A Note on Bounded Query Learning

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Abstract

We study in this note the relationships between two usual models of learning via queries: the so-called exact and bounded learning models. One of our goals is to point out under which conditions the learning dimension functions, as the abstract identification dimension, originally defined assuming the bounded model can be also used in the exact setting and how the results obtained by using them have to be interpreted.

1 Introduction

The main model of learning via queries, the exact model, was introduced by Angluin [1]. In this model, the learning algorithm obtains information about the target concept by asking queries to a teacher or expert. The goal of the algorithm is to output an exact representation of the target concept.

The bounded model is a variant of the Angluin's one proposed by Watanabe [10]. Here, instead of achieving a full knowledge of the target, it is only required to output a representation that agrees with the target concept on the words having length up to some bound. The exact value of such length bound is known in advance by the learner. Although the exact model is the most popular, one may consider the bounded one as a more suitable model when topics concerning complexity issues are in study [10, 11, 6].

We prove in this note two theorems that relate learning in these two models. Despite the fact that this subject was first studied by Gavaldà [6], some of the arguments shown there have some lack of precision. We believe that it is important to have a deep understanding of these relationships by the reason that we explain below.

Given a set of queries, a learning dimension is a function defined by a combinatorial property that characterizes the number of queries needed to learn a concept class using that set. Examples of such dimension functions are the approximate fingerprints for equivalence queries [2], the extended teaching dimension for membership queries [7], and the certificate size for the combination of membership and equivalence queries [8]. Recent work have shown new dimension notions (the abstract identification dimension and the general dimension), having a sort of universality property [4, 3]. This means that they are valid notions not only for an specific set of queries –as the approximate fingerprints– but approximate the query complexity of problems for a wide choice of queries. Both universal dimensions are defined in the bounded setting. The results in this note show how to move to the exact setting the information provided by these dimensions.

2 Representation Classes

A concept c is a subset of $\{0,1\}^\star$. This domain will be our instance space. A concept class C is a set of concepts. We denote by Σ a fixed alphabet that contains {0,1}. A *representation class* is a pair $\langle R, \phi\rangle$ that satisfies:

1. R is some language over Σ decidable in polynomial time.

2. $\phi : R \mapsto 2^{\{0,1\}^\star}$ is the semantic function. It associates a concept to any representation.

3. The set $\{\langle r, x\rangle \mid x \in \phi(r)\}$ is decidable in polynomial time.

We will deal with concept classes C defined by some representation $\langle R, \phi\rangle$, i.e. $C = \{\phi(r) \mid r \in R\}$. Abusing notation, we use c and the characteristic function $f : \{0,1\}^\star \mapsto \{0,1\}$ of c interchangeably for the concept $c \in C$ and for the syntactic representation $r \in R$ of c. Given $c \in C$ we denote by $l(c)$ the minimum length of the representations of c. We also denote by $\|X\|$ the cardinality of set X and by $X \oplus Y$ the symmetric difference of sets X and Y.

Two standard examples are DFA (deterministic finite automata) and DNF (disjunctive normal form) formulas. In the first one, the set R is formed by encodings of deterministic finite automata and the semantic function associates to each encoded automaton its accepted language. DNF formulas are a popular representation class for concepts whose elements have all the same length, say n. In this case, R is the set of encodings of DNF formulas on n variables and the semantic function assigns to each encoded formula the subset of $\{0,1\}^n$ where the formula evaluates to 1.

3 Query Learning

In query learning two players, the *learner* and the *teacher*, play a game . The learner is an algorithm and the teacher is an oracle function. Some concept class C (the *target* concept class) is known to both players and the teacher chooses a concept in C (the target concept) that is unknown to the learner. The goal of the learner is to find out what concept is, asking the teacher some type of queries.

A query is a question that the learner poses to the teacher. The most popular are *membership queries* ($\in$) and *equivalence queries* ($\equiv$). In a membership query the learner asks if some word belongs to the target concept. In an equivalence query the learner passes a hypothesis to the teacher. An hypothesis is a concept from the hypothesis class H, which is a concept class including the target class C. A teacher T replies queries consistently according to the target concept c when it works as follows:

- On a membership query with x, T replies with $c(x)$.
- On an equivalence query with $h \in H$, T replies YES if $h = c$ and some word $x \in h \oplus c$ otherwise.

Other type of queries have been defined in the literature. For instance, in [1] are introduced, among others, *subset queries* ($\subseteq$) and *superset queries* ($\supseteq$). On a subset (respectively, superset) query with hypothesis $h \in H$, a teacher consistent with the target concept c replies YES if $h \subseteq c$ (resp. $h \supseteq c$) and with a counterexample $x \in \{0,1\}^\star$ such that $h(x) = 1$ and $c(x) = 0$ (resp. $h(x) = 0$ and $c(x) = 1$) otherwise. Equivalence queries without counterexample are also defined in [1]. For these queries, called restricted equivalence queries ($\equiv_r$), the teacher answers are YES or NO.

In general, the setting of the learning game is complete when the learning protocol is defined. The *protocol* is the agreement about which the admissible queries are and, for each target concept, which the possible answers for such queries are. A teacher is consistent with the target concept c and the protocol P if it replies to each query q choosing one of the admissible answers in P for q and c.

3.1 Exact Learning

Let P be any learning protocol. A learner L identifies C under P if and only if for all $c \in C$ and for any teacher T consistent with P and c, when L is run on input l, where l denotes the length of c, and T answers its queries, it eventually halts and outputs some $h \in H$ such that $h = c$. Such algorithm is a polynomial query learner if and only if there exists a polynomial $p(l, m)$ such that at any point in the run the number and the length of the queries asked by $L(l)$ is bounded by $p(l, m)$, where m is the maximum length of any counterexample returned so far. Moreover, if $p(l, m)$ also bounds its running time, L is a polynomial time learning algorithm.

Definition 1 *Let C be a concept class. We say that C is query learnable according to protocol P if and only if a polynomial query learner identifies C under P. If the identification algorithm runs in polynomial time we say that C is learnable under P.*

There are query learning protocols where one can easily identify the target concept making few queries if arbitrarily long queries are admissible. For instance, if C is the class of DNF formulas on n variables the target formula can be identified with a polynomial number of equivalence queries using majority hypothesis (that perhaps do not have polynomial length). Because of this, we require a polynomial bound on the length of the queries.

3.2 Bounded Learning

Often, after learning a concept we want to use the resulting hypothesis to predict future instances. In many practical situations, we are not interested on instances of arbitrary length. We will be happy if our representation classifies correctly all instances up to some length bound, that we can estimate beforehand.

Inspired by this observation, Watanabe [10] (see also [11]) introduced the bounded learning model where the goal of the learner is to identify the target concept on instances of length at most n, making some kind of queries to a teacher. Here, bound n is an additional input of the learning game.

We introduce below a simplified version of the original bounded learning model proposed by Watanabe. We will denote by c_n the concept $c \cap \{0,1\}^{\leq n}$ and by C_n the concept class $\{c_n \mid c \in C\}$.

In order to define polynomial learnability in the bounded setting we need to introduce first the notions of bounded teacher and bounded learner. With respect to the queries listed above, we say that T is a bounded teacher consistent with c if and only if:

- On a membership query with $x \in \{0,1\}^{\leq n}$, T replies with $c(x)$.
- On an equivalence query with $h \in H$, T replies YES if $h_n = c_n$ and some word $x \in h_n \oplus c_n$ otherwise.
- On a subset query with hypothesis $h \in H$, T replies YES if $h_n \subseteq c_n$ and some word $x \in h_n$ such that $x \notin c_n$ otherwise.
- On a superset query with hypothesis $h \in H$, T replies YES if $c_n \subseteq h_n$ and some word $x \in c_n$ such that $x \notin h_n$ otherwise.
- On a restricted equivalence query with hypothesis $h \in H$, T replies YES if $h_n = c_n$ and NO otherwise.

In general, a bounded consistent teacher replies to queries according to the behavior of the target concept in $\{0,1\}^{\leq n}$ ignoring how the target is outside that domain.

Let P be a learning protocol. We say that L is a polynomial query (resp. time) bounded learner for C under P if and only if L satisfies the following two properties:

1. L identifies C under P: for any $c \in C$, for all n and for any bounded teacher T consistent with c and P, $L(l,n)$ outputs some $h \in H$ such that $h_n = c_n$. Here, l denotes the length of c.
2. The length and the number of queries asked by L on input (l,n) is bounded by some polynomial $p(l,n)$ (resp. the running time of $L(l,n)$ is bounded by $p(l,n)$).

The following definition is the counterpart of Definition 1 for the bounded setting.

Definition 2 *Let C be a concept class. We say that C is query bounded learnable according to protocol P if and only if a polynomial query bounded learner identifies C under P. If the identification algorithm runs in polynomial time we say that C is bounded learnable under P.*

At any intermediate stage of a query learning process, the learner knows (from the teacher's answers received so far) a set of answers S. Let C be the target class. We define the *version space* as the set of concepts from C that are consistent with the answers in S. These are all concepts being still conceivable as target concepts. We will use this notion in the proof of the next theorem. We note that according to our definition the version space does not have any inner structure, in contrast with the original definition in [9].

The two theorems below are detailed versions of a result by Gavaldà [6] that relates bounded and exact learning. However, it is worth to say that the part of the original proof that corresponds to our Theorem 5 is somewhat imprecise.

Theorem 3 *Let P be any protocol including equivalence queries. Let C be any concept class. If C is query bounded learnable (resp. bounded learnable) under P then C is also query learnable (resp. learnable) under P.*

Proof Let B be the learning algorithm that shows that C is query bounded learnable. Let p be the polynomial bounding the length and the number of queries of B. We consider the learning algorithm A described below.

On input l, where l is the length of the target concept c, algorithm A proceeds by stages. In stage n, A simulates B on input (l, n). This simulation ends when A achieves an hypothesis $h \in H$ such that $h_n = c_n$ or the teacher gives a counterexample of length greater than n. In the first case, A makes an equivalence query with hypothesis h. If $h \neq c$ then a counterexample of length greater than n is obtained. Finally, in both cases and before starting stage $n+1$, A asks as equivalence query the lexicographically first hypothesis of length at most $\log n$ in the version space, unless this set was empty. Observe that the length bound $\log n$ allows A to compute the last query in polynomial time.

The computation of A finishes when it receives a YES answer to an equivalence query. Note that at any point of the run of A makes at most $m(p(l, m)+2)$ queries, where m is the maximum length of counterexamples received so far. Moreover, if B is a polynomial time learner, A also runs in polynomial time. $\square$

We highlight that the proof of Theorem 3 can be extended to any protocol not including equivalence queries but containing queries that can simulate equivalences. For instance, statement of Theorem 3 is also valid for protocols where subset and superset are admissible queries. For protocols that cannot obtain this information from the teacher, bounded learning does not imply exact learning, as it is proved in [6].

We also note that Theorem 3 is also trivially valid without any assumption of including equivalence queries when the protocol is defined on queries which do not have examples of varying length as admissible answers (for instance, membership or restricted equivalences) and the target concept class C is such that for some integer n, $C = C_n$ (for instance, DNF formulas on n variables).

With respect to the converse of Theorem 3, we are able to show it only with some additional assumptions on the learning protocol P and the target concept class C. First, we point out a property that will allow to learners to simulate queries to a unbounded teacher by queries to a bounded one.

Definition 4 *Let P be a query learning protocol and let P_Q denote the set of queries in P. We say that P is* closed *if there exists a polynomial time computable function f, defined on P_Q and with values also in P_Q, such that for each query Q and with respect to any concept c_n in the bounded target class C_n, either:*

1. *Q is a (restricted) equivalence query, or*

2. *An answer for Q that is admissible for an unbounded teacher consistent with c_n can be computed in polynomial time from the answer to $f(Q)$ that a bounded teacher provides.*

For instance, any protocol P formed by some of the queries listed above, apart from subset queries, is always a closed protocol. Note that for a membership query of length at most n, it makes no difference whether the answer is provided by a bounded or unbounded teacher. If the length of the query is greater than n, any unbounded teacher consistent with c_n has to reply NO. For any other type of query in P (so, different from subset) either it is a (restricted) equivalence query, or the answer provided by a bounded teacher is an admissible answer for an unbounded one consistent with c_n. This shows that P is a closed protocol.

The next theorem shows that under some weak hypothesis on the target concept class C, exact learning implies bounded learning in closed protocols.

Theorem 5 *Let P be closed protocol. Let C be a concept class that satisfies:*

1. *$C_n \subseteq C$.*

2. *There exists a polynomial b such that for all $c \in C$, $l(c_n) \leq b(l(c))$.*

If C is query learnable (resp. learnable) under P then C is query bounded learnable (resp. bounded learnable) according to P.

Proof Let A be a learner that shows that C is query learnable and let q be the polynomial bounding the length and the number of queries of A. Note that A also shows that C_n is query learnable because $C_n \subseteq C$. We consider a bounded learner B that on input (l, n), where l is the length of the target concept c, simulates the computation of A when trying to learn the target c_n by stages. In stage k, B simulates A on input k as is explained below.

If A makes a query Q different from a (restricted) equivalence one, learner B instead of making query Q, asks for $f(Q)$ to the bounded teacher, where f is the function promised by Definition 4. Some answer is obtained and then B can compute an admissible answer to Q for an unbounded teacher consistent with c_n. With this answer B is able to continue the simulation of A.

If A makes a (restricted) equivalence query, B makes exactly the same query. Either, the answer provided to B is a counterexample (a NO answer for a restricted equivalence) or is a YES answer. In the first case, such answer is also valid for a unbounded teacher and the simulation goes on. In the second case, concept c_n is identified.

Stage k finishes when identification of c_n is achieved or $q(k, n)$ queries have been simulated without success. Finally, if identification has not been reached, stage $k + 1$ starts. Observe that on stage k, B keeps up the query complexity bounded by $q(k, n)$ and the running time of B is a polynomial factor of the running time of A on input k. The simulation finishes with success before the end of stage $k = l(c_n)$. Moreover, as $l(c_n) \leq b(l)$ by some polynomial b, B is a polynomial query (resp. polynomial time) bounded learner. □

Finally, we note that usual concept classes (as DFA or DNF) satisfy the hypothesis on the target class required by Theorem 5.

4 Conclusions

Given a protocol P, a learning dimension for P and C is, roughly speaking, a function defined by some combinatorial property that approximates the number of queries needed to learn C according to P. These functions must provide good upper and lower bounds on the query complexity. In a recent work by Balcázar *et al* [5, 4, 3] have been proposed learning dimension functions for several learning protocols. For instance, the general dimension [3] is a universal dimension, in the sense that is a valid dimension in any protocol (see also [4] where another dimension having a sort of universal property is defined, the abstract identification dimension).

Aforementioned learning dimensions were proposed for the bounded model of query learning. Theorems 3 and 5 in this note show how the information provided for these dimensions has to be understood in exact setting. The point is that the results concerning these learning dimension functions remain valid, up to polynomial factors, for those protocols and target concept classes that satisfy hypothesis of both theorems. Positive (resp. negative) learning results in the bounded setting remain true, up to polynomial factors, in the exact model when Theorem 3 (resp. Theorem 5) applies.

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