ON THE SLOPE OF BIELLIPTIC FIBRATIONS

Miguel A. BARJA

Abstract

Let \( \pi : S \rightarrow B \) be a bielliptic fibration. We prove \( S \) is, up to base change, a rational double cover of an elliptic fibration and that \( \pi \) is isotrivial provided it is smooth. Finally, we prove that the slope of \( \pi \) is at least four provided the genus of the fibre is at least six.

A la memoria de Fernando

0 Introduction

Let \( \pi : S \rightarrow B \) be a fibration, i.e. a surjective morphism with connected fibres, from a smooth surface \( S \) onto a smooth curve \( B \). A fibration is said to be relatively minimal when it has no vertical \((-1)\)-curve. Let \( g \) denote the genus of a general fibre and \( b \) the genus of \( B \).

Let \( \omega_{S/B} = \omega_S \otimes \pi^*(\omega_B^{-1}) \) be the relative canonical bundle and let \( \Delta(\pi) := \deg \pi_*(\omega_{S/B}) \).

It is known that \( \Delta(\pi) \geq 0 \) and that \( \Delta(\pi) = 0 \) if and only if \( \pi \) is locally trivial. Assume \( \pi \) is not locally trivial. Then we define the slope of \( \pi \) as

\[
\lambda(\pi) := \frac{\omega_{S/B}^2}{\Delta(\pi)}
\]

(see [19]). There are several results on the lower slope of relatively minimal fibrations of genus \( g \geq 2 \). First of all we have \( \lambda \geq 4 - \frac{4}{g} \) (see [8], [12], [13], [18] for the hyperelliptic case and [19] for the general case) and equality holds only in the hyperelliptic case ([9]). There are improvements in the non-hyperelliptic case for \( g \leq 5 \) (see [4], [7], [9], [11], [14]) but the presently known techniques seem to have some limitations to extend these results to higher genus.

Recently Konno is trying to find good bounds depending on some extra numerical invariants of the general fibre, such as the Clifford index. In [10], Konno finds better bounds for trigonal and plane quintic fibrations (so Clifford index 1), although they do not seem to be sharp. Also in [11] he gets general bounds depending on the Clifford index in some cases.

In this paper we deal with the case of bielliptic fibrations (i.e., when the general fibre has a 2-to-1 map onto an elliptic curve). Using the glueing results of [2] we know that,
if \( g \geq 6 \) a bielliptic fibration is a (generically) double cover of an elliptic fibration. We prove (example 1.2) that this is not true in general if \( g \leq 5 \) due to the existence of several bielliptic maps in the general fibre.

Using this we get the following sharp bound for the slope of bielliptic fibrations.

**Theorem 2.1** Let \( \pi : S \rightarrow B \) be a relatively minimal bielliptic fibration of genus \( g \geq 6 \). Let \( V \) be the relative minimal model of the elliptic fibration obtained in section 1. Then

\[
\begin{align*}
(a) & \quad \lambda(\pi) \geq 4 + \frac{2(g-5)\chi(O_V)}{\Delta(\pi)} \geq 4. \\
(b) & \quad \lambda(\pi) = 4 \text{ if and only if } S \text{ is the minimal desingularization of a double cover } S_0 \rightarrow V \text{ of a smooth elliptic surface such that} \\
& \quad \cdot \text{All the fibres of the elliptic fibration } \tau : V \rightarrow B \text{ are smooth and isomorphic}. \\
& \quad \cdot \text{The branch divisor of the double cover has only negligible singularities.}
\end{align*}
\]

In particular, the bound is sharp.

The author want to thank, among others, professor Juan Carlos Naranjo for his encouragement and interesting comments.

During the final revision of this paper the advisor of the author, professor Fernando Serrano, passed away. I would like to thank him heartfully for his support and continuous help, not only during the preparation of this work but also during the last years when I enjoyed his teachings and friendship.

All throughout this paper we work over the field of complex numbers \( \mathbb{C} \).

1 Bielliptic fibrations

Let \( F \) be a smooth curve of genus \( g \). \( F \) is called bielliptic if \( F \) admits a 2-to-1 map onto an elliptic smooth curve \( E \). Such a map is always given by the quotient by an involution \( i \in \text{Aut}(F) \), called a bielliptic involution on \( F \). It is a well known fact that such an involution is unique if \( g \geq 6 \) (see [1]).

Let \( \pi : S \rightarrow B \) be a fibration of genus \( g \). We say that \( \pi \) is bielliptic if so it is the general fibre \( F \) of \( \pi \). The following result clarifies the structure of such fibrations. Recall that the fibration \( \pi \) is said to be smooth if every fibre is smooth and it is said to be isotrivial if all the smooth fibres are mutually isomorphic.

**Proposition 1.1** Let \( \pi : S \rightarrow B \) be a bielliptic fibration of genus \( g \). Then

\[
\begin{align*}
(a) & \quad \text{Up to base change, } S \text{ is a rational double cover of an elliptic surface over the base curve.} \\
(b) & \quad \text{If } g \geq 6 \text{ the same is true without base change.} \\
(c) & \quad \text{If } \pi \text{ is smooth, then } \pi \text{ is isotrivial.}
\end{align*}
\]
Proof. (a) and (b) are consequence of general results given in [2]. We give here a sketch of proof and refer there for details.

Given \( \pi : S \to B \) we can consider \( \psi : \text{Aut}_{S/B}^{2g-2} \to B \) the scheme of relative automorphisms of \( S \) over \( B \) of order 2 having \( 2g - 2 \) base points (which corresponds fibrewise to double covers of elliptic curves) which is a quasi-projective \( B \)-scheme. After a base change \( B' \to B \) such a map has always a section defined over a non-empty Zariski open subset \( U' \subset B' \) which corresponds to a rational automorphism \( \Phi \) of the minimal desingularization \( S' \) of \( S \times_B B' \) such that \( \Phi|_{F_t} \) is a bielliptic involution for \( t \in B' \) general. If \( V \) is a desingularization of \( S'/<\Phi> \) we have a rational double cover \( S' \to V \) over \( B' \).

If \( g \geq 6 \) then \( \psi \) is clearly 1-to-1 and then base change is not needed in order to have a section.

(c) Isotrivially can be checked after base change. Following [2] section 2 we can consider after base change

\[
\begin{array}{c}
S \xymatrix{ \ar[r]^-i & J(\pi) \ar[d]^-\pi \ar[r]^-f & J(\pi) \ar[d]^-\pi & \\
& B & & \\
}
\end{array}
\]

where \( J(\pi) \) is the relative Jacobian variety of \( S \) over \( B \) and \( f \) is a rational relative endomorphism of \( J(\pi) \) such that \( f \circ i \) produces a bielliptic map on the general fibre of \( \pi \). Let \( V = (f \circ i)(S) \). Note that \( V \) is an elliptic surface over \( B \) (possibly singular). Nevertheless classification of singular fibres of a smooth elliptic surface shows, since \( J(\pi) \) is an abelian variety for every \( t \in B \), that \( V \) is smooth and the map \( \pi : V \to B \) is also smooth. Moreover, the map \( g = f \circ i : S \to V \) can be solved after some blow-ups but then exceptional curves must be contracted since \( V \subset J(\pi) \). So we have that \( S \) is a double cover of a smooth elliptic fibration (perhaps after base change). In particular every fibre of \( \pi \) is bielliptic.

Consider now the double cover \( g : S \to V \). Since \( g \) has degree two the branching divisor of \( g \) must be smooth and hence it is étale over \( B \). After new base changes we can assume that the irreducible components of the branching divisor \( D \) are sections of \( \pi \). Moreover, since \( \pi : V \to B \) is a smooth elliptic fibration it is isotrivial and then, after base change, we can assume \( V = B \times E \) (\( E \): elliptic smooth curve). Let \( D_1 \) be an irreducible component of \( D \). If \( D_1 \) is a trivial section of \( \pi \) then so must be the other components and then \( \pi \) is clearly isotrivial. Assume \( D_1 \) is not a trivial section. Then \( D_1 = \{(b, \alpha(b)) \in B \times E \mid \alpha : B \to E \text{ non constant map}\} \). Consider a fixed structure of group on \( E = (E, 0) \) and consider the automorphism of \( V \) over \( B \): \( \beta(b, x) = (b, x + \alpha(b)) \).

Note that \( \beta^{-1}(D_1) = B \times \{0\} \) and, hence, \( \beta^{-1}(D) \) is composed of trivial horizontal sections. If we change the base

\[
\begin{array}{c}
S \xymatrix{ \ar[r]^-\sim & S & \\
\ar[d]^-g \ar@{=}[r] & \ar[d]^-g \\
B \times E \ar[r]^-\beta & B \times E & \\
}
\end{array}
\]

the branching divisor of \( \bar{g} \) is just \( \beta^{-1}(D) \) which is trivial. Hence \( \pi \) is isotrivial. \qed
A bielliptic curve of genus $g \leq 5$ can have more than one bielliptic involution; the number of such involutions are in correspondence with the elliptic components of $W^1_4(F)$, the Brill-Noether locus of linear series on $F$ of type $g^1_1$. We give an example which shows that these involutions do not glue independently for a general fibration.

**Example 1.2** Take a genus five curve $F$ with exactly two bielliptic involutions $\sigma_i : F \to E_i$ such that $E_1 \not= E_2$, with $E_i$ having no exceptional automorphisms (a count of constants shows that such an $F$ can be chosen). Then we have that $\sigma_1 \times \sigma_2 : F \to E_1 \times E_2$ embeds $F$ as a smooth curve, $F \in |\ell^*_1(2p_1) \otimes \ell^*_2(2p_2)|$, being $\ell_i : E_1 \times E_2 \to E_i$ the projections and $(p_1, p_2) \in E_1 \times E_2$. Since $\text{Aut}(E_1 \times E_2)$ acts transitively on $E_1 \times E_2$ we have that for every $(q_1, q_2) \in E_1 \times E_2$ there exists $\bar{F} \in |\ell^*_1(2q_1) \otimes \ell^*_2(2q_2)|$, $\bar{F} \cong F$.

Let $B$ be any smooth curve having an involution $\iota$ and let $g : B \to \bar{B} = B/\iota$. Consider a morphism $\kappa : B \to \mathbb{P}^1$ with no factorization through $\bar{B}$. Take a fixed $\bar{t} \in \bar{B}$ such that if $g^{-1}(\bar{t}) = \{t_1, t_2\}$ then $\kappa(t_1) \neq \kappa(t_2)$. After an automorphism of $\mathbb{P}^1$ we can suppose that $\kappa(t_i)$ is the modular invariant of $E_i$ in $\mathbb{C} \subseteq \mathbb{P}^1$.

Then, by [3] p.160, there exists an elliptic fibration $\tau : V \to B$ with a section, such that $\tau^{-1}(t_i) \cong E_i$. Let $B'$ be the image in $V$ of the section of $\tau$. Consider the following pull-back

$$Z := V \times_B V \xrightarrow{\xi_2} V \xrightarrow{\ell} B$$

Then, for $t \in B$ we have $Z_t = \xi^{-1}(t) = E_{i(t)} \times E_{i'}$, where $E_m = \tau^{-1}(m)$. The natural involution on $V \times_C V$ induces a commutative diagramm

$$Z \xrightarrow{\tau} Z \xrightarrow{\xi} B$$

and then

$$Z \xrightarrow{\bar{g}} Z := Z/\iota \xrightarrow{\bar{\xi}} B$$

Note that $Z$ is a threefold fibred over $\bar{B}$ and the fibre over $g(t) \in \bar{B}$ general is $E_{i(t)} \times E_{i'}$. We can assume $Z$ is already smooth.

Let $B'' = \bar{g}(\xi^{-1}(B'))$ and $\mathcal{L} = \mathcal{O}_{Z}(2B'')$. We have that $\mathcal{L}|_{Z_t} \cong \ell^*_1(2q_1) \otimes \ell^*_2(2q_2)$ for some $(q_1, q_2) \in E_1 \times E_2$. Note that if $\mathfrak{a} \in \text{Pic} \bar{B}$ is ample enough we have an epimorphism

$$H^0(\bar{Z}, \mathcal{L} \otimes \bar{\xi}^*(\mathfrak{a})) \to H^0(E_1 \times E_2, \mathcal{L}|_{Z_t}).$$

Since by hypothesis there exists $F \in |\mathcal{L}|_{Z_t}$ we get $\bar{S} \in |\mathcal{L} \otimes \bar{\xi}^*(\mathfrak{a})|$ a surface fibred over $\bar{B}$, smooth at a general fibre and such that $\bar{S}_t = F$. Again, we can suppose $\bar{S}$ is already
Let \( \tilde{\pi} : \tilde{S} \to \tilde{B} \) and \( F_{\tilde{m}} = \tilde{\pi}^{-1}(\tilde{m}) \). For \( \tilde{m} \in \tilde{B} \) general we have that \( F_{\tilde{m}} \) is a smooth curve of genus 5 having at least two bielliptic involutions given by the inclusion \( F_{\tilde{m}} \subseteq E_{(m)} \times E_{m} \) (if \( g(m) = \tilde{m} \)) as a \((2,2)\)-divisor. We claim that for general \( \tilde{m} \in \tilde{B} \), \( F_{\tilde{m}} \) has exactly two bielliptic involutions. Since this is the case for \( F = F_{\tilde{r}} \) we only have to prove that having at most two of them is an open condition. Consider \( W_{4}(\tilde{\pi}) \to \tilde{B} \), the relative Brill-Noether locus of \( \tilde{\pi} \) (at least over an open set of \( B \), see [16]), after a base change if necessary. The number of bielliptic involutions of \( F_{\tilde{m}} \) is given by the number of elliptic components of \( W_{4}(F_{\tilde{m}}) \cong W_{4}(\tilde{\pi})_{\tilde{m}} \). Then, having at most two of such components is obviously an open condition.

We claim that \( \tilde{S} \) is not a (birational) double cover of any elliptic fibration \( \tilde{\pi} : \tilde{V} \to \tilde{B} \). Indeed, assume we have a double cover \( \tilde{f} : \tilde{S} \to \tilde{V} \) (we can suppose \( \tilde{f} \) everywhere defined after some blow-ups). Consider the base change diagram

\[
\begin{array}{ccc}
Z & \rightarrow & \tilde{Z} \\
| & | & | \\
S & \rightarrow & \tilde{S} \\
\downarrow \tilde{f} & & \downarrow \tilde{f} \\
\tilde{V} & \rightarrow & \tilde{V} \\
\downarrow \tilde{\tau} & & \downarrow \tilde{\tau} \\
B & \rightarrow & \tilde{B}
\end{array}
\]

For \( S \) we have three double covers of elliptic fibrations over \( B \):

\[
\tilde{f} : S \to \tilde{V} \\
f_i : S \to V \\
f_i = \xi_i | S \quad i = 1, 2
\]

Set \( U = \{ m \in B \mid E_{m} \not\cong E_{(m)} ; E_{m}, E_{(m)} \) and \( \tilde{E}_{m} \) are smooth and \( F_{m} \) has exactly two bielliptic involutions (where \( \tilde{E}_{m} = \tilde{\tau}^{-1}(m) \)). We have that \( U \) is a non-empty open set of \( B \). Since \( f_{1}|F_{m}, f_{2}|F_{m}, \tilde{f}|_{F_{m}} \) are double covers of \( E_{(m)} \), \( E_{m} \) and \( \tilde{E}_{m} \) respectively we have that for every \( m \in U \), \( \tilde{E}_{m} \cong E_{(m)} \) or \( \tilde{E}_{m} \cong E_{m} \).

If \( g_{1} = g \circ \eta_{U} : U \to \mathbb{P}^{1}, g_{2} = g_{U} : U \to \mathbb{P}^{1} \) and \( \tilde{g} : U \to \mathbb{P}^{1} \) are the modular morphisms induced by \( \iota \circ \tau \), \( \tau \) and \( \tilde{\tau} \) over \( U \) respectively we have that \( \tilde{g} = g_{1} \) or \( \tilde{g} = g_{2} \). Assume \( \tilde{g} = g_{2} \).

As we have \( t_{1}, t_{2} \in U \) and \( \iota(t_{1}) = t_{2} \) we get

\[
E_{t_{1}} = \tau^{-1}(t_{1}) = \tau^{-1}(t_{2}) = \tau^{-1}(t_{1}) \cong \tau^{-1}(t_{2}) = E_{t_{2}}
\]

since \( \tau \) is induced by \( \tilde{\tau} : \tilde{V} \to \tilde{B} \) and then \( \tau^{-1}(m) \cong \tau^{-1}(\iota(m)) \) for all \( m \in B \). But this is impossible since by hypothesis \( E_{t_{1}} = E_{1} \not\cong E_{2} = E_{t_{2}} \).
2 Double covers and the slope of bielliptic fibrations

We recall some basic facts about double covers (see [6], [3]).

By a double cover we mean a finite, degree two map between surfaces, $f_0 : S_0 \rightarrow V_0$. This map is determined by a divisor $Z_0$ on $V_0$ (the branch divisor) and a line bundle $\mathcal{L}_0$ such that $\mathcal{L}_0^{\otimes 2} = \mathcal{O}_{V_0}(Z_0)$. If $V_0$ is smooth, $S_0$ is normal (respectively smooth) if and only if $Z_0$ is reduced (respectively smooth).

Consider a double cover as above with $S_0$ normal and $V_0$ smooth. Then there exists a canonical resolution of singularities for $S_0$ which consists on a finite sequence of maps

$$
\begin{array}{c}
S_k \\ \sigma_k \downarrow \\
S_{k-1} \\
\vdots \\
S_1 \\ \sigma_1 \downarrow \\
S_0 \\
\end{array}
\begin{array}{c}
f_k \\
\downarrow f_{k-1} \\
\vdots \\
f_1 \\
\downarrow f_0 \\
V_k \\
\alpha_k \downarrow \\
V_{k-1} \\
\vdots \\
V_1 \\
\alpha_1 \downarrow \\
V_0 \\
\end{array}
$$

satisfying:

(i) $\alpha_j$ is the blow-up of $V_{j-1}$ at a singular point $p_{j-1}$ of $Z_{j-1}$ (the branching divisor of $f_{j-1}$).

(ii) $f_j$ is the double cover of $V_j$ defined by $\mathcal{L}_j^{\otimes 2} \cong \mathcal{O}(Z_j)$, with $Z_j = \alpha_j^*(Z_{j-1}) - 2m_{j-1}E_j$, $\mathcal{L}_j = \alpha_j^*(\mathcal{L}_{j-1}) \otimes \mathcal{O}_{V_j}(-m_{j-1}E_j)$, where $E_j$ is the exceptional divisor of $\alpha_j$ and $p_{j-1}$ is a singular point of $Z_{j-1}$ of multiplicity $2m_{j-1}$ or $2m_{j-1} + 1$.

(iii) $\sigma_j$ is a birational morphism induced by the cartesian diagram of $\alpha_j$ and $f_{j-1}$.

(iv) $Z_k$ is smooth and, hence, $S_k$ is a smooth surface.

Now we can use this as follows. Recall from section 1 that we have obtained $f : \bar{S} \rightarrow V$ a generically 2-to-1 morphism (we can suppose that $f$ is everywhere defined up to blow-ups) from a blow-up of $S$ onto an elliptic fibration $V$ over $B$ which we can suppose relatively minimal after some blow-downs. Suppose that $\pi$ is relatively minimal.

Now consider

$$
\begin{array}{c}
\bar{S} \\
\sigma \\
S \\
\sigma \\
S_k \\
\sigma \\
S_0 \\
\sigma \\
S_1 \\
\sigma \\
S_1 \\
\vdots \\
\sigma \\
V_k \\
\sigma \\
V_0 = V \\
\end{array}
\begin{array}{c}
f_k \\
\downarrow f_{k-1} \\
\vdots \\
f_1 \\
\downarrow f_0 \\
V_k \\
\alpha_k \downarrow \\
V_{k-1} \\
\vdots \\
V_1 \\
\alpha_1 \downarrow \\
V_0 \\
\end{array}
\begin{array}{c}
\bar{V}_k \\
\downarrow \bar{f}_k \\
\vdots \\
\downarrow \bar{f}_0 \\
\bar{V}_0 \\
\end{array}
\begin{array}{c}
\bar{S} \\
\bar{S} = \bar{S}_k \\
\vdots \\
\bar{S}_1 \\
\vdots \\
\bar{S}_0 \\
\end{array}
\begin{array}{c}
\bar{V} \\
\bar{V} = \bar{V}_k \\
\vdots \\
\bar{V}_1 \\
\vdots \\
\bar{V}_0 = \bar{V} \\
\end{array}
\begin{array}{c}
B \\
\end{array}
$$
where:
- $f = f_0 \circ u$ is the Stein factorization of $f$, with $u$ birational, $f_0$ finite (so it is a double cover) and $S_0$ normal.
- $f_k : S_k \to V_k$ is the canonical resolution of singularities of $f_0 : S_0 \to V_0$.
- $\sigma : S_k \to S$ is the birational morphism defined by the relative minimality of $\pi$.

**Theorem 2.1** Let $\pi : S \to B$ be a relatively minimal bielliptic fibration of genus $g \geq 6$. Let $V$ be the relative minimal model of the elliptic fibration obtained in section 1. Then

(a) $\omega_{S/B}^2 - 4\Delta(\pi) \geq 2(g - 5)\mathcal{X}^2$. In particular, if $\pi$ is not locally trivial

$$\lambda(\pi) \geq 4 + \frac{2(g - 5)\mathcal{X}^2}{\Delta(\pi)} \geq 4$$

(b) $\lambda(\pi) = 4$ if and only if $S$ is the minimal desingularization of a double cover $S_0 \to V$ of a smooth elliptic surface such that

- All the fibres of the elliptic fibration $\pi : V \to B$ are smooth and isomorphic.
- The branch divisor of the double cover has only negligible singularities (i.e., all the multiplicities $m_j$ in the above process are 2 or 3 (see [13], [17])).

In particular, the bound is sharp.

**Proof.**

(a) First of all we have

$$\omega_{S/B}^2 - 4\Delta(\pi) = (K_S^2 - 4\mathcal{X}^2) - 4(b - 1)(g - 1) \geq (K_S^2 - 4\mathcal{X}^2) - 4(b - 1)(g - 1). \quad (1)$$

For smooth double covers $f_k : \tilde{S} \to \tilde{V}$ we have (see [3], p.183):

$$\mathcal{X}^2 \tilde{S} = 2\mathcal{X}^2 \tilde{V} + \frac{1}{2}L_k K_{\tilde{V}} + \frac{1}{2}L_k L_k$$

$$K_S^2 = 2K_{\tilde{V}}^2 + 4L_k K_{\tilde{V}} + 2L_k L_k$$

so we have

$$K_S^2 - 4\mathcal{X}^2 \tilde{S} = 2[K_{\tilde{V}}^2 - 4\mathcal{X}^2 V_k] + 2L_k K_{\tilde{V}}. \quad (2)$$

Moreover, in each blow-up $\alpha_j : V_j \to V_{j-1}$ we get

$$\mathcal{X}^2 V_j = \mathcal{X}^2 V_{j-1}; \quad K_{V_j} = \alpha_j^* K_{V_{j-1}} + E_j; \quad L_j = \alpha_j^* L_{j-1} - m_{j-1}E_j.$$

Then

$$2[K_{V_j}^2 - 4\mathcal{X}^2 V_j] + 2L_j K_{V_j} = 2[K_{V_{j-1}}^2 - 4\mathcal{X}^2 V_{j-1}] + 2L_{j-1} K_{V_{j-1}} + 2(m_{j-1} - 1) \geq 2[K_{V_{j-1}}^2 - 4\mathcal{X}^2 V_{j-1}] + 2L_{j-1} K_{V_{j-1}}. \quad (3)$$
Finally as $\tau : V \to B$ is an elliptic minimal fibration, numerically we have
$$K_V \equiv \left[ 2(b-1) + \mathcal{X} \mathcal{O}_V + \sum_i \frac{(n_i-1)}{n_i} \right] E \quad ([3] \text{ p.162})$$
where $E$ denotes a smooth fibre of $\tau$ and $\{n_i\}$ are the multiplicities of singular fibres of $\tau$. In particular $K_V^2 \equiv 0$.

As $\mathcal{L}_0^{\otimes 2} = \mathcal{O}_V(Z_0)$ and $Z_0$ is the branch divisor of $f_0$ we get $\mathcal{L}_0E = (g-1)$ by Hurwitz formula. So
\begin{equation}
2[K_{\mathcal{L}_0} - 4\mathcal{X}\mathcal{O}_{\mathcal{L}_0}] + 2\mathcal{L}_0 K_{\mathcal{L}_0} = -8\mathcal{X}\mathcal{O}_{\mathcal{L}_0} + 
2\mathcal{L}_0E \left[ 2(b-1) + \mathcal{X}\mathcal{O}_{\mathcal{L}_0} + \sum_i \frac{(n_i-1)}{n_i} \right] \geq 4(b-1)(g-1) + 2(g-5)\mathcal{X}\mathcal{O}_V.
\end{equation}

Then (a) follows from (1), (2), (3) and (4) and from the fact that $\mathcal{X}\mathcal{O}_V \geq 0$ for elliptic fibrations.

(b) Looking at the proof of (a) we see that $\lambda = 4$ iff $\mathcal{X}\mathcal{O}_V = 0$ and equality holds in (1), (2), (3) and (4). So we have $\lambda = 4$ iff $S$ is the minimal desingularization of a double cover of an elliptic, relatively minimal, fibration $\tau : V \to B$ such that:

- $\tau$ has no multiple fibres ($\forall i \quad n_i = 1$).
- $\mathcal{X}\mathcal{O}_V = 0$.
- The branch divisor $Z_0$ of the double cover has only negligible singularities (see [13], [17]), i.e. all the multiplicities of the singularities of the branch divisors in the process of canonical resolution are 2 or 3.

But the first two conditions are equivalent to the fact that $\tau$ is smooth and isotrivial (see 15 thms. 6,7 Ch. IV). This allows us to construct examples with $\lambda(\tau) = 4$ which are essentially the same as in [19] example 4.3. So the bound is sharp. \hfill \Box

Remark 2.2 Although we cannot use double covers for the case of bielliptic fibrations of genus 5 we already know that $\lambda \geq 4$ also holds for such fibrations (see [9] thm.5.1, [11]).

References.


ADDRESS: Miguel A. Barja. Dept. Matemàtica Aplicada I. Universitat Politècnica de Catalunya Diagonal 647, 08028 Barcelona, Spain. e-mail: barja@mat.upc.es