Exact Consensus Controllability of Multi-agent Linear Systems

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Abstract: In this paper we study the exact controllability of multi-agent linear systems, in which all agents have an identical linear dynamic mode that can be in any order.

Key–Words: Multi-agent systems, consensus, controllability, exact consensus controllability.

1 Introduction

In the last years, the study of dynamic control multi-agents systems have attracted considerable interest, because they arise in a great number of engineering situations as for example in distributed control and coordination of networks consisting of multiple autonomous agents. There are many publications as for example ([4], [10], [12], [14]). It is due to the multi-agents appear in different fields as for example in consensus problem of communication networks ([10]), or formation control of mobile robots ([2]).

The consensus problem has been studied under different points of view, for example Jinhuan Wang, Daizhan Cheng and Xiaoming Hu in [12], analyze the case of multiagent systems in which all agents have an identical stable linear dynamics system, M.I. García-Planas in [4], generalize this result to the case where the dynamic of the agents are controllable.

Controllability is a fundamental topic in dynamic systems and it is studied under different approaches (see [1],[3],[7], for example). Given a linear system \( \dot{x} = Ax + Bu \), there are many possible control matrices \( B \) making the system \( \dot{x} = Ax + Bu \) controllable. The goal is to find the set of all possible matrices \( B \), having the minimum number of columns corresponding to the minimum number \( n_D(A) \) of independent controllers required to control the whole network. This minimum number is called exact controllability, that in a more formal manner is defined as follows.

**Definition 1** Let \( A \) be a matrix. The exact controllability \( n_D(A) \) is the minimum of the rank of all possible matrices \( B \) making the system \( \dot{x} = Ax + Bu \) controllable.

\[
 n_D(A) = \min \{ \text{rank} B, \forall B \in M_{n \times |1 \leq i \leq n} | (A, B) \text{ controllable} \}
\]

In this paper, we investigate the exact controllability of a class of multiagent systems consisting of \( k \) agents with dynamics

\[
 \dot{x}^1 = Ax^1 + Bu^1 \\
\vdots \\
\dot{x}^k = Ax^k + Bu^k
\]

where \( A \in M_{n \times (\mathbb{C})} \), and \( B \) an unknown matrix having \( n \) rows and an indeterminate number \( 1 \leq \ell \leq n \) of columns.

For this study, we need to introduce some basic concepts on Graph theory and matritial algebra.

We consider a graph \( G = (\mathcal{V}, \mathcal{E}) \) of order \( k \) with the set of vertices \( \mathcal{V} = \{ 1, \ldots, k \} \) and the set of edges \( \mathcal{E} = \{ (i,j) | i,j \in \mathcal{V} \} \subset \mathcal{V} \times \mathcal{V} \).

Given an edge \((i,j)\) \( i \) is called the parent node and \( j \) is called the child node and \( j \) is in the neighbor of \( i \), concretely we define the neighbor of \( i \) and we denote it by \( N_i \) to the set \( N_i = \{ j \in \mathcal{V} | (i,j) \in \mathcal{E} \} \).

The graph is called undirected if verifies that \((i,j) \in \mathcal{E} \) if and only if \((j,i) \in \mathcal{E} \). The graph is called connected if there exists a path between any two vertices, otherwise is called disconnected.

Associated to the graph we consider a matrix \( G = (g_{ij}) \) called (unweighted) adjacency matrix defined as follows \( g_{ii} = 0 \), \( g_{ij} = 1 \) if \((i,j) \in \mathcal{E} \), and \( g_{ij} = 0 \) otherwise.

In a more general case we can consider that a weighted adjacency matrix is \( G = (g_{ij}) \) with \( g_{ii} = 0 \), \( g_{ij} > 0 \) if \((i,j) \in \mathcal{E} \), and \( g_{ij} = 0 \) otherwise.

The Laplacian matrix of the graph is

\[
 L = (l_{ij}) = \begin{cases} 
 |N_i| & \text{if } i = j \\
 -1 & \text{if } j \in N_i \\
 0 & \text{otherwise}
\end{cases}
\]

**Remark 2** i) If the graph is undirected then the...
matrix $\mathcal{L}$ is symmetric, then there exist an orthogonal matrix $P$ such that $P \mathcal{L} P^T = D$.

ii) If the graph is undirected then 0 is an eigenvalue of $\mathcal{L}$ and $1_k = (1, \ldots, 1)^T$ is the associated eigenvector.

iii) If the graph is undirected and connected the eigenvalue 0 is simple.

For more details about graph theory see (D. West, 2007).

With respect Kronecker product, remember that $A = (a_{ij}) \in M_{m \times n}(\mathbb{C})$ and $B = (b_{ij}) \in M_{p \times q}(\mathbb{C})$ the Kronecker product is defined as follows.

Definition 3 Let $A = (a_{ij}) \in M_{m \times n}(\mathbb{C})$ and $B \in M_{p \times q}(\mathbb{C})$ be two matrices, the Kronecker product of $A$ and $B$, write $A \otimes B$, is the matrix

$$A \otimes B = \left( \begin{array}{cccc} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{array} \right) \in M_{mp \times nq}(\mathbb{C})$$

Among the properties that verifies the product of Kronecker we will make use of the following

1) $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$

2) $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$

3) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

4) If $A \in \text{GL}(n; \mathbb{C})$ and $B \in \text{GL}(p; \mathbb{C})$, then $A \otimes B \in \text{GL}(np; \mathbb{C})$ and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

5) If the products $AC$ and $BD$ are possible, then $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

See [9] for more information and properties.

Given a square matrix $A \in M_n(\mathbb{C})$, it can be reduced to a canonical reduced form (Jordan form):

$$J = \left( \begin{array}{cccc} J(\lambda_1) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & J(\lambda_n) \end{array} \right),$$

where $J(\lambda_i) = \left( \begin{array}{cccc} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{array} \right)$. (1)

See [5] for more information and properties.

2 Consensus

The consensus problem can be introduced as a collection of processes such that each process starts with an initial value, where each one is supposed to output the same value and there is a validity condition that relates outputs to inputs. It is a canonical problem that appears in the coordination of multi-agent systems. The objective is that Given initial values (scalar or vector) of agents, establish conditions under which through local interactions and computations, agents asymptotically agree upon a common value, that is to say: to reach a consensus.

The dynamic of each agent defining the system considered, is given by the following manner.

$$\dot{x}_i^1 = Ax_i^1 + Bu_i^1$$
$$\dot{x}_i^k = Ax_i^k + Bu_i^k$$

$x_i^1 \in \mathbb{R}^n$, $u_i^1 \in \mathbb{R}^\ell$, $1 \leq i \leq k$. Where matrices $A \in M_n(\mathbb{R})$ and $B \in M_{nx\ell}(\mathbb{R})$, $1 \leq \ell \leq n$.

The communication topology among agents is defined by means the undirected graph $\mathcal{G}$ with

i) Vertex set: $\mathcal{V} = \{1, \ldots, k\}$

ii) Edge set: $\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$.

an in a more specific form, we have the following definition.

Definition 4 Consider the system 2. We say that the consensus is achieved using local information if there exists a state feedback

$$u_i^j = K_i \sum_{j \in \mathcal{N}_i} (x_i^j - x_i^j), \quad 1 \leq i \leq k$$

such that

$$\lim_{t \to \infty} \|x_i^j - x_i^j\| = 0, \quad 1 \leq i, j \leq k.$$
3 Exact Consensus Controllability

We are interested in study the exact controllability of the obtained system 3. In our particular setup

**Definition 5** Let $A$ be a matrix. The exact controllability $n_D(I_k \otimes A)$ is the minimum of the rank of all possible matrices $B$ making the system 3 controllable.

$$n_D(I_k \otimes A) = \min \{ \text{rank}(B), \forall B \in M_{n \times i} : 1 \leq i \leq n \mid (I_k \otimes A, L \otimes B) \text{ controllable} \}.$$

The controllability condition depends directly on the structure of the matrix $L$.

**Proposition 7** Let $J$ be the Jordan reduced of the matrix $L$ and $P$ such that $L = P^{-1}JP$. Then, the system 3 is controllable if and only if

$$\text{rank} \left( sI_{kn} - (I_k \otimes A) \right) J \otimes B = kn$$

**Proof.** Suppose that there exist $S$ such that $P^{-1}JP = L$ and

$$\begin{align*}
\text{rank} \left( sI_{kn} - (I_k \otimes A) \right) J \otimes B &= \\
\text{rank} \left( P^{-1}I_n \right) (sI_{kn} - (I_k \otimes A)) J \otimes B &= \\
\text{rank} \left( sI_{kn} - (I_k \otimes A) \right) J \otimes B &=
\end{align*}$$

**Corollary 8** Suppose that the matrix $L$ can be reduced to the Jordan form (1), with non-zero eigenvalues $\lambda_1, \ldots, \lambda_r$. Then, the system 3 is controllable if and only if each agent is controllable.

**Proof.** Let $\lambda_i \neq 0$, $i = 1, \ldots, r$ be the eigenvalues of $L$.

$$\begin{align*}
\text{rank} \left( s(I_{k_{i,j}} \otimes I_n) - (I_k \otimes A) \right) J_j \lambda_i \otimes B &= \\
\text{rank} \left( sI_{kn} - A \right) \lambda_i B &= \\
\text{rank} \left( sI_{kn} - A \right) B &=
\end{align*}$$

$k \cdot \text{rank} \left( sI_{kn} - A \right)$

with $k_1 + \ldots + k_r = k$, $k_{i_1} + \ldots k_{i_n} = k_i$.

**Corollary 9** A necessary condition for controllability of the system 3 is that the matrix $L$ has full rank.

**Example** We consider 3 identical agents with the following dynamics of each agent

$$\begin{align*}
\dot{x}^1 &= Ax^1 + Bu^1 \\
\dot{x}^2 &= Ax^2 + Bu^2 \\
\dot{x}^3 &= Ax^3 + Bu^3
\end{align*}$$

with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B \in M_{2 \times \ell}(C)$, $1 \leq 2$.

The communication topology is defined by the undirected graph $(V, E)$:

$V = \{1, 2, 3\}$

$E = \{(i, j) \mid i, j \in V\} = \{(1, 2), (1, 3)\} \subset V \times V$

and the adjacency matrix:

$G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The neighbors of the parent nodes are $N_1 = \{2, 3\}$, $N_2 = \{1\}$, $N_3 = \{1\}$.

The Laplacian matrix of the graph is

$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

with eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$.

$$\begin{align*}
\text{rank} \left( sI_{6} - (I \otimes A) \right) L \otimes B &= \\
\text{rank} \left( sI_{k_{i,j}} - (I_k \otimes A) \right) J_j \lambda_i B &= \\
\text{rank} \left( sI_{kn} - A \right) B &=
\end{align*}$$

In fact, for all matrix $B \in M_{2 \times \ell}(C)$ for all $\ell \geq 0$

$$\begin{align*}
\text{rank} \left( sI_{6} - (I \otimes A) \right) L \otimes B &= \\
6 \text{ for all } s \neq 0 \\
5 \text{ for } s = 0
\end{align*}$$

If the matrix $L$ has full rank, then the number of columns for exact controllability of matrix $I_k \otimes A$ depends on the multiplicity of the eigenvalues of the matrix $A$ and we have the following result.
Proposition 10 Let $L$ be the Laplacian matrix of a graph having full rank. Then, the exact controllability $n_D(I_k \otimes A)$ for the system $\dot{x} = (I_k \otimes A)x + (L \otimes B)\bar{u}$ coincides with the exact controllability $n_D(A)$ for the system $\dot{x} = Ax + Bu$.

Example We consider 3 identical agents with the following dynamics of each agent

\[
\begin{align*}
\dot{x}_1^1 &= Ax_1^1 + Bu_1^1 \\
\dot{x}_2^2 &= Ax_2^2 + Bu_2^2 \\
\dot{x}_3^3 &= Ax_3^3 + Bu_3^3
\end{align*}
\]

with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B \in M_{2 \times l}(\mathbb{C})$, $1 \leq 2$.

The communication topology is defined by the undirected graph $(V, E)$:

\[
\begin{align*}
V &= \{1, 2, 3\} \\
E &= \{(i, j) \mid i, j \in V\} = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 1)\} \subset V \times V
\end{align*}
\]

and the adjacency matrix:

\[
G = \begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]

The neighbors of the parent nodes are $N_1 = \{1, 2\}, N_2 = \{1, 3\}, N_3 = \{1\}$.

The Laplacian matrix of the graph is

\[
L = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
-1 & 0 & 1
\end{pmatrix}
\]

with eigenvalues $\lambda_1 = 0.3820$, $\lambda_2 = 2$, $\lambda_3 = 2.6180$.

\[
\begin{pmatrix}
s & -1 & 0 & 0 & 0 & 0 & 2a & -a & 0 \\
0 & s & 0 & 0 & 0 & 0 & 2b & -b & 0 \\
0 & 0 & s & -1 & 0 & 0 & -a & 2a & -a \\
0 & 0 & 0 & s & 0 & 0 & -b & 2b & -b \\
0 & 0 & 0 & 0 & s & -1 & -a & 0 & a \\
0 & 0 & 0 & 0 & 0 & s & -b & 0 & b
\end{pmatrix}
\]

rank 6 for all $s$ and $b \neq 0$.

Obviously the system $\dot{x} = Ax + Bu$ with $B = \begin{pmatrix} a \\ b \end{pmatrix}$ and $b \neq 0$.

4 Conclusions

In this paper, the exact controllability for multi-agent systems where all agents have an identical linear dynamic mode are analyzed.

References:
