An educational example to the maximum power transfer objective in electric circuits using a PD-controlled DC– driver

Acho L.∗

(c-mail: leonardo.acho@ upc.edu).

Abstract: The main objective of this paper is to present an academic example of a PD controller applied to teach position control design of a DC-motor to automatically adjust a potentiometer. This adjustment is focused on to solve the maximum power transfer objective in a linear electrical circuit. This design involves the use of the extremum-seeking control. To support our proposal, numerical simulations and mathematical modeling of the main problem statement are programmed.

Keywords: PD controller; DC-motor; extremum seeking algorithm; electrical circuits; maximum power transfer.

1. INTRODUCTION

DC-motors are crucial devices in many low-powerful electromechanical systems (Morales et al. (2014)) due to they are reliable for a wide range of operation conditions (Rigatos (2009)). From the educational control point of view, DC-motors are the typical systems to be controlled by using well known classical control techniques, such as PID controllers (Kelly and Moreno (2001)). Moreover, DC motors are extensively used because their speeds, as well as their torques, can be easily manipulated over a wide grade (see Chapter 7 in Huang (2000)). On the other hand, in electrical engineering, the maximum power transfer theorem enunciates that, to obtain maximum external power from a source with a finite internal resistance, the resistance of the load must equal the resistance of the source as viewed from its output terminals (see, for instance Hayt and Kemmerly (1971)).

In control applications, this maximum power transfer theorem can be used to design, for instance, a tracking system for solar cell units (Teng et al. (2010)), where the equivalent finite internal resistance is time varying; among other applications. Finally, an strategy to search the maximum value of a function consists in to employ the extremum-seeking control theory (see, for example, Leyva et al. (2006), Dochain et al. (2011) and references there in). This strategy was proposed as a program to find the operating set point that maximize (or minimize) an objective function (Dochain et al. (2011); Leyva et al. (2006)), and its popularity comes from the fact that this scheme has been successfully applied in many engineering applications, such as in combustion process control for IC engines and gas furnace, anti-lock braking system control, among others (see, for instance, Dochain et al. (2011); Krstić and Wang (2000) and references there in).

So, the main objective of this paper is to give an educational example to instruct position control design of a DC-motor to automatic tune a potentiometer. This tuning is centered to solve the maximum power transfer objective in a linear electrical circuit. This is realized by using the extremum seeking algorithm. Numerical simulations and mathematical modeling are shown to support our project. Furthermore, our approach may be also applied to the maximum power transfer tracking control problem in solar cell systems.

The structure of this paper is as follows. Section 2 presents the problem statement. Section 3 describes the employed extremum seeking strategy, and Section 4 shows our position control design. Section 5 gives numerical experiment results. Finally, Section 6 states the conclusions.

2. PROBLEM STATEMENT

Consider the electro-mechanical system described in Fig. 1. Then, the main problem statement consists in proposing a control law \( u = u(t) \) such that the maximum power of the source be transferred to the load \( (R_L(\theta)) \). It is assumed that \( V_{Rs}(t), \theta(t), \) and \( \dot{\theta}(t) \) are available; and \( V_s \) is known. It is important to mention that the time variation assumption on \( R_s(t) \) is realistic, for instance, in solar cell applications (see Fig. 1 in Teng et al. (2010)). In addition, in this kind of applications, \( V_s \) is also a time varying element, but without lost of generality, and for simplicity, we are going to assumed it constant. In resume, manipulating \( u(t) \), we manipulate the angular position of the DC-motor \( (\theta) \). And this manipulates the
resistance value of the load \( R_L(\theta) \). Then, we require to automatically adjust it to the needed value for the maximum power transfer from the source to the load \( R_L = R_s \). If the internal resistance of the source changes, the load has to be automatically adjusted to its new required value. In other words, the control objective consists in finding a control law \( u(t) \), in the scheme shown in Fig. 1, such that:

\[
\lim_{t \to \infty} R_L(\theta) = R_s(t). \tag{1}
\]

3. EXTREMUM SEEKING ALGORITHM

The extremum seeking algorithm was proposed as a program to find the operating set point that maximize (or minimize) an objective function. The basic idea is shown in Fig. 2 (see Leyva et al. (2006)); although, other alternative exits using a form of perturbation signals (Krstić and Barb (2000); Dochain et al. (2011)). Next is the main result of the extremum seeking algorithm shown in Fig. 2.

**Theorem 1.** Given a smooth function \( f(x) \) with a global maximum at \( x = a \), which is assumed unknown, and being \( k \) a positive constant, then the algorithm shown in Figure 2 will produce finite-time convergence of \( x(t) \) to \( a \); that is, there exists a \( T_s < \infty \) such that:

\[
\lim_{t \to T_s} x(t) = a. \tag{2}
\]

**Proof.** From Fig. 2, we arrive to:

\[
\dot{x} = k \text{ sgn} \left( \frac{df(x)}{dx} \right), \tag{3}
\]

where \( \text{sgn}(\cdot) \) is the *signum function* \(^1\). By defining

\[ e = x - a, \tag{4} \]

we obtain \(^2\)

\[
\dot{e} = -k \text{ sgn}(e). \tag{5}
\]

Finally, the error dynamic (5) has finite-time convergence to \( e(t) = 0 \) (see, for instance, Chapter 2 in Perruquetti and Barbot (2002)). Proof concluded.

**Remark 1.** Observe that the parameter \( k \) controls the parameter \( T_s \). Increasing the argument \( k \) will decrease \( T_s \).

**Remark 2.** Theorem 1 is in fact a *global* finite-time asymptotic stability theorem to the equilibrium point \( e = 0 \) in (5).

**Remark 3.** Even when Theorem 1 is restricted to \( f(x) \) being a function with a global maximum point, this theorem can be also applied for local maximum points too. The local convergence to a maximum point will depend on the initial condition in (3); and, of course, on the existence of local maximum points too.

For instance, the function \( f(x) = 5x^3 + 2x^2 - 3x \) has a local maximum at \( x = -0.6 \). By realizing the block diagram given in Fig. 2, with \( k = 1 \), we obtain the numerical results shown in Fig. 3 for three different initial conditions. We observe that these trajectories converge, in finite-time, to the local optimal point \( x = -0.6 \).

4. POSITION CONTROL DESIGN

From Section 2, the control objective consists in designing a control law \( u(t) \) (see Fig. 1) such that:

\[
\lim_{t \to \infty} R_L(\theta) = R_s(t), \tag{6}
\]

with the assumption that the only available information for the control realization is the voltage \( V_{Rs}(t) \), the angular motor position \( \theta(t) \), and its angular velocity \( \dot{\theta}(t) \). It is also assumed that the resistance \( R_s(t) \) changes sufficiently-slightly on time. That is, for control design, we will take it as a constant value. And to test the proposed system

\(^2\) Note that \( \frac{df(x)}{dx} \) is positive if \( x - a < 0 \) and it is negative if \( x - a > 0 \); then, \( \text{sgn} \left( \frac{df(x)}{dx} \right) = -\text{sgn}(e) \).

---

\(^1\) The signum function exhibits the property that \( x \text{ sgn}(x) = |x| \) (see Chapter 1 in Edwards and Spurgeon (1998)).
robustness, in our numerical experiments, we are going to
time-varying it slightly (a typical practice, for instance, on
adaptive control theory). Thus, from Fig. 1, we obtain that
the electrical power on $R_L$ is:
\[ P_{RL}(R_L) = \frac{R_L V_s^2}{(R_s + R_L)^2}. \]  
(7)

Hence, we have:
\[ \frac{\partial P_{RL}}{\partial R_L} = \frac{V_s^2 (R_s - R_L)}{(R_s + R_L)^3}. \]  
(8)

Then, by using (3) with $f(\cdot) = P_{RL}(R_L)$, we arrive to:
\[ \dot{R}_L = k \sgn \left( \frac{\partial P_{RL}}{\partial R_L} \right) \]
\[ = k \sgn \left( \frac{V_s^2 (R_s - R_L)}{(R_s + R_L)^3} \right) \]
\[ = k \sgn (R_s - R_L). \]  
(9)

Now, and going back to the Fig. 1, we calculate:
\[ R_s = \frac{V_{Rs} R_L}{V_s - V_{Rs}}. \]  
(10)

Taking into account that the resistance $R_L$ is assumed
linearly dependent on the angular position of the DC-
motor $\theta$; that is, $R_L := R_L(\theta) = \theta$  
3, we find that (10) yields,
\[ R_s = \frac{V_{Rs} \theta}{V_s - V_{Rs}}. \]  
(11)

When (11) is inserted into (9), we produce:
\[ \dot{R}_L = k \sgn \left( \frac{V_{Rs} \theta}{V_s - V_{Rs}} - \theta \right) \]
\[ = k \sgn \left( \frac{\theta (2V_{Rs} - V_s)}{V_s - V_{Rs}} \right). \]  
(12)

Due to the fact that $V_s > V_{Rs}$, the over equation reduces to:
\[ \dot{R}_L = k \sgn (\theta (2V_{Rs} - V_s)). \]  
(13)

The above equation really means that:
\[ \dot{\theta} = k \sgn (\theta (2V_{Rs} - V_s)). \]  
(14)

Moreover, because $0 < R_L < \infty$, then, equation (14),
reduces to:
\[ \dot{\theta} = k \sgn (2V_{Rs} - V_s). \]  
(15)

System (15) has to be read as the desired behaviour on
the angular position of the DC-Motor. In this way, we are
going to rename it as:
\[ \dot{\theta}_d = k \sgn (2V_{Rs} - V_s), \]  
(16)

where $\theta_d(t)$ is the desired trajectory to be followed by
the angular position of the motor to fulfill our control objective.
See Fig. 4.

3 A unit gain proportionality constant is assumed.

4 For this example, these values were freely selected but positives.
ACKNOWLEDGEMENTS

The author is really grateful to the fruitful reviewers’ comments: Thank you.

REFERENCES