Modal Analysis of Coupling Problems in Optical Fibers

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Abstract—A modal analysis of the problems of excitation of the dominant mode in an optical fiber by incident plane waves and Gaussian beams has been carried out, and the results applied to the effect on transmission of misalignment in fiber junctions due to offsets, tilts, and gaps. The results in cases of matched media confirm the accuracy of previous theoretical treatments using the Born approximation, which in turn show good agreement with experimental results. In addition, the modal analysis gives more precise solutions when there is a mismatch of media and makes possible the treatment of some problems to which the Born approximation is not applicable.

I. INTRODUCTION

Over the past five years the advances made in optical signal processing techniques and in the development of low-loss glasses for optical fibers have made the utilization of the enormous bandwidth available in optical communications an attractive possibility. However, if such fibers are actually to be employed as transmission channels, in the manner of transmission lines and micro-wave waveguides, one must be able to couple, splice, bifurcate, etc., in a way analogous to that used with those older channels. This is not a trivial technical problem, because the core of the fiber, where the fields are concentrated, is typically only a few microns in diameter so that the difficulty, for example, of aligning two fibers when splicing is critical.

The excitation of propagating modes on a fiber by various types of source illumination has been studied by Snyder [1]-[3] and Marcuse [4], both using the Born approximation. Furthermore, splicing techniques have been developed [5]-[7] and some measurements made [7]-[9] of the effects of imperfect alignment at the interface between two uniform fibers. Most recently Cook et al. [9] have also given a theoretical analysis of the effects of misalignment in splicing on the transmission coefficient, again making use of the Born approximation. The importance of such a theoretical calculation lies in the fact that it would enable one to set meaningful standards of precision which must be adhered to in the practical means used for splicing.

However, all of the theoretical calculations mentioned employ the Born approximation, i.e., the assumption that the fields at the illuminated cross section of the fiber consist entirely of those in the incident wave. That there must be a discrepancy [10] resulting from the use of this approximation is indicated by the fact that Marcuse [4] calculated two values of transmission coefficient corresponding to the two possible boundary conditions to be used (continuity of tangential E or of tangential H) and arbitrarily took their geometric mean. Although this mean result was always quite reasonable, in some cases one of the two boundary conditions led to a transmission coefficient larger than unity, which is clearly impossible.

Consequently, this paper will reexamine the basic excitation problems, making use of a more rigorous modal expansion of the fields, which in turn is based on the general techniques for hybrid modes developed by Yaghjian [11], [12]. The solutions will show that, rather surprisingly, the result of taking the geometric mean of the two Born approximation coefficients was remarkably accurate. Finally, the results and techniques will be applied to the problems of the effect on efficiency of transmission at the junction of two similar optical fibers of three types of defect in their alignment: tilt of one axis with respect to the other, offset of their axes, and small gaps between the fiber cores. Numerical results will be presented for all of these problems and compared with the corresponding Born approximation results.

II. FORMULATION

The surface modes of the infinite circular dielectric rod have been extensively studied [2], [13], [14], and their orthogonality well established [15]. The fact that the modes are hybrid and the existence of a continuous spectrum make the exact solution of diffraction and scattering problems for a fiber with infinite outer diameter (od) a formidable task. In fact, there is no known exact solution even for the axially symmetric case, which reduces to a scalar problem [16], [17]. An exact solution would require solving an integral equation, and one possible means of overcoming the difficulty of its solution would be to approximate the integrals by infinite summations and solve the resulting linear system. A natural way of achieving this is to enclose the dielectric rod in a concentric perfectly conducting cylinder with radius large compared to the core radius and the wavelength. As the radius of the metallic pipe increases, the surface modes are unaffected and the nonsurface type modes become a continuum.
This method will allow us to solve the excitation and fiber coupling problems by a normal-mode analysis, as in a standard waveguide discontinuity problem, and to take into account the reflected energy.

To find the scattered and transmitted fields for a wave incident at \( z = 0 \), Fig. 1, we will have to expand the fields on both sides of the interface as an infinite summation of the modes in each structure. Continuity of the transverse components of the fields at \( z = 0 \) gives

\[
\sum_{m} a_{m} e_{m} + E_{i} = \sum_{m} a_{m} e_{m}
\]

\[
- \sum_{m} a_{m} h_{m} + H_{i} = \sum_{m} a_{m} h_{m}
\]

where \( E_{i}, H_{i} \) are the transverse incident fields, \( e_{m}, h_{m} \) the transverse fields of the surface and nonsurface modes of the fiber, and \( e_{m}^{'}, h_{m}^{'} \) are the forward traveling modes of the homogeneously filled circular waveguide [18] in the launching case or the hybrid modes of the fiber in the fiber junction case.

The set of modal coefficients \( a_{m} \), which will give the efficiencies of excitation, can be obtained from (1) by cross multiplication by \( h_{m}^{'} \) and \( e_{m}^{'} \), integration over the cross-sectional area \( A \) at the left of the interface, and making use of the orthogonality of the modes, giving

\[
a_{m} N_{m} + A_{n} = \sum_{m} a_{m} M_{m}^{n}
\]

\[
- a_{m} N_{m} + B_{n} = \sum_{m} a_{m} N_{m}^{n}
\]

these two equations can be added to obtain a linear system for the modal coefficients of the transmitted modes

\[
\sum_{m=1}^{\infty} a_{m} \left| M_{m}^{n} + N_{m}^{n} \right| = A_{n} + B_{n}, \quad n = 1, 2, \ldots
\]

where \( N_{m}^{n} = \int e_{m}^{'} \times h_{m}^{'} \hat{z} \, da \) are the normalization factors of the TE and TM modes and

\[
A_{n} = \int_{A} (E_{i} \times h_{m}^{'} \hat{z}) \, da
\]

\[
B_{n} = \int_{A} (e_{m}^{'} \times H_{i} \hat{z}) \, da
\]

\[
N_{m}^{n} = \int_{A} (e_{m}^{'} \times h_{m}^{'} \hat{z}) \, da
\]

\[
M_{m}^{n} = \int_{A} (e_{m} \times h_{m}^{'} \hat{z}) \, da.
\]

\( A_{n} \) and \( B_{n} \) can be calculated from the expressions of the incident fields, and the cross-normalization factors, \( N_{m}^{n} \) and \( M_{m}^{n} \), have been calculated by Yaghjian [12] by reducing the surface integrals to line integrals and are given in the Appendix.

The summation over \( n \) includes for the launching case all TE and TM modes of the homogeneous circular metallic waveguide arranged in order of increasing eigenvalues, and in the fiber junction case all the propagating surface modes and all the nonsurface modes (Appendix).

To obtain a numerical solution of (2b), we shall have to truncate the system and solve the resulting finite system, adding more equations until the modal coefficients obtained become stable within the accuracy of the computations, and the addition of new equations does not produce further variations.

Practical fibers are made with the radius of the cladding large compared to the core radius and usually are externally coated. This in fact reduces the continuous spectrum to a discrete one but does not affect significantly the surface modes. The preceding model can be viewed not only as a solution for the transversely infinite fiber but as an exact study of the propagation and excitation of modes in a fiber with a metallic external coating.

### III. EXCITATION COEFFICIENTS

#### A. Truncated Uniform Plane Wave

For this case, the incident fields at \( z = 0 \) are taken as those of a uniform plane wave in a medium of dielectric constant \( \varepsilon_{i} \) illuminating a circle of radius \( c \) concentric with the fiber and propagating at an angle \( \theta \) with respect to the axis of the fiber \( z \). The plane of the fiber axis and the direction of incidence is taken to be the \( x, z \) plane, so that

\[
E_{i} = E_{0} \exp \left( -jk_{z} x \sin \theta \right) \hat{E}
\]

\[
H_{i} = (\varepsilon_{i}/\mu_{0})^{1/2} \hat{n} \times E_{i}
\]

with \( k_{z}^{2} = \omega^{2}\varepsilon_{i}\mu_{0}, \hat{n} = \hat{z} \sin \theta + \hat{z} \cos \theta, \) and an \( e^{j\omega t} \) time dependence is assumed.

We will consider only \( x \)-polarized plane waves, which excite only modes with \( g_{1}(\phi), g_{2}(\phi) \) given by (A6) with \( n = 1 \). Expressions for the other polarization could be similarly derived. Also we will assume the fields incident at \( z = 0 \), but if the source were at \( z = z_{0} (z_{0} < 0) \) with \( \varepsilon_{i} \) or a different medium in the region \( z_{0} < z < 0 \), the problem could be treated in a similar way by writing the continuity equations at the \( z = z_{0} \) and \( z = 0 \) interfaces and eliminating coefficients until we are left with a system relating incident and transmitted fields, as will be done in studying the effects of gaps in fiber joints.
For an x-polarized uniform plane wave, we obtain
\[ E_z = -E_0 \exp(-j\Delta r \cos \phi) \sin \theta \] \hspace{1cm} (9a)
\[ E_r = E_0 \exp(-j\Delta r \cos \phi) \cos \theta \cos \phi \] \hspace{1cm} (9b)
\[ E_\phi = -E_0 \exp(-j\Delta r \cos \phi) \cos \theta \sin \phi \] \hspace{1cm} (9c)
with
\[ \Delta = k_z \sin \theta \] \hspace{1cm} (10)
the \textit{H} fields are obtained from (8) and (9).

We shall have to solve the system (2). Substitution into (3), (4) of the fields of the homogeneously filled circular waveguide [18] gives for \( A_n \) and \( B_n \) the following expressions.

For \( n \) odd, TE\(_{1n}\) modes
\[ A_n = -j \frac{\beta_n^2 E_0}{\omega n \gamma_n} \cos \theta \int_0^\pi \int_0^{2\pi} \exp(-j\Delta r \cos \phi) \]
\[ \left[ J_1(\gamma_n r) \cos^2 \phi + J_1'(\gamma_n r) \sin^2 \phi \right] r \, dr \, d\phi \] \hspace{1cm} (11)
with \( \gamma_n = P_{1n}'/b \) \((n = 2m - 1, m = 1, 2, \cdots)\), where \( P_{1n}' \) is the \( m \)th order zero of \( J_1' \), \( b \) is the radius of the guide, and \( \beta_n \) is the propagating constant of the \( n \)th mode. Use of the recurrence relations for the Bessel functions gives
\[ A_n = -j \frac{\beta_n^2 E_0}{2\omega n \gamma_n} \cos \theta \int_0^{2\pi} \exp(-j\Delta r \cos \phi) \left[ J_0(\gamma_n r) \right. \]
\[ + J_2(\gamma_n r) \cos 2\phi \] \hspace{1cm} (12)
Using the associated series
\[ \exp(-j\Delta r \cos \phi) = J_0(\Delta r) + 2 \sum_{k=1}^\infty (-j)^k J_k(\Delta r) \cos k\phi \] \hspace{1cm} and integrating over \( \phi \) gives
\[ A_n = -j \frac{\beta_n^2 E_0}{k_\gamma n} \frac{(\varepsilon_3/\mu_0)^{1/2} \pi}{\gamma_n} \cos \theta I_{1n}^1 \] \hspace{1cm} (13)
with
\[ I_{1n}^1 = \int_0^\pi \left[ J_0(\gamma_n r) J_0(\Delta r) - J_2(\gamma_n r) J_2(\Delta r) \right] r \, dr \] \hspace{1cm} (15)
which can be integrated analytically to give
\[ I_{1n}^1 = \left\{ \begin{array}{ll}
- [c/(\gamma_n^2 - 2)] [\Delta J_0(\gamma_n c) + J_2(\gamma_n c) J_1(\Delta c)] \\
- \gamma_n [J_0(\Delta c) + J_2(\Delta c) J_1(\gamma_n c)], & \Delta \neq \gamma_n \\
(2/\Delta^2) J_2^2(\Delta c), & \Delta = \gamma_n
\end{array} \right. \] \hspace{1cm} (16)

\( B_n \) is obtained in the same way as
\[ B_n = -j \frac{\beta_n E_0}{\gamma_n} \frac{(\varepsilon_3/\mu_0)^{1/2} \pi}{\gamma_n} I_{1n}^1 \] \hspace{1cm} (17)
for normal incidence \((\theta = 0), \Delta = 0, \) and
\[ I_{1n}^1 = \frac{\alpha J_1(\gamma_n c)}{\gamma_n}, \quad \Delta = 0. \] \hspace{1cm} (18)

Similarly for \( n \) even, TM\(_{1n}\) modes, we have
\[ A_n = -j \frac{\beta_n^2 E_0}{\gamma_n} \frac{(\varepsilon_3/\mu_0)^{1/2} \pi}{\gamma_n} \cos \theta I_{1n}^2 \] \hspace{1cm} (19)
\[ B_n = -j \frac{\beta_n E_0}{\gamma_n} \frac{(\varepsilon_3/\mu_0)^{1/2} \pi}{\gamma_n} I_{1n}^2 \] \hspace{1cm} (20)
with \( \gamma_n = P_{2n}/b \) \((n = 2m, m = 1, 2, \cdots)\), where \( P_{2n} \) is the \( m \)th order zero of \( J_1 \).

\[ I_{1n}^2 = \int_0^\pi \left[ J_0(\gamma_n r) J_0(\Delta r) + J_2(\gamma_n r) J_2(\Delta r) \right] r \, dr \] \hspace{1cm} (21)
and
\[ I_{1n}^2 = \left\{ \begin{array}{ll}
- [c/(\gamma_n^2 - 2)] [\Delta J_0(\gamma_n c) - J_2(\gamma_n c) J_1(\Delta c)] \\
- \gamma_n [J_0(\Delta c) - J_2(\Delta c) J_1(\gamma_n c)], & \Delta \neq \gamma_n \\
(c^2/2) J_2^2(\Delta c) + J_1^2(\Delta c) + J_3^2(\Delta c) - J_2(\Delta c) J_1(\Delta c), & \Delta = \gamma_n
\end{array} \right. \] \hspace{1cm} (22)
at normal incidence
\[ I_{1n}^2 = \frac{\alpha J_1(\gamma_n c)}{\gamma_n}, \quad \Delta = 0. \] \hspace{1cm} (23)
Substitution in (2) gives a system with the modal coefficients in the fiber as unknowns. Fig. 2 represents numerical
for the $a = 2\mu$ fiber which is just below cutoff of the TM01 mode [13], and at $4^\circ$ deviation from normal incidence it drops to about half that value for the matched case. As the fiber radius decreases, so does the maximum of the curves. The effect of tilts increases with the refractive index of the medium of incidence. Fibers of small radius need a much greater illuminating area, as expected, i.e., most of the power is in the cladding. The outer pipe does not have a noticeable effect in the fiber well above cutoff.

**B. Gaussian Beam**

We shall now consider an incident Gaussian beam propagating at an angle $\theta$ with respect to the axis of the fiber $z$. Again, the plane of the axis and the direction of incidence is taken to be the $x, z$ plane, and the center of the beam is displaced a distance $d$ on the positive $x$-axis.

For small angles of incidence we have

$$E_i = \frac{E_0}{(2\pi\sigma^2)^{1/2}} \exp \left( -\frac{d^2}{2\sigma^2} \right) \exp \left( -\frac{r^2}{2\sigma^2} \right) \exp (i\tau \cos \phi) E.$$  

(24)

$H_i$ is given by (8) with

$$\zeta = D - j\Delta$$  

(25)

$$D = d/\sigma^2$$  

(26)

and $\Delta$ given by (10).

We shall consider only $x$-polarized beams and treat oblique incidence and offsets separately. If both were considered, it would be necessary to perform numerical integrations with Bessel functions of the complex argument $\tau$. If the center of the beam were not on the $x$-axis but at a point $(d, \alpha)$, then the argument of $e^{i\alpha\cos \phi}$ in (24) would be replaced by $Dr \cos (\alpha - \phi)$ and both polarizations of the HE11 mode would be excited. The formalism to treat those cases is the same as that for the cases $\Delta = 0$ or $D = 0$, but the computations become much more cumbersome.

For oblique incidence ($D = 0$) we obtain the expressions for $A_n$ and $B_n$ given by (14), (17), (19), (20) but with

$$I_n^1 = \frac{1}{(2\pi\sigma^2)^{1/2}} \int_0^\theta \exp \left( -\frac{r^2}{2\sigma^2} \right) \left[ J_0(\gamma r)J_0(\Delta r) - J_2(\gamma r)J_2(\Delta r) \right] dr$$  

(27)

$$I_n^2 = \frac{1}{(2\pi\sigma^2)^{1/2}} \int_0^\theta \exp \left( -\frac{r^2}{2\sigma^2} \right) \left[ J_0(\gamma r)J_0(\Delta r) + J_2(\gamma r)J_2(\Delta r) \right] dr.$$  

(28)

Both of these expressions require numerical integration, and if both tilts and offsets were to be considered simultaneously, a multiplicative factor $\exp (-\alpha^2/2\sigma^2)$ would appear and $\gamma$ would replace $\Delta$.

Fig. 3 shows two of the curves obtained [19]. Maximum values of efficiency are 99.7 percent for $a = 2\mu$ and 99.7 percent for $a = 1.5\mu$. As the radius of the core $a$ decreases, the maximum decreases and the effect of tilts increases.

<table>
<thead>
<tr>
<th>$\sigma(\mu)$</th>
<th>Modal Solution for $n_1 = 1$</th>
<th>Modal Solution for $n_1 = 1.49$</th>
<th>Born Approximation (Geometric mean)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>33.83 percent</td>
<td>35.24 percent</td>
<td>34.16 percent</td>
</tr>
<tr>
<td>1.0</td>
<td>80.72</td>
<td>84.08</td>
<td>83.81</td>
</tr>
<tr>
<td>1.5</td>
<td>93.78</td>
<td>99.75</td>
<td>99.60</td>
</tr>
<tr>
<td>2.0</td>
<td>90.12</td>
<td>93.84</td>
<td>93.69</td>
</tr>
<tr>
<td>2.5</td>
<td>78.10</td>
<td>81.32</td>
<td>81.19</td>
</tr>
<tr>
<td>3.0</td>
<td>65.81</td>
<td>68.52</td>
<td>68.40</td>
</tr>
<tr>
<td>3.5</td>
<td>55.09</td>
<td>57.36</td>
<td>57.26</td>
</tr>
<tr>
<td>4.0</td>
<td>46.25</td>
<td>48.13</td>
<td>48.07</td>
</tr>
</tbody>
</table>

Note: HE11 mode launching efficiency in a fiber with $n_1 = 1.50$, $n_3 = 1.49$, $b = 25\mu$, $a = 2\mu$, $\lambda_0 = 0.9\mu$, excited by a Gaussian beam at oblique incidence versus Gaussian width $\sigma$. A more extensive set of curves for different fibers, outer radius and medium of incidence together with the corresponding Born approximations can be found in [19].

The transmitted HE11 mode was found to be in phase with the incident fields; values of Bessel functions were calculated to six digits accuracy and the ratio of imaginary to real part of the modal coefficient is of the order of $10^{-6}$. Solution of that system and gives the efficiency of excitation of the HE11 mode versus radius of the illuminated area for various angles of incidence. A more extensive set of curves for different fibers, outer radius and medium of incidence together with the corresponding Born approximations can be found in [19].

Fig. 3 shows two of the curves obtained [19]. Maximum values of efficiency are 99.7 percent for $a = 2\mu$ and 98.7 percent for $a = 1.5\mu$. As the radius of the core $a$ decreases, the maximum decreases and the effect of tilts increases.
Again, the efficiency for the vacuum was about 4 percent less than for $n_1 = n_2$. The results are insensitive to changes in the pipe radius except for fibers with very small core radius. A Born approximation was carried out for the same cases, and the geometric mean of the two values obtained for the modal coefficient was in very good agreement for matched media, but for incidence from the vacuum was about 4 percent higher than the corresponding modal solution (Table I). Since the theoretical calculations of Cook et al. [9] are also based on the Born approximation, the same degree of precision presumably also applies to them.

For beam offsets ($\Delta = 0$), using the associated series

$$\exp (Dr \cos \phi) = I_0(\sqrt{D}) + 2 \sum_{k=1}^{\infty} I_k(\sqrt{D}) \cos k\phi$$

we again obtain $A_k$ and $B_k$ given by (14), (17), (19), (20) but with

$$I_k = \frac{\exp \left( -\frac{d^2}{2\sigma^2} \right)}{(2\pi\sigma^2)^{1/2}} \int_0^\infty \exp \left( -\frac{r^2}{2\sigma^2} \right) \left[ J_k(\gamma_s r) I_0(\sqrt{D}) + J_k(\gamma_f r) I_0(\sqrt{D}) \right] r \, dr$$

$$I_k' = \frac{\exp \left( -\frac{d^2}{2\sigma^2} \right)}{(2\pi\sigma^2)^{1/2}} \int_0^\infty \exp \left( -\frac{r^2}{2\sigma^2} \right) \left[ J_k(\gamma_s r) I_0(\sqrt{D}) - J_k(\gamma_f r) I_0(\sqrt{D}) \right] r \, dr.$$  

Fig. 4 shows the effect of offset misalignments and indicates that they are more critical than tilts. Alignments of cores of 1 or 2 microns in fiber joints is not an easy task, especially in the field; also in the launching system there is the possibility of some misalignment. In fibers close to cutoff of the TMO1 mode an offset equal to the core radius reduces the efficiency by more than half. As the radius of the core is decreased the effect decreases, as expected. Also as $\sigma$ increases the effect becomes less important but then the efficiency drops and the effect of tilts increases. Born approximations for these cases can be found in [19].

$\sum_{l=1}^{\infty} a_l^{(p)} \sum_{n=1}^{\infty} U_{m,n} U_{l,n} \exp \left( j(\beta_n' - \beta_l) z_0 \right) - V_{m,n} V_{l,n} \exp \left( -j(\beta_n' + \beta_l) z_0 \right) \right] \right) = \sum_{l=1}^{\infty} \delta_{m,n} a_l^6 N_{hh}, \quad m = 1,2,3,\cdots$  

**IV. THE FIBER BUTT JOINT WITH A GAP**

The problem depicted in Fig. 5 is that of two identical metallic coated fibers aligned on the same axis, but whose ends are separated by the gap region $0 < z < z_0$ which has a refractive index $n_3$. In region I, $z < 0$, there is a set of forward-moving incident modes with amplitudes $a_n^{(k)}$, $k = 1, 2, \cdots, k$, and backward-moving reflected modes with coefficients $a_n^{(1)}$. Region II, $0 < z < z_0$, is characterized by a set of TE and TM circular waveguide modes with forward-directed modal coefficients $a_n^{(2)}$ and backward-directed ones $b_n^{(2)}$; and in region III, $z > z_0$, a set of transmitted hybrid modes $a_n^{(3)}$ is excited.

If one takes the cross product of the modal fields $h_m$ and $e_m$ with the equations for continuity at $z = 0$ of tangential $E$ and $H$, respectively, and then integrates over the cross section, the result is

$$\sum_{k=1}^{K} \delta_{m,n} a_k^6 N_{hh} - N_{mm} a_m^{(1)} = \sum_{n=1}^{\infty} a_n^{(2)} M_n^m - \sum_{n=1}^{\infty} b_n^{(3)} M_n^m$$

$$\sum_{k=1}^{K} \delta_{m,n} a_k^6 N_{kk} + N_{mm} a_m^{(1)} = \sum_{n=1}^{\infty} a_n^{(2)} M_n^m + \sum_{n=1}^{\infty} b_n^{(2)} M_n^m$$

where the cross-normalization factors $N_n^m$ and $M_n^m$ are defined as in (5), (6), and where

$$N_{mm} = \int_A (e_m \times h_m) \cdot \hat{z} \, da.$$  

An exactly similar procedure applied at $z = z_0$ yields

$$a_n^{(3)} N_{nn'} \exp ( -j\beta_n' z_0 ) - b_n^{(2)} N_{nn'} \exp ( j\beta_n' z_0 )$$

$$= \sum_{l=1}^{\infty} a_l^{(3)} M_n^l \exp ( -j\beta_n' z_0 )$$

$$a_n^{(2)} N_{nn'} \exp ( -j\beta_n' z_0 ) + b_n^{(2)} N_{nn'} \exp ( j\beta_n' z_0 )$$

$$= \sum_{l=1}^{\infty} a_l^{(2)} N_n^l \exp ( -j\beta_n' z_0 )$$

where $\beta_n$ is the propagation constant of the nth mode and the primes refer to region II.

The reflected mode coefficients and those in region II may be eliminated from the four sets of equations obtained, leaving
The type of coating used on the fiber seems to be irrelevant. In fact, if one takes the geometric mean there was less than 0.3 percent difference between the calculated launching efficiencies for pipes of S-P and Q₁LL compared with a value of 99.7 percent reported by Marcuse [3]. When there is only a single incident mode, the HEₘₙ solution is about 4 percent higher (Table I). The practical effect of tilts or offsets of the incident beam is quite pronounced; an offset of one core radius or a tilt of 4° reduces the coupling efficiency to less than 50 percent for the cases treated. However, a gap of 25 radii is required to produce a comparable effect when matching oil is used or about 15 radii for an air gap, so that in practice the gap is apt to be a much less serious problem. The effect of a gap of given length is insensitive to the core diameter when it is of the order of a wavelength because the lateral extent of the fields does not change very much. However, as Cook et al. [9] have pointed out, a smaller core diameter makes the effect of offsets, as measured in core diameters, less critical and of angular displacements more critical when one finally approaches the situation where the fields begin to spread laterally.

Although the gap problem was solved by rigorous application of the boundary conditions to the modal expansions, in the tilt and offset problems in fiber junctions computational simplicity prompted an approximate treatment, replacing the incident HEₘₙ mode in the fiber by a Gaussian beam. This should, however, give excellent results for the case of incidence from an actual fiber for that value of a corresponding to maximum transmission at normal incidence and no offset, since at this point the two types of incident fields are very nearly identical. That choice of a is appropriate because the two fibers were assumed identical, and coupling is maximum when the incident wave is very nearly matched to the transmitted mode. This simplification is also borne out by the experimental results of Cook et al. [9], who got good agreement with their Born approximation calculations.

A related problem, that of the radiation into a uniform medium from the termination of a fiber, has been treated, yielding a transmission coefficient of 96 percent for radiation into a vacuum and 100.0 percent for a matched medium.

In conclusion, the more rigorous modal analysis has shown that the Born approximation does give accurate results in all cases treated using matched media, and that, even though it requires somewhat greater computational effort, the modal approach would be advantageous in problems where serious mismatches of refractive index occur or where the Born approximation is inapplicable, as in the case of the gap.

APPENDIX

The set of functions $e_m$, $h_m$, are the fields of the modes of a dielectric rod of permittivity $\epsilon_1$ and radius $a$ in a medium of lower permittivity $\epsilon_2$ surrounded by a perfectly conducting pipe of radius $b$ and can be obtained from the usual longitudinal formulation \[18\].

For the range of the propagation constant $k^2 < \beta^2 < k_2^2(k_1^2 + \omega^2\mu_1\epsilon_1)$, we obtain the surface-type modes, whose existence is independent of the surrounding pipe and which have fields localized to the vicinity of the core with cutoff condition $\beta = k_2$. Their longitudinal com-
ponents are given by
\[ E_x = g_1(\phi) \begin{cases} J_n(\alpha r), & r < a \\ J_n(u) \frac{K_n'(\chi w)I_n(\chi r) - I_n(\chi w)K_n(\chi r)}{K_n'(\chi w)I_n(w) - I_n(\chi w)K_n(w)}, & a < r < b \end{cases} \] (A1a)

\[ H_x = (\epsilon_1/\mu_1)^{1/2}(\beta/k_1) P_{\theta} \gamma_1(\phi) \]

\[ \begin{cases} J_n(\alpha r), & r < a \\ J_n(u) \frac{K_n'(\chi w)I_n(\chi r) - I_n(\chi w)K_n(\chi r)}{K_n'(\chi w)I_n(w) - I_n(\chi w)K_n(w)}, & a < r < b \end{cases} \] (A1b)

where
\[ \chi = b/a \]
\[ u = \alpha a, \quad w = sa \]
\[ u^2 + w^2 = V^2 = \alpha^2\omega^2\mu_0\epsilon_0 \delta \]
\[ \delta = 1 - \epsilon_2/\epsilon_1 = 1 - n_2^2/n_1^2 \]
\[ \alpha^2 = k_2^2 - \beta^2 \quad \beta^2 = \beta_2^2 - k_2^2 \]

\[ g_1(\phi) = \begin{bmatrix} \sin n\phi \\ \cos n\phi \end{bmatrix} \quad g_2(\phi) = \begin{bmatrix} \cos n\phi \\ -\sin n\phi \end{bmatrix} \]

with \( n \) a nonnegative integer.

The eigenvalue equation is
\[ F_1 = \frac{\beta^2}{k_1^2} P_1 \] (A7)

with
\[ P_1 = \frac{V^2}{u^2w^2} n + \eta_1 \xi_1 \]
(\[ F_1 = \frac{u^2w^2}{nV^2} (n + (1 - \delta) \eta_1 \xi_1) \]
(\[ \eta_1 = \frac{J_n'(u)}{uJ_n(u)} \quad \eta_2 = \frac{K_n'(w)}{wK_n(w)} \]
\[ \xi_1 = \frac{J_n'(w)/K_n'(w) - I_n(\chi w)/K_n(\chi w)}{I_n'(w)/K_n'(w) - I_n(\chi w)/K_n(\chi w)} \]
\[ \xi_2 = \frac{I_n'(w)/K_n'(w) - I_n'(\chi w)/K_n'(\chi w)}{I_n'(w)/K_n'(w) - I_n'(\chi w)/K_n'(\chi w)} \]

As \( b \) increases, these modes become the surface modes of the open rod. In the limiting case \( \chi \to \infty ; \xi_1 \to 1, \xi_2 \to 1 \), and (A1), (A7) become the expressions for the open fiber [2]. Use of the asymptotic expressions for the Bessel functions yields for \( \chi \gg 1 \)
\[ \xi_1 \sim \frac{-1 - e^x}{1 - e^x}, \quad \xi_2 \sim \frac{-1 + e^x}{1 + e^x}, \quad \text{for} \quad w \gg 1 \]
\[ \xi_1 \sim 1 + 2/\chi^2, \quad \xi_2 \sim 1 - 2/\chi^2, \quad \text{for} \quad \chi w \ll 1 \]

The nonsurface type modes are obtained in the range \( \beta^2 \leq k_2^2 \) and the longitudinal components are given by
\[ E_x = g_1(\phi) \begin{cases} J_n(\alpha r), & r < a \\ J_n(u) \frac{J_n'(\alpha r)N_n(\chi r) - N_n(\alpha r)J_n(\chi r)}{J_n'(u)N_n(\chi r) - N_n(u)J_n(\chi r)}, & a < r < b \end{cases} \] (A2a)

\[ H_x = (\epsilon_1/\mu_1)^{1/2}(\beta/k_1) P_{\theta} \gamma_1(\phi) \]

\[ \begin{cases} J_n(\alpha r), & r < a \\ J_n(u) \frac{J_n'(\alpha r)N_n(\chi r) - N_n(\alpha r)J_n(\chi r)}{J_n'(u)N_n(\chi r) - N_n(u)J_n(\chi r)}, & a < r < b \end{cases} \] (A2b)

with
\[ \alpha^2 = k_2^2 - \beta^2 \quad \beta^2 = \beta_2^2 - k_2^2 \]
\[ u = \alpha a \quad v = \alpha a \quad u^2 - v^2 = V^2 \]

and the eigenvalue equation
\[ F_2 = (\beta_2/k_1) P_2 \] (A15)

\[ P_2 = \frac{V^2}{u^2w^2} n + \eta_1 + \eta_2 \gamma_2 \]
(\[ F_2 = \frac{u^2w^2}{nV^2} (-\eta_1 + (1 - \delta) \eta_2 \gamma_1) \]
\[ \eta_2 = \frac{N_n'(v)}{vN_n(v)} \]
(\[ \gamma_1 = \frac{J_n'(v)/N_n'(v) - J_n(\chi v)/N_n(\chi v)}{J_n'(v)/N_n(v) - J_n(\chi v)/N_n(\chi v)} \]
\[ \gamma_2 = \frac{J_n'(v)/N_n'(v) - J_n'(\chi v)/N_n'(\chi v)}{J_n'(v)/N_n(v) - J_n'(\chi v)/N_n'(\chi v)} \]
(\[ M_{m,n} = \int_A (\mathbf{e}_m \times \mathbf{h}_m') \cdot \hat{z} \, da \]
\[ = \int_A \frac{j \omega}{p_n^{1/2} - p_{n_0}^{1/2}} \oint (\mu_{n_0} \mathbf{h}_{n_0} + \epsilon_{n_0} \mathbf{e}_{n_0}) \cdot \hat{n} \, dl \
+ \int_A \frac{j}{p_n^{1/2} - p_{n_0}^{1/2}} \oint (\beta_n \mathbf{h}_n + \beta_n \mathbf{e}_n) \cdot \hat{r} \, dl \]
(\[ N_{m,n} = \int_A (\mathbf{e}_m \times \mathbf{h}_m) \cdot \hat{z} \, da \]

and \( n \geq 1 \). is obtained from (A20) with \( n \) and \( m \) interchanged:
\[ N_{mn} = \int_A (e_m \times h_m) \cdot \mathbf{e} \, da \]
\[ = -\frac{j}{2\beta_m} \left\{ \omega \int_c \left( \frac{d\mu_m h_m}{d\beta_m} - e_m \frac{d\mu_m e_m}{d\beta_m} \right) \cdot \mathbf{e} \, dl + \beta_m \int_c \left( \frac{d\mu_m h_m}{d\beta_m} e_m + e_m \frac{d\mu_m e_m}{d\beta_m} \right) \cdot \mathbf{e} \, dl \right\} \]  

where the contour \( c \) includes both sides of all the interfaces, and
\[ p_n = \omega^2 \mu_n e_n - \beta_n n \]  
\[ p_n' = \omega^2 \mu_n e_n' - \beta_n n' \]  

For a fiber with \( \epsilon_1 \) in the core and \( \epsilon_2 \) in the cladding, we have
\[ N_{an} = -j\pi a \left( \frac{1}{\gamma_n^2 - \alpha_n^2} - \frac{1}{\gamma_n^2 - \alpha_m^2} \right) \left( \omega \mu_h \mu_n h_{nr} + \beta_n h_m e_m + \beta_m h_n e_m' \right) \]
\[ + j\pi a \omega \epsilon_n \left( \frac{e_1}{\gamma_n^2 - \alpha_1^2} - \frac{e_2'}{\gamma_n^2 - \alpha_m^2} \right) e_m e_m' \]  
\[ M_{mn} = j\pi a \left( \frac{1}{\gamma_n^2 - \alpha_n^2} - \frac{1}{\gamma_n^2 - \alpha_m^2} \right) \left( \omega \mu_h \mu_n h_{nr} + \beta_n h_m e_m + \beta_m h_n e_m' \right) \]
\[ + \beta_n h_m e_m + \beta_m h_n e_m' \]
\[ - j\pi a \omega \epsilon_n \left( \frac{e_1}{\gamma_n^2 - \alpha_1^2} - \frac{e_2'}{\gamma_n^2 - \alpha_m^2} \right) e_m e_m' \]  

where
\[ \gamma_n^2 = \omega^2 \mu_n e_n - \beta_n n \]  
\[ \alpha_n^2 = \omega^2 \mu_n e_n - \beta_n n \]  
\[ \alpha_m^2 = \omega^2 \mu_m e_m - \beta_m m \]

All field components have the angular dependence removed and are evaluated at \( r = a \), except \( e_m(a) \), which denotes the radial electric field of the core at \( r = a \).

REFERENCES