Pseudo-normal form near saddle–center or saddle–focus equilibria.

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Abstract

In this paper we introduce the pseudo-normal form, which generalizes the notion of normal form around an equilibrium. Its convergence is proved for a general analytic system in a neighborhood of a saddle-center or a saddle-focus equilibrium point. If the system is Hamiltonian or reversible, this pseudo-normal form coincides with the Birkhoff normal form, so we present a new proof in these celebrated cases. From the convergence of the pseudo-normal form for a general analytic system several dynamical consequences are derived, like the existence of local invariant objects.

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§1 Introduction and main results

Since normal forms were introduced by Poincaré they have become a very useful tool to study the local qualitative behavior of dynamical systems around equilibria. Consequently, the literature devoted to this topic has been very extensive as the amount of authors involved (Poincaré, Dulac, Siegel, Birkhoff, Lyapunov, Sternberg, Arnold, Moser, Bibikov, Bruno and many others; for a general background see for instance [1, 7, 12] and references therein). In a few words, given a system

\[ \dot{X} = F(X) = \Lambda X + O(X^2), \]

(1)

around an equilibrium, say the origin \( X = 0 \), a general normal form procedure consists on looking for a (formal power series close to the identity) transformation \( X = \Phi(\chi) \) in such a way that the new system \( \dot{\chi} = \Phi^* F(\chi) \) takes its simplest form. This is called normal form and contains only the so-called resonant terms, monomials whose powers are intimately related to the characteristic exponents of system (1) at the origin.

More precisely, if \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \) is the vector formed by the characteristic exponents of system (1) at the origin, i.e. the eigenvalues of the \( m \times m \)-matrix \( \Lambda \), then \( \lambda \) is called resonant if there exist \( p_1, p_2, \ldots, p_m \in \{0, 1, 2, 3, \ldots\} \), satisfying \( |p| := p_1 + p_2 + \cdots + p_m \geq 2 \), such that

\[ \lambda_s = p_1 \lambda_1 + p_2 \lambda_2 + \cdots + p_m \lambda_m = \langle p, \lambda \rangle \]

(2)

for some \( s \in \{1, 2, \ldots, m\} \). If \( z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m \) and \( p = (p_1, p_2, \ldots, p_m) \in (\mathbb{N} \cup \{0\})^m \), we say that

\[ z^p e_s = z_1^{p_1} z_2^{p_2} \cdots z_m^{p_m} e_s \]
with \( e_i^T = (0, \ldots, 0, 1, 0, \ldots, 0) \), is a resonant monomial if \( p \) and \( s \) satisfy (2). Thus, an analytic system

\[
(3) \quad \dot{\chi} = N(\chi),
\]

with \( \chi = (\zeta_1, \ldots, \zeta_m) \) and \( N(\chi) = (n_1(\chi), n_2(\chi), \ldots, n_m(\chi)) \), is said to be in normal form if all the terms in \( N(\chi) \) are resonant.

In this work, we will focus our attention on analytic vector fields and will be specially concerned with the convergence of the normalizing transformation \( \Phi \).

There are two well-known cases where a polynomial normal form is achieved. The convergence of its normalizing transformation depends only on the location of the vector of characteristic exponents \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \) on the complex plane and on some arithmetical properties. Namely,

(i) when \( \lambda \) belongs to the Poincaré domain, that is, the convex hull of the set \( \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \) does not contain the origin;

(ii) when \( \lambda \) belongs to the complementary of this domain, the so-called Siegel’s domain, and satisfies a Diophantine condition.

In the first case, the Theorem of Poincaré–Dulac ensures the convergence of a normalizing transformation conjugating the original system to a system having only resonant terms. Since in this situation there is just a finite number of resonant monomials, the normal form is a polynomial. In the second case, the Diophantine condition permits to bound the small divisors appearing in the normalizing transformation (see, for instance, [1, Chapter 5, §24]) and its convergence is also derived (Siegel’s Theorem). The original system is conjugated to its linear part, again a polynomial.

However, resonant normal forms with an infinite number of terms do arise in some important families of dynamical systems, like the Hamiltonian or the reversible ones. In such contexts the characteristic exponents come in pairs \( \{\pm \lambda\} \) and, therefore, they always belong to the Siegel’s domain. In these cases, convergence results depend not only on the location of the characteristic exponents and their arithmetical properties but also on the kind of formal normal form they exhibit. In 1971, Bruno (see [4, Chapter II, §3, §4]) provided sufficient and, in some particular sense, necessary conditions ensuring this convergence. He denominated them conditions A and \( \omega \). The so-called condition \( \omega \) depends on arithmetical properties of the vector of characteristic exponents \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \) and can be checked explicitly. On the contrary, condition A imposes a strong restriction on the normal form forcing it (up to all order \( k \)) to depend only on one or two scalar functions (see [4, pages 173–175]). These conditions read as follows.

**Condition A:** There exist formal power series \( a(\chi), b(\chi) \) such that in equation (3)

\[
N(\chi) = \begin{pmatrix}
\lambda_1 \zeta_1 \\
\lambda_2 \zeta_2 \\
\vdots \\
\lambda_m \zeta_m
\end{pmatrix} a(\chi) + \begin{pmatrix}
\overline{\lambda_1} \zeta_1 \\
\overline{\lambda_2} \zeta_2 \\
\vdots \\
\overline{\lambda_m} \zeta_m
\end{pmatrix} b(\chi),
\]

where \( \overline{\lambda_j} \) denotes the conjugate of \( \lambda_j \).

**Condition \( \omega \):** Set

\[
\omega_k = \min |\lambda_s - \langle p, \lambda \rangle|
\]

for \( \lambda_s - \langle p, \lambda \rangle \neq 0 \) and \( 2 \leq |p| < 2^k \). Then

\[
\sum_{k \geq 1} 2^{-k} \ln \omega_k
\]
converges.

Typically, for any vector of characteristic exponents \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \), \( \omega_k \) tends to zero as \( k \) goes to infinity. Since \( \lambda_3 - \langle p, \lambda \rangle \) appear as denominators in the normalizing transformation \( \Phi \) this is the so-called small divisors phenomenon. If \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \) satisfies condition \( \omega \) then these small divisors can be bounded. Notice that this condition is imposed only on the non-resonant terms of the system. If condition \( \omega \) is satisfied it follows the existence of \( c, \nu > 0 \) such that

\[
|\lambda_3 - \langle p, \lambda \rangle| > c \exp \left( -\nu |p| \right)
\]

(see [4, page 140]). With respect to these two conditions, in [4, Theorem 4, page 186], Bruno asserts

**Theorem 1 ([Bruno])** Given a system (1), if its characteristic exponents \( (\lambda_1, \lambda_2, \ldots, \lambda_m) \) satisfy condition \( \omega \) and its (formal) normal form satisfies condition \( A \), then there exists a convergent analytic transformation \( X = \Phi(\chi) \) transforming system (1) into normal form.

In some sense conversely, if a normal form (3) can be obtained from an analytic system, such that neither of the two conditions \( \omega \) and \( A \) is satisfied, then there exists a system (1) having (3) as its normal form and such that any transformation leading it into normal form is divergent [4, page 186].

In order to show how difficult is to check condition \( A \), it is particularly interesting to consider the case of a planar analytic vector field with the origin being a linear center equilibrium point, that is, with purely imaginary characteristic exponents. Although a formal normalizing transformation can be built for such system, its convergence can be ensured a priori just in case one knows that the origin is a center. Indeed, this fact forces the corresponding formal normal form to satisfy condition \( A \). On the other hand, if the origin is a focus, we have as a rule divergence (for more details, see [7, pages 121–122]). In the case this planar vector field is polynomial, one can look for the minimal number of conditions on their coefficients ensuring the origin to be a center. This is a famous question intimately related to a local version of the 16th Hilbert’s problem called the center-focus problem, which still remains open.

There are very few cases where the fulfillment of condition \( A \) follows from the nature of the original system. Some of them arise in Hamiltonian systems, where the normal form is called the Birkhoff normal form (BNF in short). We recall that a system (1) is (locally) Hamiltonian if there exists a function \( H \in \mathcal{C}^r(U) \), where \( r = 2, \ldots, \infty \), \( \omega \) and \( U \) is a neighborhood of the origin, and a 2-form \( \omega \in \Omega^2(U) \) such that \( \omega(F, \cdot) = dH \). Note that for a Hamiltonian system condition \( A \) admits an equivalent reformulation in terms of the Hamiltonian function (the so-called condition \( H \), [5, page 225]).

Thus, let us consider a Hamiltonian system and assume it has no small divisors. This implies that condition \( \omega \) is trivially satisfied. However, this means that we can only deal with Hamiltonian systems of one or two degrees of freedom, since any Hamiltonian system having more than 2 degrees of freedom presents always small divisors. Indeed, for a 1-degree of freedom system, BNF becomes \( \dot{\chi} = N(\chi) \), with \( \chi = (\xi, \eta) \in \mathbb{C} \),

\[
N = \begin{pmatrix}
\xi a(\xi \eta) \\
-\eta a(\xi \eta)
\end{pmatrix}
\]

and characteristic exponents \( \pm \lambda \), where \( \lambda = a(0) \in \mathbb{R} \). The assumption \( \lambda \neq 0 \) makes this normal form to satisfy condition \( A \).

Consider now a 2-degrees of freedom system and denote by \( \{ \pm \lambda_1, \pm \lambda_2 \} \) its characteristic exponents at the origin. It is not difficult to check that if \( \lambda_1/\lambda_2 \notin \mathbb{R} \) then its BNF \( \dot{\chi} = N(\chi) \) satisfies condition \( A \). This means that the origin has to be

(i) a **saddle-focus**, if \( \{ \pm \lambda_1, \pm \lambda_2 \} = \{ \pm \lambda \pm \mathrm{i} \alpha \} \) with \( \lambda, \alpha \in \mathbb{R} \setminus \{0\} \), or

(ii) a **saddle-center**, if \( \{ \pm \lambda_1, \pm \lambda_2 \} = \{ \pm \lambda, \pm \mathrm{i} \alpha \} \) and \( \lambda, \alpha \in \mathbb{R} \setminus \{0\} \).
In these cases \( N \) can be written as

\[
(5) \quad (a) \quad N = \begin{pmatrix}
\xi a_1(\xi, \mu, \nu) \\
-\eta a_1(\xi, \mu, \nu) \\
\mu a_2(\xi, \mu, \nu) \\
-\nu a_2(\xi, \mu, \nu)
\end{pmatrix} \quad \text{or} \quad (b) \quad N = \begin{pmatrix}
\xi a_1(\xi, \mu^2 + \nu^2) \\
-\eta a_1(\xi, \mu^2 + \nu^2) \\
\nu a_2(\xi, \mu^2 + \nu^2) \\
-\mu a_2(\xi, \mu^2 + \nu^2)
\end{pmatrix},
\]

respectively, where \( a_j(0,0) = \lambda_j, \ j = 1, 2 \) and \( \chi = (\xi, \eta, \mu, \nu) \in \mathbb{C}^4 \).

It was Lyapunov [16] in 1907, who provided a first result in this direction. Namely, he proved that given a real analytic Hamiltonian system with characteristic exponents at the origin

\[
\{ \pm \lambda_1 (\text{pure imaginary}), \pm \lambda_2, \ldots, \pm \lambda_n \}
\]

doing that

\[
\lambda_s \neq m \lambda_1
\]

for any \( m \in \mathbb{N} \) and \( s \in \{2, 3, \ldots, n\} \), there exists always a one-parameter analytic family of periodic solutions in a neighborhood of this equilibrium point (for a detailed proof, see for instance [25]). In other words, he proved the existence of a convergent normalizing transformation leading this system into BNF with respect to the variable associated to the characteristic exponent \( \lambda_1 \).

Later, in 1958, Moser [21] extended this result to the case of the equilibrium having characteristic exponents

\[
\{ \pm \lambda_1, \pm \lambda_2, \ldots, \pm \lambda_n \}
\]

verifying that

(i) \( \lambda_1, \lambda_2 \) are independent over \( \mathbb{R} \) and that \( \lambda_s \neq m_1 \lambda_1 + m_2 \lambda_2 \) for any \( m_1, m_2 \in \mathbb{N} \), and

(ii) \( s \in \{3, 4, \ldots, n\} \).

As it has mentioned above, this corresponds to the origin being a saddle-focus or a saddle-center equilibrium point. Moser proved the existence of an analytic convergent transformation leading the original system into BNF (with respect to the variables associated to \( \lambda_1, \lambda_2 \)).

Recently, a new proof of this theorem has been provided by Giorgilli [13] putting special emphasis on the Hamiltonian character of the system (a characteristic which does not appear in Moser’s proof).

At this point, it seems natural to wonder about the convergence of a normalizing transformation \( \Phi \) in the case of a general system. The analogy with the Hamiltonian case suggests to consider 2-dimensional and 4-dimensional systems with characteristic exponents at the equilibrium point being (i) \( \pm \lambda, \lambda \in \mathbb{C} \setminus \{0\} \) and (ii) \( \{ \pm \lambda_1, \pm \lambda_2 \} \), respectively, since they have no small divisors and condition \( \omega \) is clearly satisfied. Thus, the problem becomes to investigate how far (and in which way) are these systems from the fulfillment of condition A (and, therefore, of having a convergent normalizing transformation \( \Phi \)). The case (i) was studied in [10]. The aim of the present work is to deal with case (ii), a general analytic system (1) with a saddle-focus or a saddle-center equilibrium point at the origin.

Our intention is to compare such system with a Hamiltonian one, where BNF is convergent, and to build a kind of convergent extended BNF. We will ask it to have BNF as a particular situation and we expect to obtain some interesting information even in case condition A is not satisfied.

Let us be more precise. As it has been said, it is well-known in the saddle-focus or saddle-center Hamiltonian cases the existence of a convergent transformation \( X = \Phi(\chi) \) leading system (1) into BNF, that is, the transformed system being of the form

\[
(6) \quad \dot{\chi} = (\Phi^*F)(\chi) = N(\chi),
\]
where $N$ is of the first or second type in (5) respectively. Notice that equation (6) is equivalent to say that
\[
D\Phi N = F \circ \Phi.
\]

Our approach, which comes from ideas of Moser and DeLatte [8], consists on looking for a remainder term of the form
\[
(7) \quad \hat{B} = \begin{pmatrix}
\xi \hat{b}_1(\xi \eta, \mu \nu) \\
\eta \hat{b}_1(\xi \eta, \mu \nu) \\
\mu \hat{b}_2(\xi \eta, \mu \nu) \\
\nu \hat{b}_2(\xi \eta, \mu \nu)
\end{pmatrix}
\]
or
\[
(7) \quad \hat{B} = \begin{pmatrix}
\xi \hat{b}_1(\xi \eta, \mu^2 + \nu^2) \\
\eta \hat{b}_1(\xi \eta, \mu^2 + \nu^2) \\
\mu \hat{b}_2(\xi \eta, \mu^2 + \nu^2) \\
\nu \hat{b}_2(\xi \eta, \mu^2 + \nu^2)
\end{pmatrix},
\]

depending if we are considering the saddle-focus or saddle-center case, respectively, satisfying $\hat{b}_1(0,0) = \hat{b}_2(0,0) = 0$ and such that the equality
\[
(8) \quad D\Phi N + \hat{B} = F \circ \Phi
\]
holds. Hence forward $\hat{G}$ will denote vector fields constituted by formal powers series starting with terms of order at least 2. Notice that (8) is equivalent to saying that the new system is of the form
\[
\chi = N(\chi) + (D\Phi(\chi))^{-1} \hat{B}(\chi)
\]
which is not, as a rule, a normal form. Thus, we will say that $X = \Phi(\chi)$ transforms system (1) into pseudo-normal form ($\Psi$NF in short).

The interest, we think, of this construction lies in the following facts: first, it constitutes an extension of the BNF and, therefore, in the contexts where BNF converges they must coincide; second, this procedure is convergent in some situations where BNF does not apply and, thus, it translates the problem of the existence of a convergent normalizing transformation to the one of determining if some analytic scalar-valued functions $\hat{b}_1$ and $\hat{b}_2$ vanish. Moreover, even in the case that these functions do not vanish, some interesting dynamical consequences can be derived from this pseudo-normal form.

**Theorem 2 (Main Theorem)** Given a system
\[
(9) \quad \dot{X} = F(X) = \Lambda X + \hat{F}(X),
\]
analytic around the origin (an equilibrium) and with characteristic exponents $\{\pm \lambda_1, \pm \lambda_2\}$ equal to
- $\{\pm \lambda \pm i\alpha\}$ with $\lambda, \alpha \in \mathbb{R} \setminus \{0\}$ (saddle-focus case), or
- $\{\pm \lambda, \pm i\alpha\}$ with $\lambda, \alpha \in \mathbb{R} \setminus \{0\}$ (saddle-center case),

there exist an analytic transformation $X = \Phi(\chi) = \chi + \hat{\Phi}(\chi)$ and convergent analytic vector fields $N$, as in (5), and $\hat{B}$ as in (7) in such a way that the equality
\[
D\Phi N + \hat{B} = F \circ \Phi
\]
holds.

Section §2 is devoted to the proof of this theorem.

**Remark 1** The proof of this theorem is constructive. It is based on a recurrent scheme which provides the coefficients of $\Phi$, $N$ and $\hat{B}$ order by order. Moreover, a condition for determining the radius of convergence of these vector fields is provided in equation (75).
Remark 2 As it is usual in Normal Form Theory, computations will be carried out complexifying the variables. If it is not difficult to check that the corresponding \( \Psi NF \) vector fields in the real case are of the form

\[
N(\zeta) = \begin{pmatrix}
\xi a_1(\xi, \eta, \mu^2 + \nu^2) \\
-\eta a_1(\xi, \eta, \mu^2 + \nu^2) \\
\nu a_2(\xi, \eta, \mu^2 + \nu^2) \\
-\mu a_2(\xi, \eta, \mu^2 + \nu^2)
\end{pmatrix} \quad \hat{B}(\zeta) = \begin{pmatrix}
\xi \hat{b}_1(\xi, \eta, \mu^2 + \nu^2) \\
\eta \hat{b}_1(\xi, \eta, \mu^2 + \nu^2) \\
\mu \hat{b}_2(\xi, \eta, \mu^2 + \nu^2) \\
\nu \hat{b}_2(\xi, \eta, \mu^2 + \nu^2)
\end{pmatrix},
\]

in the saddle-center case and

\[
N(\zeta) = \begin{pmatrix}
\xi a_1(\xi, \eta, \mu\nu) \\
-\eta a_1(\xi, \eta, \mu\nu) \\
\mu a_2(\xi, \eta, \mu\nu) \\
-\nu a_2(\xi, \eta, \mu\nu)
\end{pmatrix} \quad \hat{B}(\zeta) = \begin{pmatrix}
\xi \hat{b}_1(\xi, \eta, \mu\nu) \\
\eta \hat{b}_1(\xi, \eta, \mu\nu) \\
\mu \hat{b}_2(\xi, \eta, \mu\nu) \\
\nu \hat{b}_2(\xi, \eta, \mu\nu)
\end{pmatrix},
\]

in the saddle-focus one, where \((\xi, \eta, \mu, \nu) \in \mathbb{R}^4\) and the functions \(a_\ell\) and \(b_\ell, \ell = 1, 2\), are real.

A first consequence of Theorem 2 is that, if the initial system is \(hamiltonian\) then the \(\Psi NF\) becomes BNF. This is the thesis of the following proposition whose proof has been deferred to Section §3.

**Proposition H1** System (9) is \(hamiltonian\) in a neighborhood of the origin if and only if \(\hat{B}\) vanishes (and, therefore, \(\Psi NF\) becomes BNF).

In the case that system (9) is a 2-degrees of freedom \(hamiltonian\), this proposition provides a new proof for the celebrated Moser's-Lyapunov theorem

**Corollary H2** [Lyapunov, Moser] For an analytic \(hamiltonian\) system around a saddle-focus or a saddle-center equilibrium, BNF is convergent.

Some other consequences can be derived from a partial reading of Theorem 2. Namely, a linear center can be seen as a particular subsystem of the general saddle-center case. Indeed, if we write explicitly system (9),

\[
\begin{align*}
\dot{x} &= \lambda x + \hat{f}_1(x, y, q, p) \\
\dot{y} &= -\lambda y + \hat{f}_2(x, y, q, p) \\
\dot{q} &= \alpha p + \hat{f}_3(x, y, q, p) \\
\dot{p} &= -\alpha q + \hat{f}_4(x, y, q, p)
\end{align*}
\]

for \(\hat{f}_1(0, 0, q, p) = \hat{f}_2(0, 0, q, p) = 0\) and fix \(x = y = 0\), we obtain the following planar system

\[
\begin{align*}
\dot{q} &= \alpha p + \hat{f}_3(q, p) \\
\dot{p} &= -\alpha q + \hat{f}_4(q, p)
\end{align*}
\]

Here \(\hat{f}_j(q, p), j = 3, 4\) denote \(\hat{f}_j(0, 0, q, p)\) This is the framework where the previously cited \(center-focus\) problem takes place. In this case Theorem 2 provides the existence of a transformation \((q, p) = \Phi(\mu, \nu)\) and vector fields \(N(\mu, \nu)\) and \(\hat{B}(\mu, \nu)\), of the form

\[
N = \begin{pmatrix}
\nu a(\mu^2 + \nu^2) \\
-\mu a(\mu^2 + \nu^2)
\end{pmatrix}, \quad \hat{B} = \begin{pmatrix}
\mu \hat{b}(\mu^2 + \nu^2) \\
\nu \hat{b}(\mu^2 + \nu^2)
\end{pmatrix},
\]

analytic in a neighborhood of the origin, with \(a(0) = \alpha, \ b(0) = 0\), and satisfying \(D\Phi \ N + \hat{B} = F_c \circ \Phi\), where \(F_c(p, q) = (\alpha p + \hat{f}_3(q, p), -\alpha q + \hat{f}_4(q, p))\). The following corollary is a reformulation of Proposition H1.
Corollary H3 Assume $\hat{f}_3, \hat{f}_4$ analytic at the origin. Then, the following statements are equivalent.

(i) System (11) is (locally) Hamiltonian.

(ii) The origin is a center.

(iii) The function $b(\mu^2 + \nu^2)$ in (12) provided by Theorem 2 vanishes identically.

On the other hand, assuming $\hat{f}_3 \equiv \hat{f}_4 \equiv 0$ in system (10) (that is, the origin is a center in the $(q, p)$-variables), taking polar coordinates, scaling time if necessary and fixing an invariant cycle, we have a system of the form

\[ \begin{align*}
\dot{x} &= \lambda x + \hat{g}_1(x, y, \theta) \\
\dot{y} &= -\lambda y + \hat{g}_2(x, y, \theta) \\
\dot{\theta} &= 1,
\end{align*} \]

where $\gamma = \{x = y = 0\}$ is now a hyperbolic periodic orbit (of characteristic exponents $\pm \lambda$, $\lambda > 0$) and $\hat{g}_1, \hat{g}_2$ are analytic functions of $x, y$ and $\theta$. For such a system we have from Proposition H1 the following result.

Corollary H4 [Moser [20]] Assume (13) is an analytic Hamiltonian system. Then, there exists a convergent transformation leading system (13) into $\Psi\text{NF}$ in a neighborhood of $\gamma$ and this $\Psi\text{NF}$ coincides with the BNF.

Remark 3 The original result due to Moser is also valid assuming only $\hat{g}_1$ and $\hat{g}_2$ to be $C^1$ with respect to the angular variable $\theta$. With a similar scheme to the one presented in this paper, Corollary H4 can also be proved under these weaker assumptions.

Up to this point, the results already presented follow from a suitable reading of Theorem 2 in a Hamiltonian framework. However, this is not the unique context where they can be applied. Namely, these results have a counterpart in the well known setting of the reversible systems.

We say that a system $\dot{X} = F(X)$ is $\mathfrak{G}$-(time)-reversible (or simply, $\mathfrak{G}$-reversible) if it is invariant under $X \mapsto \mathfrak{G}(X)$ and a reversion in the direction of time $t \mapsto -t$, with $\mathfrak{G}$ being an involutory diffeomorphism, that is, $\mathfrak{G}^2 = \text{id}$ and $\mathfrak{G} \neq \text{id}$. From this definition, it turns out that $F$ satisfies

\[ \mathfrak{G}^* F = -F, \]

where $\mathfrak{G}^* F = (D \mathfrak{G})^{-1} F(\mathfrak{G})$. The diffeomorphism $\mathfrak{G}$ is commonly called a reversing involution of this system and is, in general, non linear. In this work we are dealing with analytic systems, so we will consider analytic involutions $\mathfrak{G}$. A set $S$ which is invariant under the action of $\mathfrak{G}$ (that is, $\mathfrak{G}(S) \subseteq S$) is called $\mathfrak{G}$-symmetric or, simply, symmetric if there is no problem of misunderstanding. Since we are dealing with systems in a neighborhood of an equilibrium point or a periodic orbit, from now on we will assume always that these elements are symmetric with respect to the corresponding involution $\mathfrak{G}$.

Important examples of reversible systems are provided by the BNF (5). For instance, the BNF around a saddle-center equilibrium point

\[ \begin{align*}
\dot{\xi} &= \xi a_1(\xi \eta, \mu^2 + \nu^2) \\
\dot{\eta} &= -\eta a_1(\xi \eta, \mu^2 + \nu^2) \\
\dot{\mu} &= \nu a_2(\xi \eta, \mu^2 + \nu^2) \\
\dot{\nu} &= -\mu a_2(\xi \eta, \mu^2 + \nu^2)
\end{align*} \]

is $\mathfrak{N}$-reversible, $\mathfrak{N}$ being the linear involution $(\xi, \eta, \mu, \nu) \mapsto (\eta, \xi, \mu, -\nu)$. Analogously, the BNF around a saddle-focus equilibrium point is reversible with respect to the linear involution $(\xi, \eta, \mu, \nu) \mapsto (\eta, \xi, \nu, \mu)$.
Proposition R1 System (9) is reversible in a neighborhood of the origin if and only if \( \hat{B} \) vanishes (and, therefore, \( \PsiNF \) becomes BNF).

Remark 4 The Reversible Lyapunov Theorem was proven by Devaney [11] in both the smooth and the analytic case, using a geometrical approach. An alternative proof for this theorem is due to Vanderbauwhede [26] (see also [24] and [17], for an extension to families of analytic reversible vector fields).

The proof of this proposition is deferred to Section §3. Notice that, in particular, it implies that locally Hamiltonian and locally reversible is the same around this equilibrium point. Like in the Hamiltonian case, we have

Corollary R2 Corollaries H3 and H4 also hold substituting Hamiltonian by reversible.

From these results, it seems natural to look for a summarizing statement connecting both contexts, the Hamiltonian and the reversible. Indeed, we can summarize the previous statements in the following theorem.

Theorem 3 Let us consider an analytic system

\[ \dot{X} = F(X) \]

and assume that one of the following three situations holds (corresponding to dimensions 2, 3 and 4, respectively),

(i) \( X = (q,p) \in \mathbb{R}^2 \) and the origin is a linear center equilibrium point (like in system (11)).

(ii) \( X = (x,y,\theta) \in \mathbb{R}^2 \times \mathbb{T} \) and \( \gamma = \{x = y = 0\} \) is a hyperbolic periodic orbit (like in system (13)).

(iii) \( X = (x,y,q,p) \in \mathbb{R}^4 \) and the origin is a saddle-center or saddle-focus equilibrium point (like in system (10)).

Then, in a neighborhood of the corresponding critical element, the following statements are equivalent

(i) System (15) is Hamiltonian (with respect to some suitable 2-form \( \omega \)).

(ii) System (15) is reversible (with respect to some suitable reversing involution \( \mathcal{G} \)).

(iii) The analytic vector field \( \hat{B} \) (as in (7)) provided by Theorem 2 vanishes.

Remark 5 This local duality around critical elements between Hamiltonian and reversible systems is quite common. As an example, see for instance [18], where it is proved this equivalence in the case of a nonsemisimple 1:1 resonance, which occurs when two pairs of purely imaginary eigenvalues of the linearized system collide. Nevertheless, there exist also counter examples of such equivalence. For instance, see the one given at [23], where it is given a class of area preserving mappings, with linear part the identity, which are not reversible.

Beyond the consequences provided by Theorem 2 in the Hamiltonian or reversible frameworks, this \( \PsiNF \)-approach can be useful to find out isolated periodic orbits in other situations.

For instance, in [10] it is shown that for the center-focus problem (case (i) in Theorem 3) each zero of the analytic function \( b \), defined in (12), gives rise to a limit cycle of system (11) close to the origin.

Now, consider system (15) with the origin being a saddle-center equilibrium point (case (iii) in Theorem 3). Let \( N \) and \( \hat{B} \), as in (5b), (7b), be the analytic vector fields provided by Theorem 2. Assume
this system (15) is not locally Hamiltonian (neither reversible, therefore). Equivalently, functions $\hat{b}_1$, $\hat{b}_2$ in equation (7b) do not vanish simultaneously. Then the transformed system becomes of the form

$$\dot{\chi} = N(\chi) + (D\Phi(\chi))^{-1}\hat{B}(\chi)$$

or, more precisely,

$$\begin{pmatrix}
    \dot{\xi} \\
    \dot{\eta} \\
    \dot{\mu} \\
    \dot{\nu}
\end{pmatrix} =
\begin{pmatrix}
    \xi a_1(\xi,\eta,\mu^2 + \nu^2) \\
    -\eta a_1(\xi,\eta,\mu^2 + \nu^2) \\
    \nu a_2(\xi,\eta,\mu^2 + \nu^2) \\
    -\mu a_2(\xi,\eta,\mu^2 + \nu^2)
\end{pmatrix} + (D\Phi(\chi))^{-1} \begin{pmatrix}
    \xi \hat{b}_1(\xi,\eta,\mu^2 + \nu^2) \\
    -\eta \hat{b}_1(\xi,\eta,\mu^2 + \nu^2) \\
    \mu \hat{b}_2(\xi,\eta,\mu^2 + \nu^2) \\
    -\nu \hat{b}_2(\xi,\eta,\mu^2 + \nu^2)
\end{pmatrix}.$$

Assume that $\hat{b}_2$ does not vanish identically but there exists, at least, a non-zero value $I_\ast > 0$ satisfying $\hat{b}_2(0,I_\ast) = 0$. If we take initial conditions $\xi^0 = \eta^0 = 0$ in (16) it follows that $\xi(t) = \eta(t) = 0 \forall t$ and, therefore, $\mu^2 + \nu^2 = I_\ast$ becomes a limit cycle of the restricted system

$$\begin{cases}
    \dot{\mu} = \nu a_2(0,I_\ast) \\
    \dot{\nu} = -\mu a_2(0,I_\ast)
\end{cases}$$

where, for small enough values of $I_\ast$, we have $a_2(0,I_\ast) = \alpha + O(I_\ast) \neq 0$. That is,

$$\Gamma_\ast = \{\mu^2 + \nu^2 = I_\ast\}$$

is a hyperbolic periodic orbit of system (17) with period $2\pi/a_2(0,I_\ast)$ and characteristic exponent $a_1(0,I_\ast) = \lambda + O(I_\ast)$. Consequently,

$$\Gamma = \Phi(\Gamma_\ast) = \{\Phi(0,0,\mu,\nu) : \mu^2 + \nu^2 = I_\ast\}$$

is a hyperbolic periodic orbit of system (15). It is also straightforward to parameterize the corresponding (local) stable and unstable invariant manifolds of $\Gamma$. Namely, there exists $\delta > 0$, given by the radius of convergence of the $\Psi$NF, such that

$$W^u_{\text{loc}}(\Gamma) = \{\Phi(0,\eta^0 e^{-\delta a_1(0,I_\ast)},\mu,\nu) : |\eta^0| < \delta, \mu^2 + \nu^2 = I_\ast\},$$

$$W^s_{\text{loc}}(\Gamma) = \{\Phi(\xi^0 e^{\delta a_1(0,I_\ast)},0,\mu,\nu) : |\xi^0| < \delta, \mu^2 + \nu^2 = I_\ast\}.$$

We finish this introduction summarizing this result.

**Corollary 1** Consider system (15) where the origin is a saddle-center equilibrium point (case (iii) in Theorem 3) and let $N$ and $\hat{B}$, as in (5b), (7b), be the analytic vector fields provided by Theorem 2. Assume that the (analytic) function $I \mapsto \hat{b}_2(0,I)$, defined in a neighborhood of the origin, does not vanish identically (so system (15) is neither Hamiltonian nor reversible). Thus, every positive zero of $\hat{b}_2(0,*)$ gives rise to a hyperbolic periodic orbit of system (15). Moreover, parameterizations for the (local) stable and unstable invariant manifolds associated to this periodic orbit are given by (18).

### §2 Proof of the Main Theorem

#### §2.1 The formal solution: a first approach

It is worth noting that both cases, the origin being a saddle-focus or being a saddle-center, can be treated formally with the same argument. Moreover, we will deal first with the case of a complex $\Psi$NF and will derive subsequently the case of a real $\Psi$NF. Indeed, let us assume that we have complexified the original variables in such a way that the new (complex) matrix $A$ is diagonal. Under this common approach, we will refer often to $\{\pm\lambda_1, \pm\lambda_2\}$ as the characteristic exponents of the origin, meaning
\{ \pm \lambda \pm i \alpha \} in the first case and \{ \pm \lambda, \pm i \alpha \} in the second one, respectively, always with \lambda, \alpha > 0. Moreover, it is not difficult to check that with such unified notation the vector fields \( \hat{N} \) and \( \hat{\mu} \) take the same form (5a) and (7a), respectively, in both cases. This will be their formal aspect along this proof if nothing against is explicitly said.

The sketch of the proof follows the standard pattern: first, we will look for a formal solution of equation

\[ D \Phi \ N + \hat{\mu} = F \circ \Phi \]  

(19)

by means of a recurrent scheme, that will consist on two steps, an initial approach and a final refinement. Later on, it will be introduced a norm which will allow us to establish the convergence of the functions involved.

Thus, let us start with the first part. We recall that \( \hat{G} \) denotes that \( G \) is formed by formal power series beginning with terms of order at least 2. Now, since the linear part of \( F(X) = \lambda X + \hat{G}(X) \) (or shorter, \( F = \lambda + \hat{G} \)) is in normal form, we have that the linear part of \( \hat{N} \) is just \( \lambda \) (notice that \( \lambda \) represents also the complex matrix of eigenvalues \( \pm \lambda_1, \pm \lambda_2 \)). Writing \( \Phi = id + \hat{\Phi} \) and \( N = \lambda + \hat{N} \), equation (19) becomes

\[ D \hat{\Phi} \ N - \lambda \hat{\Phi} = \hat{F} \circ \Phi - \hat{N} - \hat{\mu}. \]  

(20)

Assume that we already know \( \hat{\Phi}, \hat{N} \) and \( \hat{\mu} \) up to some order \( K \) and let us see which difficulties involves the computation of the terms of order \( K + 1 \) of \( \hat{\Phi} \). From equation (20) we realize that we only have to consider the terms up to order \( K + 1 \) of equation

\[ D \hat{\Phi} \ N - \lambda \hat{\Phi} = \hat{H}, \]  

(21)

where \( \hat{H} = \hat{F} \circ \Phi \) only contains terms up to order \( K \) of \( \hat{\Phi} \). The terms in \( \hat{N} \) and \( \hat{\mu} \) of order \( K + 1 \) will be determined later. By direct computation, writing

\[ \hat{\Phi} = (\hat{\phi}^{(1)}, \hat{\phi}^{(2)}, \hat{\phi}^{(3)}, \hat{\phi}^{(4)}), \quad \hat{\mu} = (\hat{\mu}^{(1)}, \hat{\mu}^{(2)}, \hat{\mu}^{(3)}, \hat{\mu}^{(4)}), \]

with

\[ \hat{\phi}^{(i)}(\xi, \eta, \mu, \nu) = \sum \phi^{(i)}_{j \kappa \lambda \mu} \xi^j \eta^k \mu^\lambda \nu^\mu, \]

\[ \hat{\mu}^{(i)}(\xi, \eta, \mu, \nu) = \sum \mu^{(i)}_{j \kappa \lambda \mu} \xi^j \eta^k \mu^\lambda \nu^\mu, \]

for \( i = 1, \ldots, 4 \), and using that

\[ N = \begin{pmatrix} \xi a_1(\xi \eta, \mu \nu) \\ -\eta a_1(\xi \eta, \mu \nu) \\ \mu a_2(\xi \eta, \mu \nu) \\ -\nu a_2(\xi \eta, \mu \nu) \end{pmatrix} = \begin{pmatrix} \xi \lambda_1 + \cdots \\ -\eta \lambda_1 + \cdots \\ \mu \lambda_2 + \cdots \\ -\nu \lambda_2 + \cdots \end{pmatrix}, \]

\[ a_1(\xi \eta, \mu \nu) = \begin{pmatrix} \xi a_1(\xi \eta, \mu \nu) \\ -\eta a_1(\xi \eta, \mu \nu) \\ \mu a_2(\xi \eta, \mu \nu) \\ -\nu a_2(\xi \eta, \mu \nu) \end{pmatrix} = \begin{pmatrix} \xi \lambda_1 + \cdots \\ -\eta \lambda_1 + \cdots \\ \mu \lambda_2 + \cdots \\ -\nu \lambda_2 + \cdots \end{pmatrix}, \]

the terms up to order \( K + 1 \) of equation (21) come from the following system,

\[ (\xi \phi^{(1)}_{\xi} - \eta \phi^{(1)}_{\eta}) a_1(\xi \eta, \mu \nu) + (\mu \phi^{(1)}_{\mu} - \nu \phi^{(1)}_{\nu}) a_2(\xi \eta, \mu \nu) - \lambda_1 \phi^{(1)} = \hat{n}^{(1)}, \]

\[ (\xi \phi^{(2)}_{\xi} - \eta \phi^{(2)}_{\eta}) a_1(\xi \eta, \mu \nu) + (\mu \phi^{(2)}_{\mu} - \nu \phi^{(2)}_{\nu}) a_2(\xi \eta, \mu \nu) + \lambda_1 \phi^{(1)} = \hat{n}^{(2)}, \]

\[ (\xi \phi^{(3)}_{\xi} - \eta \phi^{(3)}_{\eta}) a_1(\xi \eta, \mu \nu) + (\mu \phi^{(3)}_{\mu} - \nu \phi^{(3)}_{\nu}) a_2(\xi \eta, \mu \nu) - \lambda_2 \phi^{(3)} = \hat{n}^{(3)}, \]

\[ (\xi \phi^{(4)}_{\xi} - \eta \phi^{(4)}_{\eta}) a_1(\xi \eta, \mu \nu) + (\mu \phi^{(4)}_{\mu} - \nu \phi^{(4)}_{\nu}) a_2(\xi \eta, \mu \nu) + \lambda_2 \phi^{(4)} = \hat{n}^{(4)}, \]
where $\hat{\phi}_k$ represents $\frac{\partial \hat{\phi}}{\partial k}$, etc. Hence, since $a_s(\xi, \eta, \mu \nu) = \lambda_s + \cdots$, the terms of order $K + 1$ of $\hat{\Phi}$ have to satisfy

$$
\phi_{jkkm}^{(1)} = \frac{h_{jkkm}^{(1)}}{\lambda_1(j-k-1) + \lambda_2(\ell-m)}, \quad \text{if} \ j \neq k + 1 \text{ or } \ell \neq m, \\
\phi_{jkkm}^{(2)} = \frac{h_{jkkm}^{(2)}}{\lambda_1(j-k+1) + \lambda_2(\ell-m)}, \quad \text{if} \ k \neq j + 1 \text{ or } \ell \neq m, \\
\phi_{jkkm}^{(3)} = \frac{h_{jkkm}^{(3)}}{\lambda_1(j-k) + \lambda_2(\ell-m-1)}, \quad \text{if} \ j \neq k \text{ or } \ell \neq m + 1, \\
\phi_{jkkm}^{(4)} = \frac{h_{jkkm}^{(4)}}{\lambda_1(j-k) + \lambda_2(\ell-m+1)}, \quad \text{if} \ j \neq k \text{ or } m \neq \ell + 1.
$$

\hspace{1cm} (23)

It is clear from these equations that terms of the form

$$
\left( \begin{array}{c}
\xi \sum_{k \geq 0, m \geq 1} \phi_{k+1,kmm}^{(1)} (\xi \eta)^k (\mu \nu)^m \\
\eta \sum_{j \geq 0, \ell \geq 1} \phi_{j,j+1,\ell\ell}^{(2)} (\xi \eta)^j (\mu \nu)^\ell \\
\mu \sum_{k \geq 1, m \geq 0} \phi_{kk,m+1,m}^{(3)} (\xi \eta)^k (\mu \nu)^m \\
\nu \sum_{j \geq 1, \ell \geq 0} \phi_{jjj,\ell+1}^{(4)} (\xi \eta)^j (\mu \nu)^\ell
\end{array} \right)
$$

\hspace{1cm} (24)

cannot be determined and remain in principle arbitrary. In terms of simply linear algebra this amounts to say that the transformation $\Phi$ is completely determined once it has been fixed its projection on a suitable vectorial subspace, called resonant subspace.

\section*{§2.2 Definition of the projections}

The type of coefficients appearing in expression (24) and the remarks above motivate the following definition.

**Definition 1** Given a formal series $h(\xi, \eta, \mu, \nu) = \sum h_{jkkm} \xi^i \eta^j \mu^\ell \nu^m$, we define the projections

\[
P_1 h := \xi \sum_{k \geq 0, m \geq 1} h_{k+1,kmm} (\xi \eta)^k (\mu \nu)^m, \\
P_2 h := \eta \sum_{j \geq 0, \ell \geq 1} h_{j,j+1,\ell\ell} (\xi \eta)^j (\mu \nu)^\ell, \\
P_3 h := \mu \sum_{k \geq 1, m \geq 0} h_{kk,m+1,m} (\xi \eta)^k (\mu \nu)^m, \\
P_4 h := \nu \sum_{j \geq 1, \ell \geq 0} h_{jjj,\ell+1} (\xi \eta)^j (\mu \nu)^\ell.
\]

Analogously, if $H = (h^{(1)}, h^{(2)}, h^{(3)}, h^{(4)})$ is a (formal) vector field we define

\[
\mathcal{P} H := (P_1 h^{(1)}, P_2 h^{(2)}, P_3 h^{(3)}, P_4 h^{(4)})
\]

and $\mathcal{R} H := H - \mathcal{P} H$. 

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As it has been noticed before, $\mathcal{P}\hat{\Phi}$ corresponds to the terms which remain arbitrary from the solution of equation (21). Moreover, vector fields $\hat{N}$ and $\hat{B}$ are invariant under the action of $\mathcal{P}$. This property will be used in the solution of equation (20). In this sense, we have the following lemma, whose proof is omitted since it consists on straightforward computations.

**Lemma 1** Given $N = \Lambda + \hat{N}$ of the form (22), the operator $\mathcal{L}_N$ defined as

$$\mathcal{L}_N\Psi := D\Psi N - \Lambda\Psi$$

satisfies the following properties.

(i) $\mathcal{L}_N\Psi$ is linear with respect to $\Psi$ and $N$, that is

$$\mathcal{L}_N (\Psi + \Psi') = \mathcal{L}_N \Psi + \mathcal{L}_N \Psi' \quad \mathcal{L}_{N+N'} \Psi = \mathcal{L}_N \Psi + \mathcal{L}_{N'} \Psi.$$

(ii) $\mathcal{L}_N$ preserves order, that is, $\mathcal{L}_N\Psi$ and $\Psi$ start with terms in $(\xi, \eta, \mu, \nu)$ of the same order.

(iii) The projections $\mathcal{P}$ and $\mathcal{R}$ commute with $\mathcal{L}_N$, that is,

$$\mathcal{P}(\mathcal{L}_N \Psi) = \mathcal{L}_N (\mathcal{P}\Psi), \quad \mathcal{R}(\mathcal{L}_N \Psi) = \mathcal{L}_N (\mathcal{R}\Psi).$$

§2.3 The recurrent scheme

Let us come back to the solution of equation (20). Having in mind the definition of the operator $\mathcal{L}_N$, it can be written as

$$\mathcal{L}_N\hat{\Phi} = \hat{F} \circ \Phi - \hat{N} - \hat{B},$$

which is of type (21) provided we take $\hat{H} = \hat{F} \circ \Phi - \hat{N} - \hat{B}$. In a first approach to this kind of equations we have shown that they could be solved recurrently for those terms in $\Phi = \text{id} + \hat{\Phi}$ of type $\mathcal{R}\hat{\Phi}$, remaining those of the form $\mathcal{P}\hat{\Phi}$ arbitrary. This fact suggests the idea of splitting the transformation we are looking for, $\Phi$, into $\text{id} + \mathcal{P}\hat{\Phi} + \mathcal{R}\hat{\Phi}$, to determine $\mathcal{R}\hat{\Phi}$ from equation (26) and to choose a suitable value for $\mathcal{P}\hat{\Phi}$.

**Remark 6** In Normal Form theory it is standard to set $\mathcal{P}\hat{\Phi} = 0$ in order to simplify the computations. However, it could be useful to take advantage of this freedom in some concrete situations.

Applying $\mathcal{R}$ onto equation (26),

$$\mathcal{R}(\mathcal{L}_N\hat{\Phi}) = \mathcal{R}(\hat{F}(\Phi)) - \mathcal{R}\hat{N} - \mathcal{R}\hat{B},$$

using Lemma 1 and taking into account that $\mathcal{R}\hat{N} = \mathcal{R}\hat{B} = 0$ if $\hat{N}$ and $\hat{B}$ are assumed to be of the form (5a) and (7a), respectively, we obtain the equation

$$\mathcal{L}_N\left(\mathcal{R}\hat{\Phi}\right) = \mathcal{R}(\hat{F}(\Phi)).$$

On the other hand, applying now $\mathcal{P}$ onto (26), taking again into account Lemma 1, the fact that $\mathcal{P}\hat{N} = \hat{N}$, $\mathcal{P}\hat{B} = \hat{B}$ and choosing $\mathcal{P}\hat{\Phi} \equiv 0$, it follows that

$$\hat{N} + \hat{B} = \mathcal{P}(\hat{F}(\Phi)).$$

A usual way to deal with such kind of equations is to consider it as a fixed point problem. Thus, we can set

$$\mathcal{P}\hat{\Phi} \equiv 0,$$
take initial values
\[(29) \quad \Phi^{(1)} = \text{id}, \quad N^{(1)} = \Lambda, \quad \hat{B}^{(1)} = 0\]

and obtain, recurrently,
\[(30) \quad \Phi^{(K+1)} = \text{id} + \mathcal{R} \Phi^{(K+1)} \]
\[N^{(K+1)} = \Lambda + \hat{N}^{(K+1)} \]
\[\hat{B}^{(K+1)} \]

from equations
\[(31) \quad \mathcal{L}_N\left(\mathcal{R} \hat{\Psi}^{(K+1)}\right) = \mathcal{R} \left(\hat{F} \left(\Phi^{(K)}\right)\right)\]
\[(32) \quad \hat{N}^{(K+1)} + \hat{B}^{(K+1)} = \mathcal{P} \left(\hat{F} \left(\Phi^{(K)}\right)\right).\]

We will see now how these two equations can be solved formally.

§2.3.1 Solution of a $\mathcal{L}_N(\mathcal{R} \hat{\Psi}) = \mathcal{R} \hat{H}$-type equation

Assuming that we know the coefficients of $N$ and $\mathcal{R} \hat{H}$ up to a given order $K$, the coefficients of $\mathcal{R} \hat{\Psi}$ of the same order will be determined from
\[(33) \quad \mathcal{L}_N \left(\mathcal{R} \hat{\Psi}\right) = \mathcal{R} \hat{H}.\]

Indeed, writing
\[(34) \quad \mathcal{R} \hat{\Psi} = \begin{pmatrix}
\hat{\psi}_1(\xi; \eta, \mu, \nu) \\
\hat{\psi}_2(\xi; \eta, \mu, \nu) \\
\hat{\psi}_3(\xi; \eta, \mu, \nu) \\
\hat{\psi}_4(\xi; \eta, \mu, \nu)
\end{pmatrix}, \quad \mathcal{R} \hat{H} = \begin{pmatrix}
\hat{h}_1(\xi; \eta, \mu, \nu) \\
\hat{h}_2(\xi; \eta, \mu, \nu) \\
\hat{h}_3(\xi; \eta, \mu, \nu) \\
\hat{h}_4(\xi; \eta, \mu, \nu)
\end{pmatrix},\]

where
\[\hat{\psi}_w(\xi; \eta, \mu, \nu) = \sum_{j+k+l+m \geq 2} \psi_{jk\ell m}^{(w)} \xi^j \eta^k \mu^l \nu^m\]
\[\hat{h}_w(\xi; \eta, \mu, \nu) = \sum_{j+k+l+m \geq 2} \hat{h}_{jk\ell m}^{(w)} \xi^j \eta^k \mu^l \nu^m\]

for $w = 1, \ldots, 4$, and taking into account that
\[N(\xi; \eta, \mu, \nu) = \begin{pmatrix}
\xi a_1(\xi \eta \mu \nu) \\
-\eta a_1(\xi \eta \mu \nu) \\
\mu a_2(\xi \eta \mu \nu) \\
-\nu a_2(\xi \eta \mu \nu)
\end{pmatrix}\]
with \( a_i(\xi \eta, \mu \nu) = \lambda_i + \tilde{a}_i(\xi \eta, \mu \nu) \), it follows that the left-hand side of (33) is equivalent to

\[
D \left( \mathcal{R} \hat{\Psi} \right) N - \Lambda \mathcal{R} \Psi = \begin{pmatrix}
\hat{\psi}_1, \xi \\
\hat{\psi}_2, \xi \\
\hat{\psi}_3, \xi \\
\hat{\psi}_4, \xi \\
\hat{\psi}_1, \eta \\
\hat{\psi}_2, \eta \\
\hat{\psi}_3, \eta \\
\hat{\psi}_4, \eta \\
\hat{\psi}_1, \mu \\
\hat{\psi}_2, \mu \\
\hat{\psi}_3, \mu \\
\hat{\psi}_4, \mu \\
\hat{\psi}_1, \nu \\
\hat{\psi}_2, \nu \\
\hat{\psi}_3, \nu \\
\hat{\psi}_4, \nu \\
\end{pmatrix}
\begin{pmatrix}
\xi a_1 \\
-\eta a_1 \\
-\mu a_2 \\
-\nu a_2 \\
\end{pmatrix} = \begin{pmatrix}
\lambda_1 \hat{\psi}_1 \\
-\lambda_1 \hat{\psi}_2 \\
\lambda_2 \hat{\psi}_3 \\
-\lambda_2 \hat{\psi}_4 \\
\end{pmatrix} =
\begin{pmatrix}
(\xi \hat{\psi}_1, \xi - \eta \hat{\psi}_1, \eta) a_1 + (\mu \hat{\psi}_1, \mu - \nu \hat{\psi}_1, \nu) a_2 - \lambda_1 \hat{\psi}_1 \\
(\xi \hat{\psi}_2, \xi - \eta \hat{\psi}_2, \eta) a_1 + (\mu \hat{\psi}_2, \mu - \nu \hat{\psi}_2, \nu) a_2 + \lambda_2 \hat{\psi}_2 \\
(\xi \hat{\psi}_3, \xi - \eta \hat{\psi}_3, \eta) a_1 + (\mu \hat{\psi}_3, \mu - \nu \hat{\psi}_3, \nu) a_2 - \lambda_2 \hat{\psi}_3 \\
(\xi \hat{\psi}_4, \xi - \eta \hat{\psi}_4, \eta) a_1 + (\mu \hat{\psi}_4, \mu - \nu \hat{\psi}_4, \nu) a_2 + \lambda_2 \hat{\psi}_4 \\
\end{pmatrix}.
\]

We can refer to this vector field, in short, as

\[
\left( L_N^{(1)} \hat{\psi}_1, L_N^{(2)} \hat{\psi}_2, L_N^{(3)} \hat{\psi}_3, L_N^{(4)} \hat{\psi}_4 \right)
\]

and write its components, in formal power series expansions, as

\[
L_N^{[w]}\hat{\psi}_w(\xi, \eta, \mu, \nu) = \sum_{j+k+\ell+m \geq 2} g_{jk\ell m}^{[w]}(\xi \eta, \mu \nu) \psi_{jk\ell m}^{[w]}(\xi \eta, \mu \nu) \xi^j \eta^k \mu^\ell \nu^m,
\]

being

\[
g_{jk\ell m}^{[w]}(\xi \eta, \mu \nu) := \gamma_{jk\ell m}^{[w]}(\lambda) + (j - k) \alpha_1(\xi \eta, \mu \nu) + (\ell - m) \alpha_2(\xi \eta, \mu \nu),
\]

with

\[
\gamma_{jk\ell m}^{[w]}(\lambda) := \begin{cases} 
(j - k - 1) \lambda_1 + (\ell - m) \lambda_2 & \text{if } w = 1 \\
(j - k + 1) \lambda_1 + (\ell - m) \lambda_2 & \text{if } w = 2 \\
(j - k) \lambda_1 + (\ell - m - 1) \lambda_2 & \text{if } w = 3 \\
(j - k) \lambda_1 + (\ell - m + 1) \lambda_2 & \text{if } w = 4.
\end{cases}
\]

Notice, from equation (35), that \( L_N \) acts on \( \mathcal{R} \hat{\Psi} \) **multiplying** each coefficient \( \psi_{jk\ell m} \) by a function of the products \( \xi \eta \) and \( \mu \nu \). To take advantage of this feature we will express our formal series expansions in a more convenient way which will highlight those terms of the form \((\xi \eta)^p\) and \((\mu \nu)^q\). A similar idea was suggested in [9]. In our case it works as follows. For any component \( \hat{\psi}_w \) of \( \mathcal{R} \hat{\Psi} \) we have

\[
\hat{\psi}_w(\xi, \eta, \mu, \nu) = \sum_{j+k+\ell+m \geq 2} \psi_{jk\ell m}^{(w)}(\xi \eta, \mu \nu)^{j-k} (\xi \eta)^{k} \mu^\ell \nu^m = \sum_{j+k+\ell+m \geq 2} \psi_{jk\ell m}^{(w)}(\xi \eta)^{j-k} (\xi \eta)^{k} \mu^\ell \nu^m.
\]
Defining \( p = j - k, q = \ell - m \) and taking into account that \( j + k + \ell + m \geq 2, p + k \geq 0 \) and \( q + m \geq 0 \), this expansion is equivalent to

\[
\sum_{p,q \in \mathbb{Z}} \psi_{pq}^{(u)}(\xi \eta, \mu \nu) \xi^p \mu^q
\]

where

\[
\psi_{pq}^{(u)}(\xi \eta, \mu \nu) = \sum_{(k,m) \in Q_{pq}} \psi_{p+k,k,q+m,m}^{(u)}(\xi \eta)^k (\mu \nu)^m
\]

and

\[
Q_{pq} := \left\{ (k,m) \in (\mathbb{N} \cup \{0\})^2 : \begin{array}{c}
k \geq \max \{0, -p\} \\
m \geq \max \{0, -q\}
\end{array}, \ k + m \geq 1 - \frac{p + q}{2} \right\}.
\]

In the same way, for \( \mathcal{H} \) we get

\[
\hat{h}_w(\xi, \eta, \mu, \nu) = \sum_{p,q \in \mathbb{Z}} h_{pq}^{(w)}(\xi \eta, \mu \nu) \xi^p \mu^q
\]

where

\[
h_{pq}^{(w)}(\xi, \eta, \mu, \nu) = \sum_{(k,m) \in Q_{pq}} h_{p+k,k,q+m,m}^{(w)}(\xi \eta)^k (\mu \nu)^m.
\]

With this notation formula (35) becomes

\[
\sum_{p,q \in \mathbb{Z}} g_{pq}^{(w)}(\xi \eta, \mu \nu) \psi_{pq}^{(w)}(\xi \eta, \mu \nu) \xi^p \mu^q
\]

where now

\[
g_{pq}^{(w)}(\xi \eta, \mu \nu) := \Gamma_{pq}^{(w)}(\lambda) + p \hat{a}_1(\xi \eta, \mu \nu) + q \hat{a}_2(\xi \eta, \mu \nu)
\]

being

\[
\Gamma_{pq}^{(w)}(\lambda) := \begin{cases}
(p-1)\lambda_1 + q \lambda_2 & \text{if } w = 1 \\
(p+1)\lambda_1 + q \lambda_2 & \text{if } w = 2 \\
p\lambda_1 + (q-1)\lambda_2 & \text{if } w = 3 \\
p\lambda_1 + (q+1)\lambda_2 & \text{if } w = 4.
\end{cases}
\]

Thus, equality

\[
\mathcal{L}_N(\mathcal{H}) = \mathcal{H}
\]

gives rise to the equations

\[
L_N^{(w)} \hat{\psi}_w(\xi, \eta, \mu, \nu) = \hat{h}_w(\xi, \eta, \mu, \nu)
\]

or, in formal series expansions,

\[
\sum_{p,q \in \mathbb{Z}} g_{pq}^{(w)}(\xi \eta, \mu \nu) \psi_{pq}^{(w)}(\xi \eta, \mu \nu) \xi^p \mu^q = \sum_{p,q \in \mathbb{Z}} h_{pq}^{(w)}(\xi \eta, \mu \nu) \xi^p \mu^q,
\]

whose formal solution is given by

\[
\hat{\psi}_w(\xi, \eta, \mu, \nu) = \sum_{p,q \in \mathbb{Z}} \psi_{pq}^{(w)}(\xi \eta, \mu \nu) \xi^p \mu^q
\]

with the functions \( \psi_{pq}^{(w)}(\xi \eta, \mu \nu) \) coming from

\[
\psi_{pq}^{(w)}(\xi \eta, \mu \nu) = \frac{h_{pq}^{(w)}(\xi \eta, \mu \nu)}{g_{pq}^{(w)}(\xi \eta, \mu \nu)} = \frac{h_{pq}^{(w)}(\xi \eta, \mu \nu)}{\Gamma_{pq}^{(w)}(\lambda) + p \hat{a}_1(\xi \eta, \mu \nu) + q \hat{a}_2(\xi \eta, \mu \nu)},
\]

for \( w = 1, 2, \ldots, 4 \) and \( p, q \in \mathbb{Z} \). With this notation coefficients with \( p = \pm 1 \) and \( q = 0 \) or \( p = 0 \) and \( q = \pm 1 \) are those belonging to the projection \( \mathcal{P} \hat{\Psi} \).
\section{Solution of a $\hat{N} + \hat{B} = \mathcal{P}\hat{H}$-type equation}

As it has been done for equations of type $\mathcal{L}_N(\mathcal{R}\hat{\Phi}) = \mathcal{R}\hat{H}$ we are going to prove that equation $\hat{N} + \hat{B} = \mathcal{P}\hat{H}$ determines uniquely the coefficients of $\hat{N}$ and $\hat{B}$ provided they are of type (5a) and (7a), respectively, and that $\hat{H}$ is known. Thus, writing

\begin{align}
\hat{N} &= (\xi\hat{a}_1, -\eta\hat{a}_1, \mu\hat{a}_2, -\nu\hat{a}_2) \\
\hat{B} &= (\xi\hat{b}_1, \eta\hat{b}_1, \mu\hat{b}_2, \nu\hat{b}_2), \\
\mathcal{P}\hat{H} &= (\xi\hat{h}_1, \eta\hat{h}_2, \mu\hat{h}_3, \nu\hat{h}_4),
\end{align}

where $\hat{a}_i$, $\hat{b}_i$ and $\hat{h}_w$ are functions of $\xi\eta$ and $\mu\nu$, for $i = 1, 2$ and $w = 1, 2, \ldots, 4$, the solution of this equation is given explicitly by

\begin{align}
\hat{a}_1 &= \frac{1}{2} \left( \hat{h}_1 - \hat{h}_2 \right), \\
\hat{b}_1 &= \frac{1}{2} \left( \hat{h}_1 + \hat{h}_2 \right), \\
\hat{a}_2 &= \frac{1}{2} \left( \hat{h}_3 - \hat{h}_4 \right), \\
\hat{b}_2 &= \frac{1}{2} \left( \hat{h}_3 + \hat{h}_4 \right).
\end{align}

Notice that the form of the functions $\hat{a}_i$, $\hat{b}_i$ and $\hat{h}_w$ implies that $\mathcal{P}\hat{H}$, $\hat{N}$ and $\hat{B}$ only contain terms of odd order.

\section{The recurrent scheme: an improvement}

One of the features of this procedure is that it provides a constructive (and, therefore, implementable on a computer) way to determine $\hat{\Phi}$, $\hat{N}$ and $\hat{B}$. To do it we need to define (and allocate memory for them) data vectors representing these vector fields. Unfortunately, the scheme above implies to handle (and to recompute) the complete vectors storing $\hat{\Phi}$, $\hat{N}$ and $\hat{B}$, at any step of the process. This makes it slow and not much efficient. In this sense it is easy to refine it by paying attention on the order of the solutions of equations (31)–(32).

Before going on with this refinement, let us introduce some notation. We will denote $G = O_{[K]}$ if $G$ is a homogeneous polynomial in the spatial variables $\xi, \eta, \mu, \nu$ of order exactly $K$. Besides, we will write $G = O_K$ if $G$ contains only terms of order greater or equal than $K$ in these variables and $G = O_{\leq K}$ if all the terms in $G$ are of order less or equal than $K$. Thus, we have

\textbf{Lemma 2} At any step $K \geq 1$ of the process (29)–(32) the following estimates hold

\begin{align}
\mathcal{R}\hat{\Phi}^{(K+1)} - \mathcal{R}\hat{\Phi}^{(K)} &= O_{K+1}, \\
\hat{N}^{(K+1)} - \hat{N}^{(K)} &= O_{K+1}, \\
\hat{B}^{(K+1)} - \hat{B}^{(K)} &= O_{K+1}.
\end{align}

\textbf{Proof.} We proceed inductively.

- For $K = 1$, from the initial values (29)

\begin{align}
\hat{\Phi}^{(1)} &= \text{id}, \\
\hat{N}^{(1)} &= \Lambda, \\
\hat{B}^{(1)} &= 0
\end{align}

one has that $\mathcal{R}\hat{\Phi}^{(1)} = \hat{N}^{(1)} = \hat{B}^{(1)} = 0$. In this case formula (29) reads

\begin{align}
\mathcal{L}_{\hat{N}^{(1)}} \left( \mathcal{R}\hat{\Phi}^{(2)} \right) = \mathcal{R} \left( \hat{\Phi} \left( \hat{\Phi}^{(1)} \right) \right).
\end{align}

Its right-hand side becomes

\begin{align}
\mathcal{R} \left( \hat{\Phi} \left( \hat{\Phi}^{(1)} \right) \right) = \mathcal{R} \left( \hat{\Phi} \left( \text{id} \right) \right) = \mathcal{R}\hat{F}
\end{align}
and its left-hand side
\[ L_{N(1)}(\mathcal{R}\hat{\Phi}^{(2)}) = D(\mathcal{R}\hat{\Phi}^{(2)}) \Lambda - \Lambda \mathcal{R}\hat{\Phi}^{(2)} = \left[ \Lambda, \mathcal{R}\hat{\Phi}^{(2)} \right], \]
where \([G,H] = (DH) - (DG)H\) stands for the Lie bracket of the vector fields \(G\) and \(H\). We recall also that, abusing of the notation, we denote with the same symbol \(\Lambda\) the matrix \(\Lambda\) and the vector field \(\Lambda \text{id}\). Thus formula (44) becomes
\[ \left[ \Lambda, \mathcal{R}\hat{\Phi}^{(2)} \right] = \mathcal{R}\hat{F}. \]
Using that \(\mathcal{R}\hat{F} = O_2\) and that the Lie bracket preserves the order it follows that \(\mathcal{R}\hat{\Phi}^{(2)} = O_2\) and, consequently, \(\mathcal{R}\hat{\Phi}^{(2)} - \mathcal{R}\hat{\Phi}^{(1)} = O_2\).

With respect to \(\hat{N}\) and \(\hat{B}\), we have now that
\[ \hat{N}^{(2)} + \hat{B}^{(2)} = \mathcal{P} \left( \hat{F} \left( \Phi^{(1)} \right) \right) = \mathcal{P} \left( \hat{F} \left( \text{id} \right) \right) = \mathcal{P} \hat{F}. \]
From formulas (43) and having in mind that \(\mathcal{P} \hat{F} = O_3\) it turns out that \(\hat{N}^{(2)}, \hat{B}^{(2)} = O_3\) so, in particular,
\[ \hat{N}^{(2)} - \hat{N}^{(1)} = O_2, \quad \hat{B}^{(2)} - \hat{B}^{(1)} = O_2. \]
- Assume now, as induction hypothesis, that
\[ \begin{align*}
\mathcal{R}\hat{\Phi}^{(K)} & - \mathcal{R}\hat{\Phi}^{(K-1)} = O_K, \\
\hat{N}^{(K)} & - \hat{N}^{(K-1)} = O_K, \\
\hat{B}^{(K)} & - \hat{B}^{(K-1)} = O_K.
\end{align*} \tag{45} \]
hold for any arbitrary step \(K - 1\). To avoid a cumbersome notation, let us denote
\[ \begin{align*}
\mathcal{R}\Delta\hat{\Phi}^{(K)} & := \mathcal{R}\hat{\Phi}^{(K+1)} - \mathcal{R}\hat{\Phi}^{(K)} \\
\Delta\hat{N}^{(K)} & := \hat{N}^{(K+1)} - \hat{N}^{(K)} \\
\Delta\hat{B}^{(K)} & := \hat{B}^{(K+1)} - \hat{B}^{(K)}.
\end{align*} \]
With such notation, hypothesis (45) becomes just
\[ \mathcal{R}\Delta\hat{\Phi}^{(K-1)} = O_K, \quad \Delta\hat{N}^{(K-1)} = O_K, \quad \Delta\hat{B}^{(K-1)} = O_K \]
and we want to prove that
\[ \mathcal{R}\Delta\hat{\Phi}^{(K)} = O_{K+1}, \quad \Delta\hat{N}^{(K)} = O_{K+1}, \quad \Delta\hat{B}^{(K)} = O_{K+1}. \]
Subtracting equation (31) for two consecutive steps \(K - 1\) and \(K\) we get
\[ \begin{align*}
L_{N(K)} \left( \mathcal{R}\hat{\Phi}^{(K+1)} \right) - L_{N(K-1)} \left( \mathcal{R}\hat{\Phi}^{(K)} \right) = \mathcal{R} \left( \hat{F} \left( \Phi^{(K)} \right) \right) - \mathcal{R} \left( \hat{F} \left( \Phi^{(K-1)} \right) \right). \tag{46}
\end{align*} \]
Using the linearity of the operator \(L\) (see Lemma 1) we have
\[ \begin{align*}
L_{N(K)} \left( \mathcal{R}\hat{\Phi}^{(K+1)} \right) & = L_{N(K)} \left( \mathcal{R}\hat{\Phi}^{(K)} \right) + L_{N(K)} \left( \mathcal{R}\Delta\hat{\Phi}^{(K)} \right) = \\
L_{N(K-1)} \left( \mathcal{R}\hat{\Phi}^{(K)} \right) + L_{\Delta\hat{N}^{(K-1)}} \left( \mathcal{R}\hat{\Phi}^{(K)} \right) + L_{N(K)} \left( \mathcal{R}\Delta\hat{\Phi}^{(K)} \right),
\end{align*} \]
so the left-hand side of equation \((46)\) becomes

\[
\mathcal{L}_{N(K)} \left( \hat{\mathcal{R}}^{\hat{\Phi}^{[K+1]}} \right) - \mathcal{L}_{N(K-1)} \left( \hat{\mathcal{R}}^{\hat{\Phi}^{[K]}} \right) =
\]

\[
\mathcal{L}_{\Delta N(K-1)} \left( \hat{\mathcal{R}}^{\hat{\Phi}^{[K]}} \right) + \mathcal{L}_{N(K)} \left( \mathcal{R} \Delta \hat{\Phi}^{[K]} \right) =
\]

\[
D \left( \hat{\mathcal{R}}^{\hat{\Phi}^{[K]}} \right) \Delta \hat{\Phi}^{[K-1]} + \mathcal{L}_{N(K)} \left( \mathcal{R} \Delta \hat{\Phi}^{[K]} \right) =
\]

\[
\mathcal{L}_{N(K)} \left( \mathcal{R} \Delta \hat{\Phi}^{[K]} \right) + \mathcal{O}_{K+1},
\]

where we have taken into account that \(\Delta \hat{\Phi}^{[K-1]} \) has null linear part. With respect to the right-hand side of \((46)\), expanding it in Taylor series, it turns out that

\[
\hat{F} \left( \phi^{(K)} \right) - \hat{F} \left( \phi^{(K-1)} \right) = \]

\[
D \hat{F} \left( \phi^{(K-1)} \right) \mathcal{R} \Delta \hat{\Phi}^{[K-1]} + \sum_{j \geq 2} \frac{1}{j!} D^j \hat{F} \left( \phi^{(K-1)} \right) \left( \mathcal{R} \Delta \hat{\Phi}^{[K-1]} \right)^j.
\]

Since \(D \hat{F} \left( \phi^{(K-1)} \right) = \mathcal{O}_1 \) and \(\mathcal{R} \Delta \hat{\Phi}^{[K-1]} = \mathcal{O}_K \), by induction hypothesis, it turns out that

\[
\mathcal{R} \left( \hat{F} \left( \phi^{(K)} \right) \right) - \mathcal{R} \left( \hat{F} \left( \phi^{(K-1)} \right) \right) = \mathcal{O}_{K+1}.
\]

Consequently, equation \((46)\) becomes of type

\[
\mathcal{L}_{N(K)} \left( \mathcal{R} \Delta \hat{\Phi}^{[K]} \right) = \mathcal{O}_{K+1}.
\]

Since \(\mathcal{L}\) preserves the order (see Lemma 1) it follows that \(\mathcal{R} \Delta \hat{\Phi}^{[K]} = \mathcal{O}_{K+1}\).

Concerning \(\Delta \hat{N}^{[K]} \) and \(\Delta \hat{B}^{[K]} \) we proceed in the same way. Subtracting formula \((32)\) for \(K-1\) and \(K\) one obtains that

\[
\Delta \hat{N}^{[K]} + \Delta \hat{B}^{[K]} = \mathcal{P} \left( \hat{F} \left( \phi^{(K)} \right) - \hat{F} \left( \phi^{(K)} \right) \right).
\]

From \((47)\) we know that

\[
\mathcal{P} \left( \hat{F} \left( \phi^{(K)} \right) - \hat{F} \left( \phi^{(K)} \right) \right) = \mathcal{O}_{K+1}
\]

and therefore, using \((43)\), it follows that \(\Delta \hat{N}^{[K]}, \Delta \hat{B}^{[K]} = \mathcal{O}_{K+1}, \) which concludes the proof.

\[\square\]

An important consequence of this lemma is the reduction of the computational effort of the recurrent scheme: in the \(K\)-th step of our recurrent scheme the coefficients of order less or equal than \(K\) computed from the previous iteration will remain invariant. Therefore, from now onwards we will consider

\[
\hat{\Phi}^{[K+1]} = \mathcal{O}_{\leq K+1}, \quad \hat{\mathcal{N}}^{[K+1]} = \mathcal{O}_{\leq K+1}, \quad \hat{\mathcal{B}}^{[K+1]} = \mathcal{O}_{\leq K+1},
\]

obtained from the equations \((31)-(31)\) taken only up to order \(K+1\)

\[
\left\{ \mathcal{L}_{N(K)} \left( \hat{\mathcal{R}}^{\hat{\Phi}^{[K+1]}} \right) \right\}_{\leq K+1} = \left\{ \mathcal{R} \left( \hat{\mathcal{F}} \left( \phi^{(K)} \right) \right) \right\}_{\leq K+1}
\]

(48)

\[
\hat{\mathcal{N}}^{[K+1]} + \hat{\mathcal{B}}^{[K+1]} = \left\{ \mathcal{P} \left( \hat{\mathcal{F}} \left( \phi^{(K)} \right) \right) \right\}_{\leq K+1}.
\]

(49)
This implies in particular that
\[ \Delta \hat{\Phi}^{(K)} = O_{[K+1]}, \quad \Delta \hat{N}^{(K)} = O_{[K+1]}, \quad \Delta \hat{B}^{(K)} = O_{[K+1]). \]

From a computational point of view, at any step \( K \) of this recurrent scheme it would be just necessary to compute these incremental terms. Besides, notice that since \( \hat{N}^{(K)} \) and \( \hat{B}^{(K)} \) contain only terms of odd order, it follows that
\[
\hat{N}^{(2J)} - \hat{N}^{(2J-1)} = \hat{B}^{(2J)} - \hat{B}^{(2J-1)} = 0
\]

or, equivalently,
\[
\Delta \hat{N}^{(2J-1)} = \Delta \hat{B}^{(2J-1)} = 0,
\]

for any \( J \geq 2 \).

§2.5 Convergence of the recurrent scheme

§2.5.1 Definition of the norm, estimates and technical lemmas

The domains we consider are those of type
\[ \overline{D}_\sigma = \{ z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_j| \leq \sigma \quad j = 1, 2, \ldots, n \}, \]

where \( r > 0 \) and \( |\cdot| \) denotes the standard modulo. By an analytic function \( f(z) \) on \( \overline{D}_\sigma \) we mean a function with Taylor expansion
\[
f(z) = \sum_{\alpha \in (\mathbb{N} \cup \{0\})^n} f_\alpha z^\alpha
\]

(absolutely) convergent for any \( z \in \overline{D}_\sigma \). We use the standard multi-index notation: if
\[ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \quad \text{and} \quad z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \]

one sets
\[
|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n
\]
\[ \alpha! = \alpha_1! \alpha_2! \cdots \alpha_n! \]
\[ z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \]
\[ D^\alpha = \frac{\partial |\alpha|}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \cdots \partial z_n^{\alpha_n}} \]

and \( 1 = (1, 1, \ldots, 1) \), \( 0 = (0, 0, \ldots, 0) \). Moreover, in \( (\mathbb{N} \cup \{0\})^n \) we consider the following partial ordering:
\[ \alpha \geq \beta \quad \text{whenever} \quad \alpha_j \geq \beta_j \quad \text{for} \quad j = 1, 2, \ldots, n. \]

Given a function \( f \) analytic on \( \overline{D}_\sigma \) we consider the following norms: the supremum norm
\[ \| f \|_{\infty, \sigma} = \sup_{z \in \overline{D}_\sigma} |f(z)| \]

and the 1-norm
\[
\| f \|_{1, \sigma} = \sum_{|\alpha| \geq 0} |f_\alpha| \sigma^{|\alpha|}.
\]
For a vector field $F = (f_1, f_2, \ldots, f_n): D_{\sigma} \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^n$ we define

$$(53) \quad \|F\|_{\infty, \sigma} = \sup_{i=1, \ldots, n} \|f_i\|_{\infty, \sigma}, \quad \|F\|_{1, \sigma} = \frac{1}{n} \sum_{i=1, \ldots, n} \|f_i\|_{1, \sigma}$$

and analogously if $F: D_{\sigma} \subseteq \mathbb{C}^n \rightarrow \mathbb{M}_{n,n} (\mathbb{C}^n)$. The next lemma list some properties of these norms. We omit its proof since it is standard.

**Lemma 3** Let $f$ be an analytic function on $D_{\sigma_1}$ satisfying that $f(0) = 0$ and assume $0 < \sigma_2 \leq \sigma_1$. Then, the following properties hold:

(i) $\|f\|_{\infty, \sigma_2} \leq \|f\|_{1, \sigma_1}$.

(ii) Let $\Phi = (\phi_1, \phi_2, \ldots, \phi_n): D_{\sigma_2} \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^n$ be analytic on $D_{\sigma_2}$ and satisfying that $\|\Phi\|_{\infty, \sigma_2} \leq \sigma_1$ (that is, $\Phi(D_{\sigma_2}) \subseteq D_{\sigma_1}$). Then we have

$$\|f \circ \Phi\|_{1, \sigma_2} \leq \|f\|_{1, \sigma_1}.$$ 

If $F = (f_1, \ldots, f_n)$ is an analytic vector field on $D_{\sigma_1}$ the same estimate holds for $\|F \circ \Phi\|_{1, \sigma_2}$.

(iii) Let $g$ be an analytic function on $D_{\sigma}$ satisfying that $|g(z)| \geq C \forall z \in D_{\sigma}$. Then, one has that

$$\left\| \frac{1}{g} \right\|_{1, \sigma} \leq \frac{1}{C}.$$ 

(iv) If $G_{[K]} = O_{[K]}$ and $H_{[L]} = O_{[L]}$ are homogeneous polynomials of orders $K$ and $L$, respectively, with $K \neq L$, then

$$\|G_{[K]} + H_{[L]}\|_{1, \sigma_2} = \|G_{[K]}\|_{1, \sigma_2} + \|H_{[L]}\|_{1, \sigma_2}.$$ 

From this point up to the end of this section we will prove some technical results which will be used during the proof of the convergence of the recurrent scheme introduced in Sections §2.3 and §2.4. In particular, next lemma provides a lower bound for $|q_1\lambda_1 + q_2\lambda_2|$ which works in both cases, when the equilibrium point is a saddle-center or a saddle-focus (whose characteristic exponents are given by $\{\pm \lambda, i\alpha\}$ and $\{\pm \lambda, i\alpha\}$, respectively).

**Lemma 4** Let us define

$$(54) \quad \omega_\infty = \omega_\infty (\Lambda) := \min \{\lambda, \alpha\}$$

where we assume $\lambda, \alpha > 0$. Then, we have that

$$|q_1\lambda_1 + q_2\lambda_2| \geq \left(\sqrt{q_1^2 + q_2^2}\right) \omega_\infty$$

for any $q_1, q_2 \in \mathbb{Z}$.

**Proof.** We proceed separately. Thus,

(a) **Saddle-center case:** as it has been mentioned above, we have $\lambda_1 = \lambda$ and $\lambda_2 = i\alpha$ so

$$|q_1\lambda_1 + q_2\lambda_2| = \sqrt{q_1^2 \lambda^2 + q_2^2 \alpha^2} \geq \left(\sqrt{q_1^2 + q_2^2}\right) \min \{\lambda, \alpha\} = \left(\sqrt{q_1^2 + q_2^2}\right) \omega_\infty.$$ 

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(b) \textbf{Saddle-focus case}: now we have $\lambda_1 = \lambda + \imath \alpha$ and $\lambda_2 = \lambda - \imath \alpha$. Then,

$$|q_1 \lambda_1 + q_2 \lambda_2| = |(q_1 + q_2)\lambda + (q_1 - q_2)\imath \alpha| = \sqrt{(q_1 + q_2)^2 \lambda^2 + (q_1 - q_2)^2 \alpha^2}.$$

If $q_1 q_2 > 0$, using that $|q_1| + |q_2| \geq \sqrt{q_1^2 + q_2^2}$, one obtains that

$$\sqrt{(q_1 + q_2)^2 \lambda^2 + (q_1 - q_2)^2 \alpha^2} \geq \sqrt{(q_1 + q_2)^2 \lambda^2} = (|q_1| + |q_2|) \lambda \geq \left(\sqrt{q_1^2 + q_2^2}\right) \omega_\infty.$$

On the other hand, if $q_1 q_2 < 0$ then

$$\sqrt{(q_1 + q_2)^2 \lambda^2 + (q_1 - q_2)^2 \alpha^2} \geq \sqrt{(q_1 - q_2)^2 \alpha^2} = (|q_1| + |q_2|) \alpha \geq \left(\sqrt{q_1^2 + q_2^2}\right) \omega_\infty,$$

which proves the lemma.

\[ \square \]

\textbf{Remark 7} In fact, $\omega_\infty$ constitutes a lower bound for the values $\omega_k$ introduced by Bruno in condition $\omega$ (see Section §1). Moreover, notice that, in the saddle-center case, one has that

$$\rho(A^{-1}) = \omega_\infty^{-1},$$

where $\rho(M)$ is the spectral radius of the matrix $M$, defined as the maximum of the modulus of their eigenvalues.

Now, we present a basic result which provides estimates for the vector fields $\mathcal{R} \hat{\Psi}$, $\hat{N}$ and $\hat{B}$ that are solution of the equations

$$\mathcal{N} + \mathcal{B} = \mathcal{P} \mathcal{H}$$

$$\mathcal{L}_N \left( \mathcal{R} \hat{\Psi} \right) = \mathcal{R} \mathcal{H}$$

and whose formal approach has been derived in Sections §2.3.2 and §2.3.1, respectively.

\textbf{Proposition 1} Let us consider a vector field $\mathcal{H}$ analytic on $\mathcal{T}_{\sigma}$ and let $\mathcal{R} \hat{\Psi}$ and $\hat{N}$, $\hat{B}$ (of the form (5a) and (7a), respectively) be the solutions of equations (55), (formally) derived in Sections §2.3.2 and §2.3.1. Then, the following estimates hold.

\begin{enumerate}
\item[(i)] First, we have

$$\left\| \hat{N} \right\|_{1,\sigma}, \left\| \hat{B} \right\|_{1,\sigma} \leq \left\| \mathcal{P} \mathcal{H} \right\|_{1,\sigma}.$$

\item[(ii)] Moreover,

$$\left\| \mathcal{R} \hat{\Psi} \right\|_{1,\sigma} \leq \frac{\left\| \mathcal{R} \mathcal{H} \right\|_{1,\sigma}}{\omega_\infty \left( 1 - \frac{1}{\sigma \omega_\infty} \left\| \mathcal{R} \mathcal{H} \right\|_{1,\sigma} \right)}$$

provided we assume that the bound

$$\left\| \mathcal{R} \mathcal{H} \right\|_{1,\sigma} < \frac{\sigma \omega_\infty}{4}$$

is satisfied.
\end{enumerate}
Proof.

(i) From equation (42) and formulas (43) it follows straightforwardly that \( \| \hat{N} \|_{1,\sigma} \) and \( \| \hat{B} \|_{1,\sigma} \) are both bounded by \( \| \mathcal{P} \hat{H} \|_{1,\sigma} \).

(ii) In Section §2.3.1 we dealt with equation

\[
\mathcal{L}_N \left( \mathcal{R} \hat{\Psi} \right) = \mathcal{R} \hat{H}
\]

where \( N \) had the form (5a) and we wrote \( \mathcal{R} \hat{H} = (\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_4) \) where

\[
\hat{h}_w(\xi, \eta, \mu, \nu) = \sum_{p, q \in \mathbb{Z}} h_{p+q}^{(w)}(\xi, \eta, \mu) \xi^p \mu^q
\]

for \( w = 1, 2, \ldots, 4 \), with

\[
h_{p+q}^{(w)}(\xi, \eta, \mu) = \sum_{(k, m) \in Q_{pq}} h_{p+k, k, q+m, m}^{(w)}(\xi, \eta, \mu) \xi^k (\mu \nu)^m
\]

and

\[
Q_{pq} := \left\{ (k, m) \in (\mathbb{N} \cup \{0\})^2 : \begin{array}{l}
k \\
m
\end{array} \geq \max \left\{ 0, -p \right\}, \begin{array}{l}
k \\
m
\end{array} \geq \max \left\{ 0, -q \right\}, k + m \geq 1 - \frac{p + q}{2} \right\}.
\]

We note that, by definition of the 1-norm (52), we have that

\[
\left\| \hat{h}_w \right\|_{1,\sigma} = \sum_{j+k+\ell+m \geq 2} \left| h_{j+k+\ell+m} \right| \sigma^{-j-k-\ell-m} =
\]

\[
\sum_{p, q \in \mathbb{Z}} \sum_{(k, m) \in Q_{pq}} \left| h_{p+k, k, q+m, m}^{(w)} \right| \sigma^{p+q+2(k+m)} =
\]

\[
\sum_{p, q \in \mathbb{Z}} \left\| h_{p+q}^{(w)}(\xi, \eta, \mu) \xi^p \mu^q \right\|_{1,\sigma}.
\]

From that section we also know that its solution \( \mathcal{R} \hat{\Psi} = (\hat{\psi}_1, \hat{\psi}_2, \ldots, \hat{\psi}_4) \) is given, in terms of formal power series, by

\[
\hat{\psi}_w(\xi, \eta, \mu, \nu) = \sum_{p, q \in \mathbb{Z}} \psi_{pq}^{(w)}(\xi, \eta, \mu) \xi^p \mu^q
\]

where \( \psi_{pq}^{(w)}(\xi, \eta, \mu) \) are obtained from

\[
\psi_{pq}^{(w)}(\xi, \eta, \mu) = \frac{h_{pq}^{(w)}(\xi, \eta, \mu)}{\Gamma_{pq}^{(w)}(\lambda) + p \hat{a}_1(\xi, \eta, \mu) + q \hat{a}_2(\xi, \eta, \mu)},
\]

for \( w = 1, 2, \ldots, 4 \), \( p, q \in \mathbb{Z} \) and the coefficients \( \Gamma_{pq}^{(w)}(\lambda) \) as defined in (39). Notice that the functions \( \psi_{pq}^{(w)} \) in (61) are rational functions of \( \xi, \eta, \mu \). Therefore, equation (60) is not an standard representation in power series, that is, formula (58) does not apply to \( \psi_{pq}^{(w)} \).

To estimate the 1-norm of \( \mathcal{R} \hat{\Psi} \) on \( \mathfrak{D}_\sigma \) we have to bound their components. Indeed, using Lemma 3(i), we can write

\[
\left\| \hat{\psi}_w \right\|_{1,\sigma} \leq \sum_{p, q \in \mathbb{Z}} \left\| \psi_{pq}^{(w)}(\xi, \eta, \mu) \xi^p \mu^q \right\|_{1,\sigma} =
\]

\[
\sum_{p, q \in \mathbb{Z}} \left\| \frac{h_{pq}^{(w)}(\xi, \eta, \mu)}{\Gamma_{pq}^{(w)}(\lambda) + p \hat{a}_1(\xi, \eta, \mu) + q \hat{a}_2(\xi, \eta, \mu)} \xi^p \mu^q \right\|_{1,\sigma} \leq
\]

\[
\sum_{p, q \in \mathbb{Z}} \left\| \frac{h_{pq}^{(w)}(\xi, \eta, \mu) \xi^p \mu^q}{\Gamma_{pq}^{(w)}(\lambda) + p \hat{a}_1(\xi, \eta, \mu) + q \hat{a}_2(\xi, \eta, \mu)} \right\|_{1,\sigma} \frac{1}{\Gamma_{pq}^{(w)}(\lambda) + p \hat{a}_1(\xi, \eta, \mu) + q \hat{a}_2(\xi, \eta, \mu)}
\]

\[22\]
The next lemma gives an upper bound for the second norm appearing in this formula (62).

**Lemma 5** Consider $\Gamma_{pq}^{(w)}(\lambda)$ as defined in (39) and $\hat{a}_1(\xi,\eta,\mu,\nu)$, $\hat{a}_2(\xi,\eta,\mu,\nu)$ coming from (5a). Then, for any $p, q \in \mathbb{Z}$ and $(\xi, \eta, \mu, \nu) \in \overline{D}_\sigma$, we have that

$$\left| \Gamma_{pq}^{(w)}(\lambda) + p \hat{a}_1(\xi,\eta,\mu,\nu) + q \hat{a}_2(\xi,\eta,\mu,\nu) \right| \geq \omega_{\infty} \left( 1 - \frac{4}{\sigma \omega_{\infty}} \| \mathcal{R} \hat{H} \|_{1,\sigma} \right)$$

provided estimate (56) is satisfied.

**Proof.** (lemma) We will distinguish two cases:

(a) If $|p| + |q| \geq 1$ it follows that

$$\left| \Gamma_{pq}^{(w)}(\lambda) + p \hat{a}_1(\xi,\eta,\mu,\nu) + q \hat{a}_2(\xi,\eta,\mu,\nu) \right| \geq \left| \Gamma_{pq}^{(w)}(\lambda) \right| - \| p \hat{a}_1 + q \hat{a}_2 \|_{\infty,\sigma}$$

From the definition of $\Gamma_{pq}^{(w)}$ in (39) and applying Lemma 4 it turns out that

$$\left| \Gamma_{pq}^{(w)}(\lambda) \right| \geq M_{pq} \omega_{\infty},$$

where we define

$$M_{pq} := \min \left\{ \sqrt{(|p|-1)^2 + q^2}, \sqrt{p^2 + (|q|-1)^2} \right\}.$$

We recall that the terms $h_{pq}^{(w)}(\xi,\eta,\mu,\nu)$ with $|p| = 1$ and $q = 0$ or $p = 0$ and $|q| = 1$ vanish since they belong to the projection $\mathcal{P} \hat{H}$, so, in particular, this implies that

$$M_{pq} \geq 1.$$

Moreover, it is clear that

$$|p|, |q| \leq 2M_{pq}.$$

Coming back to equation (63) we have that

$$\left| \Gamma_{pq}^{(w)}(\lambda) \right| - \| p \hat{a}_1 + q \hat{a}_2 \|_{\infty,\sigma} \geq M_{pq} \omega_{\infty} \left( 1 - \frac{1}{M_{pq} \omega_{\infty}} \| p \hat{a}_1 + q \hat{a}_2 \|_{\infty,\sigma} \right) \geq \omega_{\infty} \left( 1 - \frac{4}{\sigma \omega_{\infty}} \| \mathcal{R} \hat{H} \|_{1,\sigma} \right),$$

where it has been taken into account the assumption (56) and, by (65), (43) and Lemma 3(i), that

$$\frac{1}{M_{pq} \omega_{\infty}} \| p \hat{a}_1 + q \hat{a}_2 \|_{\infty,\sigma} \leq \frac{1}{\omega_{\infty}} \left( \frac{|p|}{M_{pq}} \| \hat{a}_1 \|_{\infty,\sigma} + \frac{|q|}{M_{pq}} \| \hat{a}_2 \|_{\infty,\sigma} \right) \leq \frac{2}{\omega_{\infty}} \left( \| \hat{a}_1 \|_{\infty,\sigma} + \| \hat{a}_2 \|_{\infty,\sigma} \right) = \frac{2}{\sigma \omega_{\infty}} \left( \| \sigma \hat{a}_1 \|_{\infty,\sigma} + \| \sigma \hat{a}_2 \|_{\infty,\sigma} \right) \leq \frac{4}{\sigma \omega_{\infty}} \| \mathcal{R} \hat{H} \|_{\infty,\sigma} \leq \frac{4}{\sigma \omega_{\infty}} \| \mathcal{R} \hat{H} \|_{1,\sigma}.$$
(b) If \( p = q = 0 \) one has that

\[
(66) \quad \left| \Gamma_{pq}^{(w)} (\lambda) + p \hat{a}_1 (\xi, \mu \nu) + q \hat{a}_2 (\xi, \mu \nu) \right| = \left| \Gamma_{00}^{(w)} (\lambda) \right| \geq \omega_\infty,
\]

and, in particular, assuming again (56),

\[
\left| \Gamma_{00}^{(w)} (\lambda) \right| \geq \omega_\infty \geq \omega_\infty \left( 1 - \frac{4}{\sigma \omega_\infty} \left\| \mathcal{R} \hat{H} \right\|_{1,\sigma} \right).
\]

This concludes the proof of this lemma. ♦

Since we are assuming that (56) holds, we can apply this lemma together with lemma 3(iii) and, therefore, it follows that

\[
\left\| \Gamma_{pq}^{(w)} (\lambda) + p \hat{a}_1 (\xi, \mu \nu) + q \hat{a}_2 (\xi, \mu \nu) \right\|_{1,\sigma} \leq \frac{1}{\omega_\infty \left( 1 - \frac{4}{\sigma \omega_\infty} \left\| \mathcal{R} \hat{H} \right\|_{1,\sigma} \right)}.
\]

Thus, estimate (62) jointly with (59) gives

\[
\left\| \hat{\Psi}_{\omega} \right\|_{1,\sigma} \leq \frac{1}{\omega_\infty \left( 1 - \frac{4}{\sigma \omega_\infty} \left\| \mathcal{R} \hat{H} \right\|_{1,\sigma} \right)} \sum_{p,q \in \mathbb{Z}} \left\| h_{pq}^{(w)} (\xi, \mu \nu) \xi^p \mu^q \right\|_{1,\sigma} = \frac{\left\| \hat{h}_{\omega} \right\|_{1,\sigma}}{\omega_\infty \left( 1 - \frac{4}{\sigma \omega_\infty} \left\| \mathcal{R} \hat{H} \right\|_{1,\sigma} \right)}
\]

for \( w = 1, 2, \ldots, 4 \). Finally, using (53), it turns out that

\[
\left\| \mathcal{R} \hat{\Psi} \right\|_{1,\sigma} \leq \frac{\left\| \mathcal{R} \hat{H} \right\|_{1,\sigma}}{\omega_\infty \left( 1 - \frac{4}{\sigma \omega_\infty} \left\| \mathcal{R} \hat{H} \right\|_{1,\sigma} \right)}.
\]

\[\square\]

\section{Proof of the convergence}

To ease the reading of this proof, let us recall briefly the problem we are dealing with. Let consider a system

\[
(67) \quad \dot{X} = F(X) = \Lambda + \hat{F}(X)
\]

where \( F \) is analytic on a domain \( \overline{D_R} \) and having at \( X = 0 \) a saddle-focus or saddle-center equilibrium point with characteristic exponents \( \{ \pm \lambda_1, \pm \lambda_2 \} \) equal to \( \{ \pm \lambda, \pm i \alpha \} \) and \( \{ \pm \lambda, \pm i \alpha \} \), respectively. As it has been seen at the beginning of Section \( \S 2.1 \), we can assume the matrix \( \Lambda \) to be written in (complex) diagonal form. This allows us to deal with both cases using a unified approach. We also recall that, again in Section \( \S 2.1 \), we introduced the notation \( \Lambda \) to denote both the matrix \( \Lambda \) and the vector field \( \Lambda d \). We will only use explicitly the second expression in cases of possible misunderstanding.
Our aim is the following: we are looking for an analytic transformation \( X = \Phi(\chi) = \chi + \hat{\Phi}(\chi) \) and analytic vector fields \( N \) and \( \hat{B} \) (that we can assume to be of the form \((5a)\) and \((7a)\), respectively) such that the equality

\[
D\Phi N + \hat{B} = F(\Phi)
\]

is satisfied. We say in that case that \( \Phi \) leads system \((67)\) into \( \PsiNF \). To get such transformation and vector fields we have developed in Sections \( \S2.3 \) and \( \S2.4 \) the following recurrent scheme to whose convergence proof is devoted this section. Setting the following condition on \( \hat{\Phi} \),

\[
\mathcal{P}\hat{\Phi} \equiv 0,
\]

we take initial values

\[
\Phi^{(1)} = \text{id}, \quad N^{(1)} = \Lambda, \quad \hat{B}^{(1)} = 0
\]

and obtain, recurrently,

\[
\begin{align*}
\Phi^{(K+1)} &= \text{id} + \mathcal{R}\hat{\Phi}^{(K+1)} \\
N^{(K+1)} &= \Lambda + \hat{N}^{(K+1)}
\end{align*}
\]

with

\[
\begin{align*}
\hat{\Phi}^{(K+1)} &= \mathcal{O}_{\leq K+1}, \quad \hat{N}^{(K+1)} = \mathcal{O}_{\leq K+1}, \quad \hat{B}^{(K+1)} = \mathcal{O}_{\leq K+1},
\end{align*}
\]

from equations

\[
\begin{align*}
\left\{ \mathcal{L}_{X^{(K)}} \left( \mathcal{R}\hat{\Phi}^{(K+1)} \right) \right\}_{\leq K+1} &= \left\{ \mathcal{R} \left( \hat{F} \left( \Phi^{(K)} \right) \right) \right\}_{\leq K+1} \\
\hat{N}^{(K+1)} + \hat{B}^{(K+1)} &= \left\{ \mathcal{P} \left( \hat{F} \left( \Phi^{(K)} \right) \right) \right\}_{\leq K+1}.
\end{align*}
\]

Once set our problem, let us start with the proof. Let us consider a positive constant \( 0 < \gamma < 1 \) (in order to simplify the estimates, we can assume \( \gamma \geq 1/2 \), which is not restrictive). As it is commonly done in Normal Form Theory, we can scale our system by means of a change \( X = \alpha Z \), where \( \alpha > 0 \) is a constant to determine. Thus we have a new system

\[
\dot{Z} = F_\alpha(Z) := \Lambda + \alpha^{-1}\hat{F}(\alpha Z),
\]

with \( F_\alpha \) analytic on \( \overline{D}_r \), where \( r := \alpha^{-1}R \). Let us consider a positive constant \( 0 < \gamma < 1 \). In order to simplify the estimates, we can assume \( \gamma \geq 1/2 \), which is not restrictive. Then, since \( \hat{F}_\alpha \) starts with terms of order at least 2, we can choose \( \alpha \) big enough (so \( r \) small enough) in such a way that the following estimate holds

\[
\|\hat{F}\|_{1,r} \leq \left( \frac{(1 - \gamma)\omega_\infty}{8} \right) r.
\]

Calling again \( Z \) and \( F_\alpha \) as \( X \) and \( F \), respectively, we can assume our system \((67)\) to be analytic on \( \overline{D}_r \) and satisfying \((75)\). We are going to prove that the limit vector fields \( \Phi \), \( N \) and \( \hat{B} \) obtained from this recurrent scheme satisfy \((68)\) and are analytic on \( \overline{D}_{\gamma R} \) (and therefore, reversing the scaling, on \( \overline{D}_{\gamma R} \)).

For ease of reading we will itemize the proof in several parts: the first one will provide some estimates on the approximations provided by the recurrent scheme; in the second one, their convergence will be derived.
(i) Consider system (67) having $F$ analytic on a domain $\overline{B}_r$ and satisfying the assumption (75). Apply onto it the recurrent scheme (69)–(73) and consider the sequences

$$
\begin{align*}
\left\{ \left\| \Phi^{(K)} \right\|_{1,s} \right\}_K, & \quad \left\{ \left\| N^{(K)} \right\|_{1,s} \right\}_K, & \quad \left\{ \left\| \hat{B}^{(K)} \right\|_{1,s} \right\}_K,
\end{align*}
$$

defined for $K \geq 1$ and being $s = \gamma r$. Then, the following properties are satisfied.

(a) They increase monotonically, that is,

$$
\begin{align*}
\left\| \Phi^{(K+1)} \right\|_{1,s} & \geq \left\| \Phi^{(K)} \right\|_{1,s}, \\
\left\| N^{(K+1)} \right\|_{1,s} & \geq \left\| N^{(K)} \right\|_{1,s}, \\
\left\| \hat{B}^{(K+1)} \right\|_{1,s} & \geq \left\| \hat{B}^{(K)} \right\|_{1,s}.
\end{align*}
$$

(b) All these sequences are uniformly upper-bounded. Precisely, for all $K \geq 1$ we have that

$$
\left\| \Phi^{(K)} \right\|_{1,s} \leq r
$$

and that

$$
\left\| N^{(K)} \right\|_{1,s}, \left\| \hat{B}^{(K)} \right\|_{1,s} \leq \left\| F \right\|_{1,r}.
$$

Let us prove these assertions.

(a) From Lemma 2 we have that

$$
\begin{align*}
\Phi^{(K+1)} & = \Phi^{(K)} + R \Delta \hat{\Phi}^{(K)}, \\
N^{(K+1)} & = N^{(K)} + \Delta N^{(K)}, \\
\hat{B}^{(K+1)} & = \hat{B}^{(K)} + \Delta \hat{B}^{(K)},
\end{align*}
$$

where $R \Delta \hat{\Phi}^{(K)}$, $\Delta N^{(K)}$ and $\Delta \hat{B}^{(K)}$ are all three $C_{[K+1]}$, except in the case of odd $K$ where one has that

$$
\Delta N^{(2J-1)} = \Delta \hat{B}^{(2J-1)} = 0.
$$

Therefore, taking into account Lemma 3(iv), it turns out that

$$
\begin{align*}
\left\| \Phi^{(K+1)} \right\|_{1,s} = \left\| \Phi^{(K)} + R \Delta \hat{\Phi}^{(K)} \right\|_{1,s} = \\
\left\| \Phi^{(K)} \right\|_{1,s} + \left\| R \Delta \hat{\Phi}^{(K)} \right\|_{1,s} \geq \left\| \Phi^{(K)} \right\|_{1,s}.
\end{align*}
$$

In the same way it can be proved for $\left\| N^{(K+1)} \right\|_{1,s}$ and $\left\| \hat{B}^{(K+1)} \right\|_{1,s}$.

(b) To see it we proceed inductively. Thus, for $K = 1$ equation (72) becomes

$$
\left\{ \mathcal{L}_{N^{(1)}} \left( \mathcal{R} \hat{\Phi}^{(2)} \right) \right\}_{L^2} = \left\{ \mathcal{R} \left( \hat{F}(\Phi^{(1)}) \right) \right\}_{L^2}.
$$

Having in mind that $N^{(1)} = \Lambda$ (so $\hat{N}^{(1)} = 0$), $\Phi^{(1)} = \text{id}$ and the definition (25) of the operator $\mathcal{L}$, this equation is equivalent to

$$
D \left( \mathcal{R} \hat{\Phi}^{(2)} \right) \Lambda - \Lambda \mathcal{R} \hat{\Phi}^{(2)} = \mathcal{R} F^{[2]}.
$$
and to

$$\left[ \Lambda, \mathcal{R}\hat{\Phi}^{(2)} \right] = F_{[2]}$$

where $[H,G] = (DG)H - (DH)G$ stands for the Lie bracket of the vector fields $H$ and $G$. Now, from Proposition 1(ii), taking into account that $\hat{a}_1^{(1)} = \hat{a}_2^{(1)} = 0$ (the functions appearing in $\hat{N}^{(1)}$) and using estimate (66) it follows that

$$\left\| \mathcal{R}\hat{\Phi}^{(2)} \right\|_{1,s} \leq \frac{\|F_{[2]}\|_{1,s}}{\omega_{\infty}}$$

and, in particular,

$$\left\| \mathcal{R}\hat{\Phi}^{(2)} \right\|_{1,s} \leq \frac{\|\hat{F}\|_{1,r}}{\omega_{\infty}}.$$  \hspace{1cm} (79)

Thus, applying Lemma 3(iv), the assumption $\mathcal{P}\hat{\Phi} = 0$ and the estimate (75), one obtains that

$$\left\| \Phi^{(2)} \right\|_{1,s} \leq s + \frac{\|\hat{F}\|_{1,r}}{\omega_{\infty}} \leq \gamma r + \frac{1 - \gamma}{8} r \leq \gamma r + (1 - \gamma)r = r.$$

Concerning vector fields $N^{(2)}$ and $\hat{B}^{(2)}$ we have that

$$N^{(2)} = N^{(1)} = \Lambda, \quad \hat{B}^{(2)} = \hat{B}^{(1)} = 0$$

and, therefore, estimate (78) is trivially satisfied.

Thus, by induction hypothesis, assume that the following bounds

$$\left\| \Phi^{(K)} \right\|_{1,s} \leq r \quad \left\| N^{(K)} \right\|_{1,s} \leq \left\| \hat{B}^{(K)} \right\|_{1,s} \leq \left\| F \right\|_{1,r}$$

hold. We are going to show that they are also true for $K + 1$. We start dealing with equation (72), namely,

$$\left\{ \mathcal{L}_{N^{(K)}} \left( \mathcal{R}\hat{\Phi}^{(K+1)} \right) \right\}_{K+1} = \left\{ \mathcal{R} \left( \hat{F} \left( \Phi^{(K)} \right) \right) \right\}_{K+1}.$$

This equation is of type $\mathcal{L}_N(\mathcal{R}\hat{\Phi}) = \mathcal{R}\hat{H}$ provided we take

$$N = N^{(K)}, \quad \mathcal{R}\hat{\Phi} = \mathcal{R}\hat{\Phi}^{(K+1)}, \quad \mathcal{R}\hat{H} = \mathcal{R} \left( \hat{F} \left( \Phi^{(K)} \right) \right)$$

and consider just terms up to order $K + 1$. Setting $\sigma = s$ and taking into account estimate (75), the induction hypothesis and Lemma 3(i,ii) it follows that

$$\left\| \mathcal{R}\hat{H} \right\|_{1,s} = \left\| \mathcal{R} \left( \hat{F} \left( \Phi^{(K)} \right) \right) \right\|_{1,s} \leq \left\| \hat{F} \right\|_{1,r} \leq \left( \frac{(1 - \gamma)\omega_{\infty}}{8} \right) r.$$

Using that $1/2 \leq \gamma < 1$ and that $s = \gamma r$, this estimate reads

$$\left\| \mathcal{R}\hat{H} \right\|_{1,s} \leq \left( \frac{(1 - \gamma)\omega_{\infty}}{8} \right) r \leq \frac{\gamma}{8} \omega_{\infty} r = \frac{s \omega_{\infty}}{8} < \frac{s \omega_{\infty}}{4},$$

27
which is assumption (56). Applying Proposition 1(ii) and that
\[
1 - \frac{4}{\gamma r \omega_\infty} \| \mathcal{R} \left( \hat{F}(\Phi^{(K)}) \right) \|_{1, \gamma r} = 1 - \frac{4}{\gamma r \omega_\infty} \left\| \mathcal{R} \hat{H} \right\|_{1, \gamma r} \geq
1 - \left( \frac{4}{s \omega_\infty} \right) \left( \frac{\omega_\infty}{8} \right) = 1 - \frac{1}{2} = \frac{1}{2}
\]
we obtain
\[
\| \mathcal{R} \hat{\Phi}^{(K+1)} \|_{1, \gamma r} \leq \frac{\| \mathcal{R} \left( \hat{F}(\Phi^{(K)}) \right) \|_{1, \gamma r}}{\omega_\infty \left( 1 - \frac{4}{\gamma r \omega_\infty} \right) \| \mathcal{R} \left( \hat{F}(\Phi^{(K)}) \right) \|_{1, \gamma r}} \leq \frac{(1 - \gamma) r}{\omega_\infty / 2} = \frac{(1 - \gamma) r}{4}.
\]

Finally, from Lemma 3(iv) one obtains that
\[
\| \Phi^{(K+1)} \|_{1, s} = \| \Phi^{(K+1)} \|_{1, \gamma r} = \| \text{id} \|_{1, \gamma r} + \| \mathcal{R} \hat{\Phi}^{(K+1)} \|_{1, \gamma r} \leq \gamma r + \frac{(1 - \gamma) r}{4} \leq \gamma r + (1 - \gamma) r = r.
\]

Concerning \( N^{(K+1)} \) and \( \hat{B}^{(K+1)} \), having in mind the induction hypothesis \( \| \Phi^{(K)} \|_{1, s} \leq r \), equation (73) and section \( \S 2.3.2 \), one obtains that
\[
\| \hat{N}^{(K+1)} \|_{1, s} \leq \| \hat{F} \|_{1, r}
\]
and
\[
\| \hat{B}^{(K+1)} \|_{1, s} \leq \| \hat{F} \|_{1, r} \leq \| F \|_{1, r}.
\]

Since \( N^{(K+1)} = \Lambda + \hat{N}^{(K+1)} \) and \( F = \Lambda + \hat{F} \) it turns out that
\[
\| N^{(K+1)} \|_{1, s} \leq \| F \|_{1, r}
\]
which concludes the proof of (b).

(ii) At (i) it has been proved that the sequences
\[
\left\{ \| \Phi^{(K)} \|_{1, s} \right\}_K, \quad \left\{ \| N^{(K)} \|_{1, s} \right\}_K, \quad \left\{ \| \hat{B}^{(K)} \|_{1, s} \right\}_K,
\]
increase monotonically and are uniformly upper-bounded. Applying onto them the Ascoli-Arzelà theorem it follows that they admit convergent subsequences
\[
\left\{ \| \Phi^{(K_j)} \|_{1, s} \right\}_j, \quad \left\{ \| N^{(K_j)} \|_{1, s} \right\}_j, \quad \left\{ \| \hat{B}^{(K_j)} \|_{1, s} \right\}_j.
\]
Therefore, if we define a vector field \( \Phi \) given by
\[
\Phi(\chi) := \lim_{J \to \infty} \Phi^{(K_j)}(\chi)
\]
for any \( \chi \in \mathcal{D}_s \), it follows that the limit
\[
\| \Phi \|_{1, s} = \lim_{J \to \infty} \| \Phi^{(K_j)} \|_{1, s}
\]
28
exists and is finite. From Weierstrass theorem it follows that $\Phi$ is an analytic vector field on $\overline{C} = \overline{C_{\gamma^r}}$. Moreover, since the recurrent scheme (69)-(73) and Lemma 2, provide vector fields $\Phi^{(K+1)}$ of the form 

$$\Phi^{(K+1)} = \Phi^{(K)} + R \Delta \hat{\Phi}^{(K)}$$

where $R \Delta \hat{\Phi}^{(K)} = O^{(K+1)}$, it can be derived that the subsequence $\left\{ \|\Phi^{(K)}\|_{1,s} \right\}_K$ is, in fact, the complete sequence $\left\{ \|\Phi^{(K)}\|_{1,s} \right\}_K$. In a similar way one obtains $N$ and $\hat{B}$, analytic vector fields on $\overline{C_{\gamma^r}}$ defined as

$$N := \lim_{K} N^{(K)}, \quad \hat{B} := \lim_{K} \hat{B}^{(K)}.$$ 

Together with $\Phi$, they satisfy that

$$D \Phi \ N + \hat{B} = F(\Phi)$$

and therefore, they lead system (67) into $\PsiNF$. This concludes the proof of the Main Theorem.

§3  Proof of Propositions H1 and R1

§3.1 Proof of Proposition H1

It is clear that if $\hat{B} \equiv 0$ then $\PsiNF$ is just $BNF$ so, let consider the converse situation. To fix ideas, let us deal with a 4-dimensional Hamiltonian system with the origin being a saddle-center equilibrium point. The saddle-focus case can be done in a similar way. Assume moreover that the center variables have been complexified (becoming complex conjugated). Applying Moser’s Theorem [21], we know the existence of an analytic convergent transformation $\Psi$, close to the identity, leading it into $BNF$,

$$\left\{ \begin{array}{l} \dot{\xi} = \xi a_1(\xi, \eta, \mu) \\ \dot{\eta} = -\eta a_1(\xi, \eta, \mu) \\ \dot{\mu} = \mu a_2(\xi, \eta, \mu) \\ \dot{\nu} = -\nu a_2(\xi, \eta, \mu) \end{array} \right. \quad (80)$$

with $a_1(\xi, \eta, \mu) = \lambda + \cdots$ and $a_2(\xi, \eta, \mu) = \alpha + \cdots$. It is clear that $h_1(\xi, \eta) = \xi \eta a_1(\xi, \eta, 0) = \lambda \xi \eta + \cdots$ and $h_2(\mu) = \mu \nu a_2(0, \mu, \nu) = i \alpha \mu \nu + \cdots$ are independent first integrals of system (80) and, therefore,

$$\begin{array}{l} h_1 = \tilde{h}_1 \circ \Psi^{-1} = \lambda \xi \eta + \cdots \\ h_2 = \tilde{h}_2 \circ \Psi^{-1} = i \alpha \mu \nu + \cdots \end{array}$$

are independent first integrals of the original one. Let $\Phi$ be the convergent analytic transformation leading the initial system into $\PsiNF$, that is, such that the new system is of the form

$$\dot{\chi} = N(\chi) + (D \Phi(\chi))^{-1} \hat{B}(\chi) \quad (81)$$

where $\chi = (\xi, \eta, \mu, \nu)$ denotes now the $\PsiNF$-variables. Since $\Phi$ starts with the identity and $h_1$, $h_2$ are independent first integrals of the original system, it follows that $\hat{h}_1 = h_1 \circ \Phi$ and $\hat{h}_2 = h_2 \circ \Phi$ are first integrals of (81) and, moreover, they begin with $\lambda \xi \eta + \cdots$ and $i \alpha \mu \nu + \cdots$, respectively. Indeed, they satisfy

$$D \hat{h}_j \left( N + (D \Phi)^{-1} \hat{B} \right) \equiv 0 \quad (82)$$

for $j = 1, 2$. Assume now that $\hat{B} \neq 0$ so its minimal order terms are

$$\begin{pmatrix} \xi b_1^{[1]}(\xi, \eta)^{\nu}(\mu)^{s} + \cdots \\ \eta b_1^{[1]}(\xi, \eta)^{\nu}(\mu)^{s} + \cdots \\ \mu b_2^{[2]}(\xi, \eta)^{\nu}(\mu)^{s'} + \cdots \\ \nu b_2^{[2]}(\xi, \eta)^{\nu}(\mu)^{s'} + \cdots \end{pmatrix}$$
with \( b^{(1)}_{r,s} \neq 0 \) or \( b^{(2)}_{r,s'} \neq 0 \) (and \( r + s \) not necessarily equal to \( r' + s' \)). Using that \( \hat{h}_1 = \lambda \xi + \cdots \) and \( (D\Phi)^{-1} = I - (D\hat{\Phi}) + \cdots \), the term of type \((\xi \eta)^j (\mu \nu)^m\) of minimal order corresponding to the left-hand side of equation (82), for \( j = 1 \), is given by

\[
(\lambda \eta + \cdots \lambda \xi + \cdots 0 + \cdots 0 + \cdots) \begin{pmatrix}
\xi b^{(1)}_{r,s} (\xi \eta)^r (\mu \nu)^s + \cdots \\
\eta b^{(1)}_{r,s} (\xi \eta)^r (\mu \nu)^s + \cdots
\end{pmatrix} = \\
2\lambda b^{(1)}_{r,s} (\xi \eta)^{r+1} (\mu \nu)^s + \cdots
\]

Since \( \lambda \neq 0 \) it implies that \( b^{(1)}_{r,s} = 0 \). Applying the same argument to equation (82) with \( j = 2 \), and using that \( \alpha \neq 0 \), it follows that \( b^{(2)}_{r,s'} = 0 \), which contradicts the assumption of \( \hat{B} \neq 0 \). Consequently, \( \hat{B} \) vanishes.

§3.2 Proof of Proposition R1

The problem of the convergence of the \( \Psi_{\mathbb{NF}} \) (and \( \mathbb{BNF} \)) around an equilibrium is certainly a local problem. In the reversible setting, this implies that both the linearized system and the reversing involution can be taken in suitable way. Namely,

**Lemma 6** Let us consider a system

\[ \dot{X} = F(X) \]

analytic around the origin, a saddle-center or a saddle-focus equilibrium, and assume it is reversible with respect to an (in principle, non linear) involutive diffeomorphism \( \mathcal{G} \). Suppose that the origin is a fixed point of \( \mathcal{G} \). Then there exists an analytic change of variables \( X \mapsto Z \), defined in a neighborhood of the origin, such that in the new coordinates the linearized system becomes

\[
\dot{Z} = \begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & -\lambda & 0 & 0 \\
0 & 0 & \mathbf{i} \alpha & 0 \\
0 & 0 & 0 & -\mathbf{i} \alpha
\end{pmatrix} Z
\]

or

\[
\dot{Z} = \begin{pmatrix}
\lambda + \mathbf{i} \alpha & 0 & 0 & 0 \\
0 & \lambda - \mathbf{i} \alpha & 0 & 0 \\
0 & 0 & -(\lambda + \mathbf{i} \alpha) & 0 \\
0 & 0 & 0 & -(\lambda - \mathbf{i} \alpha)
\end{pmatrix} Z,
\]

depending if we are in the saddle-center or saddle-focus case, respectively, and assuming \( \lambda, \alpha > 0 \). Moreover, in these coordinates and for both cases, the symmetry \( \mathcal{G} \) can be taken of the form

\[ Z \mapsto \mathcal{R} Z, \]

where \( \mathcal{R} \) is given by the matrix

\[
(83) \quad \mathcal{R} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]
Proof. The proof of this lemma is essentially contained in [24]. For the saddle-center case, it is proved there the existence of a coordinate system, with center at the origin, in whose variables, say \( Z = (u_1, u_2, z, \overline{z}) \), the linearized system is given by

\[
\begin{pmatrix}
\dot{u}_1 \\
\dot{u}_2 \\
\dot{z} \\
\dot{\overline{z}}
\end{pmatrix} = \begin{pmatrix}
0 & \lambda & 0 & 0 \\
\lambda & 0 & 0 & 0 \\
0 & 0 & i\alpha & 0 \\
0 & 0 & 0 & -i\alpha
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
z \\
\overline{z}
\end{pmatrix},
\]

and the linear part of the involution \( \mathcal{G} \) has the form

\[
\begin{pmatrix}
u_1 \\
u_2 \\
z \\
\overline{z}
\end{pmatrix} \mapsto \begin{pmatrix}
u_1 \\
-u_2 \\
\overline{z} \\
z
\end{pmatrix},
\]

with \( z \in \mathbb{C} \) and \( u_1, u_2 \in \mathbb{R} \). In that proof, it has been used that the linearization of an involution around one of its fixed points is also an involution. Performing the linear transformation

\[
(v_1, v_2, z, \overline{z}) = \left( \frac{u_1 + u_2}{2}, \frac{u_1 - u_2}{2}, z, \overline{z} \right)
\]

we reach the claimed result about the form of the linear parts of \( F \) and \( \mathcal{G} \). Now, by Bochner’s theorem (see [2, 19]) there exists an analytic change of variables, defined in a neighborhood of the fixed point, the origin, which conjugates the symmetry \( \mathcal{G} \) to its linear part

\[
(v_1, v_2, z, \overline{z}) \mapsto (v_2, v_1, \overline{z}, z).
\]

With respect to the saddle-focus case, it works in a similar way. In this situation the first change of variables takes \( X \) to

\[
Z = (z_1, z_2, z_3, z_4) = (z_1, z_1, \overline{z}_3, \overline{z}_3),
\]

the linearization of the system \( \dot{X} = F(X) \) to

\[
\begin{pmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\dot{z}_4
\end{pmatrix} = \begin{pmatrix}
\lambda + i\alpha & 0 & 0 & 0 \\
0 & \lambda - i\alpha & 0 & 0 \\
0 & 0 & -(\lambda + i\alpha) & 0 \\
0 & 0 & 0 & -(\lambda - i\alpha)
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4
\end{pmatrix},
\]

and the linearization of \( \mathcal{G} \) to \( Z \mapsto \overline{Z} \), which is the involution given by the matrix (83). The local analytic conjugacy with \( \mathcal{G} \) is again provided by Bochner’s theorem.

\( \square \)

Therefore, it is not restrictive to assume that our system is written, in a neighborhood of the origin, in the form

\[
\dot{X} = \Lambda + \tilde{F}(X),
\]

with

\[
\Lambda = \begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & -\lambda & 0 & 0 \\
0 & 0 & i\alpha & 0 \\
0 & 0 & 0 & -i\alpha
\end{pmatrix} X
\]

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in the saddle-center case, or
\[
\Lambda = \begin{pmatrix}
    \lambda + i\alpha & 0 & 0 & 0 \\
    0 & \lambda - i\alpha & 0 & 0 \\
    0 & 0 & -(\lambda + i\alpha) & 0 \\
    0 & 0 & 0 & -(\lambda - i\alpha)
\end{pmatrix} X,
\]
in the case of a saddle-focus, respectively. Moreover, we can assume it to be reversible with respect to the linear involution
\[
\mathcal{R} : (x_1, x_2, x_3, x_4) \mapsto (x_2, x_1, x_4, x_3).
\]
Thus, the reversibility condition (14) reads
\[
(84) \quad \mathcal{R} F(\mathcal{R} X) = -F(X).
\]
Once we have set the linear framework, we present a property which characterizes those transformations that preserve a given linear reversibility.

**Lemma 7** Let \( \Psi \) be a diffeomorphism satisfying
\[
(85) \quad \mathcal{R} \Psi (\mathcal{R} \chi) = \Psi (\chi).
\]
Then the transformation \( X = \Psi (\chi) \) preserves the \( \mathcal{R} \)-reversibility, that is, the new system
\[
\dot{\chi} = G(\chi) := (\Psi^* F)(\chi)
\]
is also \( \mathcal{R} \)-reversible.

**Proof.** To see that \( \dot{\chi} = G(\chi) \) is \( \mathcal{R} \)-reversible we have to check that \( \mathcal{R} G(\mathcal{R} \chi) = -G(\chi) \). Differentiating both sides of equation (85) we get
\[
\mathcal{R} D\Psi (\mathcal{R} \chi) \mathcal{R} = D\Psi (\chi).
\]
Using this property and equations (84), (85) it follows that
\[
\mathcal{R} G(\mathcal{R} \chi) = \mathcal{R} \left((D\Psi (\mathcal{R} \chi))^{-1} F(\Psi (\mathcal{R} \chi))\right) = \\
\mathcal{R} \left((D\Psi (\mathcal{R} \chi))^{-1} F(\Psi (\chi))\right) = -\mathcal{R} \left((D\Psi (\mathcal{R} \chi))^{-1} \mathcal{R} F(\Psi (\chi))\right) = \\
-\mathcal{R} \left(\mathcal{R} (D\Psi (\chi))^{-1} \mathcal{R} F(\Psi (\chi))\right) = -(D\Psi (\chi))^{-1} F(\Psi (\chi)) = \\
-(\Psi^* F)(\chi) = -G(\chi),
\]
which concludes the proof of this lemma. 

The proof of Proposition R1 is based on the following two points:

- Applying Theorem 2, there exist an analytic transformation \( X = \Phi(\chi) \) and analytic vector fields \( N(\chi), \dot{B}(\chi) \) leading the original system into \( \Psi \mathcal{F} \), provided the origin is a saddle-center or saddle-focus equilibrium point. That is, satisfying the equality
  \[
  D\Phi N + \dot{B} = F \circ \Phi.
  \]
- We will prove that the vector fields obtained from the recurrent scheme satisfy: (a) the transformation \( X = \Phi(\chi) \) verifies relation (85), so it preserves \( \mathcal{R} \)-reversibility; (b) \( N \) and \( \dot{B} \) are \( \mathcal{R} \)-reversible. This last property will imply that \( \dot{B} \) has to vanish and, therefore, \( \Psi \mathcal{F} \) will become BNF.
Lemma 8 Let us consider an $\mathcal{R}$-reversible system

$$\dot{X} = F(X) = \Lambda + \hat{F}(X)$$

with $\mathcal{R} : (x_1, x_2, x_3, x_4) \mapsto (x_2, x_1, x_4, x_3)$, analytic on a neighborhood of the origin, that we assume to be a saddle-center or a saddle-focus equilibrium point. Let us take $\Phi^{(K)}, N^{(K)}$ and $\hat{B}^{(K)}$, the vector fields provided by the $\Psi\text{NF}$-recurrent scheme: set $\mathcal{P}\hat{\Phi} = 0$, take initial values

$$\Phi^{(1)} = \text{id}, \quad N^{(1)} = \Lambda, \quad \hat{B}^{(1)} = 0$$

and obtain, recurrently,

$$\Phi^{(K+1)} = \text{id} + \mathcal{R}\hat{\Phi}^{(K+1)}$$
$$N^{(K+1)} = \Lambda + \hat{N}^{(K+1)}$$
$$\hat{B}^{(K+1)}$$

with

$$\hat{\Phi}^{(K+1)} = O_{\leq K+1}, \quad \hat{N}^{(K+1)} = O_{\leq K+1}, \quad \hat{B}^{(K+1)} = O_{\leq K+1},$$

from equations

$$\left\{ \mathcal{L}_{\mathcal{N}^{(K)}} \left( \mathcal{R}\hat{\Phi}^{(K+1)} \right) \right\}_{\leq K+1} = \left\{ \mathcal{R} \left( \hat{F} \left( \Phi^{(K)} \right) \right) \right\}_{\leq K+1} \tag{86}$$
$$\hat{N}^{(K+1)} + \hat{B}^{(K+1)} = \left\{ \mathcal{P} \left( \hat{F} \left( \Phi^{(K)} \right) \right) \right\}_{\leq K+1}. \tag{87}$$

Then, the following assertions hold,

(i) For any $K \geq 1$, the vector field $\Phi^{(K)}$ satisfies (85) and the vector fields $N^{(K)}$ and $\hat{B}^{(K)}$ are $\mathcal{R}$-reversible, that is,

$$\mathcal{R}\Phi^{(K)}(\mathcal{R}x) = \Phi^{(K)}(x),$$

and

$$\mathcal{R}N^{(K)}(\mathcal{R}x) = -N^{(K)} \quad \text{and} \quad \mathcal{R}\hat{B}^{(K)}(\mathcal{R}x) = -\hat{B}^{(K)}.$$

(ii) The vector fields $\Phi$, $N$ and $\hat{B}$ provided by Theorem 2 and defined as

$$\Phi = \lim_{K \to \infty} \Phi^{(K)}, \quad N = \lim_{K \to \infty} N^{(K)}, \quad \hat{B} = \lim_{K \to \infty} \hat{B}^{(K)},$$

verify that

- The change of variables $X = \Phi(x)$ satisfies relation (85) and, therefore, it preserves the $\mathcal{R}$-reversibility;
- $N$ and $\hat{B}$ are $\mathcal{R}$-reversible.

(iii) Since $\hat{B}$ is $\mathcal{R}$-reversible it turns out that $\hat{B}$ vanishes.

Remark 8 As it happened in the general case, $\Phi$ is also convergent if we fix $\mathcal{P}\hat{\Phi}$ equal to any analytic function, convergent in the same domain as $\mathcal{R}\hat{\Phi}$ and verifying (85).

Proof. It is based in some statements that we list and prove separately. Namely,
(a) If a vector field $H$ is $\mathcal{R}$-reversible then its projections $\mathcal{P}H$ and $\mathcal{R}H$ are also $\mathcal{R}$-reversible.

Having in mind the definition of the projections $\mathcal{P}$ and $\mathcal{R}$ we can write

\[
H(\chi) = \mathcal{P}H(\chi) + \mathcal{R}H(\chi) = \begin{pmatrix}
\xi h_1(\xi, \eta, \mu, \nu) \\
\eta h_2(\xi, \eta, \mu, \nu) \\
\mu h_3(\xi, \eta, \mu, \nu) \\
\nu h_4(\xi, \eta, \mu, \nu)
\end{pmatrix} + \begin{pmatrix}
\tilde{h}_1(\xi, \eta, \mu, \nu) \\
\tilde{h}_2(\xi, \eta, \mu, \nu) \\
\tilde{h}_3(\xi, \eta, \mu, \nu) \\
\tilde{h}_4(\xi, \eta, \mu, \nu)
\end{pmatrix}.
\]

Since $H$ is $\mathcal{R}$-reversible we have that $\mathcal{R}H(\chi) + H(\chi) = 0$. With respect the first term it turns out that

\[
\mathcal{R}H(\chi) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
\xi h_1(\xi, \eta, \mu, \nu) \\
\eta h_2(\xi, \eta, \mu, \nu) \\
\mu h_3(\xi, \eta, \mu, \nu) \\
\nu h_4(\xi, \eta, \mu, \nu)
\end{pmatrix} + \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
\tilde{h}_1(\eta, \xi, \nu, \mu) \\
\tilde{h}_2(\eta, \xi, \nu, \mu) \\
\tilde{h}_3(\eta, \xi, \nu, \mu) \\
\tilde{h}_4(\eta, \xi, \nu, \mu)
\end{pmatrix} = \begin{pmatrix}
\xi h_2(\xi, \eta, \mu, \nu) \\
\eta h_1(\xi, \eta, \mu, \nu) \\
\mu h_4(\xi, \eta, \mu, \nu) \\
\nu h_3(\xi, \eta, \mu, \nu)
\end{pmatrix} + \begin{pmatrix}
\tilde{h}_2(\eta, \xi, \nu, \mu) \\
\tilde{h}_1(\eta, \xi, \nu, \mu) \\
\tilde{h}_4(\eta, \xi, \nu, \mu) \\
\tilde{h}_3(\eta, \xi, \nu, \mu)
\end{pmatrix} = 0
\]

Therefore, from

\[
\mathcal{R}H(\chi) + H(\chi) = \begin{pmatrix}
\xi (h_2(\xi, \eta, \mu, \nu) + h_1(\xi, \eta, \mu, \nu)) \\
\eta (h_1(\xi, \eta, \mu, \nu) + h_2(\xi, \eta, \mu, \nu)) \\
\mu (h_4(\xi, \eta, \mu, \nu) + h_3(\xi, \eta, \mu, \nu)) \\
\nu (h_3(\xi, \eta, \mu, \nu) + h_4(\xi, \eta, \mu, \nu))
\end{pmatrix}
\]

and using that

\[
\mathcal{P} \begin{pmatrix}
\tilde{h}_2(\eta, \xi, \nu, \mu) \\
\tilde{h}_1(\eta, \xi, \nu, \mu) \\
\tilde{h}_4(\eta, \xi, \nu, \mu) \\
\tilde{h}_3(\eta, \xi, \nu, \mu)
\end{pmatrix} = \mathcal{P} \begin{pmatrix}
\tilde{h}_1(\xi, \eta, \mu, \nu) \\
\tilde{h}_2(\xi, \eta, \mu, \nu) \\
\tilde{h}_3(\xi, \eta, \mu, \nu) \\
\tilde{h}_4(\xi, \eta, \mu, \nu)
\end{pmatrix} = 0
\]

it follows that

\[
h_2(\xi, \eta, \mu, \nu) + h_1(\xi, \eta, \mu, \nu) = 0
\]

\[
h_4(\xi, \eta, \mu, \nu) + h_3(\xi, \eta, \mu, \nu) = 0
\]

and

\[
\tilde{h}_2(\eta, \xi, \nu, \mu) + \tilde{h}_1(\eta, \xi, \nu, \mu) = 0
\]

\[
\tilde{h}_4(\eta, \xi, \nu, \mu) + \tilde{h}_3(\eta, \xi, \nu, \mu) = 0.
\]

That is,

\[
\mathcal{P}H(\chi) = \begin{pmatrix}
\xi h_1(\xi, \eta, \mu, \nu) \\
-\eta h_1(\xi, \eta, \mu, \nu) \\
\mu h_3(\xi, \eta, \mu, \nu) \\
-\nu h_3(\xi, \eta, \mu, \nu)
\end{pmatrix}, \quad \mathcal{R}H(\chi) = \begin{pmatrix}
\tilde{h}_1(\xi, \eta, \mu, \nu) \\
-\tilde{h}_1(\eta, \xi, \nu, \mu) \\
\tilde{h}_3(\xi, \eta, \mu, \nu) \\
-\tilde{h}_3(\eta, \xi, \nu, \mu)
\end{pmatrix}.
\]
From this expression, it is straightforward to check that
\[
\mathfrak{R} (PH)(\mathfrak{R} \chi) = -PH(\chi), \quad \mathfrak{R} (RH)(\mathfrak{R} \chi) = -RH(\chi).
\]

(b) Let \( \hat{\mathfrak{R}} \hat{\Psi} \) be the solution of an equation of type
\[
\mathcal{L}_N \left( \hat{\mathfrak{R}} \hat{\Psi} \right) = \hat{\mathfrak{R}} \hat{H}.
\]
Then, if \( \hat{\mathfrak{R}} \hat{H} \) is \( \mathfrak{R} \)-reversible it follows that
\[
\mathfrak{R} \left( \hat{\mathfrak{R}} \hat{\Psi} \right)(\mathfrak{R} \chi) = \hat{\mathfrak{R}} \hat{\Psi}(\chi),
\]
this is, the transformation \( X = \chi + \hat{\mathfrak{R}} \hat{\Psi}(\chi) \) preserves the \( \mathfrak{R} \)-reversibility.

To see it, let us consider
\[
\hat{\mathfrak{R}} \hat{\Psi}(\chi) = \begin{pmatrix}
\hat{\psi}_1(\xi, \eta, \mu, \nu) \\
\hat{\psi}_2(\xi, \mu, \mu, \nu) \\
\hat{\psi}_3(\xi, \mu, \mu, \nu) \\
\hat{\psi}_4(\xi, \mu, \mu, \nu)
\end{pmatrix}, \quad \hat{\mathfrak{R}} \hat{H}(\chi) = \begin{pmatrix}
\hat{h}_1(\xi, \eta, \mu, \nu) \\
\hat{h}_2(\xi, \eta, \mu, \nu) \\
\hat{h}_3(\xi, \eta, \mu, \nu) \\
\hat{h}_4(\xi, \eta, \mu, \nu)
\end{pmatrix}
\]
and write them in the form
\[
\hat{\psi}_w(\xi, \eta, \mu, \nu) = \sum_{j+k+\ell+m \geq 2} \psi_{j,k,m}^{[w]} \xi^j \eta^k \mu^\ell \nu^m
\]
\[
\hat{h}_w(\xi, \eta, \mu, \nu) = \sum_{j+k+\ell+m \geq 2} h_{j,k,m}^{[w]} \xi^j \eta^k \mu^\ell \nu^m
\]
for \( w = 1, 2, \ldots, 4 \). Since \( \hat{\mathfrak{R}} \hat{H} \) is \( \mathfrak{R} \)-reversible one has that
\[
\hat{h}_2(\xi, \eta, \mu, \nu) = -\hat{h}_1(\xi, \eta, \mu, \nu), \quad \hat{h}_4(\xi, \eta, \mu, \nu) = -\hat{h}_3(\xi, \eta, \mu, \nu),
\]
which is equivalent to
\[
h_{j,k,m}^{[2]} = -h_{k,j,m}^{[1]}, \quad h_{j,k,m}^{[4]} = -h_{k,j,m}^{[3]}.
\]
In a similar way it follows that to prove that \( \mathfrak{R}(\hat{\mathfrak{R}}\hat{\Psi})(\mathfrak{R} \chi) = \hat{\mathfrak{R}} \hat{\Psi}(\chi) \) it is enough to check that
\[
\hat{\psi}_2(\xi, \eta, \mu, \nu) = \hat{\psi}_1(\xi, \eta, \mu, \nu), \quad \hat{\psi}_4(\xi, \eta, \mu, \nu) = \hat{\psi}_3(\xi, \eta, \mu, \nu)
\]
or, as before, that
\[
\psi_{j,k,m}^{[2]} = \psi_{k,j,m}^{[1]}, \quad \psi_{j,k,m}^{[4]} = \psi_{k,j,m}^{[3]}.
\]
We will see that the first condition in (89) holds and, therefore, that \( h_{j,k,m}^{[2]} = h_{k,j,m}^{[1]} \) is also satisfied. With respect to the second condition in (89), and consequently
\[
\hat{\psi}_4(\xi, \eta, \mu, \nu) = \hat{\psi}_3(\xi, \eta, \mu, \nu),
\]
it works in the same way. Thus, we recall, from Section 2.3.1, that equation
\[
\mathcal{L}_N(\hat{\mathfrak{R}} \hat{\Psi}) = \hat{\mathfrak{R}} \hat{H}
\]
can be written, in a vectorial form, as

\[
\begin{pmatrix}
L_N^{(1)} \hat{\psi}_1,
L_N^{(2)} \hat{\psi}_2,
L_N^{(3)} \hat{\psi}_3,
L_N^{(4)} \hat{\psi}_4
\end{pmatrix} = \begin{pmatrix}
\hat{h}_1, 
\hat{h}_2, 
\hat{h}_3, 
\hat{h}_4
\end{pmatrix}
\]

where

\[
L_N^{(w)} \hat{\psi}_w(\xi, \eta, \mu, \nu) = \sum_{j+k+\ell+m \geq 2} \frac{(w)}{g_{jk\ell m}}(\xi, \eta, \mu, \nu) \psi^{(w)}_{jk\ell m} \xi^{j} \eta^{k} \mu^{\ell} \nu^{m},
\]

being

\[
\frac{(w)}{g_{jk\ell m}}(\xi, \eta, \mu, \nu) := \frac{(w)}{\gamma_{jk\ell m}}(\lambda) + (j-k) \hat{a}_1(\xi, \eta, \mu, \nu) + (\ell-m) \hat{a}_2(\xi, \eta, \mu, \nu),
\]

with

\[
\frac{(w)}{\gamma_{jk\ell m}}(\lambda) = \begin{cases} 
(j-k+1)\lambda_1 + (\ell-m)\lambda_2 & \text{if } w = 1, \\
(j-k+1)\lambda_1 + (\ell-m)\lambda_2 & \text{if } w = 2, \\
(j-k)\lambda_1 + (\ell-m-1)\lambda_2 & \text{if } w = 3, \\
(j-k)\lambda_1 + (\ell-m+1)\lambda_2 & \text{if } w = 4.
\end{cases}
\]

It is also derived from the same section that, provided

\[N = (\xi a_1(\xi, \eta, \mu, \nu), -\eta a_1(\xi, \eta, \mu, \nu), \mu a_2(\xi, \eta, \mu, \nu), -\nu a_2(\xi, \eta, \mu, \nu))\]

and \(R\hat{H}\) are known, the formal solution \(\hat{\psi}_w(\xi, \eta, \mu, \nu)\) of equations (90) is uniquely determined. Thus, writing (90) for \(w = 1\) and applying the involution \(R\), one obtains

\[
L_N^{(1)} \hat{\psi}_1(\eta, \xi, \nu, \mu) = \hat{h}_1(\eta, \xi, \mu, \nu).
\]

From the left-hand side of (91) it follows

\[
L_N^{(1)} \hat{\psi}_1(\eta, \xi, \nu, \mu) = \sum_{j+k+\ell+m \geq 2} \frac{(1)}{g^{(1)}_{jk\ell m}}(\eta, \xi, \nu, \mu) \psi^{(1)}_{jk\ell m} \eta^{j} \xi^{k} \nu^{\ell} \mu^{m} - \sum_{j+k+\ell+m \geq 2} \frac{(1)}{g^{(1)}_{jk\ell m}}(\xi, \eta, \mu, \nu) \psi^{(1)}_{jk\ell m} \xi^{j} \eta^{k} \mu^{\ell} \nu^{m},
\]

where it has been used that

\[
\frac{(1)}{\gamma^{(1)}_{jk\ell m}}(\lambda) = (k-j)\lambda_1 + (m-\ell)\lambda_2 = -(j-k+1)\lambda_1 + (\ell-m)\lambda_2 = -\frac{(2)}{\gamma^{(2)}_{jk\ell m}}(\lambda)
\]

and

\[
\frac{(1)}{g^{(1)}_{jk\ell m}}(\xi, \eta, \mu, \nu) = \frac{(1)}{\gamma^{(1)}_{jk\ell m}}(\lambda) + (k-j) \hat{a}_1(\xi, \eta, \mu, \nu) + (m-\ell) \hat{a}_2(\xi, \eta, \mu, \nu) = -\frac{(2)}{g^{(2)}_{jk\ell m}}(\xi, \eta, \mu, \nu).
\]

Concerning the right-hand side of (91), having in mind (88), we have that

\[\hat{h}_1(\eta, \xi, \mu, \nu) = -\hat{h}_2(\xi, \eta, \mu, \nu)
\]

so, finally, equation (91) becomes equivalent to the equation

\[
\sum_{j+k+\ell+m \geq 2} \frac{(2)}{g^{(2)}_{jk\ell m}}(\xi, \eta, \mu, \nu) \psi^{(1)}_{jk\ell m} \xi^{j} \eta^{k} \mu^{\ell} \nu^{m} = \hat{h}_2(\xi, \eta, \mu, \nu).
\]
Since the unique solution of
\[ L^{(2)}_{X_N} \hat{\psi}(\xi, \eta, \mu, \nu) = \sum_{j+k+\ell+m \geq 2} \hat{g}^{[2]}_{jkm}(\xi \eta \mu \nu) \psi^{(2)}_{jkm} \xi^j \eta^k \mu^\ell \nu^m = \hat{h}_2(\xi, \eta, \mu, \nu) \]
is given by
\[ \hat{\psi}(\xi, \eta, \mu, \nu) = \sum_{j+k+\ell+m \geq 2} \psi^{(2)}_{jkm} \xi^j \eta^k \mu^\ell \nu^m \]
it follows, comparing with (92), that
\[ \psi_{1} = \psi_{jkm}^{(2)} \]
for \( j + k + \ell + m \geq 2 \).

(c) If \( H \) is \( \mathcal{R} \)-reversible then \( H \leq K \) (constituted by its terms of order less or equal than \( K \)) is also \( \mathcal{R} \)-reversible, for any \( K \geq 1 \).

This assertion comes directly from the fact that the equality \( \mathcal{R} \ H(\mathcal{R} \chi) = -H(\chi) \) must be satisfied at any order.

We are now in conditions of proving the assertions (i), (ii) and (iii).

(i) First, note that from its form, \( \hat{N}^{(K)} \) satisfies that
\[ \mathcal{R} \hat{N}^{(K)}(\mathcal{R} \chi) = -N^{(K)}(\chi) \]
and, therefore, so does \( N^{(K)} \),
\[ \mathcal{R} N^{(K)}(\mathcal{R} \chi) = -N^{(K)}(\chi) . \]
Thus, \( \hat{N}^{(K)} \) is \( \mathcal{R} \)-reversible for any \( K \geq 1 \). We are going to prove that \( \Phi^{(K)} \) verifies condition (85) and \( \hat{B}^{(K)} \) is \( \mathcal{R} \)-reversible using an inductive argument.

For \( K = 2 \) (the case \( K = 1 \) is trivial) we have that
\[ \left\{ L_{N^{(1)}} \left( \mathcal{R} \hat{\Phi}^{(2)} \right) \right\}_{\leq 2} = \left\{ \mathcal{R} \left( \hat{F} \left( \Phi^{(1)} \right) \right) \right\}_{\leq 2} \]
or, simplifying,
\[ L_{\mathcal{A}} \left( \mathcal{R} \hat{\Phi}^{(2)} \right) = F_{[2]} . \]

Applying properties (c) and (b) above one obtains that \( \mathcal{R} \hat{\Phi}^{(2)} \) preserves \( \mathcal{R} \)-reversibility. On the other hand, \( \hat{B}^{(2)} = 0 \) so it is trivially a \( \mathcal{R} \)-reversible vector field.

Assume, by induction hypotheses that, for a given \( K \geq 1 \),
\[ - \Phi^{(K)} = \text{id} + \mathcal{R} \hat{\Phi}^{(K)} \text{ satisfies } (85) \text{ (so it preserves } \mathcal{R} \text{-reversibility)}, \]
\[ - \hat{B}^{(K)} \text{ is a } \mathcal{R} \text{-reversible vector field}. \]

Using these induction hypotheses and applying properties (a), (c) and (b) on equation (86) it follows that \( \mathcal{R} \hat{\Phi}^{(K+1)} \) preserves \( \mathcal{R} \)-reversibility. Consequently, so does \( \Phi^{(K+1)} = \text{id} + \mathcal{R} \hat{\Phi}^{(K+1)} \).

Besides, from equation (87) we have
\[ \hat{B}^{(K+1)} = \left\{ \mathcal{P} \left( \hat{F}(\Phi^{(K)}) \right) \right\}_{\leq K+1} - \hat{N}^{(K+1)} . \]

By induction hypothesis \( \Phi^{(K)} = \text{id} + \mathcal{R} \hat{\Phi}^{(K)} \) preserves \( \mathcal{R} \)-reversibility so, taking into account (a), (c) and the fact that \( \hat{N}^{(K+1)} \) is \( \mathcal{R} \)-reversible, it turns out that \( \hat{B}^{(K+1)} \) is also \( \mathcal{R} \)-reversible.
(ii) We have to prove now that the limit transformation $X = \Phi(\chi)$, whose convergence comes from the proof of Theorem 2, preserves $\mathcal{R}$-reversibility. Using the result above, we have that

$$\mathcal{R} \Phi^K(\mathcal{R}\chi) = \Phi^K(\chi)$$

holds for any $K \geq 1$. Letting $K$ tend to infinity it follows that

$$\mathcal{R} \Phi(\mathcal{R}\chi) = \Phi(\chi),$$

so $\Phi$ preserves $\mathcal{R}$-reversibility. With respect to $\tilde{N}$, its $\mathcal{R}$-reversibility comes straightforwardly from its form and its convergence. Concerning $\tilde{B}$ a similar argument to the one used for $\Phi$ applies. Thus $\tilde{B}$ is $\mathcal{R}$-reversible and consequently we have that

$$\mathcal{R} \tilde{B}(\mathcal{R}\chi) = -\tilde{B}(\chi).$$

(iii) In particular, formula (93) implies that

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\eta \hat{b}_1(\xi \eta, \mu \nu) \\
\xi \hat{b}_1(\xi \eta, \mu \nu) \\
\nu \hat{b}_2(\xi \eta, \mu \nu) \\
\mu \hat{b}_2(\xi \eta, \mu \nu)
\end{pmatrix}
= 
\begin{pmatrix}
\xi \hat{b}_1(\xi \eta, \mu \nu) \\
\eta \hat{b}_1(\xi \eta, \mu \nu) \\
\mu \hat{b}_2(\xi \eta, \mu \nu) \\
\nu \hat{b}_2(\xi \eta, \mu \nu)
\end{pmatrix}
= 
\begin{pmatrix}
\xi \bar{b}_1(\xi \eta, \mu \nu) \\
\eta \bar{b}_1(\xi \eta, \mu \nu) \\
\mu \bar{b}_2(\xi \eta, \mu \nu) \\
\nu \bar{b}_2(\xi \eta, \mu \nu)
\end{pmatrix}
$$

and, therefore,

$$\hat{b}_1(\xi \eta, \mu \nu) = -\bar{b}_1(\xi \eta, \mu \nu) \quad \text{and} \quad \hat{b}_2(\xi \eta, \mu \nu) = -\bar{b}_2(\xi \eta, \mu \nu),$$

so $\hat{b}_1(\xi \eta, \mu \nu) = \hat{b}_2(\xi \eta, \mu \nu) = 0$ and the lemma is proved.

From this lemma the proof of Proposition R1 follows straightforwardly. The transformation $\Phi$ preserves $\mathcal{R}$-reversibility, the vector field $N$ is $\mathcal{R}$-reversible and $\tilde{B} = 0$ so, in fact, the WNF is nothing else but the BNF.

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