycles (see [40], for example). Orient each cycle either clockwise or counterclockwise and apply the above theorem. □

7.2 Magical and Antimagical Product Labelings

7.2.1 Introduction

An $n \times n$ magic square is an $n \times n$ array consisting of all integers $1, 2, \ldots, n^2$ such that the sum of any row or column in the array is constant. It is known that there is an $n \times n$ magic square for every integer $n \geq 3$, see [2]. Steward [38] was motivated by the notion of magic squares to define vertex magic labelings. A graph G of size q is said to be vertex magic if there is a labeling from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that, at each vertex v, the sum of the labels on the edges incident with v is constant. Such a labeling is called a vertex magic labeling. It is interesting to notice that if an $n \times n$ magic square is given, then it is possible to construct a vertex magic labeling of a complete bipartite graph $K_{n,n}$ for every integer $n \geq 3$, and vice versa.

Hartsfield and Ringel [24] introduced antimagic labelings as follows. A graph G of size q is said to be antimagic if there is a bijective labeling $f : E(G) \to \{1, 2, \ldots, q\}$ such that the sum of all the labels incident with each vertex are distinct given that the vertices are distinct. Such a labeling is called an antimagic labeling. Among the graphs known to be antimagic we find paths, cycles, complete graphs, and wheels. It is also easy to see, that K_2 is not antimagic. In fact, Hartsfield and Ringel [24] conjectured that all graphs other than K_2 are antimagic.

The last definition that will be presented in this introduction is the one given by Ringel and Lladó in [34]. A (p,q)-graph G is defined to be edge antimagic if there exists a bijective labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that if v_1u_1 and v_2u_2 are any two different edges of G, then $f(v_1) + f(v_1u_1) + f(u_1) \neq f(v_2) + f(v_2u_2) + f(v_2)$. In their paper they included the result that every connected graph other than K_2 is edge-antimagic. In this section we shall investigate, the analogous concepts to magic, antimagic, vertex magic and edge antimagic labelings in terms of products. Some characterizations will be presented and in order to obtain them, some classical number theoretical results will be used. The following two lemmas are known as Bertrand's Postulate and Ingham's Theorem respectively.

Lemma 7.9. *(Bertrand's Postulate) For any real number $x > 1$, there exists a prime p such that $x < p < 2x$.*

Bertrand's Postulate was first proved by Pafnuty Chebyshev in 1850, and then Ingham [25] extended Bertrand's Postulate in the following way.

Lemma 7.10. *(Ingham's Theorem) For any positive integer k, there is a positive integer n_k such that if $n \geq n_k$, then there are at least k primes between n and $2n$.*

In his first published paper, Paul Erdős [9] gave a short elegant proof of the above two lemmas; see [1] for a brief exposition on the history of Bertrand's Postulate and Erdős' proof.

For our study, the value of n_2, as defined in the previous lemma, will be useful, and Erdős' proof provides the means to compute its exact value. In order to do this, we first consider the prime numbers

$$7, 11, 13, 19, 23, 37, 43, 73, 83, 139, 163, 277, 317,$$
$$547, 631, 1093, 1259, 2179, 2503, 4357 \text{ and } 5003.$$

They have the property that for every integer n with $6 \leq n \leq 4000$, there are at least two of these primes in the interval $(n, 2n)$. Second, from Erdős' proof, it is easy to deduce that for $n \geq 4000$, there are at least $n/(30 \log_2 n)$ primes in the interval $(n, 2n)$; see [1]. Therefore, $n_2 = 6$.

¿From now on, unless otherwise stated, the results and definitions found in the rest of this chapter, were first introduced by Figueroa et al. in [11].

7.2.2 Product magic graphs

A graph G of size q is said to be the product magic if there is a bijective labeling $f : E(G) \to \{1, 2, \ldots, q\}$ such that at each vertex v, the product of the labels on the edges incident with v is constant, and this product is called the valence of f. Such a labeling is said to be a product magic labeling. In this section, and with the aid of Bertrand's Postulate we completely characterize product magic graphs.

Theorem 7.11. *A graph G of size q is product magic if and only if $q \leq 1$.*

Proof.

For the necessity, let G be a graph of size $q \geq 2$, and let $f : E(G) \rightarrow \{1, 2, \ldots, q\}$ be a bijective function. We will show that f is not a product magic labeling of G. By Bertrand's Postulate, there exists a prime number l in the set $\{\lceil q/2 \rceil, \lceil q/2 \rceil + 1, \ldots, q\}$. Let $e = uv$ be the edge of G such that $f(e) = l$, and assume that $w \in V(G) \setminus \{u, v\}$ (w exists since $q \geq 2$). Then the integers

$$\alpha = \prod \{f(a) \mid a \in E(G) \text{ and is incident with } v\}$$

and

$$\beta = \prod \{f(b) \mid b \in E(G) \text{ and is incident with } w\}$$

are different, since $l \mid \alpha$ and $l \nmid \beta$. Therefore f is not a product magic labeling of G.

For the sufficiency, observe that if G has size $q \leq 1$, then G is isomorphic to either K_2, $\overline{K_n}$ or $K_2 \cup \overline{K_n}$ for some $n \in \mathbb{N}$, and clearly these graphs are product magic. □

7.2.3 Product Antimagic Graphs

A graph G of size q is said to be product magic if there exists a bijective function $f : E(G) \rightarrow \{1, 2, \ldots, q\}$, such that all products of the labels on the edges incident with the same vertex are distinct. Such a labeling is called a product antimagic labeling. Next, we present some results about product antimagic graphs.

It is certainly true that, for a 1-regular graph of size $q \geq 1$, any labeling of the edges using the elements of the set $\{1, 2, \ldots, q\}$ will result in the same product on every pair of adjacent vertices. Therefore, we have the following theorem.

Theorem 7.12. *Any 1-regular graph is not product antimagic.*

Hartsfield and Ringel [23] stated that every path P_n and cycle C_n is antimagic for all integers $n \geq 3$. Analogously, we have the following two results

Theorem 7.13. *Every path P_n of order $n (\geq 4)$ is product antimagic.*

Proof.

Suppose first that G is isomorphic to the path P_n of order $n(\geq 4)$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_i = v_i v_{i+1} \mid 1 \leq i \leq n-1\}$. Then the next two cases will be considered.
Case 1: Assume that n is even and define the labeling

$$f : E(G) \to \{1, 2, \ldots, n\}$$

so that $f(e_i) = i$ for every integer i with $1 \leq i \leq n-1$.

We next show that f is a product antimagic labeling of g. Let

$$\pi_i = \begin{cases} 1, & \text{if } i = 1; \\ n, & \text{if } i = n; \\ i(i-1) & \text{if } 1 < i < n-1. \end{cases}$$

Now, we observe the facts that $\pi_1 \neq \pi_n$, π_n is odd, π_i is even for $1 < i < n$, and $\pi_i < \pi_j$ for $1 \leq i < j \leq n-1$. Thus, from the preceding observations, we conclude that $\pi_i \neq \pi_j$ for every possible integers i and j with $1 \leq i < j \leq n$.
Case 2: Assume that n is odd and define the labeling

$$f : E(G) \to \{1, 2, \ldots, n\}$$

so that

$$f(e) = \begin{cases} n-1, & \text{if } e = e_{n-2}; \\ n-2, & \text{if } e = e_{n-1}; \\ i & \text{if } e = e_i \text{ for } 1 \leq i \leq n-3. \end{cases}$$

We next show that f is a product antimagic labeling of G.

Let π_i denote the product of the labels on the edges incident with each vertex v_i. Then

$$\pi_i = \begin{cases} 1, & \text{if } i = 1; \\ n-2, & \text{if } i = n; \\ (n-2)(n-1), & \text{if } i = n-1; \\ i(i-1), & \text{if } 1 < i < n-1. \end{cases}$$

Thus, $\pi_1 \neq \pi_n, \pi_1 \neq \pi_{n-1}, \pi_n \neq \pi_{n-1}, \pi_n$ is odd, π_i is even for $1 < i < n-1$, and $\pi_i < \pi_j$ if $1 < i < j < n$. Hence, the preceding observations lead to conclude that $\pi_i \neq \pi_j$ for every possible pair of integers i and j with $1 \leq i < j \leq n$.

Therefore, the proof of this theorem is completed. □

We point out that although P_3 is antimagic, it is not product antimagic.

Theorem 7.14. *Every cycle C_n is product antimagic.*

Proof.

Let $E(C_n) = \{e_1, e_2, \ldots, e_n\}$ and suppose then that e_i is incident with e_{i+1}, where the subscripts are taken modulo n. We now consider the labeling $f : E(C_n) \to \{1, 2, \ldots, n\}$ so that

$$f(e_i) = \begin{cases} 2i-1, & \text{if } 1 \le i \le \lceil \frac{n}{2} \rceil; \\ 2n-2i+2, & \text{if } \lceil \frac{n+2}{2} \rceil \le i \le n. \end{cases}$$

In order to complete the proof, we need to show that f is an antimagic labeling of C_n. First, notice that $f(e_i) < f(e_{i+1})$ and $f(e_i)$ is odd for each integer i with $1 \le i \le \lceil n/2 \rceil$. Then $f(e_i) \cdot f(e_{i+1}) < f(e_j) \cdot f(e_{j+1})$ for every possible pair of integers i and j with $1 \le i < j \le \lceil n/2 \rceil - 1$. Consequently, $f(e_i) \cdot f(e_{i+1})$ is odd for every integer i with $1 \le i \le \lceil n/2 \rceil - 1$.

On the other hand, $f(e_i) > f(e_{i+1})$ and $f(e_i)$ is even for every integer i such that $\lceil (n+2)/2 \rceil \le i \le n$. Hence $f(e_i) \cdot f(e_{i+1}) > f(e_j) \cdot f(e_{j+1})$ for every possible pair of integers i and j such that $\lceil (n+2)/2 \rceil \le i < j \le n-1$, which implies that $f(e_i) \cdot f(e_{i+1})$ is even for every integer i with $\lceil (n+2)/2 \rceil \le i \le n-1$. Thus, all product of labels adjacent with different vertices are distinct and therefore we conclude that f is a product antimagic labeling of C_n. □

The next corollary is a generalized version of the preceding theorem.

Corollary 7.15. *Every 2-regular graph is product antimagic.*

Proof.

Suppose first that G is a 2-regular graph having $k(\ge 2)$ components. Then G is isomorphic to $\bigcup_{i=1}^{k} C_{m_i}$, where C_{m_i} is a cycle of order m_i for every integer i with $1 \le i \le k$. Let $E(C_{m_i}) = \{e_{i,r} \mid r = 1, 2, \ldots, m_i\}$ and assume that $e_{i,r}$ is incident with $e_{i,s}$ if and only if $r \equiv s+1 \pmod{m_i}$. We further let $f_1, f_2, \ldots, f_k$ be the labeling of $C_{m_1}, C_{m_2}, \ldots C_{m_k}$ as described in the proof of Theorem 7.14, respectively.

We next define a labeling $f : E(G) \to \{1, 2, \ldots, m_1 + m_2 + \ldots + m_k\}$ in the following manner. If $e_i \in E(C_{m_j})$, then $f(e_i) = f_i(e_i) + \sum_{k=1}^{i} m_k$.

Notice that any product of the labels on the edges incident with any vertex of C_{m_i} is greater than the product of the labels on the edges incident with any vertex of C_{m_j} if and only if $i > j$.

Finally, we mention that a similar argument to the one used in the proof of Theorem 7.14 can be developed to verify that all resulting products associated with each vertex in the same component are different from each other. □

A way of constructing new product antimagic graphs using the join operation from known product antimagic graphs is presented in the next theorem.

Theorem 7.16. *If G is a product antimagic graph, then $G + K_1$ is product antimagic.*

Proof.

Suppose first that G is a product antimagic (p, q)-graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$ and product antimagic labeling f. Then, without loss of generality, the resulting products at vertices can be sorted in ascending order, say $\pi_1 < \pi_2 < \ldots < \pi_p$, where π_i is the resulting product at each vertex v_i. In addition, we consider the graph $G + K_1$ with $V(G + K_1) = V(G) \cup \{u\}$ and $E(G + K_1) = E(G) \cup \{uv_i \mid i = 1, 2, \ldots, p\}$. We next define the labeling $g : E(G + K_1) \to \{1, 2, \ldots, p + q\}$ so that

$$g(e) = \begin{cases} f(e), & \text{if } e \in E(G); \\ i + q, & \text{if } e = uv_i \text{ and } 1 \leq i \leq p. \end{cases}$$

Then we observe that the resulting product at each vertex v_i of $G + K_1$ is given by $(i + q) \cdot \pi_i$. Further, note that $(i + q) \cdot \pi_i < (j + q) \cdot \pi_j$ for every possible pair of integers i and j with $1 \leq i < j \leq p$. It remains to be shown that $\rho > (p + q) \cdot \pi_p$, where $\rho = \prod_{i=1}^{p}(i + q)$. Now, in order to finish the proof, notice that π_p is the product of at most $p - 1$ positive integers less than or equal to q since $0 \leq \deg_G v_p \leq p - 1$. Hence, $\pi_p < \prod_{i=1}^{p-1}(i + q)$, which produces the desired result. □

It was stated by Hartsfield and Ringel [23] that wheels and complete graphs are antimagic. The analogous results for product antimagic graphs will be presented next, as immediate consequences of Theorems 7.14 and 7.16.

Corollary 7.17. *Every wheel W_n of order n is product antimagic.*

Corollary 7.18. *A complete graph K_n is product antimagic if and only if $n \geq 3$.*

Another way of constructing product antimagic graphs from known product antimagic graphs is presented in the next theorem.

Theorem 7.19. *Let G be a product antimagic (p, q)-graph, then the graph $G \circ \overline{K}_n$ is product antimagic for every positive integer n.*

Proof.

Suppose that G is a product antimagic (p, q)-graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $E(G) = \{e_1, e_2, \ldots e_q\}$. Also, let the function $f : E(G) \to \{1, 2, \ldots, q\}$ be a product antimagic labeling of G and π_i denote the product of the labels on the edges incident with the vertex v_i for each i. Since f is a product antimagic labeling of G, we assume, without loss of generality, that $\pi_i < \pi_j$ for every possible pair of integers i and j with $1 \leq i < j \leq q$. Let $H \cong G \circ \overline{K}_n$, then H can be described with

$$V(H) = V(G) \cup \{v_{i,j} \mid 1 \le i \le p \text{ and } 1 \le j \le n\}$$

and

$$E(H) = E(G) \cup \{v_i v_{i,j} \mid 1 \le i \le p \text{ and } 1 \le j \le n\}.$$

Further we define the labeling $g : E(H) \to \{1, 2, \ldots, q + np\}$ so that

$$g(e) = \begin{cases} f(e), & \text{if } e \in E(G); \\ q + n(i-1) + j, & \text{if e} \in E(H) \cap \overline{E(G)}. \end{cases}$$

We observe that g is a product antimagic labeling of $G \circ \overline{K}_n$, completing the proof. □

Finally we will conclude this section with two conjectures.

Conjecture 7.20. *A connected graph G of size q is product antimagic if and only if $q \ge 3$.*

Conjecture 7.21. *A graph G of size q is product antimagic if and only if $q \ge 3$.*

7.2.4 Product Edge-Magic Graphs

For a (p, q)-graph G, a bijective labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ is said to be a product edge-magic labeling of G if $f(u) \cdot f(uv) \cdot f(v)$ is a constant k, independent of the choice of any edge uv of G. If such a labeling exists, then k is called the valence of the labeling and G is said to be a product edge-magic graph. The objective in this section is to support the intuitive feeling that almost all graphs are not product edge-magic. However, our first result shows that there are indeed some graphs that are product edge-magic.

Theorem 7.22. *The graph $K_2 \cup \overline{K}_n$ is product edge-magic for every positive integer n. Moreover, there exists exactly $(n+3)!$ product edge-magic labelings of $K_2 \cup \overline{K}_n$.*

Proof.

Let $G \cong K_2 \cup \overline{K}_n$. Then the set of labels is $\{1, 2, \ldots, n + 3\}$. Since there is only one product to consider, it follows that any arbitrary labeling of G with all the elements of the set $\{1, 2, \ldots, n + 3\}$ produces a product edge-magic labeling of G.

Finally notice that once the elements of G are fixed, there exists exactly $(n + 3)!$ different product edge-magic labelings of G. □

Our next objective is to prove a characterization of product edge-magic graphs without isolated vertices. However in order to do this, we will first state and prove the following two lemmas.

Lemma 7.23. *Let G be a graph without isolated vertices that is not isomorphic to $K_{1,n}$ for any $n \in \mathbb{N}$, then G is not product magic.*

Proof.
Let G be any (p,q)-graph not isomorphic to $K_{1,n}$ for any $n \in \mathbb{N}$, and let $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ be a bijective function. We will show that f is not a product edge-magic labeling of G. To do so, we recall that, by Bertrand's Postulate, there exists at least a prime number l in the set

$$\left\{ \left\lceil \frac{p+q}{2} \right\rceil, \left\lceil \frac{p+q}{2} \right\rceil + 1, \ldots, p+q \right\}.$$

Next, we will consider two cases.
Case 1: Assume that there exists an edge $e = uv$ in $E(G)$ such that $f(e) = l$. Then if $e' = u'v'$ is an edge of $E(G)$ different from e, (such edge exists since G has no isolated vertices and is not isomorphic to $K_{1,1}$), we have that $f(u) \cdot f(e) \cdot f(v) \neq f(u') \cdot f(e') \cdot f(v')$ since $l \mid f(u) \cdot f(e) \cdot f(v)$ but $l \nmid f(u') \cdot f(e') \cdot f(v')$. Therefore f is not a product edge-magic labeling of G.
Case 2: Assume that there exists a vertex v in $V(G)$ such that $f(v) = l$ and consider the set $E_v = \{e \in E(G) \mid e \text{ is incident with } v\}$. Of course, since G does not contain isolated vertices,we have that $E_v \neq \emptyset$. Also since $G \ncong K_{1,n}$ for any n and does not contain isolated vertices, we know that the set $E(G) \setminus E_v \neq \emptyset$. Thus, let $e = uv$ and $e' = u'v'$ belong to E_v and $E(G) \setminus E_v$ respectively. Then, we have that $f(u) \cdot f(e) \cdot f(v) \neq f(u') \cdot f(e') \cdot f(v')$ since $l \mid f(u) \cdot f(e) \cdot f(v)$ but $l \nmid f(u') \cdot f(e') \cdot f(v')$. Therefore f is not a product edge-magic labeling of G. □

Lemma 7.24. *The graph $K_{1,p}$ is not product edge-magic for any integer $p \geq 2$.*

Proof. First of all, an exhaustive computer search for the cases where $p = 2, 3, 4, 5$ shows that the statement holds for these small values of p. Next, let $G \cong K_{1,p}$ where $p \geq 6$ and assume that $f : V(G) \cup E(G) \to \{1, 2, \ldots, 2p+1\}$ is a bijective function. Then, by the comments following Lemma 7.10 we know that $n_2 = 6$, so there exist at least two primes l_1 and l_2 in the set $\{p+1, p+2, \ldots, 2p+1\}$. Thus if u is the vertex of degree p in $V(G)$, necessarily there exists an edge $e = uv$ in $E(G)$ such that either $f(e) = l_1$ or l_2 or $f(v) = l_1$ or l_2. Without loss of generality assume that $f(e) = l_1$. Then if $e' = u'v' \in E(G) \setminus \{e\}$, we have that $f(u) \cdot f(e) \cdot f(v) \neq$

$f(u') \cdot f(e') \cdot f(v')$ since $l_1 \mid f(u) \cdot f(e) \cdot f(v)$ but $l_1 \nmid f(u') \cdot f(e') \cdot f(v')$. Therefore, f is not a product edge-magic labeling of G. □

Therefore, putting together the previous two lemmas, we obtain the following characterization of the product edge-magic graphs without isolated vertices.

Theorem 7.25. *A graph G of size q without isolated vertices is product edge magic if and only if $q \leq 1$.*

Kotzig and Rosa [27] proved that a 1-regular graph G is magic if and only if the size of G is even. Furthermore, they proposed the question of finding necessary and sufficient conditions for an r-regular graph to be magic when $r = 2$, 3 and 4. The next result constitutes an answer for the analogous question in terms of product edge-magic graphs.

Corollary 7.26. *An r-regular graph G is product edge-magic if and only if $G \cong K_2$.*

In [28], Kotzig and Rosa also mentioned that every graph can be embedded into an edge magic graph. The next theorem is the analogous result for product edge-magic graphs. Notice that in light of the previous theorem, this embedding can only be obtained by adding isolated vertices.

Theorem 7.27. *For every graph G, there exists a product edge-magic graph H containing G as an induced subgraph.*

Proof.

For every graph G, there exists a magic (p, q)-graph H_1 containing G as a subgraph (as noted by Kotzig and Rosa [28]). Suppose now that f is an magic labeling of H_1 and then we define the labeling $g : E(H_1) \cup V(H_1) \to \{1, 2, \ldots, p + q\}$ so that $g(x) = 2^{f(x)}$ for every element x in $E(H_1) \cup V(H_1)$. Since f is an magic labeling of H_1, it follows that $g(u_1) \cdot g(v_1) \cdot g(u_1v_1) = g(u_2) \cdot g(v_2) \cdot g(u_2v_2)$ for any pair of distinct edges u_1v_1 and u_2v_2 of H_1.

In order to finish the proof, we consider the graph H that is isomorphic to $H_1 \cup (2^{p+q} - p - q)K_1$ for which the product edge-magic labeling $h : V(H) \cup E(H) \to \{1, 2, \ldots, 2^{p+q} - p - q\}$ is constructed in the following manner. If x is any element in $E(H_1) \cup V(H_1)$, then $h(x) = g(x)$; otherwise, assign any remaining integer from the set $\{1, 2, \ldots, 2^{p+q} - p - q\}$. Certainly, h is a product edge-magic labeling of H and therefore the proof is completed. □

The previous theorem suggests to define the concept of product edge-magic deficiency as follows. The product edge-magic deficiency of a graph G, $\mu_p(G)$, is defined to be the minimum number of isolated vertices that we have to union G with so that we get a product edge-magic graph.

7.2.5 Product Edge-Antimagic Graphs

A (p, q)-graph G, we define G is product edge-antimagic if there exists a labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ satisfying the condition that $f(u_1) \cdot f(v_1) \cdot f(u_1v_1) \neq f(u_2) \cdot f(v_2) \cdot f(u_2v_2)$ for any pair of distinct edges u_1v_1, u_2v_2 of G. Such a labeling is called a *product edge-antimagic labeling.*

Ringel and Lladó in [34] stated that each graph is edge-antimagic. Our next theorem is the analogous result for product edge-antimagic graphs.

Theorem 7.28. *Every graph other than K_2 and $K_2 \cup \overline{K}_n$ is product edge-antimagic for each positive integer n.*

Proof.

We first note that the graphs K_2 and $K_2 \cup \overline{K}_n$ are trivially not product edge-antimagic for every positive integer n.

We next suppose that G is a (p, q)-graph other than K_2 and $K_2 \cup \overline{K}_n$ for some positive integer n and let $f : V(G) \to \{1, 2, \ldots, p\}$ be any vertex labeling. Then let

$$\{\pi_i \mid i = 1, 2, \ldots, q\} = \{f(u) \cdot f(v) \mid uv \in E(G)\}$$

so that $\pi_i \leq \pi_j$ if $1 \leq i < j \leq q$. Now we define $g : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ to be the labeling such that $g(u) = f(u)$ for every $u \in V(G)$ and $g(e) = p + i$ if $e = uv \in E(G)$ and $f(u) \cdot f(v) = \pi_i$.

Finally we notice that g is a product edge-antimagic labeling of G and therefore the desired result follows. □