problem in Rayleigh fast-fading channels. Application of
equalization techniques should also be considered to combat
the performance degradation due to intersymbol interference
in these cases.

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Comments on and Extensions of Wolf's Signal-to-Channel
Noise Formulas for Delta-Modulated Systems

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Abstract—The channel noise effects on linear delta modulation (LDM)
systems have not yet been adequately analyzed. This paper presents a new
and general formulation of these effects, based on the theoretical work by
Wolf [1]. A comparative discussion of our formulas with previous results is
also included. Finally, the application of our methods and the validity
of our comments are illustrated by some numerical examples.

I. INTRODUCTION

This paper gives a method for calculating the signal-to-channel
noise power ratio in a linear delta modulation (LDM) sys-
tem. Although LDM systems do not offer a valid alternative to
PCM for applications requiring a wide dynamic range [2], the
technique described can be applied to other robust DM meth-
ods, especially digitally syllabic-companded delta modulation
(DSCDM) systems.

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In many cases, the channel noise effect is negligible with respect to the quantization noise (granular and excess slope errors), but, in some applications, where the quantization noise is made relatively negligible in an otherwise noisy channel, this effect can be a critical quality parameter. Even in cases where the two noise components are of equal importance, one can calculate the channel noise effects by the techniques described herein.

The quality criterion employed in this paper is the classical mean-squared error measure. Although in most applications of voice or image transmission subjective criteria may be more appropriate, the mean-squared expressions are a first-order indicator. Furthermore, they can be modified easily into a weighted mean-squared error criterion, which is in closer agreement with subjective evaluations.

Finally, it should be pointed out that extensions of the given formulation are easily obtained for those cases in which the independent channel error model cannot be used (e.g., in commercial telephony) in the same way that Wolf indicated [11] for the Gilbert burst-noise model. The only modification necessary would be in the digital autocorrelation function $E(n_i n_{i+m})$ (see Section 11).

II. SIGNAL AND CHANNEL NOISE POWERS

Let $b_i$ indicate the $i$th transmitted symbol, where $b_i = \pm 1$, and $n_i$ the channel noise effect on this symbol, where $n_i = 0$ if there is no error and $n_i = \pm 2$ if there is an error in the decision on $b_i$.

The receiver detects and regenerates the incoming channel signal; if $p(t)$ is the basic shape of the regenerated pulses, we will have, at the input to the LD demodulator,

$$s(t) = \sum_{j=-\infty}^{\infty} b_p(t - iT)$$

and a noise term:

$$n(t) = \sum_{j=-\infty}^{\infty} n_p(t - iT)$$

where $T^{-1}$ is the symbol rate. The LD demodulator can be considered a linear system, having an equivalent impulse response $h(t)$ and a transfer function $H(f)$. Figure 1 shows the general situation just described.

Since the output of the LD demodulator is a continuous process whose power level does not depend on the time origin, we can randomize the reference time for the pulses [3], obtaining the signal and noise terms:

$$s(t) = \sum_{j=-\infty}^{\infty} b_p(t + \theta - iT) \quad (3)$$

$$n(t) = \sum_{j=-\infty}^{\infty} n_p(t + \theta - iT) \quad (4)$$

where $\theta$ is a random variable, uniformly distributed over $[0, \ T]$. These signal and noise terms are stationary processes, with respective autocorrelation functions [3]:

$$R_{se}(t) = \frac{1}{T} \sum_{m=-\infty}^{\infty} E(b_i b_{i+m}) R_{pp}(t + mT) \quad (5)$$

$$R_{nn}(t) = \frac{1}{T} \sum_{m=-\infty}^{\infty} E(n_i n_{i+m}) R_{pp}(t + mT) \quad (6)$$

and the power spectral densities:

$$G_{es}(f) = \frac{1}{T} \sum_{m=-\infty}^{\infty} E(b_i b_{i+m}) S_{pp}(f) \exp(j2\pi mTf) \quad (7)$$

$$G_{nn}(f) = \frac{1}{T} \sum_{m=-\infty}^{\infty} E(n_i n_{i+m}) S_{pp}(f) \exp(j2\pi mTf). \quad (8)$$

Here $R_{pp}(t)$ and $S_{pp}(f)$ are the autocorrelation function and the energy spectral density of $p(t)$, respectively. Then, the signal and noise powers at the output can be calculated with the help of the following formulas:

$$P(s_0) = \frac{1}{T} \sum_{m=-\infty}^{\infty} E(b_i b_{i+m}) [R_{pp}(t) \ast R_{hh}(t)] \int_{-\infty}^{\infty} S_{pp}(f) \ |H(f)|^2 \ \exp(j2\pi mTf) df; \quad (9)$$

$$P(n_0) = \frac{1}{T} \sum_{m=-\infty}^{\infty} E(n_i n_{i+m}) [R_{pp}(t) \ast R_{hh}(t)] \int_{-\infty}^{\infty} S_{pp}(f) \ |H(f)|^2 \ \exp(j2\pi mTf) df; \quad (10)$$

where $\ast$ indicates convolution, and $R_{hh}(t)$ is the autocorrelation function of the impulse response of the LD demodulator.

We have thus found the signal-to-channel noise formula:

$$\frac{P(s_0)}{P(n_0)} = \frac{\sum_{m=-\infty}^{\infty} E(b_i b_{i+m}) I(mT)}{\sum_{m=-\infty}^{\infty} E(n_i n_{i+m}) I(mT)} \quad (11)$$

where

$$I(mT) = \int_{-\infty}^{\infty} S_{pp}(f) \ |H(f)|^2 \ \exp(j2\pi mTf) df. \quad (12)$$

This signal-to-channel noise formula is absolutely general for LDM systems. It can also be extended to adaptive (variable step) DM systems, using an appropriate redefinition of the signal and noise terms, although the digital source model will have to be changed. It is also possible to consider line coding effects, but we will restrict our discussion to direct transmission.
III. WOLF'S MODEL FOR INDEPENDENT CHANNEL ERRORS

The only problem remaining to be solved is the evaluation of $E(b_i b_{i+m})$ and $E(n_i n_{i+m})$. Wolf's model [1] is appropriate for evaluating these mathematical expectations assuming independent channel errors. This model consists of two synchronized Markov chains for the digitized source and the channel errors; Figure 2 shows the model. The transition probabilities $P_T$ are equal, since we assume symmetry in this regard. Only $P_T$ is required for the evaluation of the digital autocorrelation functions, and we can obtain it from more structured (and realistic) Markov models for the quantized signal.

We derive the signal-to-channel noise formula in Appendix I. The obtained result is:

$$P(s_0) = \frac{1}{4P_e} \int_{-\infty}^{\infty} \left[ \frac{P_T}{P_T^2 + (1 - 2P_T) \sin^2 \pi fT} \right] S_{pp}(f) / |H(f)|^2 df.$$  (13)

In practical systems, the final step in the LD demodulator is a bandpass filter, with lower and upper cutoff frequencies $f_{c1}$, $f_{c2}$, respectively, and $1/T \gg f_{c2}$. A first approximation would be to assume $\sin^2 \pi f T \approx 0$ in the effective band of integration. Then:

$$\sum_{m=-\infty}^{\infty} E(b_i b_{i+m}) \exp(i2\pi m fT) \approx (1 - P_T)/P_T$$  (14)

$$\sum_{m=-\infty}^{\infty} E(n_i n_{i+m}) \exp(i2\pi m fT) \approx 4P_e[1 - 2P_e(1 + P_T)/P_T]$$  (15)

and

$$P(s_0) \approx \frac{1}{4P_e} \frac{1 - P_T}{P_T + P_e(1 - 2P_T)}.$$  (16)

This is Wolf's formula. Note that Wolf did not consider the final bandpass filter. We note that the approximation is acceptable when $1/T$ is large with respect to $f_{c2}$. If we accept the approximation, the resulting formula will apply independent of the shape of the regenerated pulses. Wolf assumed a perfect integrator, but his result is also useful for leaky and double integration systems and for delta-sigma systems. In the latter case, $P_T$ would correspond to a model of the digitized integrated message signal. The same model previously indicated would apply to the process resulting from the integration of the analog message; we will denote the corresponding transition probability by $P_T'$.

Wolf's formula is exact when $P_T = 0.5$, i.e., when $P(s_0)/P(n_0) = 1/4P_e$, but it will be more and more inaccurate as $P_T$ approaches zero or unity. This appears to be the main reason for the progressive separation of Wolf's theoretical curve from Braun's experimental values [1]: the fitting for low $P_e$ implied deviation for high $P_e$ values.

If $P_T$ approaches unity, $P(s_0)$ will increase appreciably in practice with respect to the value obtained assuming $\sin^2 \pi f T = 0$, due to the term $(1 - 2P_T) \sin^2 \pi f T$ in the denominator of (1.3), considering that, in practice, the effective bandwidth corresponding to $H(f)$ is less than $1/T$. $P(n_0)$ has a small variation because of the presence of the factor $P_e$ in the second term of (1.4). Consequently, $P(s_0)/P(n_0)$ exceeds Wolf's value. Just the contrary occurs when $P_T$ approaches zero. The variations are larger when $P_e$ is small, i.e., in "good" channel cases.

It is easy to see that our formulation gives $P(s_0)/P(n_0) = 0$ when $P_T = 0$ or $P_T = 1$ (signal absent), but it also implies that $P(s_0)/P(n_0) \neq 0$ when $P_e = 1/2$. This incorrect result is due to the problem present other difficulties [1]. Nevertheless, the cases in which $P_e \approx 1/2$ correspond to transmissions over unusable channels (having a capacity near zero), and the previous $P(s_0)/P(n_0)$ formulas are decreasing functions of $P_e$ which indicate the system performance in all practical situations.

IV. EXAMPLES

When the information consists of a voice signal and the system is approximately optimized with respect to granular and excess slope noises, Kikkert's simulation results [4] allow us to conclude that $P_e$ will not be very far from $0.5$. In this case, the application of Wolf's formula will be acceptable. This remains true as long as $1/T$ is large enough to maintain $\sin^2 \pi f T \approx 0$ in the integration band. Nevertheless, we will present more general calculations to illustrate the departure from Wolf's results.

Let $P_T$ be the approximation of $P_T$, the probability of a symbol being transmitted. From the above expression of $p(t)$:

$$S_{pp}(f) = V^2 T^2 \sin^2 \pi f T.$$  (19)

where:

$$\text{sinc} \ x \triangleq \sin \pi x / \pi x.$$  (20)
Fig. 3: Perfect integration linear delta demodulator (T: delay; the final block is an ideal bandpass filter).

TABLE I
SIGNAL-TO-CHANNEL NOISE POWER RATIOS (IN dB), PERFECT INTEGRATION (AND WOLF'S FORMULA VALUES)

<table>
<thead>
<tr>
<th>$P_T$</th>
<th>$10^1$</th>
<th>$10^3$</th>
<th>$10^5$</th>
<th>$10^7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>11.57</td>
<td>34.68</td>
<td>54.92</td>
<td>74.91</td>
</tr>
<tr>
<td>0.20</td>
<td>6.73</td>
<td>29.86</td>
<td>49.91</td>
<td>69.91</td>
</tr>
<tr>
<td>0.35</td>
<td>6.26</td>
<td>26.63</td>
<td>46.87</td>
<td>66.67</td>
</tr>
<tr>
<td>0.50</td>
<td>3.98</td>
<td>23.96</td>
<td>43.96</td>
<td>63.96</td>
</tr>
<tr>
<td>0.65</td>
<td>1.51</td>
<td>21.30</td>
<td>41.30</td>
<td>61.30</td>
</tr>
<tr>
<td>0.80</td>
<td>4.69</td>
<td>17.98</td>
<td>37.97</td>
<td>57.97</td>
</tr>
<tr>
<td>0.95</td>
<td>8.36</td>
<td>12.21</td>
<td>32.21</td>
<td>52.21</td>
</tr>
</tbody>
</table>

Table I presents numerical results from our formulation, showing the small differences with respect to Wolf’s values (in brackets) if $P_T$ is not very different from 0.5, and the increasing differences when $P_T$ goes to zero or 1.

If a single RC (leaky) integrator is used, we will have:

$$| H(f) |^2 = \left[ \Pi \left( \frac{f - f_c}{b} \right) + \Pi \left( \frac{f + f_c}{b} \right) \right] / 4 \sin^2 \pi f T$$

(21)

where $f_c = (f_{c1} + f_{c2})/2$ is the central frequency, and $b = f_{c2} - f_{c1}$ is the bandwidth of the final bandpass filter. Table II presents numerical results corresponding to a typical $f_3$ equal to 150 Hz.

If $P_T = 0.5$ and we put $\text{sinc}^2 T f \approx 1$ in $[f_{c1}, f_{c2}]$, we will obtain Johnson's formula for $P(n_0)$ [6] [7].

A double integrator is shown in Figure 4. Typical parameters allow us to write:

$$| H(f) |^2 \approx \left[ \Pi \left( \frac{f - f_c}{b} \right) + \Pi \left( \frac{f + f_c}{b} \right) \right] / [1 + (f/f_3)^2]$$

(22)

where $f_3$ is the cutoff frequency of the RC filter. Table III shows numerical values corresponding to a typical $f_3$ equal to 150 Hz.

If $P_T = 0.5$ and we put $\text{sinc}^2 T f \approx 1$ in $[f_{c1}, f_{c2}]$, we will obtain Johnson's formula for $P(n_0)$ [6] [7].

A double integrator is shown in Figure 4. Typical parameters allow us to write:

$$| H(f) |^2 \approx \left[ \Pi \left( \frac{f - f_c}{b} \right) + \Pi \left( \frac{f + f_c}{b} \right) \right] f_c^2 / f_3^2 (1 + (f/f_3)^2)$$

(23)

Table III presents numerical results corresponding to a typical $f_3$ equal to 1 kHz. If $P_T = 0.5$ and we approximate $\text{sinc}^2 T f$ by unity, our formulation will give an already known result for $P(n_0)$ [2]:

$$P(n_0) \approx 8 P_e V^2 f_1^2 T (f_{c1} - 1/f_{c2})$$

(24)

$$f_2 = 1/2 \pi R_2 C_2 (1 - R_2 C_2 (1 + R_1/R_2)^2) / 4 R_1 C_1.$$  

(25)

In the case of delta-sigma modulation, the results obtained will not be directly comparable with the previous ones, since...


<table>
<thead>
<tr>
<th>$P_T$</th>
<th>$10^1$</th>
<th>$10^3$</th>
<th>$10^5$</th>
<th>$10^7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>9.98</td>
<td>31.54</td>
<td>51.56</td>
<td>71.56</td>
</tr>
<tr>
<td>0.20</td>
<td>6.36</td>
<td>23.28</td>
<td>42.29</td>
<td>62.29</td>
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<tr>
<td>0.35</td>
<td>6.20</td>
<td>26.53</td>
<td>46.53</td>
<td>66.53</td>
</tr>
<tr>
<td>0.50</td>
<td>3.98</td>
<td>23.98</td>
<td>43.98</td>
<td>63.98</td>
</tr>
<tr>
<td>0.65</td>
<td>1.53</td>
<td>21.33</td>
<td>41.33</td>
<td>61.33</td>
</tr>
<tr>
<td>0.80</td>
<td>-1.65</td>
<td>18.02</td>
<td>38.01</td>
<td>58.01</td>
</tr>
<tr>
<td>1.00</td>
<td>-8.32</td>
<td>11.25</td>
<td>21.25</td>
<td>41.25</td>
</tr>
</tbody>
</table>

Some signal-to-channel noise ratio values are included in Table IV.

$P_T' \neq P_T$ in a general case. The receiver has:

$$|H(f)|^2 = \Pi \left( \frac{f - f_c}{b} \right) + \Pi \left( \frac{f + f_c}{b} \right).$$

(27)

Some signal-to-channel noise ratio values are included in Table IV.

$P_T' = 0.5$ and $\sin^2 TF \approx 1$ in $[f_{c1}, f_{c2}]$ will lead to Johnson's formula for $P(n_0)$ [5] [6].

The case of linear delta-sigma modulation can be solved by calculating the corresponding $I(mT)$ in the time domain. Some integral functions appear in the resulting expression, which is difficult to manipulate and to interpret. Making $\sin^2 TF \approx 1$, we obtain:

$$I(mT) \approx 2V^2 T^2 \int_{f_{c1}}^{f_{c2}} \cos 2\pi mT df$$

$$= 2V^2 T^2 \left( f_{c2} \sin 2mT f_{c2} - f_{c1} \sin 2mT f_{c1} \right)$$

and the resulting $P(n_0)/P(n_0)$ can be expressed with the help of (11). The same approximate method can be applied in other cases, but the integrals $I(mT)$ have to be calculated numerically.

**V. SUMMARY AND CONCLUSIONS**

The general formulation of the signal-to-channel noise power ratio in (direct) LDM systems has been presented and compared with earlier results. The main conclusions are:

1) A slight modification of Wolf's work allows us to extend it to all LDM systems;

2) Wolf's formula is accurate enough in practical systems having $P_T$ not very different from 0.5, but it fails when $P_T$ goes to zero or $P_T$ goes to 1, and especially when $P_e$ is small or when $T$ is large.

Some numerical results are comparatively presented.

Additional comments are:

3) The method can be easily extended to correlated channel errors (e.g., to the case of burst errors accordingly to the Gilbert model [7], considered by Wolf), the only modification being the alteration of $E(n_0 n_{i+1})$ values (not very problematic in the case of the Gilbert model);

4) By properly redifining signal and noise terms, the method can be extended to coded LDM systems and to other practical DM systems (adaptive delta modulation, ADM, and, particularly, digitally syllabic companded delta modulation, DSCDM). We are obtaining results along these lines at the present time.

**APPENDIX I**

From the model, it is easy to obtain [11]:

$$E(b_i b_{i+m}) = (1 - 2P_T)^{|m|}$$

(1.1)

$$E(n_i n_{i+m}) = \begin{cases} 4P_e, & \text{if } m = 0 \\ 4P_e^2(1 - 2P_T)^{|m|}, & \text{if } m \neq 0 \end{cases}$$

(1.2)

where $P_e$ is the error probability. Then, rearranging the power expressions, we can write:

$$\sum_{m=-\infty}^{\infty} E(b_i b_{i+m}) \exp (j2\pi mTf)$$

$$= \sum_{m=-\infty}^{\infty} (1 - 2P_T)^{|m|} \exp (j2\pi mTf) =$$

$$= \frac{P_T(1 - P_T)}{P_T^2 + (1 - 2P_T) \sin^2 \pi Tf}$$

$$= 4P_e \left[ 1 + P_e \sum_{m=-\infty}^{\infty} (1 - 2P_T)^{|m|} \exp (j2\pi mTf) \right]$$

$$= 4P_e \left[ 1 + P_e \frac{P_T(1 - 2P_T) - (1 - 2P_T) \sin^2 \pi Tf}{P_T^2 + (1 - 2P_T) \sin^2 \pi Tf} \right]$$

(1.3)

(1.4)

where the prime sign in the summation indicates exclusion of the zero index term. The ratio of the integrals of these functions multiplied by $S_{pp}(f) \cdot |H(f)|^2$ will be the signal-to-channel noise ratio, formula (13) in the main text.

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