Eigenvalues and eigenvectors of monomial matrices

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**Abstract**— Spectral properties of special matrices matrices have been widely applied. We focus on monomial matrices over a finite field $F_p$, $p \neq 2$. We describe a way to find the minimal annihilating polynomial, a set of linearly independent eigenvectors.

**Keywords**: Monomial matrices, eigenvalues, eigenvectors.

1 Introduction

Eigenvalues and eigenvectors have many applications in pure and many areas in applied mathematics. Monomial matrices occur in the explicit description of the isometry class of a linear code (2, 4, 5). Also they appear when you want to construct self-dual codes (see [3] for example).

In this paper we determine eigenvalues and eigenvectors of monomial matrices. The method of finding eigenvalues by obtaining the roots of the characteristic polynomial is not workable many times. Here we present a method which relies on the cycle type of the permutation associated to the monomial matrix. More concretely, in Section 2 we state the basic results and present a short summary of all well-known definitions which will be used later.

Throughout the paper, we will denote by $F_p$ the finite field of $p$ elements, with $p$ a prime number, $p \neq 2$. Some of the results obtained depend on the characteristic of the finite field.

For any matrix $M \in M_n(F_p)$, let us denote by $Q_M(t) = \det(tI_n - M)$ the characteristic polynomial of $M$ and by $P_M(t)$ the minimal annihilating polynomial of $M$ (the monic polynomial of least degree which annihilates all vectors in $E$).

2 Preliminaries

We first recall the basic definitions about monomial matrices.

**Definition 1** A monomial matrix of order $n$ is a regular $n \times n$-matrix which has in each row and in each column exactly one non-zero component.

Monomial matrices form a group. The product of monomial matrices is a monomial matrix. The inverse of a monomial matrix is again a monomial matrix.

Permutation matrices are monomial matrices in which all non-zero components are equal to 1. Its rows are a permutation of the rows of the identity matrix. We will denote by $P(\sigma)$ the permutation matrix associated to the permutation $\sigma$; that is to say, the permutation matrix in which the non-zero components are in columns $i_1, \ldots, i_n$. Equivalently, the permutation matrix in which the permutation applied to the rows of the identity matrix $I_n$ is $i_1, \ldots, i_n$.

Unlike permutation matrices, monomial matrices are not necessarily orthogonal.

The following property of monomial matrices is well-known and will be useful for our purposes.

**Lemma 2** Every monomial matrix is a product of a diagonal matrix with a permutation matrix.

In general, we will make use of the following notation. We will write any monomial matrix as:

$$M(a_1, \ldots, a_n; i_1, \ldots, i_n) = \text{diag}(a_1, \ldots, a_n)P(i_1, \ldots, i_n)$$

**Example:** $M(a_1, a_2, a_3; 3, 1, 2) = \left(\begin{array}{ccc} 0 & 0 & a_1 \\ a_2 & 0 & 0 \\ 0 & a_3 & 0 \end{array}\right)$

where $P(3, 1, 2) = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$.

Any permutation $\sigma$ of $\{1, \ldots, n\}$ can be written as a product of disjoint cycles (also called “orbits”). The usual notation $(i_1, \ldots, i_k)$ of a $k$-cycle means that $i_1$ is replaced by $i_2$, $i_2$ by $i_3$, and so on being the last replacement $i_k$ by $i_1$. A 1-cycle will be denoted by $(i)$ and it means that this element remains
unchanged (it is a fixed point of the permutation). There is not an only possibility of the decomposition since being the cycles disjoint they can be written in any order and, moreover, any rotation of a given cycle specifies the same cycle. The cycle type of a permutation is the data of how many cycles of each length are present in the cycle decomposition of the cycle. If the cycle is a product of \(m_1 k_1\)-cycles, \(m_2 k_2\)-cycles, ..., \(m_r k_r\)-cycles, then we will write that its cycle type is \(k_1 + \cdots + m_1 k_1 + k_2 + \cdots + m_2 k_2 + \cdots + k_r + \cdots + m_r k_r\). Two permutations are conjugate in the symmetric group if and only if they have the same cycle type. See, for example, [I] for further reading about this topic.

3 Characteristic polynomial

The decomposition of the permutation associated to the matrix in disjoint cycles and, more concretely, the cycle type, determines the characteristic polynomial. The eigenvalues are the roots of this characteristic polynomial.

We list below the characteristic polynomial of all monomial matrices of order \(n\), for \(n = 2, 3, 4, 5\) and \(n = 4\). For \(n > 4\), analogous tables can be constructed.

The factorization of the characteristic polynomial into irreducible factors depends not only on the cycle type of the permutation but on the finite field.

For \(n = 2\):

<table>
<thead>
<tr>
<th>Monomial matrix</th>
<th>Cycle type</th>
<th>Characteristic polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M(a_1, a_2; 1, 2))</td>
<td>1 + 1</td>
<td>((t - a_1)(t - a_2))</td>
</tr>
<tr>
<td>(M(a_1, a_2; 2, 1))</td>
<td>2</td>
<td>(t^2 - a_1 a_2)</td>
</tr>
</tbody>
</table>

For \(n = 3\):

<table>
<thead>
<tr>
<th>Monomial matrix</th>
<th>Cycle type</th>
<th>Characteristic polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M(a_1, a_2, a_3; 1, 2, 3))</td>
<td>1 + 1 + 1</td>
<td>((t - a_1)(t - a_2)(t - a_3))</td>
</tr>
<tr>
<td>(M(a_1, a_2, a_3; 2, 1, 3))</td>
<td>1 + 2</td>
<td>((t^2 - a_1 a_2)(t - a_3))</td>
</tr>
<tr>
<td>(M(a_1, a_2, a_3; 3, 2, 1))</td>
<td>1 + 2</td>
<td>((t^2 - a_1 a_3)(t - a_2))</td>
</tr>
<tr>
<td>(M(a_1, a_2, a_3; 1, 3, 2))</td>
<td>1 + 2</td>
<td>((t^2 - a_2 a_3)(t - a_1))</td>
</tr>
<tr>
<td>(M(a_1, a_2; a_3; 2, 3, 1))</td>
<td>3</td>
<td>(t^3 - a_1 a_2 a_3)</td>
</tr>
<tr>
<td>(M(a_1, a_2; a_3; 3, 1, 2))</td>
<td>3</td>
<td>(t^3 - a_1 a_2 a_3)</td>
</tr>
</tbody>
</table>

For \(n = 4\):

<table>
<thead>
<tr>
<th>Monomial matrix</th>
<th>Cycle type</th>
<th>Characteristic polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M(a_1, a_2, a_3; 1, 2, 3, 4))</td>
<td>1 + 1 + 1 + 1</td>
<td>((t - a_1)(t - a_2)(t - a_3)(t - a_4))</td>
</tr>
<tr>
<td>(M(a_1, a_2, a_3; 2, 1, 3, 4))</td>
<td>1 + 1 + 2</td>
<td>((t^2 - a_1 a_2)(t - a_3)(t - a_4))</td>
</tr>
<tr>
<td>(M(a_1, a_2, a_3; 3, 2, 1, 4))</td>
<td>1 + 1 + 2</td>
<td>((t^2 - a_1 a_3)(t - a_2)(t - a_4))</td>
</tr>
<tr>
<td>(M(a_1, a_2, a_3; 4, 3, 2, 1))</td>
<td>1 + 1 + 2</td>
<td>((t^2 - a_2 a_3)(t - a_1)(t - a_4))</td>
</tr>
<tr>
<td>(M(a_1, a_2, a_3; 1, 2, 4, 3))</td>
<td>1 + 1 + 2</td>
<td>((t^2 - a_1 a_2 a_3)(t - a_4))</td>
</tr>
<tr>
<td>(M(a_1, a_2, a_3; 2, 1, 4, 3))</td>
<td>1 + 1 + 2</td>
<td>((t^2 - a_1 a_3 a_4)(t - a_2))</td>
</tr>
<tr>
<td>(M(a_1, a_2, a_3; 3, 2, 1, 4))</td>
<td>1 + 1 + 2</td>
<td>((t^2 - a_2 a_3 a_4)(t - a_1))</td>
</tr>
<tr>
<td>(M(a_1, a_2, a_3; 1, 3, 2, 4))</td>
<td>1 + 1 + 2</td>
<td>((t^2 - a_1 a_2 a_3 a_4))</td>
</tr>
<tr>
<td>(M(a_1, a_2, a_3; 2, 3, 1, 4))</td>
<td>1 + 1 + 2</td>
<td>((t^2 - a_1 a_3 a_4)(t - a_2))</td>
</tr>
<tr>
<td>(M(a_1, a_2, a_3; 3, 1, 2, 4))</td>
<td>1 + 1 + 2</td>
<td>((t^2 - a_2 a_3 a_4)(t - a_1))</td>
</tr>
<tr>
<td>(M(a_1, a_2; a_3; 1, 2, 3, 4))</td>
<td>3</td>
<td>(t^3 - a_1 a_2 a_3 a_4)</td>
</tr>
<tr>
<td>(M(a_1, a_2; a_3; 2, 1, 3, 4))</td>
<td>3</td>
<td>(t^3 - a_1 a_2 a_3 a_4)</td>
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<tr>
<td>(M(a_1, a_2; a_3; 3, 2, 1, 4))</td>
<td>3</td>
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<td>(M(a_1, a_2; a_3; 4, 3, 2, 1))</td>
<td>3</td>
<td>(t^3 - a_1 a_2 a_3 a_4)</td>
</tr>
</tbody>
</table>

In general, we have that the characteristic polynomial of a monomial matrix \(M(a_1, \ldots, a_n; i_1, \ldots, i_n)\) can be deduced from the decomposition into disjoint cycles of the permutation \(i_1 \ldots i_n\) and the coefficients \(a_1, \ldots, a_n\).

**Theorem 3** The characteristic polynomial of each cycle of length \(k\) \(j_1 \ldots j_k\) in the decomposition of the permutation \(i_1 \ldots i_n\) into disjoint cycles, being \(a_{j_1}, \ldots, a_{j_k}\) the coefficients of the matrix in columns \(j_1, \ldots, j_k\), \(t^k - a_{j_1} \cdots a_{j_k}\) is a divisor of the characteristic polynomial of \(M = M(a_1, \ldots, a_n; i_1, \ldots, i_n)\).

**Proof.** The proof follows of the following facts.

1. The characteristic polynomial of \(M\) is a product of factors, each of them corresponding to one of the disjoint cycles in the decomposition of \(i_1 \ldots i_k\).

2. For each cycle in such a decomposition, the corresponding polynomial is as in the statement.

**Examples.** 1) Let us consider the matrix \(M\)

\[
M(a_1, a_2, a_3, \ldots, a_n; n, 1, 2, \ldots, n - 1) = \\
\begin{pmatrix}
0 & 0 & \cdots & 0 & a_1 \\
0 & a_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & a_n
\end{pmatrix}
\]

The permutation consists of one single cycle of length \(n\), being the corresponding non-zero coefficients \(a_1, a_2, a_3, \ldots, a_n\). Then the characteristic polynomial of this matrix is

\[
Q_M(t) = t^n - \prod_{i=1}^{n} a_i
\]
Let us consider the particular case where \( n = 7, a_1 = a_2 = 1, a_3 = \cdots = a_7 = 2 \).

In \( \mathbb{F}_5[t] \), \( Q_M(t) = t^7 - 2 \) and the only eigenvalue of \( M \) in \( \mathbb{F}_5 \) is \( \lambda = 3 \).

In \( \mathbb{F}_7[t] \), \( Q_M(t) = t^7 - 4 \) and the only eigenvalue of \( M \) in \( \mathbb{F}_5 \) is \( \lambda = 4 \).

In \( \mathbb{F}_{11}[t] \), \( Q_M(t) = t^7 - 10 \) and the only eigenvalue of \( M \) in \( \mathbb{F}_{11} \) is \( \lambda = 10 \).

2) Let us consider the matrix \( M \)

\[
M(a_1, a_2, \ldots, a_{n-1}, a_n; 2, 3, \ldots, i, 1, i+2, i+3, \ldots, n, i+1) =
\begin{pmatrix}
0 & a_1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
& \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{i-1} & 0 & 0 & \cdots & 0 \\
a_i & 0 & \cdots & 0 & 0 & \cdots & \vdots & \vdots \\
& \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & a_{i+1} & \cdots & 0 & 0 \\
& \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{n-1} \\
0 & 0 & \cdots & 0 & 0 & \cdots & a_n & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

In this case, the permutation is a product of two cycles of lengths \( i \) and \( n-i \), being the corresponding non-zero coefficients \( a_1, \ldots, a_{i-1}, a_i \) and \( a_{i+1}, \ldots, a_{n-1}, a_n \), respectively. Then

\[
Q_M(t) = \left( t^i - \prod_{j=1}^{i-1} a_j \right) \left( t^n - \prod_{j=i+1}^{n} a_j \right)
\]

Let us consider again the particular case where \( n = 7, a_1 = a_2 = 1, a_3 = \cdots = a_7 = 2 \) and \( i = 3 \).

In \( \mathbb{F}_5[t] \), \( Q_M(t) = (t^3 - 2)(t^4 - 1) \) and the eigenvalues of \( M \) in \( \mathbb{F}_5 \) are \( \lambda_1 = 3 \) (with multiplicity 2), \( \lambda_2 = 1 \), \( \lambda_3 = 2 \) and \( \lambda_4 = 4 \).

In \( \mathbb{F}_7[t] \), \( Q_M(t) = (t^3 - 2)(t^4 - 2) \) and the only eigenvalues of \( M \) in \( \mathbb{F}_5 \) are \( \lambda_1 = 2 \) and \( \lambda_2 = 5 \).

In \( \mathbb{F}_{11}[t] \), \( Q_M(t) = (t^3 - 2)(t^4 - 5) \) and the only eigenvalues of \( M \) in \( \mathbb{F}_{11} \) are \( \lambda_1 = 7 \), \( \lambda_2 = 2 \) and \( \lambda_3 = 9 \).

The minimal polynomial \( P_M(t) \) of a matrix \( M \) is a divisor of the characteristic polynomial. It can be used, for example, to know whether a matrix is diagonalizable or not.

In order to state the main result, we need to introduce some notation.

Let us denote by \( G_k(t) \) the least common multiple of all factors in the characteristic polynomial of \( M = M(a_1, \ldots, a_n; i_1, \ldots, i_n) \) of degree \( k \) corresponding to cycles of length \( k, 1 \leq k \leq n \). Then we can obtain the minimal annihilating polynomial of \( M \) from \( G_1(t), \ldots, G_k(t) \) as follows.

**Theorem 4** \( P_M(t) = LCM(G_1(t), \ldots, G_k(t)) \).

**Example.**

Let us consider the matrix \( M \in M_9(\mathbb{F}_5) \)

\[
M(2, 3, 1, 1, 3, 4, 1, 4, 2, 3, 1, 5, 4, 7, 6, 8, 9)
\]

\[
= \begin{pmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix}
\]

Then \( 231547689 = (2, 3, 1)(5, 4)(7, 6)(8)(9) \) and \( Q_M(t) = (t^3 - 6)(t^2 - 3)(t^2 - 4)(t - 1)(t - 4) \), according to Theorem 1. Therefore \( G_1(t) = (t - 1)(t - 4), G_2(t) = (t^2 - 3)(t^2 - 4) \) and \( G_3(t) = t^3 - 1 \). Thus

\[
P_M(t) = LCM((t - 1)(t - 4), (t^2 - 3)(t^2 - 4), t^3 - 1)
\]

\[
= LCM((t - 1)(t - 4), (t^2 - 3)(t - 2)(t - 3), (t - 1)(t^2 + t + 1))
\]

\[
= (t - 1)(t - 2)(t - 3)(t - 4)(t^2 - 3)(t^2 + t + 1)
\]

**4 Eigenvectors**

Let us consider a monomial matrix

\[
M = M(a_1, \ldots, a_n; i_1, \ldots, i_n).
\]

Our goal is to determine a maximal set of linearly independent of eigenvectors. Let us assume that the permutation \( i_1, \ldots, i_n \) splits into disjoint cycles: \( m_1 \) cycles of length \( k_1, \ldots, m_\ell \) cycles of length \( k_\ell \).

For any irreducible cycle \( (j) \) of length 1, we have an eigenvector \( e_j \).

We know after Section 2 that for each cycle \( a_{j_1} \ldots a_{j_k} \) of length \( k \geq 2 \) there exists a factor \( t^k - a_{j_1} \ldots a_{j_k} \) in the decomposition of the characteristic polynomial of \( M \) into irreducible factors. Depending on the finite field we consider, this polynomial has a number \( l \) of roots, \( 0 \leq l \leq k \).

**Theorem 5** For each root \( \lambda \) (in the case where there exists any) we obtain an eigenvector:

\[
(\lambda^{k-1}a_{j_1}a_{j_2} \ldots a_{j_k}, \lambda a_{j_1}a_{j_2} \ldots a_{j_k}, \ldots, \lambda^{k-3}a_{j_{k-1}}a_{j_k}, \lambda^{k-2}a_{j_k})
\]
This statement can be directly checked by straightforward calculations.

Example.

Let us consider the matrix $M$

$$M(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9; 2, 3, 4, 1, 6, 7, 8, 9, 5) =
\begin{pmatrix}
0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 \\
a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 \\
0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0
\end{pmatrix}$$

The eigenvalues of $M$ have the following form:

- For each root $\lambda$ of the polynomial $t^4 - a_1a_2a_3a_4$ in $\mathbb{F}_p$ we have the linearly independent vectors:
  $$(\lambda^3, a_2a_3a_4, \lambda a_3a_4, \lambda^2a_4, 0, 0, 0, 0)$$

- For each root $\mu$ of the polynomial $t^5 - a_5a_6a_7a_8a_9$ in $\mathbb{F}_p$ we have the linearly independent vectors:
  $$(\mu^4, a_5a_7a_8a_9, \mu a_7a_8a_9, \mu^2a_8a_9, \mu^3a_9)$$

References


