Lower bounds on the maximum number of non-crossing acyclic graphs

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Abstract

This paper is a contribution to the problem of counting geometric graphs on point sets. More concretely, we look at the maximum numbers of non-crossing spanning trees and forests. We show that the so-called double chain point configuration of N points has $\Omega(12.52^N)$ non-crossing spanning trees and $\Omega(13.61^N)$ non-crossing forests. This improves the previous lower bounds on the maximum number of non-crossing spanning trees and of non-crossing forests among all sets of N points in general position given by Dumitrescu, Schulz, Sheffer and Tóth (SIAM J. Discr. Math 27(2), 2013). Our analysis relies on the tools of analytic combinatorics, which enable us to count certain families of forests on points in convex position, and to estimate their average number of components. A new upper bound of $O(22.12^N)$ for the number of non-crossing spanning trees of the double chain is also obtained.

1 Introduction

A geometric graph on a point set S (throughout, S has no three collinear points) is a graph with vertex set S and whose edges are straight-line segments with endpoints in S. A geometric graph is called *non-crossing* (nc- for short) if no two edges intersect except at common endpoints. Counting nc-geometric graphs is a prominent problem in combinatorial geometry, since Ajtai et al. [3] showed in 1982 that there exists a constant $c > 0$ such that the number of nc-geometric graphs on sets S of N points is bounded from above by $O(c^N)$. Here we focus on non-crossing acyclic graphs, that is, spanning trees and forests. Hoffmann et al. [9] proved that no set S of N points has more than $O(141.07^N)$ nc-spanning trees. The maximum number of nc-spanning trees (among all sets of N points) is very likely much smaller.

The point set with most nc-spanning trees known so far is the so-called *double chain*. The double chain of $N = 2n$ points consists of two sets of n points each, one forming a convex chain and one forming a concave chain. We denote them by the upper and the lower chain. Furthermore, each straight-line defined by two points from the upper chain leaves all the points from the lower chain on the same side, and reversely; see Figure 1. We will refer to the left and to the right side of the double chain, where the point of the double chain with smallest abscissa is on the left and the point with largest abscissa is on the right.

Counting nc-geometric graphs on the double chain was initiated by García et al. [8] who proved that it has $\Theta^*(8^N)$ triangulations[1], $\Omega(9.35^N)$ nc-spanning trees and $\Omega(4.64^N)$ nc-polygonizations, where the latter bound also is the current best lower bound on the maximum number of nc-polygonizations among all sets S of N points. The lower bound for the number of nc-spanning trees of the double chain was subsequently improved to $\Omega(10.42^N)$ [4] and to $\Omega(12.0026^N)$ [5].

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[1]We use the O^*-, Θ^*-, and Ω^*-notation to describe the asymptotic growth of the number of geometric graphs as a function of the number N of points, neglecting polynomial factors. If a class of nc-graphs has $\Theta^*(c^N)$ elements on N points, we say that c is the *growth constant* of the class.

Our main contribution is an improvement of this bound to $\Omega(12.52^N)$, which also improves our previous bound of $\Omega(12.31^N)$ presented in [10]. We also give a new lower bound of $\Omega(13.61^N)$ for the maximum number of nc-forests among all sets S of N points. Along the way we also obtain the asymptotic growth for the number of nc-spanning trees of a point set similar to the double chain, the so-called *single chain*. In this set the upper chain is replaced by a single point; see Figure 2. The single chain was considered by Aichholzer et al. [2] to count the number of pseudo-triangulations. We prove that it has $\Theta^*(9.5816\ldots^n)$ nc-spanning trees.

The paper is organized as follows. In Section 2 we describe a construction on the double chain that produces $\Omega(12.52^N)$ nc-spanning trees, and we include a variation for generating nc-forests. The key counting ingredient for obtaining our lower bounds is being able to enumerate certain classes of forests on points in convex position, and, more crucially, to estimate their average number of components. This is achieved through the methods of analytic combinatorics, with a spirit and techniques similar to those used by Flajolet and Noy [6], and it is the content of Section 3. A curious consequence of these calculations for points in convex position, is that the growth constant of nc-forests is one more than the growth constant of nc-forests that have no isolated vertex. In Section 4 we give an explanation of this fact and show that it also applies to other families of nc-graphs on points in convex position. We also prove that the growth constant of nc-forests of the double chain is at least one more than the growth constant of those forests without isolated vertices, and from this fact we prove our lower bound of $\Omega(13.61^N)$ nc-forests of the double chain. Finally, in Section 5 we provide a new upper bound of $O(22.12^N)$ for the number of nc-spanning trees of the double chain.

2 Lower bounds

In this section we give lower bounds for the numbers of spanning trees and forests of the double chain. For spanning trees, the result is the following.

Theorem 2.1. *The double chain on N points has $\Omega(12.52^N)$ non-crossing spanning trees.*

We next describe a family of trees that gives the desired bound. Our construction depends on some parameters that are determined later.

For any spanning tree of the double chain on $N = 2n$ points, the vertices on the upper and lower chains induce two forests F_U and F_L on a set of n points in convex position; there are also some edges with one endpoint on each chain (the *interior* edges). The first restriction is that we consider trees where only one vertex in each component of F_U is incident to interior edges; this vertex will be called the *mark* of the component and we call F_U a *marked forest*.

Of the several interior edges that are incident to a mark v, let e_v be the rightmost one. The second restriction we impose is that for each component C of F_L that is not incident to an edge e_v, there is a unique edge joining this component to a mark m_C in F_U, and this edge has as endpoint in F_L the leftmost vertex v_C in C. Moreover, m_C is as to the right as possible.

The set of edges $M_1 = \{e_v : v \text{ is a mark}\}$ induces a forest. Observe that this forest is uniquely determined by the leftmost interior edge in each component (assuming the set of marks is known). The set of these edges is denoted M_2; note that M_2 is a matching.

See Figure 1 for an example of a spanning tree of the double chain satisfying the conditions above.

We claim that the following data are enough to construct one such tree with $|M_2| = k$:

a1) A marked nc-forest F_U with at least k components on a set of n points in convex position,

a2) a subset of k of the marks in F_U, always including the leftmost mark,

a3) a nc-forest F_L on a set of n points in convex position, and

a4) a subset M_L of k vertices in F_L.

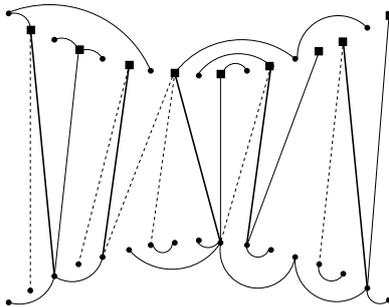Indeed, it suffices to do the following:

Figure 1: A spanning tree of the double chain. Squares indicate the marks of F_U, the bold edges are the edges of M_2 and the thin solid interior edges are the other edges of M_1. We have $k = 5$ edges in the matching M_2. The edges in F_U and F_L are drawn as arcs for clarity.

t1) Match the k marks from a2) with the k vertices in F_L (corresponding to the edges of M_2);

t2) join the other marks in F_U to the leftmost visible vertex in M_L (corresponding to the edges of $M_1 \backslash M_2$);

t3) for each component of F_L that has no vertex in M_L, take its leftmost vertex and join it to the rightmost visible mark;

t4) if the result is not connected, for each component C let l_C be its leftmost mark. Label the connected components as $C_1, \ldots, C_r$ in such a way that the marks $l_{C_1}, \ldots, l_{C_r}$ are ordered from left to right. Now, for $i \geq 2$, join l_{C_i} to the rightmost visible vertex of M_L in C_{i-1}.

Note that in this way we do not generate all the spanning trees satisfying the restrictions imposed above, but that all the trees that are generated are different, which is enough for our purposes.

We next count in how many ways we can choose the forests and subsets in items a1)–a4), which essentially amounts to enumerating forests in points in convex position. As stated in Proposition 2.2 below, the number of nc-forests on n points in convex position where each component has a mark is $\Theta^*(9.5816\ldots^n)$, and for n sufficiently large, at least 40% of them have $0.2237n$ or more components each. However, there is another way of choosing the forest F_U from a1) that gives us more choice. First we select ℓ points out of the n in the upper chain and we mark all of them, and then on the remaining $n - \ell$ points, we choose a marked forest F'_U such that all components in this forest have at least two vertices.

Of the estimates below, (i) is well-known [6] and the other two will be proved in Section 3.

Proposition 2.2. *(i) The number of non-crossing forests on n points in convex position is $bn^{-3/2}\omega_F^n(1 + O(1/n))$, where $\omega_F = 8.2246\ldots$ and b is a constant.*

(ii) The number of marked non-crossing forests on n points in convex position is $cn^{-3/2}\omega_M^n(1 + O(n^{-1/2}))$, where $\omega_M = 9.5816\ldots$ and c is a constant. Moreover, for n large enough at least 40% of these forests have $0.2237n$ or more components.

(iii) The number of marked non-crossing forests on n points in convex position such that no component is an isolated vertex is $dn^{-3/2}\omega_U^n(1 + O(n^{-1/2}))$, where $\omega_U = 8.5816\ldots$ and d is a constant. Moreover, for n large enough at least 40% of these forests have $0.1332n$ or more components.

With this knowledge, setting $k = \alpha n$ and $\ell = \beta n$ (for $\alpha, \beta \in (0, 1)$ to be determined), and ignoring subexponential terms, we get the following lower bound on the number of nc-spanning trees of the double chain:

$$b(\alpha, \beta) = \binom{n}{\beta n} 8.5816^{n - \beta n} \binom{0.1332(n - \beta n) + \beta n}{\alpha n} 8.2246^n \binom{n}{\alpha n}.$$
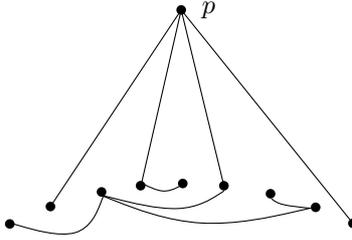
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Figure 2: A spanning tree of the single chain, obtained from a marked forest on the lower chain.

Note that when choosing the $k = \alpha n$ marks from the upper forest F_U we are actually not using all the available marks, but only the ones that come from isolated vertices and the first $0.1332(n - \beta n)$ marks of the marked forest F'_U.

Using the binary entropy function $H(x) = -x \log_2(x) - (1-x) \log_2(1-x)$ and Stirling's formula, we can estimate the binomial coefficient $\binom{\epsilon n}{\delta n}$ as $2^{\epsilon H(\frac{\delta}{\epsilon})n}$, ignoring again subexponential terms.

It thus is enough to maximize

$$
e(\alpha, \beta) = H(\beta) + (1 - \beta) \log_2(8.5816) + (0.1332(1 - \beta) + \beta) H\left(\frac{\alpha}{0.1332(1 - \beta) + \beta}\right)
$$
$$
+ \log_2(8.2246) + H(\alpha).
$$

With the help of a a computer algebra software we find that the values $\alpha = 0.267, \beta = 0.267$ give $\Omega^*(2^{(7.293063.../2)N}) = \Omega^*(12.5232\ldots^N)$ nc-spanning trees on the double chain on N vertices, thus proving Theorem 2.1.

We remark that the bound of $\Theta^*(9.5816\ldots^n)$ marked forests on n points in convex position gives the number of nc-spanning trees of the single chain. This point set has triangular convex hull and all but one point p of the set are in convex position; see Figure 2. To count the number of nc-spanning trees of this set, notice that the deletion of p gives a marked nc-forest on the lower chain (the mark of each component being the vertex that was adjacent to p).

Corollary 2.3. *The single chain on N points has $\Theta^*(9.5816\ldots^N)$ non-crossing spanning trees.*

Using a similar construction, we next prove a lower bound of $\Omega(13.40^N)$ for the number of nc-forests of the double chain. This is not our strongest result on the number of nc-forests, as in Section 4 we further improve this bound to $\Omega(13.61^N)$. We nevertheless present it here to show how far the ideas in this section can be pushed in the case of forests, and because the improvement in Section 4 needs as a first step the number of nc-forests of a particular kind, which will be bounded using a variation of the following construction.

Theorem 2.4. *The double chain on N points has $\Omega(13.40^N)$ non-crossing forests.*

The restriction of a nc-forest of the double chain to each of the chains gives two forests F_U and F_L. Again, we consider only forests where in each component of F_U there is at most one vertex incident with interior edges. Thus, F_U is a forest where some components have a mark and some others do not. As before, let e_v be the rightmost interior edge incident with a mark v, let $M_1 = \{e_v : v \text{ is a mark}\}$, and let M_2 be the set consisting of the leftmost edge in each component of M_1. We impose the further restriction that no edge e_v is incident with an isolated vertex of F_L.

From the following data we can construct a forest satisfying the conditions above with $|M_2| = k$ and where the forest F_L has exactly m isolated vertices.

b1) A nc-forest F_U where at least k of its components have a mark, on a set of n points in convex position,

b2) a subset of k of the marks in F_U, always including the leftmost mark,

b3) a nc-forest F_L on a set of n points in convex position, with exactly m isolated vertices,

b4) a subset M_L of k vertices in F_L, none of which is isolated, and

b5) a subset of the m isolated vertices of F_L.

Indeed, it suffices to do the following:

f1) Match the k marks from b2) with the k vertices in F_L (corresponding to the edges of M_2);

f2) join the other marks in F_U to the leftmost visible vertex in M_L (corresponding to the edges of $M_1 \backslash M_2$);

f3) join each vertex of the set in b5) with the rightmost visible mark of F_U.

We use the following estimates, also proved in Section 3.

Proposition 2.5. *(i) The number of nc-forests with no isolated vertex on a set of n points in convex position is $bn^{-3/2}\omega_L^n(1 + O(n^{-1/2}))$, where $\omega_L = 7.2246\ldots$ and b is a constant.*

(ii) On a set of n points in convex position, the number of nc-forests where some of the connected components that are not isolated vertices bear a mark is $cn^{-3/2}\omega_S^n(1 + O(n^{-1/2}))$, where $\omega_S = 9.8643\ldots$ and c is a constant. Moreover, at least 40% of these forests have $0.1106n$ or more connected components that bear marks.

As we did for trees, to choose the forest F_U we first pick ℓ vertices that will be isolated and will all bear a mark, and then on the remaining $n - \ell$ vertices we take a forest where some of the components that are not isolated are marked. Let $k = \alpha n$, $\ell = \beta n$ and $m = \gamma n$. Using the estimates from Proposition 2.5, the number of nc-forests of the double chain on $2n$ points is at least

$$f(\alpha, \beta, \gamma) = \binom{n}{\beta n} 9.8643^{n-\beta n} \binom{0.1106(n - \beta n) + \beta n}{\alpha n} \binom{n}{\gamma n} \binom{n - \gamma n}{\alpha n} 7.2246^{n-\gamma n} 2^{\gamma n}.$$

We estimate this quantity as before and the values $\alpha = 0.235$, $\beta = 0.245$ and $\gamma = 0.166$ give $\Omega^*(13.4025\ldots^N)$ nc-forests on the double chain on N points, as needed.

Before moving to the proofs of Propositions 2.2 and 2.5, observe that, in the notation of those propositions, one has
$$\omega_F = \omega_L + 1, \qquad \omega_M = \omega_U + 1.$$
That is, for points in convex position, the growth constant for nc-forests is one more than the growth constant for nc-forests without isolated vertices, and similarly for marked nc-forests.

We are not aware that such a relationship has been observed before, and we give a combinatorial proof of it in Section 4. Thus, if we are only interested in growth rates and not in the exact asymptotic behaviour, one need not carry out the calculations explained in Section 3 for the classes that have no isolated vertices.

Furthermore, we prove in Section 4 a related inequality between the growth constants of nc-forests in the double chain, which allows us to improve the bound in Theorem 2.4.

3 Non-crossing forests of points in convex position

In this section we use generating functions and the techniques of analytic combinatorics to prove Propositions 2.2 and 2.5.

Consider a set S_n of n points in convex position, labelled counterclockwise $p_1, \ldots, p_n$; the vertex p_1 is called the root vertex. A systematic study of several classes of non-crossing graphs
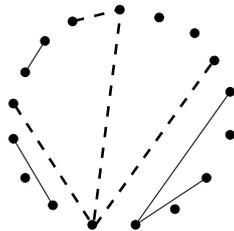
Figure 3: A nc-forest decomposes as a nc-tree (dashed edges) with a (possibly empty) nc-forest between any two of its consecutive vertices.

with generating functions was undertaken by Flajolet and Noy [6]; the results in this section are an extension of theirs using similar techniques (see also the book *Analytic combinatorics* [7] by Flajolet and Sedgewick).

For $n \geq 1$, let $\mathcal{T}^{(n)}$ be a possibly empty set of spanning nc-trees on S_n. Let $\mathcal{T} = \bigcup_{n \geq 1} \mathcal{T}^{(n)}$ and let $\mathcal{F}_{\mathcal{T}}$ be the set of those nc-forests such that its connected components belong to $\mathcal{T}$ (by taking as the root of each component the vertex with smallest label and relabelling the other vertices suitably). Let $T(z)$ and $F_T(z)$ be the corresponding generating functions, that is,

$$T(z) = \sum_{n \geq 1} t_n z^n, \qquad F_T(z) = \sum_{n \geq 0} f_n z^n,$$

where t_n and f_n denote the number of n-vertex graphs in $\mathcal{T}$ and $\mathcal{F}_{\mathcal{T}}$, respectively. For technical convenience, we set $f_0 = 1$ (but $t_0 = 0$).

We have the following key relation

$$F_T(z) = 1 + T(zF_T(z)). \tag{1}$$

The combinatorial explanation is as follows (see Figure 3). Given a forest in $\mathcal{F}_{\mathcal{T}}$, let t_1 be the connected component that contains the root vertex; this component is of course a tree of $\mathcal{T}$. Now, the vertices (if any) that lie strictly between any two consecutive vertices of t_1 induce a nc-forest, which belongs to $\mathcal{F}_T$. Thus, to recursively construct a forest in $\mathcal{F}_{\mathcal{T}}$, we start with a tree $t_1 \in \mathcal{T}$ and after each vertex of t_1 we may insert an arbitrary nc-forest. In terms of generating functions, each appearance of z in $T(z)$ (a vertex) must be substituted by $zF_T(z)$ (a pair of a vertex and a nc-forest). The fact that the constant term of $F_T(z)$ is 1 accounts for the case where no forest is inserted after a vertex of t_1. (For more details about translating combinatorial decompositions into generating functions, we refer to [7, Section I].) Thus, if an equation for $T(z)$ is known, we immediately get from (1) an equation for $F_T(z)$. We now follow this scheme and find equations for the generating functions for the different classes of forests under consideration.

Let $\mathcal{T}_0$ be the set of all nc-trees. It is well-known that the generating function $T_0(z)$ satisfies

$$T_0(z)^3 - zT_0(z) + z^2 = 0. \tag{2}$$

Let now $\mathcal{T}^*$ be the class of marked trees, that is, trees with one vertex distinguished; as a tree with n vertices gives rise to n marked trees, we have $T^*(z) = zT_0'(z)$. By differentiating equation (2) and eliminating $T_0(z)$ we get an equation for $T_0'(z)$, namely,

$$(27z^2 - 4z)T_0'(z)^3 + (1 - 6z)T_0'(z) - 1 + 8z = 0. \tag{3}$$

From this and (1) we obtain an equation for the generating function for marked forests, $U = F_{T^*}(z)$:

$$27zU^4 + (8z^3 - 6z^2 - 81z - 4)U^3 + (5z^2 + 82z + 12)U^2 - (28z + 12)U + 4 = 0.$$

6

Now let $\mathcal{T}_1$ be the class of nc-trees with more than one vertex; clearly $T_1(z) = T_0(z) - z$. From equations (2) and (1) it follows that $V = F_{T_1}(z)$ satisfies

$$(1 + z)^3 V^3 - (3z^2 + 7z + 3)V^2 + (4z + 3)V - 1 = 0.$$

As for the class $\mathcal{T}_1^*$ of marked trees with at least two vertices, the corresponding generating function is $T_1^*(z) = zT'(z) - z$. From (3) and (1) we obtain the following equation for $Y = F_{T_1^*}(z)$:

$$27z(1 + z)^3 Y^4 - (83z^3 + 180z^2 + 93z + 4)Y^3 + (99z^2 + 106z + 12)Y^2 - (12 + 40z)Y + 4 = 0. \quad (4)$$

The last class of forests we need is the one where some of the components that are not isolated vertices have a mark. This class is $\mathcal{F}_{\mathcal{T}_0 \cup \mathcal{T}_1^*}$. We obtain the corresponding equation for its generating function as above, with $W = F_{T_0 + T_1^*}(z)$:

$$27z(1 + z)^3 W^4 - (20z^3 + 234z^2 + 93z + 4)W^3 + 6(3z + 3)(8z + 1)W^2 - 12(4z + 1)W + 4 = 0.$$

Once an algebraic equation for $F_T(z)$ is known, it is usually routine to obtain an asymptotic estimate of the coefficients of $F_T(z)$. The method we apply is the one described in [6, Section 4] or, more generally, in [7, Sections VI and VII]. The main idea is that the singularity of $F_T(z)$ with smallest modulus (the *dominant* singularity) and the behaviour around this singularity determine the asymptotic behaviour of the coefficients of $F_T(z)$. More concretely, we summarize in the following theorem the results from [7] that are sufficient in our setting (see Lemma VII.3 and Theorems VI.1 and VI.3 for more details). The subindices indicate derivatives with respect to that variable.

Theorem 3.1. *Let $Y(z)$ be defined by the implicit equation $G(z, Y(z)) = 0$ and let ρ be the dominant singularity of $Y(z)$. Suppose that $G(z, y)$ is a polynomial and that τ is such that $G(\rho, \tau) = 0$, $G_z(\rho, \tau) \neq 0$, $G_y(\rho, \tau) = 0$ and $G_{yy}(\rho, \tau) \neq 0$. Then*

$$[z^n]Y(z) = \gamma \rho^n n^{-3/2} \left(1 + O(n^{-1/2})\right) \qquad \text{with } \gamma = \sqrt{\frac{\rho G_z(\rho, \tau)}{2\pi G_{yy}(\rho, \tau)}}.$$

Now, to find the dominant singularity ρ we use the fact that it must be one of the roots of the discriminant of the equation satisfied by $F_T(z)$, as explained in [7, Section VII.7.1]. Also, Pringsheim's theorem [7, Theorem IV.6] asserts that ρ is real positive.

Carrying out the calculations for the equations given for the different classes of forests proves the parts in Propositions 2.2 and 2.5 that do not deal with the number of components. We give the details for the case of marked forests without isolated vertices given by equation (4).

The corresponding discriminant is

$$-16z^3(1 - 20z + 67z^2)^2(4778z^5 + 7640z^4 + 793z^3 - 1454z^2 + 283z - 16).$$

Its real positive roots are $(10 - \sqrt{33})/67 = 0.063513\ldots$ and $(10 - \sqrt{33})/67 = 0.234993\ldots$ from the quadratic factor, and one root $0.116527\ldots$ of the degree 5 factor. From these candidates, we need to identify which one is the smallest singularity. We follow the methods of [7, Section VII.7.1] to study the behaviour of $F_{T_1^*}(z)$ in a neighbourhood of each of the candidate singularities, starting with the smallest one. It turns out that at $(10 - \sqrt{33})/67$ there is a multiple point, but no singularity, whereas the next candidate $0.1165\ldots$ is indeed a singularity. Now we solve equation (4) for $z = \rho = 0.1165\ldots$; of the four solutions, one is double, which corresponds to the value at the singular point. So $\tau = 1.1537\ldots$. Now it is only a matter of checking the hypotheses of Theorem 3.1.

We need next to refine our generating functions in order to take into account the number of components. We consider the bivariate generating function

$$F_T(z) = \sum_{n \geq 0} f_{n,k} z^n w^k,$$

where $f_{n,k}$ stands for the number of forests in $\mathcal{F}_{\mathcal{T}}$ with n vertices and k connected components. It is easy to see that equation (1) becomes

$$F_T(z, w) = 1 + wT(zF_T(z, w)).$$

Let $X_{n,k} = [z^n w^k]F_T(z, w)/[z^n]F_T(z)$, that is, the probability that a uniformly chosen forest in $\mathcal{F}_{\mathcal{T}}$ with n vertices has k components. As explained in [6, Section 5] and [7, Section IX.7], the singular behaviour of $F_T(z, w)$ gives information about the probability $X_{n,k}$. More concretely, for the generating functions we are considering, the singularity ρ of $F_T(z)$ "lifts" to a singularity $\rho(w)$ of $F_T(z, w)$ (that means, in particular, $\rho(1) = \rho$). It can be shown in this situation that the mean of $X_{n,k}$ is $\kappa n + O(1)$, where $\kappa = -\rho'(1)/\rho$. Moreover, $X_{n,k}$ converges in law to a Gaussian law. This implies that for each positive ε, $1/2 - \varepsilon$ of the forests in $\mathcal{F}_{\mathcal{T}}$ with n vertices have at least $\mu_n n$ components, for sufficiently large n.

Again, we provide some detail for the case of marked forests without isolated vertices. The generating function $Y = F_{T_1^*}(z, w)$ satisfies the equation

$$27z(zw + 1)^3 Y^4 - (2z^3 w^3 + 81z^3 w^2 + 18z^2 w^2 + 162z^2 w + 12zw + 81z + 4)Y^3 +$$
$$(18z^2 w^2 + 81z^2 w + zw^2 + 24zw + 81z + 12)Y^2 - (zw^2 + 12zw + 27z + 12)Y + 4 = 0.$$

The discriminant of this equation (with respect to Y) is a polynomial in z and w; it has one double factor which is quadratic in z and a factor that has degree 5 in z. Of the five roots of this last factor, we consider the one that at $w = 1$ gives the known value of ρ, and find $\rho'(1)$ by differentiating with respect to w the degree 5 factor of the discriminant.

Similar calculations give the remaining statements from Propositions 2.2 and 2.5. We just mention that for the case of forests where only some of the non-isolated components have marks, the equation that gives the bivariate generating function is

$$F_{T \cup T_1^*}(z, w) = 1 + T(zF_{T \cup T_1^*}(z, w)) + wT_1^*(zF_{T \cup T_1^*}(z, w)),$$

as we only want to estimate the number of components that have a mark and not the total number of components.

4 Relationship between growth constants

In this section we explain in a simple way the relationship between the growth constants observed at the end of Section 2, and also prove that the average numbers of components of the corresponding forests are also related. Moreover, we use similar ideas to prove a relationship between the growth constants of nc-forests in the double chain, allowing or not allowing isolated vertices. This relationship enables us to improve the bound $\Omega(13.40^N)$ of Theorem 2.4 to $\Omega(13.61^N)$.

We first focus on nc-forests on points in convex position, allowing or not allowing isolated vertices. Let $f(n)$ be the number of forests on n points in convex position, and let $\tilde{f}(n)$ be the number of such forests that have no isolated vertices. Although we actually know the values of the growth constants, we assume in this section no knowledge about the numbers $f(n)$ and $\tilde{f}(n)$. So first we show that they behave asymptotically as exponentials. For this we use the following lemma on superadditive functions.

Lemma 4.1. ([11, Lemma 11.6]) Let $g : \mathbb{N} \to \mathbb{N}$ be such that $g(i + j) \geq g(i)g(j)$ for all $i, j \in \mathbb{N}$. Then $\lim_{n \to \infty} g(n)^{1/n}$ exists.

It is clear that f and $\tilde{f}$ satisfy the hypotheses of this lemma, as the union of a forest on the points $p_1, \ldots, p_i$ and a forest on the points $p_{i+1}, \ldots, p_{i+j}$ gives a forest on $i + j$ points. Now observe that any nc-forest can be constructed by choosing first some vertices to be isolated and then choosing a nc-forest without isolated vertices; we thus have the relation

$$f(n) = \sum_{i=0}^{n} \binom{n}{i} \tilde{f}(n - i),$$

where we set $\tilde{f}(0) = 1$ for convenience. The analysis of this equation gives the relationship between the growth constants, as we next show. Since there is nothing particular to forests in the argument, we state it in general terms.

Proposition 4.2. *Let $f : \mathbb{N} \to \mathbb{N}$ and $\tilde{f} : \mathbb{N} \to \mathbb{N}$ be such that there exist constants b and c such that $\lim_{n \to \infty} f(n)^{1/n} = b$ and $\lim_{n \to \infty} \tilde{f}(n)^{1/n} = c$. If for all $n \geq 1$ it holds that*

$$f(n) = \sum_{i=0}^{n} \binom{n}{i} \tilde{f}(n-i),$$

then $b = 1 + c$.

Proof. Let us start by writing $f(n) = h(n)b^n$ and $\tilde{f}(n) = \tilde{h}(n)c^n$ for some functions $h : \mathbb{N} \to \mathbb{R}$ and $\tilde{h} : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n \to \infty} h(n)^{1/n} = \lim_{n \to \infty} \tilde{h}(n)^{1/n} = 1$. Then,

$$h(n)b^n = \sum_{i=0}^{n} \binom{n}{i} \tilde{h}(n-i)c^{n-i}. \tag{5}$$

We now give upper and lower bounds for the right-hand side that imply $b = c + 1$. For the upper bound, take j such that $\tilde{h}(n-j) \geq \tilde{h}(n-i)$ for all $i \leq n$ (note that j depends on n). Then

$$\sum_{i=0}^{n} \binom{n}{i} \tilde{h}(n-i)c^{n-i} \leq \tilde{h}(n-j) \sum_{i=0}^{n} \binom{n}{i} c^{n-i} = \tilde{h}(n-j)(c+1)^n.$$

By taking the limit of the n-th roots of both sides of the inequality $h(n)b^n \leq \tilde{h}(n-j)(c+1)^n$ we immediately get $b \leq c + 1$.

For the lower bound, fix any $\varepsilon > 0$. Then there is N such that $\tilde{h}(n) \geq (1-\varepsilon)^n$ for all $n \geq N$. For α with $0 < \alpha < 1$ and $n \geq N/(1-\alpha)$ we bound the sum in (5) by the term corresponding to $i = \alpha n$, resulting in

$$\sum_{i=0}^{n} \binom{n}{i} \tilde{h}(n-i)c^{n-i} \geq \binom{n}{\alpha n} (c(1-\varepsilon))^{n-\alpha n}.$$

Using the entropy function as in Section 2, we have

$$\lim_{n \to \infty} \left(\binom{n}{\alpha n} (c(1-\varepsilon))^{n-\alpha n} \right)^{1/n} = 2^{H(\alpha)} (c(1-\varepsilon))^{1-\alpha},$$

and thus $b \geq 2^{H(\alpha)} (c(1-\varepsilon))^{1-\alpha}$ for all α. The maximum is achieved at $\alpha = (1+c(1-\varepsilon))^{-1}$ with value $1 + c(1-\varepsilon)$. As this holds for all $\varepsilon > 0$, we conclude that $b \geq 1 + c$, as needed. $\square$

The relationship between the growth constants in items (ii) and (iii) from Proposition 2.2 also follows from the proposition above. Based on results of Flajolet and Noy [6], we immediately get that the growth constant for non-crossing graphs without isolated vertices is $5 + 4\sqrt{2}$ and that the growth constant for non-crossing partitions without singleton sets is 3.

There is also a relationship between the average number of components in forests and forests without isolated vertices. Let $\mathcal{F}_n$ and $\tilde{\mathcal{F}}_n$ be the sets of nc-forests with n vertices and of nc-forests with n vertices, none of them being isolated. Given any forest F, write $k(F)$ for the number of components of F, and let μ_n and $\tilde{\mu}_n$ denote the mean of the number of components in forests from $\mathcal{F}_n$ and $\tilde{\mathcal{F}}_n$, respectively. Then

$$\mu_n = \frac{\sum_{F \in \mathcal{F}_n} k(F)}{f(n)} = \frac{1}{f(n)} \sum_{i=0}^{n} \binom{n}{i} \sum_{\tilde{F} \in \tilde{\mathcal{F}}_{n-i}} (k(\tilde{F}) + i) = \frac{1}{f(n)} \sum_{i=0}^{n} \binom{n}{i} \tilde{f}(n-i)(\tilde{\mu}_{n-i} + i).$$

As in the case of the growth constants, this equation determines the limit of μ_n/n, provided we assume that $\lim_{n \to \infty} \tilde{\mu}_n/n$ exists.

Proposition 4.3. *Let f, $\tilde{f}$, b, and c, be defined as in Proposition 4.2, and let (μ_n) and $(\tilde{\mu}_n)$ be real-valued sequences such that*

$$\mu_n = \frac{1}{f(n)} \sum_{i=0}^{n} \binom{n}{i} \tilde{f}(n-i)(\tilde{\mu}_{n-i} + i). \tag{6}$$

If there exists a constant $\tilde{\mu}$ such that $\lim_{n\to\infty} \tilde{\mu}_n/n = \tilde{\mu}$, then

$$\lim_{n\to\infty} \frac{\mu_n}{n} = \frac{1 + c\tilde{\mu}}{b}.$$

Proof. We first rewrite the right-hand side of (6). Easy manipulation gives that the term $\sum_{i=0}^{n} \binom{n}{i} \tilde{f}(n-i)i$ equals $n \sum_{i=0}^{n-1} \binom{n-1}{i} \tilde{f}(n-1-i) = nf(n-1)$. As for the term $\sum_{i=0}^{n} \binom{n}{i} \tilde{f}(n-i)\tilde{\mu}_{n-i}$, it can be rewritten as

$$n \sum_{i=0}^{n-1} \binom{n-1}{i} \tilde{f}(n-1-i) \frac{\tilde{f}(n-i)}{\tilde{f}(n-1-i)} \frac{\tilde{\mu}_{n-i}}{n-i}.$$

We now compute $\lim_{n\to\infty} \mu_n/n$. From the calculations above,

$$\lim_{n\to\infty} \frac{\mu_n}{n} = \frac{1}{b} + \lim_{n\to\infty} \frac{1}{f(n)} \sum_{i=0}^{n-1} \binom{n-1}{i} \tilde{f}(n-1-i) \frac{\tilde{f}(n-i)}{\tilde{f}(n-1-i)} \frac{\tilde{\mu}_{n-i}}{n-i}.$$

As

$$\lim_{n\to\infty} \frac{\tilde{f}(n)}{\tilde{f}(n-1)} \frac{\tilde{\mu}_n}{n} = c\tilde{\mu},$$

for every $\varepsilon > 0$ there exists N such that

$$c\tilde{\mu} - \varepsilon < \frac{\tilde{f}(n)}{\tilde{f}(n-1)} \frac{\tilde{\mu}_n}{n} < c\tilde{\mu} + \varepsilon \text{ for } n \geq N.$$

Therefore,

$$\frac{1}{f(n)} \sum_{i=0}^{n-1} \binom{n-1}{i} \tilde{f}(n-1-i) \frac{\tilde{f}(n-i)}{\tilde{f}(n-1-i)} \frac{\tilde{\mu}_{n-i}}{n-i} \leq (c\tilde{\mu} + \varepsilon) \frac{1}{f(n)} \sum_{i=0}^{n-N} \binom{n-1}{i} \tilde{f}(n-1-i)$$

$$+ \frac{1}{f(n)} \sum_{i=n-N+1}^{n} \binom{n-1}{i} \tilde{f}(n-i) \frac{\tilde{\mu}_{n-i}}{n-i}. \tag{7}$$

We claim that the right-hand side tends to $(c\tilde{\mu} + \varepsilon)/b$. First notice that the numerator of the second summand is a sum of N terms that is easily seen to be $O(n^N)$. Similarly, in the first summand the sum of the missing terms for i from $n - N + 1$ to n is also $O(n^N)$. Hence, the right-hand side of inequality (7) is

$$(c\tilde{\mu} + \varepsilon) \frac{1}{f(n)} \left(\sum_{i=0}^{n} \binom{n-1}{i} \tilde{f}(n-1-i) + O(n^N) \right) = (c\tilde{\mu} + \varepsilon) \frac{f(n-1)}{f(n)} + (c\tilde{\mu} + \varepsilon) \frac{O(n^N)}{f(n)},$$

which tends to $(c\tilde{\mu} + \varepsilon)/b$ as desired.

Thus,

$$\lim_{n\to\infty} \frac{\mu_n}{n} \leq \frac{1}{b} + \frac{c\tilde{\mu} + \varepsilon}{b},$$

for all $\varepsilon > 0$. As an analogous argument shows that

$$\lim_{n\to\infty} \frac{\mu_n}{n} \geq \frac{1}{b} + \frac{c\tilde{\mu} - \varepsilon}{b}$$

for all $\varepsilon > 0$, we conclude that $\lim_{n\to\infty} \mu_n/n$ exists and equals $(1 + c\tilde{\mu})/b$. $\square$

The reader can check that this relationship between μ and $\tilde{\mu}$ holds for the numbers given in Proposition 2.2.

We now turn our attention to the number of nc-forests of the double chain, allowing or not allowing isolated vertices. We cannot prove that the respective growth constants differ exactly by one (although we believe this is the case), but only that they differ by at least one, which is enough to obtain a lower bound better than the one in Theorem 2.4.

Proposition 4.4. *Let $g(n)$ be the number of nc-forests of the double chain with $2n$ points, and let $\tilde{g}(n)$ be the number of those that have no isolated vertices. There exist constants γ and $\tilde{\gamma}$ such that $\lim_{n\to\infty} g(n)^{1/n} = \gamma^2$ and $\lim_{n\to\infty} \tilde{g}(n)^{1/n} = \tilde{\gamma}^2$, and these constants satisfy $\gamma \geq \tilde{\gamma} + 1$.*

Proof. That the growth constants exist is an immediate consequence of Lemma 4.1. We claim that the following bound holds

$$g(n) \geq \sum_{i=0}^{n} \binom{n}{i}^2 \tilde{g}(n-i). \tag{8}$$

Indeed, the right-hand side counts the number of nc-forests in the double chain that have the same number of isolated vertices in each chain. As before, write $\tilde{g}(n) = \tilde{h}(n)\tilde{\gamma}^{2n}$ with $\lim_{n\to\infty} \tilde{h}(n)^{1/n} = 1$. For every $\varepsilon > 0$ there exists N such that $\tilde{h}(n) \geq (1-\varepsilon)^n$ if $n \geq N$. Thus, for any α and $n \geq N/(1-\alpha)$ we have the lower bound

$$\sum_{i=0}^{n} \binom{n}{i}^2 \tilde{g}(n-i) \geq \binom{n}{\alpha n}^2 (1-\varepsilon)^{n-\alpha n} \tilde{\gamma}^{2(n-\alpha n)}.$$

The n-th root of the right-hand side tends to $2^{2H(\alpha)}(\tilde{\gamma}^2(1-\varepsilon))^{1-\alpha}$, which is maximized at $\alpha = (1+\tilde{\gamma}\sqrt{1-\varepsilon})^{-1}$ with value $(1+\tilde{\gamma}\sqrt{1-\varepsilon})^2$. As this holds for all $\varepsilon > 0$, we conclude that $\gamma \geq \tilde{\gamma}+1$. $\quad\square$

We remark that $\left(\sum_{i=0}^{n} \binom{n}{i}^2 \tilde{\gamma}^{2(n-i)}\right)^{1/n}$ tends indeed to $(1+\tilde{\gamma})^2$. For this, we write the sum as an evaluation of the Legendre polynomial $P_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 (x-1)^{n-k}(x+1)^k/2^n$ [1], that is,

$$\sum_{i=0}^{n} \binom{n}{i}^2 \tilde{\gamma}^{2(n-i)} = (1-\tilde{\gamma}^2)^n P_n\left(\frac{1+\tilde{\gamma}^2}{1-\tilde{\gamma}^2}\right).$$

As the generating function $\sum_{n\geq 0} P_n(x)t^n$ is $(1-2tx+t^2)^{-1/2}$ and the inverse of the dominant singularity of the generating function gives the exponential growth of its coefficients (see [7, Section IV.3.2]), for a fixed value of x we have $\lim_{n\to\infty} P_n(x)^{1/n} = (x-\sqrt{x^2-1})^{-1} = x + \sqrt{x^2-1}$, which for $x = (1+\tilde{\gamma}^2)(1-\tilde{\gamma}^2)$ gives the claimed result.

We actually believe that relation (8) is asymptotically an equality, and we conjecture that $\gamma = \tilde{\gamma}+1$.

We can apply Proposition 4.4 by taking the number of nc-trees as a lower bound for the number of nc-forests without isolated vertices. Thus, from Theorem 2.1 we immediately get that there are $\Omega(13.52^N)$ nc-forests in the double chain. We can do slightly better by using a construction similar to the ones in Section 2.

Corollary 4.5. *The double chain on N points has $\Omega(13.61^N)$ non-crossing forests.*

Proof. We modify the construction that proves the bound in Theorem 2.4 so that no vertex in the resulting forest is isolated. For this, take

c1) a set of ℓ vertices from the upper chain, all of them marked,

c2) a nc-forest on $n - \ell$ points in convex position such that none of them is isolated and where at least $k - \ell$ components are marked,

c3) a subset of k vertices among the ℓ from c1) and the marks from c2), always including the leftmost mark,

c4) a nc-forest F_L on n points in convex position, and

c5) a subset M_L of k vertices from F_L.

Then do the following to obtain a forest without isolated vertices:

f1) Match the k vertices from c3) with the ones from c5);

f2) join the other marks from c1) and c2) to the leftmost visible vertex on M_L;

f3) join each vertex in F_L that still is isolated with the rightmost visible mark on the upper chain.

We now need to compute the number of nc-forests on n points in convex position required in item c2), and also the average number of marked components. This is done as in Section 3, giving that the growth constant for the number of such forests is $8.8643\ldots$ and that the average number of marked components is κn with $\kappa = 0.1231\ldots$, the limiting distribution being again Gaussian. Actually, the value of the growth constant follows immediately from Proposition 4.2, but observe that the average number of components needs to be computed with analytic methods, as relation (6) does not hold in this case.

Now, setting $k = \alpha n$ and $\ell = \beta n$, we conclude that the number of nc-forests of the double chain without isolated vertices is at least

$$\binom{n}{\beta n} 8.8643^{n-\beta n} \binom{0.1231(n-\beta n)+\beta n}{\alpha n} 8.2246^n \binom{n}{\alpha n},$$

which for $\alpha = 0.263$ and $\beta = 0.267$ gives $\Omega^*(12.6108^N)$. Finally, from Proposition 4.4 we conclude that the double chain has $\Omega^*(13.6108^N)$ nc-forests. $\square$

Another consequence of Proposition 4.4 is that the number of nc-spanning trees of the double chain is $O(23.68^N)$. Indeed, Dumitrescu et al. [5] proved that the corresponding number of nc-forests is $O(24.68^N)$, so the growth constant for spanning trees is at least one less. In the following section we improve this upper bound on the number of nc-spanning trees to $O(22.12^N)$.

5   The upper bound

Recall that for any nc-spanning tree of the double chain, the vertices on the upper and lower chains induce two forests F_U and F_L on a set of n points in convex position and a forest F_I formed by interior edges, i.e., edges with one endpoint on each chain. We first count the number of forests F_I with a given number of edges.

Proposition 5.1. *Let $F_I(N,k)$ be the set of nc-forests in the interior of the double chain on $N = 2n$ vertices with $k \geq 1$ edges. The number of forests $|F_I(N,k)|$ is*

$$|F_I(N,k)| = \sum_{\ell=1}^{\min(k,n)} \sum_{i=1}^{n-\ell+1} \sum_{j=1}^{n-\ell+1} \binom{n-i}{\ell-1} \binom{n-j}{\ell-1} \binom{2n-\ell-i-j+1}{k-\ell}. \tag{9}$$

Proof. We assign each forest of $F_I(N,k)$ to a unique matching described by the following algorithm: Scan the edges of the forest from left to right. Let u_1 be the vertex on the upper chain incident to the first edge encountered. Let $i-1$ be the number of points to the left of u_1 on the upper chain. Similarly, let $j-1$ be the number of points to the right of the vertex d_ℓ on the lower chain incident to the last edge encountered. The rightmost edge incident to u_1 is the first edge of the matching. Let d_1 be the other endpoint of this first edge of the matching on the lower chain. See Figure 4. Assume the first $r \geq 1$ edges of the matching are determined and let d_r be the endpoint on the
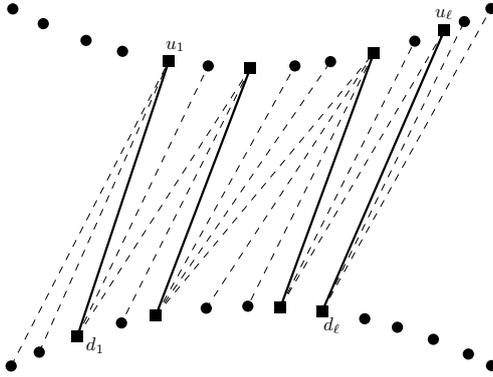
Figure 4: A nc-forest of $F_I(N, k)$ consists of a matching (bold edges) and a subset of the optional edges (dashed edges). The first $i - 1$ points from the left on the upper chain and the last $j - 1$ points from the right on the lower chain have no incident optional edges.

lower chain of the last edge of the matching encountered so far. Let u_{r+1} be the vertex on the upper chain incident to the next edge of the forest which does not have d_r as its other endpoint. Then the rightmost edge incident to u_{r+1} is the next edge of the matching. The algorithm stops when all the edges of the forest are scanned. The resulting matching has between 1 and n edges. For the forest in Figure 1 (in fact, a tree) the matching corresponds to the bold edges (the edges of M_2).

Now, to count $|F_I(N, k)|$, we first consider all matchings M_2 with a fixed number ℓ of edges, and then sum over all possible values of ℓ. We extend each matching M_2 in all possible ways to a forest $F_I(M_2)$ which has M_2 as its uniquely assigned matching. This can be done by scanning the interior of the double chain from left to right again. Denote the ℓ edges of M_2 by $u_i d_i$, $1 \leq i \leq \ell$; u_i is the vertex on the upper chain. First examine the points of the double chain to the left of $u_1 d_1$, the first edge of M_2. The vertices on the upper chain before u_1 can not have incident edges in $F_I(M_2)$. The vertices on the lower chain before d_1 can (optionally) be connected to u_1. Assume we have scanned all points to the left of the edge $u_r d_r$, $r \geq 1$. Then d_r can (optionally) be connected to all vertices on the upper chain from u_r to u_{r+1}. And u_{r+1} can (optionally) be connected to all vertices from d_r to d_{r+1} on the lower chain. We continue in this way until we reach the last edge $u_\ell d_\ell$ of M_2. Then all the vertices on the upper chain on the right of u_ℓ can (optionally) be connected to d_ℓ, but the vertices on the lower chain on the right of d_ℓ can not have incident edges. Thus, we can build a forest with k edges $F_I(M_2)$, assigned to M_2, by adding a subset of $k - \ell$ edges from these optional edges to M_2. Altogether there are $2n - 1 - \ell - (i - 1) - (j - 1)$ optional edges, where $2n - 1$ is the number of edges of a spanning tree of the double chain, the matching has ℓ edges, $i - 1$ points are on the upper chain to the left of u_1 and $j - 1$ points are on the lower chain to the right of d_ℓ.

This gives the factor $\binom{2n - \ell - i - j + 1}{k - \ell}$ of Equation (9). The terms $\binom{n - i}{\ell - 1}$ and $\binom{n - j}{\ell - 1}$ correspond to the selection of points for building a matching M_2 with ℓ edges. $\qquad \square$

Theorem 5.2. *The double chain on N points has $O(22.12^N)$ non-crossing spanning trees.*

Proof. The product of the numbers of forests F_U, F_L and F_I gives an upper bound on the number of nc-spanning trees of the double chain. This asymptotic counting can be refined by only counting spanning trees with forests from the set $F_I(N, k)$ for a certain value of k. Still, we also will count graphs that contain cycles.

First, we partition the set of nc-spanning trees of the double chain into $2n - 1$ classes, according to the number of edges of F_I. For asymptotic counting, it is sufficient to only consider the one class of nc-spanning trees, with k edges in forest F_I, that contains most spanning trees, over-counting by a factor of at most $2n - 1$. Given a non-crossing spanning tree, removing the k edges of the forest F_I in the interior, the graph breaks into a forest consisting of $k + 1$ components; a forest

on the lower chain of k_1 components, and a forest on the upper chain of k_2 components, with $k_1 + k_2 = k + 1$. It is sufficient to only count the number of spanning forests for a certain value of k_1, that maximizes the product of the three terms. This is, again, because we can ignore polynomial factors. For the same reason, we also can simplify some terms in the following, neglecting some constants. In particular we can assume that the forests on the two chains have k components instead of $k + 1$, and set $k = \alpha n$ and $k_1 = \beta n$, with $0 < \alpha < 2$ and $0 < \beta < \min\{\alpha, 1\}$. We shall show that the maximum is attained for $k_1 = k_2 = \frac{k}{2}$.

The number of nc-forests with c components on a set of n points in convex position is given by the formula [6]

$$F_{n,c} = \frac{1}{2n-c}\binom{n}{c-1}\binom{3n-2c-1}{2n-c-1}.$$

In our setting, once fixed k, the factor for the forests in the interior is independent from the other two factors. We get for the product of the number of forests in the two chains

$$f_{n,k_1,k_2} = \frac{1}{2n-k_1}\binom{n}{k_1-1}\binom{3n-2k_1-1}{2n-k_1-1}\frac{1}{2n-k_2}\binom{n}{k_2-1}\binom{3n-2k_2-1}{2n-k_2-1}.$$

Then,

$$f_{n,k_1,k_2} \approx \binom{n}{\beta n}\binom{3n-2\beta n}{2n-\beta n}\binom{n}{\alpha n - \beta n}\binom{3n-2(\alpha n - \beta n)}{2n-(\alpha n - \beta n)}.$$

As in Section 2 we use the binary entropy function to estimate a binomial coefficient, which gives

$$f_{n,k_1,k_2} \approx 2^{\left(H(\beta)+(3-2\beta)H(\frac{2-\beta}{3-2\beta})+H(\alpha-\beta)+(3-2\alpha+2\beta)H(\frac{2-\alpha+\beta}{3-2\alpha+2\beta})\right)n}.$$

For n and α fixed, this is a function of β we want to maximize. Equivalently we can maximize the logarithm of basis 2 of this function and we can ignore the factor n. Hence, we maximize

$$g_\alpha(\beta) = H(\beta) + (3-2\beta)H(\frac{2-\beta}{3-2\beta}) + H(\alpha-\beta) + (3-2\alpha+2\beta)H(\frac{2-\alpha+\beta}{3-2\alpha+2\beta}).$$

Then standard calculations show that $g_\alpha(\beta)$ attains its maximum at $\beta = \frac{\alpha}{2}$.

Next, we determine the asymptotic growth of $|F_I(N,k)|$. Ignoring polynomial factors, Equation (9) can be bounded from above by

$$\sum_{\ell=1}^{\min(k,n)}\binom{n}{\ell}^2\binom{2n-\ell}{k-\ell}.$$

This can be seen from the proof of Proposition 5.1, when, for each i and j, we allow to choose the ℓ vertices on a chain for the matching among all the n points of the chain; and the number of optional edges is always less than $2n - \ell$.

Then, for asymptotic counting it is sufficient to only consider the largest term in this sum. We determine the value of ℓ that gives the largest summand. We set $k = \alpha n$ and $\ell = \lambda n$, with $0 < \alpha < 2$ and $0 < \lambda < 1$. Then a summand has the form

$$\approx \binom{n}{\lambda n}^2\binom{(2-\lambda)n}{(\alpha-\lambda)n} \approx 2^{\left(2H(\lambda)+(2-\lambda)H(\frac{\alpha-\lambda}{2-\lambda})\right)n}.$$

For α fixed, we maximize

$$h_\alpha(\lambda) = 2H(\lambda) + (2-\lambda)H(\frac{\alpha-\lambda}{2-\lambda}).$$

Solving $h'_\alpha(\lambda) = 0$, we obtain

$$\lambda = \frac{1+2\alpha \pm \sqrt{1+4\alpha}}{2\alpha}$$

14

and verify that h_α indeed attains the maximum at $\frac{1+2\alpha-\sqrt{1+4\alpha}}{2\alpha}$.

We finally can write the product of the three terms (number of forests in lower chain, number of forests in upper chain, number of forests in the interior) as a function of n and $k = \alpha n$. We already have seen that the first two terms are equal. The number of non-crossing trees of the double chain on $2n$ points is asymptotically bounded from above by

$$2^{t(\alpha)n} = 2^{\left(2(H(\frac{\alpha}{2})+(3-\alpha)H(\frac{2-\alpha/2}{3-\alpha}))+h_\alpha(\frac{1+2\alpha-\sqrt{1+4\alpha}}{2\alpha})\right)n}.$$

The function $t(\alpha)$ is maximized for $\alpha = 0.750614$ with $2^{t(\alpha)} = 8.93341$. Therefore, the number of non-crossing spanning trees of the double chain on a set of $N = 2n$ points is at most $2^{8.93341N/2} = 22.1112^N$.

$\square$

Acknowledgements

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