HOMOTOPY LINEAR ALGEBRA

IMMA GÁLVEZ-CARRILLO, JOACHIM KOCK, AND ANDREW TONKS

ABSTRACT. By homotopy linear algebra we mean the study of linear functors between slices of the ∞-category of ∞-groupoids, subject to certain finiteness conditions. After some standard definitions and results, we assemble said slices into ∞-categories to model the duality between vector spaces and profinite-dimensional vector spaces, and set up a global notion of homotopy cardinality à la Baez–Hoffnung–Walker compatible with this duality. We needed these results to support our work on incidence algebras and Möbius inversion over ∞-groupoids; we hope that they can also be of independent interest.

CONTENTS

0. INTRODUCTION

0.1. Vector spaces and spans (Yoneda and Bénabou). It has been known since the early days of category theory [33], [6] that the category of sets and spans behaves a lot like the category of vector spaces. Its objects are sets; morphisms from S to T are spans

$$S \leftarrow M \rightarrow T,$$

and composition is given by pullback. The pullback formula for composition can in fact be written as a matrix multiplication.

Further vector-space flavour can be brought out by a slight reinterpretation of the objects in the span category. Since the sets S and T index the 'columns' and 'rows' in the 'matrix' M, they play the role of bases of vector spaces. The vector space is

the scalar-multiplication-and-sum completion of the basis set S. The corresponding categorical construction is the slice category $\boldsymbol{Set}_{/S}$ whose objects are maps $V \to S$. The fibre V_s over an element $s \in S$ plays the role of the coefficient of the basis element s in a linear combination. The way a span acts on a 'vector' $V \to S$ is now just a special case of matrix multiplication: by pullback, and then composition. These two operations are the most basic functors between slice categories, and they are adjoint: for a given set map $f : S \to T$, pullback along f is denoted f^* and composition denoted $f_!$, and we have the adjunction

$$\boldsymbol{Set}_{/S} \underset{f^*}{\overset{f_!}{\underset{\perp}{\rightleftarrows}}} \boldsymbol{Set}_{/T}$$

The category of sets is locally cartesian closed, hence f^* preserves all colimits. In particular it preserves 'linear combinations', and it is appropriate to call composites of lowershrieks and upperstars *linear functors*.

The promised stronger vector-space flavour thus comes from considering the category whose objects are slice categories and whose morphisms are linear functors.

0.2. Cardinality. The constructions above work with arbitrary sets, but in order to maintain our 'linear combination' interpretation of an object $V \to S$ of a slice category we must impose certain finiteness conditions. A linear combination is a finite sum of finite scalar multiples of vectors from the (possibly infinite) basis; we should thus require V (but not S) to be a finite set, and further finiteness conditions should also be imposed on spans. These finiteness conditions are also needed in order to be able to take cardinality and recover results at the level of vector spaces. The cardinality of $V \to S$ is a vector in the vector space spanned by S, namely the linear combination $\sum_{s \in S} |V_s| \, \delta_s$, where δ_s is the basis vector indexed by s, and $|V_s|$ denotes the usual cardinality of the set V_s.

0.3. Objective algebraic combinatorics (Joyal, Lawvere, Lawvere–Menni). Linear algebra with sets and spans is most useful in the coordinatised situation (since the slice categories are born with a 'basis'), and in situations where the coefficients are natural numbers. In practice it is therefore mostly algebraic combinatorics that can benefit from *objective linear algebra* as we call it, following Lawvere, who has advocated the *objective method* in combinatorics and number theory.

In a nutshell, algebraic combinatorics is the study of combinatorial objects via algebraic structures associated to them. Most basically these algebraic structures are vector spaces. Further algebraic structure, such as coalgebra structure, is induced from the combinatorics of the objects. Generating functions and incidence algebras are two prime examples of this mechanism. While these algebraic techniques are very powerful, for enumerative purposes for example, it is also widely acknowledged that bijective proofs represent deeper understanding than algebraic identities; this is one motivation for wishing to objectify combinatorics. A highlight in this respect is Joyal's theory of species [19], which reveals the objective origin of many operations with power series. A species is a $\boldsymbol{Set}$-valued functor on the groupoid of finite sets and bijections; the value on an n-element set is the objective counterpart to the nth coefficient in the corresponding exponential generating function.

The present work was motivated by incidence algebras and Möbius inversion. A Möbius inversion formula is classically an algebraic identity in the incidence algebra (of a locally finite poset, say, or more generally a Möbius category in the sense of Leroux [25]), hence an equation between two linear maps. So by realising the two linear maps as spans and establishing a bijection between the sets representing these spans, a bijective proof can be obtained. An objective Möbius inversion principle for Möbius 1-categories was established by Lawvere and Menni [24]; the ∞-version of these results [11], [12], [13] required the developments of homotopy linear algebra of the present paper.

0.4. From sets to groupoids (Baez–Dolan, Baez–Hoffnung–Walker). Baez and Dolan [3] discovered that the theory of species can be enhanced by considering groupoid-valued species instead of set-valued species. The reason is that most combinatorial objects have symmetries, which are not efficiently handled with **Set**-coefficients. Their paper [3] illustrated this point by showing that the exponential generating function corresponding to a species is literally the cardinality of the associated analytic functor, provided this analytic functor is taken with groupoid coefficients rather than set coefficients. They also showed how the annihilation and creation operators in Fock space can be given an objective combinatorial interpretation in this setting.

A subsequent paper by Baez, Hoffnung and Walker [4] developed in detail the basic aspects of linear algebra over groupoids, under the name 'groupoidification'. One important contribution was to check that the symmetry factors that arise behave as expected and cancel out appropriately in the various manipulations. A deeper insight in their paper is to clarify the notion of (groupoid) cardinality by deriving all cardinality assignments, one for each slice category, from a single global prescription, defined as a functor from groupoids and spans to vector spaces. This does not work for all groupoids; the ones that admit a cardinality are called *tame*, a notion akin to square-integrability, and convergence plays a role. There is a corresponding notion of tame span.

We now come to the new contributions of the present paper, and as a first approximation we explain them in terms of three contrasts with Baez–Hoffnung–Walker:

0.5. From groupoids to ∞-groupoids. We work with coefficients in ∞-groupoids, so as to incorporate more homotopy theory. The abstraction step from 1-groupoids to ∞-groupoids is actually not so drastic, since the theory of ∞-categories is now so well developed that one can deal with elementary aspects of ∞-groupoids with almost the same ease as one deals with sets (provided one deals with sets in a good categorical way).

0.6. Homotopy notions. In fact, because of the abstract viewpoints forced upon us by the setting of ∞-groupoids, we are led to some conceptual simplifications, valuable even when our results are specialised to the 1-groupoid level. The main point is the consistent homotopy approach. We work consistently with homotopy fibres, while in [4] 'full' fibres are employed. We exploit homotopy sums, where [4] spells out the formulae in ordinary sums, at the price of carrying around symmetry

factors. Just as an ordinary sum is a colimit indexed over a set, a homotopy sum is a colimit indexed over an ∞-groupoid. The advantage of working with homotopy fibres and homotopy sums is that homotopy sums (an example of $f_!$) are left adjoint to homotopy fibres (an example of f^*) in exactly the same way as, over sets, sums are left adjoint to fibres. As a consequence, with the correct notation, no symmetry factors appear — they are hidden and take care of themselves. (See [9] for efficient exploitation of this viewpoint.)

0.7. Finiteness conditions and duality issues. A more substantial difference to the Baez–Hoffnung–Walker approach concerns the finiteness conditions. Their motivating example of Fock space led them to the tameness notion which is self-dual: if a span is tame then so is the transposed (or adjoint) span (i.e. the same span read backwards). One may say that they model Hilbert spaces rather than plain vector spaces.

Our motivating examples are incidence coalgebras and incidence algebras; these are naturally vector spaces and profinite-dimensional vector spaces, respectively, and a fundamental fact is the classical duality between vector spaces and profinite-dimensional vector spaces. Recall that if V is a vector space, the linear dual V^* is naturally a profinite-dimensional vector space, and that in turn the continuous linear dual of V^* is naturally isomorphic to V. In the fully coordinatised situation characteristic of algebraic combinatorics, S is some set of (isoclasses of) combinatorial objects, the vector space spanned by S is the set of finite linear combinations of elements in S, which we denote by $\mathbb{Q}_S$, and the dual can naturally be identified with the space of $\mathbb{Q}$-valued functions, $\mathbb{Q}^S$. (Some further background on this duality is reviewed in 4.1 below.)

The appropriate finiteness condition to express these notions is simply homotopy finiteness: an ∞-groupoid is called homotopy finite, or just *finite*, when it has finitely many components, all homotopy groups are finite, and there is an upper bound on the dimension of nontrivial homotopy groups. A morphism of ∞-groupoids is called finite when all its fibres are finite. Letting $\boldsymbol{grpd}$ denote the ∞-category of finite ∞-groupoids, the role of vector spaces is played by finite-∞-groupoid slices $\boldsymbol{grpd}_{/S}$, while the role of profinite-dimensional vector spaces is played by finite-presheaf ∞-categories $\boldsymbol{grpd}^S$, where S is only required to be locally finite. Linear maps are given by spans of *finite type*, meaning $S \xleftarrow{p} M \xrightarrow{q} T$ in which p is a finite map. Prolinear maps are given by spans of *profinite type*, where instead q is a finite map. We set up two ∞-categories: the ∞-category $\underrightarrow{\boldsymbol{lin}}$ whose objects are the slices $\boldsymbol{grpd}_{/S}$ and whose mapping spaces are ∞-groupoids of finite-type spans, and the ∞-category $\underleftarrow{\boldsymbol{lin}}$ whose objects are finite-presheaf ∞-categories $\boldsymbol{grpd}^S$ and whose mapping spaces are ∞-groupoids of profinite-type spans; we show that these are dual. We introduce a global notion of cardinality such that the classical duality becomes the cardinality of the $\underrightarrow{\boldsymbol{lin}}$ - $\underleftarrow{\boldsymbol{lin}}$ duality.

We proceed to outline the paper, section by section.

The finiteness conditions are needed to be able to take homotopy cardinality. However, as long as we are working at the objective level, it is not necessary to impose the finiteness conditions, and in fact, the theory is simpler without them. Furthermore, the notion of homotopy cardinality is not the only notion of size: Euler characteristic and various multiplicative cohomology theories are other potential alternatives, and it is reasonable to expect that the future will reveal more comprehensive and unified notions of size and measures. For these reasons, we begin in Section 2 with 'linear algebra' without finiteness conditions.

Let $\textbf{Grpd}$ denote the ∞-category of ∞-groupoids. We define formally the ∞-category $\textbf{LIN}$, whose objects are slices $\textbf{Grpd}_{/S}$ and whose morphisms are linear functors. We show that the ∞-category $\textbf{Grpd}_{/S}$ is the homotopy-sum completion of S, and interpret scalar multiplication and homotopy sums as special cases of the lowershriek operation. The canonical basis is given by the 'names', functors $\ulcorner x \urcorner : 1 \to S$. We show that linear functors can be presented canonically as spans. We exploit results already proved by Lurie [28] to establish that $\textbf{LIN}$ is symmetric monoidal closed. The tensor product is given by

$$\textbf{Grpd}_{/S} \otimes \textbf{Grpd}_{/T} = \textbf{Grpd}_{/S \times T}.$$

In Section 3 we start getting into finiteness conditions. An ∞-groupoid X is *locally finite* if at each base point x the homotopy groups $\pi_i(X, x)$ are finite for $i \geq 1$ and are trivial for i sufficiently large. It is called *finite* if furthermore it has only finitely many components. The cardinality of a finite ∞-groupoid X is defined as

$$|X| := \sum_{x \in \pi_0 X} \prod_{i > 0} |\pi_i(X, x)|^{(-1)^i}.$$

We work out the basic properties of this notion, notably how it interacts with pullbacks in special cases. We check that the ∞-category $\textbf{grpd}$ of finite ∞-groupoids is locally cartesian closed.

In Section 4 we first recall the duality between vector spaces and profinite-dimensional vector spaces, on which the $\underrightarrow{\textbf{lin}}$ - $\underleftarrow{\textbf{lin}}$ duality is modelled.

The basis S must be required locally finite in order to have pullback stability of finite ∞-groupoids over it, but it is essential that we do not require S to be finite, as the vector spaces we wish to model are not finite dimensional. To the category of vector spaces corresponds the ∞-category $\textbf{grpd}_{/S}$ of finite ∞-groupoids over S. To the category of profinite-dimensional vector spaces corresponds the ∞-category $\textbf{grpd}^S$ of finite-∞-groupoid-valued presheaves. We also introduce the variants $\textbf{grpd}^S_{\text{fin.sup.}}$ of presheaves with finite support, and $\textbf{Grpd}^{\text{rel.fin.}}_{/S}$ of finite maps to S; the latter can be thought of as a space of measures on S (in view of 7.2). These two ∞-categories are naturally equivalent to the previous pair, but live on the opposite side of the duality we are setting up.

We proceed to assemble these collections of finite slices into the following ∞-categories.

There is an ∞-category $\textbf{lin}$ whose objects are ∞-categories of the form $\textbf{grpd}_{/\alpha}$ where α is a finite ∞-groupoid, and with morphisms given by finite spans $\alpha \leftarrow \mu \to \beta$. This ∞-category corresponds to the category $\textbf{vect}$ of finite-dimensional vector

spaces. We need infinite indexing, so the following two extensions are introduced, referring to a locally finite ∞-groupoid S. There is an ∞-category $\underrightarrow{\boldsymbol{lin}}$ whose objects are ∞-categories of the form $\boldsymbol{grpd}_{/S}$, and whose morphisms are spans of finite type (i.e. the left leg has finite fibres). This ∞-category corresponds to the category $\underrightarrow{\boldsymbol{vect}}$ of general vector spaces (allowing infinite-dimensional ones). Finally we have the ∞-category $\underleftarrow{\boldsymbol{lin}}$ whose objects are ∞-categories of the form $\boldsymbol{grpd}^S$ with S a locally finte ∞-groupoid, and whose morphisms are spans of profinite type (i.e. the right leg has finite fibres). This ∞-category corresponds to the category $\underleftarrow{\boldsymbol{vect}}$ of profinite-dimensional vector spaces.

In order actually to define $\boldsymbol{lin}$, $\underrightarrow{\boldsymbol{lin}}$ and $\underleftarrow{\boldsymbol{lin}}$ as ∞-categories, in Section 5 we take an intermediate step up in the realm of presentable ∞-categories — so to speak extending scalars from $\boldsymbol{grpd}$ to $\boldsymbol{Grpd}$ — to be able to leverage our work from Section 2.

So, within the ambient ∞-category $\boldsymbol{LIN}$ we define the following subcategories: the ∞-category $\boldsymbol{Lin}$ with objects of the form $\boldsymbol{Grpd}_{/\alpha}$ and morphisms given by finite spans; the ∞-category $\underrightarrow{\boldsymbol{Lin}}$ consisting of $\boldsymbol{Grpd}_{/S}$ and spans of finite type; and the ∞-category $\underleftarrow{\boldsymbol{Lin}}$ consisting of $\boldsymbol{Grpd}^S$ and spans of profinite type.

We characterise profinite spans by the following pleasant 'analytic' continuity condition (5.1):

A linear functor $F : \boldsymbol{Grpd}^T \to \boldsymbol{Grpd}^S$ is given by a profinite span if and only if for all $\varepsilon \subset S$ there exists $\delta \subset T$ and a factorisation

$$
\begin{array}{ccc}
\boldsymbol{Grpd}^T & \longrightarrow & \boldsymbol{Grpd}^\delta \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F_\delta} \\
\boldsymbol{Grpd}^S & \longrightarrow & \boldsymbol{Grpd}^\varepsilon
\end{array}
$$

where ε and δ denote finite ∞-groupoids, and the horizontal maps are the projections of the canonical pro-structures.

The three ∞-categories constructed with $\boldsymbol{Grpd}$ coefficients are in fact equivalent to the three ∞-categories with $\boldsymbol{grpd}$ coefficients introduced heuristically.

In Section 7 we establish that the pairing $\boldsymbol{grpd}_{/S} \times \boldsymbol{grpd}^S \to \boldsymbol{grpd}$ is perfect. In Section 8 we prove that upon taking cardinality this yields the pairing $\mathbb{Q}_{\pi_0 S} \times \mathbb{Q}^{\pi_0 S} \to \mathbb{Q}$. To define the cardinality notions, we follow Baez–Hoffnung–Walker [4] and introduce a 'meta cardinality' functor, which induces cardinality notions in all slices and in all presheaf ∞-categories. In our setting, this amounts to a functor

$$
\begin{array}{rcl}
\| \, \| : \underrightarrow{\boldsymbol{lin}} & \longrightarrow & \boldsymbol{Vect} \\
\boldsymbol{grpd}_{/S} & \longmapsto & \mathbb{Q}_{\pi_0 S}
\end{array}
$$

and a dual functor

$$
\begin{array}{rcl}
\| \, \| : \underleftarrow{\boldsymbol{lin}} & \longrightarrow & \underleftarrow{\boldsymbol{vect}} \\
\boldsymbol{grpd}^S & \longmapsto & \mathbb{Q}^{\pi_0 S}.
\end{array}
$$

For each fixed ∞-groupoid S, this gives an individual notion of cardinality $|\ |$: $\mathbf{grpd}_{/S} \to \mathbb{Q}_{\pi_0 S}$ (and dually $|\ | : \mathbf{grpd}^S \to \mathbb{Q}^{\pi_0 S}$), since vectors are just linear maps from the ground field.

The vector space $\mathbb{Q}_{\pi_0 S}$ is spanned by the elements $\delta_s := |^\ulcorner s^\urcorner|$. Dually, the profinite-dimensional vector space $\mathbb{Q}^{\pi_0 S}$ is spanned by the characteristic functions $\delta^t = \frac{|h^t|}{|\Omega(S,t)|}$ (the cardinality of the representable functors divided by the cardinality of the loop space).

0.8. Related work.

Part of the material developed here may be considered either folklore, or straightforward generalisations of well-known results in 1-category theory, or special cases of fancier machinery.

$(\infty, 1)$-categories of spans have been studied by many people in different contexts and with different goals, e.g. Lurie [28], Dyckerhoff–Kapranov [8] and Barwick [5]. Lurie [29] studies an $(\infty, 2)$ version relevant for the present purposes; Dyckerhoff and Kapranov [8] study a different $(\infty, 2)$-category of spans; and Haugseng [18], motivated by topological field theory [27], studies an (∞, n)-category of iterated spans (which for $n = 2$ is different from both the previous).

Finally, the theory of slices and linear functors is subsumed into the theory of polynomial functors, where a further right adjoint enters the picture, the right adjoint to pullback. The theory of polynomial functors over ∞-categories is developed in [17]; see [16] for the classical case.

Note. This paper was originally written as an appendix to [10], to provide precise statements and proofs of the results in homotopy linear algebra needed in the theory of decomposition spaces, an ∞-groupoid setting for incidence algebras and Möbius inversion. That manuscript has now been split into smaller papers [11], [12], [13], [14], [15], its appendix becoming the present paper.

1. Preliminaries on ∞-groupoids and ∞-categories

We work with ∞-categories, in the sense of Joyal [21] and Lurie [26], but we can get away with working model-independently, since our undertakings are essentially elementary: our objects of study are the ∞-category of ∞-groupoids and its slices, and many of the arguments (for example concerning pullbacks) can be carried out almost as if we were working with the category of sets — with a few homotopy caveats.

In the implementation of ∞-categories as quasi-categories, ∞-groupoids are precisely Kan complexes, and serve as a model for topological spaces up to homotopy. For example, to each object x in an ∞-groupoid X, there are associated homotopy groups $\pi_n(X, x)$ for $n > 0$; a map $X \to Y$ of ∞-groupoids is an equivalence if and only if it induces a bijection on the level of π_0 and isomorphisms on all homotopy groups, and so on.

The great insight of Joyal [20] was to fit this into a theory of ∞-categories, in which ∞-groupoids play the role that *sets* play in category theory. For example, for any two objects x, y in an ∞-category $\mathcal{C}$ there is (instead of a hom set) a mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ which is an ∞-groupoid. Universal properties, such as limits, colimits and adjoints can be expressed as equivalences of mapping spaces. Presheaves take values in ∞-groupoids, and constitute the colimit completion.

1.1. Slices and Beck–Chevalley. Maps of ∞-groupoids with codomain S form the objects of a slice ∞-category $\boldsymbol{Grpd}_{/S}$, which behaves very much like a slice category in ordinary category theory. (We should mention here that since we work model-independently, when we refer to $\boldsymbol{Grpd}_{/S}$ we refer to an ∞-category determined up to equivalence. In contrast, [21] and [26] often refer to two different specific models in the category of simplicial sets with the Joyal model structure, which while of course equivalent, have different technical advantages.)

Pullback along a morphism $f : T \to S$ defines a functor $f^* : \boldsymbol{Grpd}_{/S} \to \boldsymbol{Grpd}_{/T}$. This functor is right adjoint to the functor $f_! : \boldsymbol{Grpd}_{/T} \to \boldsymbol{Grpd}_{/S}$ given by post-composing with f. The following Beck–Chevalley rule (push-pull formula) [17] holds for ∞-groupoids: given a pullback square

$$
\begin{array}{ccc}
\cdot & \xrightarrow{\ f\ } & \cdot \\
{\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\
\cdot & \xrightarrow[\ g\ ]{} & \cdot
\end{array}
$$

there is a canonical equivalence of functors

$$
(1) \qquad\qquad\qquad p_! \circ f^* \simeq g^* \circ q_!.
$$

1.2. Defining ∞-categories and sub-∞-categories. In this work we are concerned in particular with defining certain ∞-categories, a task often different in nature than that of defining ordinary categories: while in ordinary category theory one can define a category by saying what the objects and the arrows are (and how they compose), this from-scratch approach is more difficult for ∞-categories, as one would have to specify the simplices in all dimensions and verify the filler condition (that is, describe the ∞-category as a quasi-category). In practice, ∞-categories are constructed from existing ones by general constructions that automatically guarantee that the result is again an ∞-category, although the construction typically uses universal properties in such a way that the resulting ∞-category is only defined up to equivalence. To specify a sub-∞-category of an ∞-category $\mathcal{C}$, it suffices to specify a subcategory of the homotopy category of $\mathcal{C}$ (i.e. the category whose hom sets are π_0 of the mapping spaces of $\mathcal{C}$), and then pull back along the components functor. What this amounts to in practice is to specify the objects (closed under equivalences) and specifying for each pair of objects x, y a full sub-∞-groupoid of the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$, also closed under equivalences, and closed under composition.

We will use the terms subcategory and subgroupoid rather than the more clumsy 'sub-∞-category' and 'sub-∞-groupoid'.

1.3. Fundamental equivalence. Fundamental to many constructions and arguments in this work is the canonical equivalence

$$\mathbf{Grpd}_{/S} \simeq \mathbf{Grpd}^S$$

which is the homotopy version of the equivalence $\mathbf{Set}_{/S} \simeq \mathbf{Set}^S$ (for S a set), expressing the two ways of encoding a family of sets $\{X_s \mid s \in S\}$: either regarding the members of the family as the fibres of a map $X \to S$, or as a parametrisation of sets $S \to \mathbf{Set}$. To an object $X \to S$ one associates the functor $S^{\mathrm{op}} \to \mathbf{Grpd}$ sending $s \in S$ to the ∞-groupoid X_s. The other direction is the Grothendieck construction, which works as follows: to any presheaf $F : S^{\mathrm{op}} \to \mathbf{Grpd}$, which sits over the terminal presheaf $*$, one associates the object $\mathrm{colim}(F) \to \mathrm{colim}(*)$. It remains to observe that $\mathrm{colim}(*)$ is equivalent to S itself. More formally, the Grothendieck construction equivalence is a consequence of a finer result, namely Lurie's straightening theorem ([26, Theorem 2.1.2.2]), as has also been observed in [2, Remark 2.6]. Lurie constructs a Quillen equivalence between the category of right fibrations over S and the category of (strict) simplicial presheaves on $\mathfrak{C}[S]$. Combining this result with the fact that simplicial presheaves on $\mathfrak{C}[S]$ is a model for the functor ∞-category $\mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Grpd})$ (see [26], Proposition 5.1.1.1), the Grothendieck construction equivalence follows. Note that when S is just an ∞-groupoid (i.e. a Kan complex), $X \to S$ is a right fibration if and only if X itself is an ∞-groupoid. Hence altogether $\mathbf{Grpd}_{/S} \simeq \mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Grpd})$, and since S^{op} is canonically equivalent to S (since it is just an ∞-groupoid), this establishes the fundamental equivalence from this fancier viewpoint.

2. Homotopy linear algebra without finiteness conditions

2.1. Scalar multiplication and homotopy sums. The 'lowershriek' operation

$$f_! : \mathbf{Grpd}_{/I} \to \mathbf{Grpd}_{/J}$$

along a map $f : I \to J$ has two special cases, which play the role of scalar multiplication (tensoring with an ∞-groupoid) and vector addition (homotopy sums):

The ∞-category $\mathbf{Grpd}_{/I}$ is *tensored* over $\mathbf{Grpd}$. Given $g : X \to I$ in $\mathbf{Grpd}_{/I}$ then for any $S \in \mathbf{Grpd}$ we consider the projection $p_S : S \times X \to X$ in $\mathbf{Grpd}_{/X}$ and put

$$S \otimes g \quad := \quad g_!(p_S) : S \times X \to I \text{ in } \mathbf{Grpd}_{/I}.$$

It also has *homotopy sums*, by which we mean colimits indexed by an ∞-groupoid. The colimit of a functor $F : B \to \mathbf{Grpd}_{/I}$ is a special case of the lowershriek. Namely, the functor F corresponds by adjunction to an object $g : X \to B \times I$ in $\mathbf{Grpd}_{/B \times I}$, and we have

$$\mathrm{colim}(F) = p_!(g)$$

where $p : B \times I \to I$ is the projection. We interpret this as the homotopy sum of the family $g : X \to B \times I$ with members

$$g_b : X_b \longrightarrow \{b\} \times I = I,$$

and we denote the homotopy sum by an integral sign:

$$(2) \qquad \int^{b \in B} g_b := p_! g \quad \text{in } \mathbf{Grpd}_{/I}.$$

(The use of an integral sign, with superscript, is standard notation for colimits that arise as coends [30].)

Example 2.2. With $I = 1$, this gives the important formula

$$\int^{b \in B} X_b = X,$$

expressing the total space of $X \to B$ as the homotopy sum of its fibres.

Using the above, we can define the B-indexed *linear combination* of a family of vectors $g : X \to B \times I$ and scalars $f : S \to B$,

$$\int^{b \in B} S_b \otimes g_b = p_!(g_!(f')) \; : S \times_B X \to I \quad \text{in } \mathbf{Grpd}_{/I},$$

as illustrated in the first row of the following diagram

(3)

$$
\begin{array}{c}
S \times_B X \xrightarrow{\;f'\;} X \xrightarrow{\;\;pg\;\;} I \\
\end{array}
$$

Note that the members of the family $g_!(f')$ are just $(g_!(f'))_b = S_b \otimes g_b$.

2.3. Basis. In $\mathbf{Grpd}_{/S}$, the names $\ulcorner s \urcorner : 1 \to S$ play the role of a basis. Every object $X \to S$ can be written uniquely as a linear combination of basis elements; or, by allowing repetition of the basis elements instead of scalar multiplication, as a homotopy sum of basis elements:

Lemma 2.4. *For any $f : S \to B$ in $\mathbf{Grpd}_{/B}$ we have*

$$f = \int^{s \in S} \ulcorner f(s) \urcorner = \int^{b \in B} S_b \otimes \ulcorner b \urcorner.$$

Proof. The first equality is an example of the definition of homotopy sum (2), applied to the family $S \xrightarrow{(\mathrm{id}, f)} S \times B$ with members $\ulcorner f(s) \urcorner : 1 = S_s \to \{s\} \times B = B$. For the final expression, consider the family $g : B \xrightarrow{(\mathrm{id}, \mathrm{id})} B \times B$ with members the names $\ulcorner b \urcorner$, and the scalars given by $f : S \to B$ itself. Then calculating the linear combination $\int^{b \in B} S_b \otimes \ulcorner b \urcorner$ by (3) gives just f, since pg and qg are the identity. $\qquad\square$

The name $\ulcorner b \urcorner : 1 \to B$ corresponds under the Grothendieck construction to the representable functor

$$
\begin{array}{rcl}
B & \longrightarrow & \mathbf{Grpd} \\
x & \longmapsto & \mathrm{Map}(b, x).
\end{array}
$$

Thus, interpreted in the presheaf category $\mathbf{Grpd}^B$, the Lemma is the standard result expressing any presheaf as a colimit of representables.

Proposition 2.5. $\mathbf{Grpd}_{/S}$ *is the homotopy-sum completion of* S. *Precisely, for* $\mathcal{C}$ *an* ∞-*category admitting homotopy sums, precomposition with the Yoneda embedding* $S \to \mathbf{Grpd}_{/S}$ *induces an equivalence of* ∞-*categories*

$$\mathrm{Fun}^{\int}(\mathbf{Grpd}_{/S}, \mathcal{C}) \overset{\sim}{\to} \mathrm{Fun}(S, \mathcal{C}),$$

where the functor category on the left consists of homotopy-sum preserving functors.

Proof. Since every object in $\mathbf{Grpd}_{/S}$ can be written as a homotopy sum of names, to preserve homotopy sums is equivalent to preserving all colimits, so the natural inclusion $\mathrm{Fun}^{\mathrm{colim}}(\mathbf{Grpd}_{/S}, \mathcal{C}) \to \mathrm{Fun}^{\int}(\mathbf{Grpd}_{/S}, \mathcal{C})$ is an equivalence. It is therefore enough to establish the equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\mathbf{Grpd}_{/S}, \mathcal{C}) \overset{\sim}{\to} \mathrm{Fun}(S, \mathcal{C}).$$

In the case where $\mathcal{C}$ is cocomplete, this is true since $\mathbf{Grpd}_{/S} \simeq \mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Grpd})$ is the colimit completion of S. The proof of this statement (Lurie [26], Theorem 5.1.5.6) goes as follows: it is enough to prove that left Kan extension of any functor $S \to \mathcal{C}$ along the Yoneda embedding exists and preserves colimits. Existence follows from [26, Lemma 4.3.2.13] since $\mathcal{C}$ is assumed cocomplete, and the fact that left Kan extensions preserve colimits [26, Lemma 5.1.5.5 (1)] is independent of the cocompleteness of $\mathcal{C}$. In our case $\mathcal{C}$ is not assumed to be cocomplete but only to admit homotopy sums. But since S is just an ∞-groupoid in our case, this is enough to apply Lemma 4.3.2.13 of [26] to guarantee the existence of the left Kan extension. $\square$

2.6. Linear functors. A span

$$I \overset{p}{\leftarrow} M \overset{q}{\to} J$$

defines a *linear functor*

(4) $$\mathbf{Grpd}_{/I} \overset{p^*}{\longrightarrow} \mathbf{Grpd}_{/M} \overset{q_!}{\longrightarrow} \mathbf{Grpd}_{/J}.$$

Lemma 2.7. *Linear functors preserve linear combinations,*

$$L\left(\int^{b \in B} S_b \otimes g_b\right) = \int^{b \in B} S_b \otimes L(g_b).$$

Proof. This follows from the Beck–Chevalley rule (1), since linear combinations (that is, scalar multiplication and homotopy sums) are colimits defined using lowershriek operations (see 2.1). $\square$

2.8. Matrices. Coming back to the span

$$I \overset{p}{\longleftarrow} M \overset{q}{\longrightarrow} J$$

and the linear functor

$$q_! p^* : \mathbf{Grpd}_{/I} \longrightarrow \mathbf{Grpd}_{/J},$$

consider an element $\ulcorner i \urcorner : 1 \to I$. Then we have, by Lemma 2.4,

$$q_! p^* \ulcorner i \urcorner = (M_i \to J) = \int^{j \in J} M_{i,j} \otimes \ulcorner j \urcorner$$

For a more general element $f : X \to I$ we have $f = \int^i X_i \otimes \ulcorner i \urcorner$ and so by homotopy linearity 2.7

$$q_! p^* f = \int^{i,j} X_i \otimes M_{i,j} \otimes \ulcorner j \urcorner.$$

2.9. The symmetric monoidal closed ∞-category $\boldsymbol{Pr}^{\mathrm{L}}$. There is an ∞-category $\boldsymbol{Pr}^{\mathrm{L}}$, defined and studied in [26, Section 5.5.3], whose objects are the presentable ∞-categories, and whose morphisms are the left adjoint functors, or equivalently colimit-preserving functors. The ∞-category $\boldsymbol{Pr}^{\mathrm{L}}$ has an 'internal hom' (see [26, 5.5.3.8]): if $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞-categories, $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$, defined as the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the colimit-preserving functors, is again presentable. The mapping spaces in $\boldsymbol{Pr}^{\mathrm{L}}$ are $\mathrm{Map}_{\boldsymbol{Pr}^{\mathrm{L}}}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})^{\mathrm{eq}}$. Finally, $\boldsymbol{Pr}^{\mathrm{L}}$ has a canonical symmetric monoidal structure, left adjoint to the closed structure. See Lurie [28], subsection 4.8.1, and in particular 4.8.1.14 and 4.8.1.17. The tensor product can be characterised as universal recipient of functors in two variables that preserve colimits in each variable, and we have an evaluation functor

$$\mathcal{C} \otimes \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$$

which exhibits $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ as an exponential of $\mathcal{D}$ by $\mathcal{C}$.

This tensor product has an easy description in the case of presheaf categories (cf. [28, 4.8.1.12]): if $\mathcal{C} = \mathcal{P}(\mathcal{C}_0)$ and $\mathcal{D} = \mathcal{P}(\mathcal{D}_0)$ for small ∞-categories $\mathcal{C}_0$ and $\mathcal{D}_0$, then we have

$$(5) \qquad\qquad \mathcal{P}(\mathcal{C}_0) \otimes \mathcal{P}(\mathcal{D}_0) \simeq \mathcal{P}(\mathcal{C}_0 \times \mathcal{D}_0).$$

2.10. The ∞-category $\boldsymbol{LIN}$. We define $\boldsymbol{LIN}$ to be the full subcategory of $\boldsymbol{Pr}^{\mathrm{L}}$ spanned by the slices $\boldsymbol{Grpd}_{/S}$, for S a locally finite ∞-groupoid. We call the functors *linear*. The mapping spaces in $\boldsymbol{LIN}$ are

$$
\begin{aligned}
\boldsymbol{LIN}(\boldsymbol{Grpd}_{/I}, \boldsymbol{Grpd}_{/J}) &= \mathrm{Fun}^{\mathrm{L}}(\boldsymbol{Grpd}_{/I}, \boldsymbol{Grpd}_{/J})^{\mathrm{eq}} \\
&\simeq \mathrm{Fun}^{\mathrm{L}}(\boldsymbol{Grpd}^I, \boldsymbol{Grpd}^J)^{\mathrm{eq}} \\
&\simeq \mathrm{Fun}(I, \boldsymbol{Grpd}^J)^{\mathrm{eq}} \\
&\simeq (\boldsymbol{Grpd}^{I \times J})^{\mathrm{eq}} \\
&\simeq (\boldsymbol{Grpd}_{/I \times J})^{\mathrm{eq}}.
\end{aligned}
$$

This shows in particular that the linear functors are given by spans. Concretely, tracing through the chain of equivalences, a span defines a left adjoint functor as described above in 2.6. Composition in $\mathbf{LIN}$ is given by composing spans, i.e. taking a pullback. This amounts to the Beck–Chevalley condition.

The ∞-category $\mathbf{LIN}$ inherits a symmetric monoidal closed structure from $\mathbf{Pr}^{\mathrm{L}}$. For the 'internal hom':

$$
\begin{aligned}
\underline{\mathbf{LIN}}(\mathbf{Grpd}_{/I}, \mathbf{Grpd}_{/J}) \; &:= \; \mathrm{Fun}^{\mathrm{L}}(\mathbf{Grpd}_{/I}, \mathbf{Grpd}_{/J}) \\
&\simeq \; \mathrm{Fun}(I, \mathbf{Grpd}^{J}) \\
&\simeq \; \mathrm{Fun}(I \times J, \mathbf{Grpd}) \\
&\simeq \; \mathbf{Grpd}_{/I \times J}.
\end{aligned}
$$

Also the tensor product restricts, and we have the convenient formula

$$
\mathbf{Grpd}_{/I} \otimes \mathbf{Grpd}_{/J} = \mathbf{Grpd}_{/I \times J}
$$

with neutral object $\mathbf{Grpd}$. This follows from formula (5) combined with the fundamental equivalence $\mathbf{Grpd}_{/S} \simeq \mathbf{Grpd}^{S}$.

Clearly we have

$$
\mathbf{LIN}(\mathbf{Grpd}_{/I} \otimes \mathbf{Grpd}_{/J}, \mathbf{Grpd}_{/K}) \simeq \mathbf{LIN}(\mathbf{Grpd}_{/I}, \underline{\mathbf{LIN}}(\mathbf{Grpd}_{/J}, \mathbf{Grpd}_{/K}))
$$

as both spaces are naturally equivalent to $(\mathbf{Grpd}_{/I \times J \times K})^{\mathrm{eq}}$.

2.11. The linear dual. 'Homming' into the neutral object defines a contravariant autoequivalence of $\mathbf{LIN}$:

$$
\begin{aligned}
\mathbf{LIN} \; &\longrightarrow \; \mathbf{LIN}^{\mathrm{op}} \\
\mathbf{Grpd}_{/S} \; &\longmapsto \; \underline{\mathbf{LIN}}(\mathbf{Grpd}_{/S}, \mathbf{Grpd}) \simeq \mathbf{Grpd}_{/S} \simeq \mathbf{Grpd}^{S}.
\end{aligned}
$$

Here there right-hand side should be considered the dual of $\mathbf{Grpd}_{/S}$. (Since our vector spaces are fully coordinatised, the difference between a vector space and its dual is easily blurred. We will see a clearer difference when we come to the finiteness conditions, in which situation the dual of a 'vector space' $\mathbf{grpd}_{/S}$ is $\mathbf{grpd}^{S}$ which should rather be thought of as a profinite-dimensional vector space.)

For a span $S \xleftarrow{p} M \xrightarrow{q} T$ defining a linear functor $F := q_{!} \circ p^{*} : \mathbf{Grpd}_{/S} \to \mathbf{Grpd}_{/T}$, the same span read backwards defines the *dual functor* $F^{\vee} := p_{!} \circ q^{*} : \mathbf{Grpd}^{T} \to \mathbf{Grpd}^{S}$. Under the fundamental equivalence, this can also be considered a linear functor $F^{t} : \mathbf{Grpd}_{/T} \to \mathbf{Grpd}_{/S}$, called the *transpose* of F.

2.12. Remark. It is clear that there is actually an $(\infty, 2)$-category in play here, with the $\underline{\mathbf{LIN}}(\mathbf{Grpd}_{/S}, \mathbf{Grpd}_{/T})$ as hom ∞-categories. This can be described as a Rezk-category object in the 'distributor' $\mathbf{Cat}$, following the work of Barwick and Lurie [29]. Explicitly, let Λ_{k} denote the full subcategory of $\Delta_{k} \times \Delta_{k}$ consisting of the pairs (i, j) with $i + j \leq k$. These are the shapes of diagrams of k composable spans. They form a cosimplicial category. Define Sp_{k} to be the full subcategory of $\mathrm{Fun}(\Lambda_{k}, \mathbf{Grpd})$ consisting of those diagrams $S : \Lambda_{k} \to \mathbf{Grpd}$ for which for all $i' < i$

and $j' < j$ (with $i + j \leq k$) the square

$$
\begin{array}{ccc}
S_{i',j'} & \longrightarrow & S_{i,j'} \\
\downarrow & \lrcorner & \downarrow \\
S_{i',j} & \longrightarrow & S_{i,j}
\end{array}
$$

is a pullback. Then we claim that

$$
\begin{aligned}
\Delta^{\mathrm{op}} & \longrightarrow & \mathbf{Cat} \\
[k] & \longmapsto & \mathrm{Sp}_k
\end{aligned}
$$

defines a Rezk-category object in $\mathbf{Cat}$ corresponding to $\underline{\mathbf{LIN}}$. We leave the claim unproved, as the result is not necessary for our purposes.

3. Cardinality of finite ∞-groupoids

3.1. Finite ∞-groupoids. An ∞-groupoid X is called *locally finite* if at each base point x the homotopy groups $\pi_i(X, x)$ are finite for $i \geq 1$ and are trivial for i sufficiently large. An ∞-groupoid is called *finite* if it is locally finite and has finitely many components. An example of a non locally finite ∞-groupoid is $B\mathbb{Z}$.

Let $\mathbf{grpd} \subset \mathbf{Grpd}$ be the full subcategory spanned by the finite ∞-groupoids. For S any ∞-groupoid, let $\mathbf{grpd}_{/S}$ be the 'comma ∞-category' defined by the following pullback diagram of ∞-categories:

$$
\begin{array}{ccc}
\mathbf{grpd}_{/S} & \longrightarrow & \mathbf{Grpd}_{/S} \\
\downarrow & \lrcorner & \downarrow \\
\mathbf{grpd} & \longrightarrow & \mathbf{Grpd}.
\end{array}
$$

3.2. Cardinality. [3] The *(homotopy) cardinality* of a finite ∞-groupoid X is the nonnegative rational number given by the formula

$$
|X| := \sum_{x \in \pi_0 X} \prod_{i>0} |\pi_i(X, x)|^{(-1)^i}.
$$

Here the norm signs on the right refer to order of homotopy groups.

If X is a 1-groupoid, that is, an ∞-groupoid having trivial homotopy groups $\pi_i(X) = 0$ for $i > 1$, its cardinality is

$$
|X| = \sum_{x \in \pi_0 X} \frac{1}{|\mathrm{Aut}_X(x)|}.
$$

The notion and basic properties of homotopy cardinality have been around for a long time. See in particular Baez–Dolan [3] and also Toën [32]. The first printed reference we know of is Quinn [31, p.340].

Lemma 3.3. *A finite sum of finite ∞-groupoids is again finite, and cardinality is compatible with finite sums:*

$$
\left| \sum_{i=1}^{n} X_i \right| = \sum_{i=1}^{n} |X_i|.
$$

This is clear from the definition.

Lemma 3.4. *Suppose B is connected. Given a fibre sequence*

$$
\begin{array}{ccc}
F & \longrightarrow & E \\
\downarrow & \lrcorner & \downarrow \\
1 & \longrightarrow & B,
\end{array}
$$

if two of the three spaces are finite then so is the third, and in that case

$$|E| \;=\; |F| \, |B| .$$

Proof. This follows from the homotopy long exact sequence of a fibre sequence. $\square$

For $b \in B$, we denote by $B_{[b]}$ the connected component of B containing b. Thus an ∞-groupoid B is locally finite if and only if each connected component $B_{[b]}$ is finite.

Lemma 3.5. *Suppose B locally finite. Given a map $E \to B$, then E is finite if and only if all fibres E_b are finite, and are nonempty for only finitely many $b \in \pi_0 B$. In this situation,*

$$|E| \;=\; \sum_{b \in \pi_0(B)} |E_b| \, \big|B_{[b]}\big| .$$

Proof. Write E as the sum of the full fibres $E_{[b]}$, and apply Lemma 3.4 to the fibrations $E_b \to E_{[b]} \to B_{[b]}$ for each $b \in \pi_0(B)$. Finally sum (3.3) over those $b \in \pi_0 B$ with non-empty E_b. $\square$

Corollary 3.6. *Cardinality preserves (finite) products.*

Proof. Apply the Lemma 3.5 to a projection. $\square$

3.7. Notation. Given any ∞-groupoid B and a function $q : \pi_0 B \to \mathbb{Q}$, we write

$$\int^{b \in B} q_b \;:=\; \sum_{b \in \pi_0 B} q_b \, \big|B_{[b]}\big|$$

if the sum is finite. Then the previous lemma says

$$|E| \;=\; \int^{b \in B} |E_b|$$

for any finite ∞-groupoid E and a map $E \to B$. Two important special cases are given by fibre products and loop spaces:

Lemma 3.8. *In the situation of a pullback*

$$
\begin{array}{ccc}
X \times_B Y & \longrightarrow & X \times Y \\
\downarrow & \lrcorner & \downarrow \\
B & \xrightarrow{\;diag\;} & B \times B,
\end{array}
$$

if X and Y are finite, and B is locally finite, then $X \times_B Y$ is finite and

$$|X \times_B Y| \;=\; \int^{b \in B} |X_b|\,|Y_b|\,.$$

Proposition 3.9. *The ∞-category **grpd** is closed under finite limits.*

Proof. It is closed under pullbacks by the previous lemma, and it also contains the terminal object, hence it is closed under all finite limits. $\square$

Lemma 3.10. *In the situation of a loop space*

$$
\begin{array}{ccc}
\Omega(B,b) & \longrightarrow & 1 \\
\downarrow & \lrcorner & \downarrow {}^{\ulcorner b \urcorner} \\
1 & \xrightarrow[{\ulcorner b \urcorner}]{} & B_{[b]}\,.
\end{array}
$$

we have that B is locally finite if and only if each $\Omega(B,b)$ is finite, and in that case

$$|\Omega(B,b)| \cdot \left| B_{[b]} \right| = 1.$$

3.11. Finite maps. We say that a map $p : E \to B$ is *finite* if any pullback to a finite base X has finite total space X', as in the diagram

(6)
$$
\begin{array}{ccc}
X' & \longrightarrow & E \\
\downarrow & \lrcorner & \downarrow {}^{p} \\
X & \xrightarrow[c]{} & B.
\end{array}
$$

The following two results are immediate.

Lemma 3.12. *If B is finite and $E \to B$ is finite, then E is finite.* $\square$

Lemma 3.13. *Finite maps are stable under base change.* $\square$

Lemma 3.14. *A map $E \to B$ is finite if and only if each fibre E_b is finite.*

Proof. The 'only if' implication is a special case of 3.13. If $p : E \to B$ has finite fibres, then also the map $X' \to X$ in the pullback diagram (6) has finite fibres $X'_x = E_{c(x)}$. But since also X is finite, Lemma 3.5 then implies that X' is finite. Hence p is finite. $\square$

Lemma 3.15. *Suppose $p : E \to B$ has locally finite base.*
  (1) *If p is finite then E is locally finite.*
  (2) *If E is finite then p is finite.*

Proof. A full fibre $E_{[b]}$ of p is finite if and only if E_b is, by Lemma 3.4. If each full fibre $E_{[b]}$ is finite, then each component $E_{[e]}$ is, and if E is finite then each full fibre is. $\square$

Lemma 3.16. *B is locally finite iff each name $1 \to B$ is a finite map.*

Proposition 3.17. *The ∞-category **grpd** is cartesian closed.*

Proof. We already know that **Grpd** is cartesian closed. We need to show that for X and Y finite ∞-groupoids, the mapping space $\mathrm{Map}(X, Y)$ is again finite. We can assume X and Y connected: indeed, if we write them as sums of their connected components, $X = \sum X_i$ and $Y = \sum Y_j$, then we have

$$\mathrm{Map}(X, Y) = \mathrm{Map}(\sum X_i, Y) = \prod_i \mathrm{Map}(X_i, Y) = \prod_i \sum_j \mathrm{Map}(X_i, Y_j)$$

Since these are finite products and sums, if we can prove that each $\mathrm{Map}(X_i, Y_j)$ is finite, then we are done. Since Y is finite, $\mathrm{Map}(S^k, Y)$ is finite for all $k \geq 0$, and there is $r \geq 0$ such that $\mathrm{Map}(S^k, Y) = *$ for all $k \geq r$. This is to say that Y is r-truncated. On the other hand, since X is finite, it has the homotopy type of a CW complex with finitely many cells in each dimension. Write

$$X = \underset{i \in I}{\mathrm{colim}}\, E_i$$

for its realisation as a cell complex. Write $X' = \mathrm{colim}_{i \in I'} E_i$ for the colimit obtained by the same prescription but omitting all cells of dimension $> r$; this is now a finite colimit, and the comparison map $X \to X'$ is r-connected. Since Y is r-truncated, we have

$$\mathrm{Map}(X', Y) \overset{\sim}{\to} \mathrm{Map}(X, Y),$$

and the first space is finite: indeed,

$$\mathrm{Map}(X', Y) = \mathrm{Map}(\underset{i \in I'}{\mathrm{colim}}\, E_i, Y) = \underset{i \in I'}{\lim}\, \mathrm{Map}(E_i, Y)$$

is a finite limit of finite spaces, hence is finite by Proposition 3.9. $\square$

Theorem 3.18. *For each locally finite ∞-groupoid S, the comma ∞-category $\mathbf{grpd}_{/S}$ is cartesian closed.*

Proof. This is essentially a corollary of Proposition 3.17 and the fact that the bigger ∞-category $\mathbf{Grpd}_{/S}$ is cartesian closed. We just need to check that the internal mapping object in $\mathbf{Grpd}_{/S}$ actually belongs to $\mathbf{grpd}_{/S}$. Given $a : A \to S$ and $b : B \to S$, the internal mapping object is

$$\underline{\mathrm{Map}}_{/S}(a, b) \to S$$

given fibrewise by

$$\underline{\mathrm{Map}}_{/S}(a, b)_s = \mathrm{Map}(A_s, B_s)$$

Since A_s and B_s are finite spaces, also the mapping space is finite, by 3.17. $\square$

Corollary 3.19. *The ∞-category $\mathbf{grpd}$ is locally cartesian closed.*

4. Finiteness conditions on ∞-groupoid slices

In this section, after some motivation and background from linear algebra, we first explain the finiteness conditions imposed on slice categories in order to model vector spaces and profinite-dimensional vector spaces. Then afterwards we assemble all this into ∞-categories using more formal constructions.

4.1. Duality in linear algebra. There is a fundamental duality

$$\textbf{\textit{Vect}} \simeq \underrightarrow{\textbf{\textit{vect}}}^{\,\mathrm{op}}$$

between vector spaces and profinite-dimensional vector spaces: given any vector space V, the linear dual V^* is a profinite-dimensional vector space, and conversely, given a profinite-dimensional vector space, its continuous dual is a vector space. This equivalence is a formal consequence of the observation that the category **vect** of finite-dimensional vector spaces is self-dual: $\textbf{vect} \simeq \textbf{vect}^{\mathrm{op}}$, and the fact that $\textbf{\textit{Vect}} = \underrightarrow{\textbf{vect}}$, the ind completion of **vect**.

In the fully coordinatised situation typical to algebraic combinatorics, the vector space arises from a set S (typically an infinite set of isoclasses of combinatorial objects): the vector space is then

$$V = \mathbb{Q}_S = \left\{ \sum_{s \in S} c_s\, \delta_s \ : \ c_s \in \mathbb{Q} \text{ almost all zero} \right\},$$

with basis the symbols δ_s for each $s \in S$. The linear dual is the function space $V^* = \mathbb{Q}^S$, with canonical pro-basis consisting of the functions δ^s, taking the value 1 on s and 0 elsewhere.

Vectors in $\mathbb{Q}_S$ are finite linear combinations of the δ_s, and we represent a vector as an infinite column vector $\vec{v}$ with only finitely many non-zero entries. A linear map $f : \mathbb{Q}_S \to \mathbb{Q}_T$ is given by matrix multiplication

$$\vec{v} \mapsto A \cdot \vec{v}.$$

for A an infinite 2-dimensional matrix with T-many rows and S-many columns, and with the crucial property that it is *column finite*: in each column there are only finitely many non-zero entries. More generally, the matrix multiplication of two column-finite matrices makes sense and is again a column-finite matrix. The identity matrix is clearly column finite. A basis element δ_s is identified with the column vector all of whose entries are zero, except the one of index s.

On the other hand, elements in the function space $\mathbb{Q}^S$ are represented as infinite row vectors. The continuous linear map $\mathbb{Q}^T \to \mathbb{Q}^S$, dual to the linear map f, is represented by the *same matrix A*, but viewed now as sending a row vector $\vec{w}$ (indexed by T) to the matrix product $\vec{w} \cdot A$. Again the fact that A is column finite ensures that this matrix product is well defined.

There is a canonical perfect pairing

$$\begin{array}{ccc} \mathbb{Q}_S \times \mathbb{Q}^S & \longrightarrow & \mathbb{Q} \\ (\vec{v}, f) & \longmapsto & f(\vec{v}) \end{array}$$

given by evaluation. In matrix terms, it is just a matter of multiplying $f \cdot \vec{v}$.

This duality has a very neat description in homotopy linear algebra. While the vector space $\mathbb{Q}_{\pi_0 S}$ is modelled by the ∞-category $\textbf{\textit{grpd}}_{/S}$, the function space $\mathbb{Q}^{\pi_0 S}$ is modelled by the ∞-category $\textbf{\textit{grpd}}^S$.

The classical duality results from taking cardinality of a duality on the categorical level, that we proceed to explain. For the most elegant definition of cardinality we

first need to introduce the objective versions of **Vect** and $\underleftarrow{\textbf{vect}}$. These will be ∞-categories $\underrightarrow{\textbf{lin}}$ whose objects are of the form $\textbf{grpd}_{/S}$, and $\underleftarrow{\textbf{lin}}$ whose objects are of the form $\textbf{grpd}^S$.

We shall need also the following variations. For S a locally finite ∞-groupoid, we are concerned with the following ∞-categories.

- $\textbf{grpd}_{/S}$: the slice ∞-category of morphisms $\sigma \to S$, with σ finite.
- $\textbf{grpd}^S$: the full subcategory of $\textbf{Grpd}^S$ spanned by the presheaves $S \to \textbf{Grpd}$ whose images lie in $\textbf{grpd}$.
- $\textbf{Grpd}_{/S}^{\text{rel.fin.}}$: the full subcategory of $\textbf{Grpd}_{/S}$ spanned by the finite maps $p: X \to S$.
- $\textbf{grpd}_{\text{fin.sup.}}^S$: the full subcategory of $\textbf{Grpd}^S$ spanned by presheaves with finite values and finite support. By the support of a presheaf $F: S \to \textbf{Grpd}$ we mean the full subgroupoid of S spanned by the objects x for which $F(x) \neq \emptyset$.

Proposition 4.2. *The fundamental equivalence $\textbf{Grpd}_{/S} \simeq \textbf{Grpd}^S$ restricts to equivalences*

$$\textbf{Grpd}_{/S}^{\text{rel.fin.}} \simeq \textbf{grpd}^S \qquad and \qquad \textbf{grpd}_{/S} \simeq \textbf{grpd}_{\text{fin.sup.}}^S$$

Proof. The inclusions $\textbf{Grpd}_{/S}^{\text{rel.fin.}} \subset \textbf{Grpd}_{/S}$ and $\textbf{grpd}^S \subset \textbf{Grpd}^S$ are both full, and the objects characterising them correspond to each other under the fundamental equivalence because of Lemma 3.14. Similarly, the inclusions $\textbf{grpd}_{/S} \subset \textbf{Grpd}_{/S}$ and $\textbf{grpd}_{\text{fin.sup.}}^S \subset \textbf{Grpd}^S$ are both full, and the objects characterising them correspond to each other under the fundamental equivalence, this time in virtue of Lemma 3.5. $\square$

Proposition 4.3. *For a span $S \xleftarrow{p} M \xrightarrow{q} T$ of locally finite ∞-groupoids, the following are equivalent:*

(1) *p is finite*

(2) *The linear functor $F := q_! \circ p^* : \textbf{Grpd}_{/S} \to \textbf{Grpd}_{/T}$ restricts to*

$$\textbf{grpd}_{/S} \xrightarrow{p^*} \textbf{grpd}_{/M} \xrightarrow{q_!} \textbf{grpd}_{/T}$$

(3) *The transpose $F^t := p_! \circ q^* : \textbf{Grpd}_{/T} \to \textbf{Grpd}_{/S}$ restricts to*

$$\textbf{Grpd}_{/T}^{\text{rel.fin.}} \xrightarrow{q^*} \textbf{Grpd}_{/M}^{\text{rel.fin.}} \xrightarrow{p_!} \textbf{Grpd}_{/S}^{\text{rel.fin.}}$$

(4) *The dual functor $F^\vee : \textbf{Grpd}^T \to \textbf{Grpd}^S$ restricts to*

$$\textbf{grpd}^T \to \textbf{grpd}^S$$

(5) *The dual of the transpose, $F^{t\vee} : \textbf{Grpd}^S \to \textbf{Grpd}^T$ restricts to*

$$\textbf{grpd}_{\text{fin.sup.}}^S \to \textbf{grpd}_{\text{fin.sup.}}^T$$

Proof. The biimplications (1)$\Leftrightarrow$(2) and (1)$\Leftrightarrow$(3) follow from the definition of finite map. The biimplications (2)$\Leftrightarrow$(5) and (3)$\Leftrightarrow$(4) follow from the equivalences in Proposition 4.2. $\square$

4.4. Finite homotopy sums. The ∞-category $\mathbf{grpd}_{/S}$ has finite homotopy sums: for I finite and $F : I \to \mathbf{grpd}_{/S}$ we have $\operatorname{colim} F = p_!(X \to I \times S)$, where $p : I \times S \to S$ is the projection. A family $X \to I \times S$ comes from some $F : I \to \mathbf{grpd}_{/S}$ and admits a homotopy sum in $\mathbf{grpd}_{/S}$ when for each $i \in I$, the partial fibre X_i is finite. Since already I was assumed finite, this is equivalent to having X finite.

The following is the finite version of Proposition 2.5

Lemma 4.5. *The ∞-category $\mathbf{grpd}_{/S}$ is the finite-homotopy-sum completion of S.*

5. ∞-CATEGORIES OF LINEAR MAPS WITH INFINITE ∞-GROUPOID COEFFICIENTS

Our main interest is in the linear ∞-categories with finite ∞-groupoid coefficients, but it is technically simpler to introduce first the infinite-coefficients version of these ∞-categories, since they can be defined as subcategories in $\mathbf{LIN}$, and can be handled with the ease of presentable ∞-categories.

Recall that a span $(S \xleftarrow{p} M \xrightarrow{q} T)$ defines a linear functor

$$L : \mathbf{Grpd}_{/S} \xrightarrow{p^*} \mathbf{Grpd}_{/M} \xrightarrow{q_!} \mathbf{Grpd}_{/T}.$$

Let $\mathbf{Lin} \subset \mathbf{LIN}$ be the ∞-category whose objects are the slices $\mathbf{Grpd}_{/\sigma}$, with σ finite, and whose morphisms are those linear functors between them which preserve finite objects. Clearly these are given by the spans of the form $\sigma \leftarrow \mu \to \tau$ where σ, τ and μ are finite. Note that there are equivalences of ∞-categories $\mathbf{Grpd}_{/\sigma} \simeq \mathbf{Grpd}^{\sigma}$ for each finite σ.

From now on we adopt the blanket convention that Greek letters denote finite ∞-groupoids.

Let $\underrightarrow{\mathbf{Lin}}$ be the ∞-category whose objects are the slices $\mathbf{Grpd}_{/S}$ with S locally finite, and whose morphisms are the linear functors between them that preserve finite objects. These correspond to the spans of the form $S \xleftarrow{p} M \to T$ with p finite.

Let $\underleftarrow{\mathbf{Lin}}$ be the ∞-category whose objects are the presheaf categories $\mathbf{Grpd}^S$ with S locally finite, and whose morphisms are the continuous linear functors:

A linear functor $F : \mathbf{Grpd}^T \to \mathbf{Grpd}^S$ is called *continuous* when for all $\varepsilon \subset S$ there exists $\delta \subset T$ and a factorisation

$$
\begin{array}{ccc}
\mathbf{Grpd}^T & \longrightarrow & \mathbf{Grpd}^\delta \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F_\delta} \\
\mathbf{Grpd}^S & \longrightarrow & \mathbf{Grpd}^\varepsilon.
\end{array}
$$

Here we quantify over finite groupoids ε and δ with full inclusions into S and T; the horizontal maps are the projections of the canonical pro-structures.

Proposition 5.1. *For a linear functor $F : \mathbf{Grpd}^T \to \mathbf{Grpd}^S$ in $\mathbf{LIN}$, represented by a span*

$$S \xleftarrow{p} M \xrightarrow{q} T,$$

the following are equivalent.

(1) *The span is of finite type (i.e. p is a finite map).*
(2) *F is continuous.*

Proof. It is easy to see that if the span is of finite type then F is continuous: for any given finite $\varepsilon \subset S$ with inclusion j, the pullback μ is finite, and we can take δ to be the essential full image of the composite $q \circ m$:

$$(7) \qquad \begin{array}{ccccc} \varepsilon & \xleftarrow{\bar{p}} & \mu & \xrightarrow{\bar{q}} & \delta \\ {\scriptstyle j}\downarrow & {\scriptstyle \llcorner} & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle i} \\ S & \xleftarrow{p} & M & \xrightarrow{q} & T. \end{array}$$

Now by Beck-Chevalley,

$$j^* p_! q^* = \bar{p}_! m^* q^* = \bar{p}_! \bar{q}^* i^*$$

which is precisely the continuity condition.

Conversely, if the factorisation in the continuity diagram exists, let $\varepsilon \leftarrow \mu \to \delta$ be the span (of finite ∞-groupoids) representing f_{δ_ε}. Then we have the outer rectangle of the diagram (7) and an isomorphism

$$j^* p_! q^* = \bar{p}_! \bar{q}^* i^*$$

Now a standard argument implies the existence of m completing the diagram: namely take the pullback of j and p, with the effect of interchanging the order of upperstar and lowershriek. Now both linear maps are of the form upperstars-followed-by-lowershriek, and by uniqueness of this representation, the said pullback must agree with μ and in particular is finite. Since this is true for every ε, this is precisely to say that p is a finite map. $\qquad \square$

The continuity condition is precisely continuity for the profinite topology, as we proceed to explain. Every locally finite ∞-groupoid S is canonically the filtered colimit of its finite full subgroupoids:

$$S = \operatorname*{colim}_{\alpha \subset S} \alpha.$$

Similarly, $\mathbf{Grpd}^S$ is a cofiltered limit of ∞-categories $\mathbf{Grpd}^\alpha$:

$$\mathbf{Grpd}^S = \lim_{\alpha \subset S} \mathbf{Grpd}^\alpha.$$

This leads to the following 'categorical' description of the mapping spaces (compare SGA4 [1], Exp.1):

$$\underleftarrow{\mathbf{Lin}}(\mathbf{Grpd}^T, \mathbf{Grpd}^S) := \lim_{\varepsilon \subset S} \operatorname*{colim}_{\delta \subset T} \mathbf{Lin}(\mathbf{Grpd}^\delta, \mathbf{Grpd}^\varepsilon).$$

6. ∞-CATEGORIES OF LINEAR MAPS WITH FINITE ∞-GROUPOID COEFFICIENTS

6.1. The ∞-category *lin*. Let $\widehat{\mathbf{Cat}}$ denote the (very large) ∞-category of possibly large ∞-categories. We define ***lin*** to be the subcategory of $\widehat{\mathbf{Cat}}$ whose objects are those ∞-categories equivalent to $\mathbf{grpd}_{/\sigma}$ for some finite ∞-groupoid σ, and whose mapping spaces are the full subgroupoids of those of $\widehat{\mathbf{Cat}}$ given by the functors which are restrictions of functors in $\mathbf{Lin}(\mathbf{Grpd}_{/\sigma}, \mathbf{Grpd}_{/\tau})$. Note that the latter

mapping space was exactly defined as those linear functors in **LIN** that preserved finite objects. Hence, by construction there is an equivalence of mapping spaces

$$lin(grpd_{/\sigma}, grpd_{/\tau}) \simeq Lin(Grpd_{/\sigma}, Grpd_{/\tau}),$$

and in particular, the mapping spaces are given by spans of finite ∞-groupoids. The maps can also be described as those functors that preserve finite homotopy sums. By construction we have an equivalence of ∞-categories

$$lin \simeq Lin.$$

6.2. The ∞-category $\underrightarrow{lin}$. Analogously, we define $\underrightarrow{lin}$ to be the subcategory of $\widehat{\textbf{Cat}}$, whose objects are the ∞-categories equivalent to $grpd_{/S}$ for some locally finite ∞-groupoid S, and whose mapping spaces are the full subgroupoids of the mapping spaces of $\widehat{\textbf{Cat}}$ given by the functors that are restrictions of functors in $\textbf{Lin}(\textbf{Grpd}_{/S}, \textbf{Grpd}_{/T})$; in other words (by 4.3), they are the ∞-groupoids of spans of finite type. Again by construction we have

$$\underrightarrow{lin} \simeq \underrightarrow{Lin}.$$

6.3. ∞-categories of prolinear maps. We denote by $\underleftarrow{lin}$ the ∞-category whose objects are the ∞-categories $grpd^S$, where S is locally finite, and whose morphisms are restrictions of continuous linear functors. We have seen that the mapping spaces are given by spans of finite type:

$$\underleftarrow{lin}(grpd^T, grpd^S) \;\; = \;\; \left\{ (T \xleftarrow{q} M \xrightarrow{p} S) \; : \; p \text{ finite} \right\}.$$

As in the ind case we have

$$\underleftarrow{lin} \simeq \underleftarrow{Lin},$$

and by combining the previous results we also find

$$\underleftarrow{lin}(grpd^T, grpd^S) := \lim_{\varepsilon \subset S} \operatorname*{colim}_{\delta \subset T} lin(grpd^\delta, grpd^\varepsilon).$$

6.4. Mapping ∞-categories. Just as $\widehat{\textbf{Cat}}$ has internal mapping ∞-categories, whose maximal subgroupoids are the mapping spaces, we also have internal mapping ∞-categories in **lin**, denoted $\underline{lin}$:

$$\underline{lin}(grpd_{/\sigma}, grpd_{/\tau}) \simeq grpd_{/\sigma \times \tau}.$$

Also $\underrightarrow{lin}$ and $\underleftarrow{lin}$ have mapping ∞-categories, but due to the finiteness conditions, they are not internal. The mapping ∞-categories (and mapping spaces) are given in each case as ∞-categories (respectively ∞-groupoids) of spans of finite type. Denoting the mapping categories with underline, we content ourselves to record the important case of 'linear dual':

Proposition 6.5.

$$\underline{\underrightarrow{lin}}(grpd_{/S}, grpd) \;\; = \;\; grpd^S$$

$$\underline{\underleftarrow{lin}}(grpd^T, grpd) \;\; = \;\; grpd_{/T}.$$

6.6. Remark. It is clear that the correct viewpoint here would be that there is altogether a 2-equivalence between the $(\infty, 2)$-categories

$$\underrightarrow{\boldsymbol{lin}}{}^{\mathrm{op}} \cong \underleftarrow{\boldsymbol{lin}}$$

given on objects by $\boldsymbol{grpd}_{/S} \mapsto \boldsymbol{grpd}^S$, and by the identity on homs. It all comes formally from an ind-pro like duality starting with the anti-equivalence

$$\boldsymbol{lin} \simeq \boldsymbol{lin}^{\mathrm{op}}.$$

(Since we only (co)complete over filtered diagrams of monomorphisms, this is not precisely ind-pro duality).

Taking $S = 1$ we see that $\boldsymbol{grpd}$ is an object of both ∞-categories, and mapping into it gives the duality isomorphisms of Proposition 6.5.

6.7. Monoidal structures. The ∞-category $\underrightarrow{\boldsymbol{lin}}$ has two monoidal structures: $\oplus$ and $\otimes$, where $\boldsymbol{grpd}_{/S} \oplus \boldsymbol{grpd}_{/T} = \boldsymbol{grpd}_{/S+T}$ and $\boldsymbol{grpd}_{/S} \otimes \boldsymbol{grpd}_{/T} = \boldsymbol{grpd}_{/S \times T}$. The neutral object for the first is clearly $\boldsymbol{grpd}_{/0} = 1$ and the neutral object for the second is $\boldsymbol{grpd}_{/1} = \boldsymbol{grpd}$. The tensor product distributes over the direct sum. The direct sum is both the categorical sum and the categorical product (i.e. is a biproduct). There is also the operation of infinite direct sum: it is the infinite categorical sum but not the infinite categorical product. This is analogous to vector spaces.

Similarly, also the ∞-category $\underleftarrow{\boldsymbol{lin}}$ has two monoidal structures, $\oplus$ and $\otimes$, given as $\boldsymbol{grpd}^S \oplus \boldsymbol{grpd}^T = \boldsymbol{grpd}^{S+T}$ and $\boldsymbol{grpd}^S \otimes \boldsymbol{grpd}^T = \boldsymbol{grpd}^{S \times T}$. The $\otimes$ should be considered the analogue of a completed tensor product. Again $\oplus$ is both the categorical sum and the categorical product, and $\otimes$ distributes over $\oplus$. Again the structures allow infinite versions, but this times the infinite direct sum is a categorical infinite product but is not an infinite categorical sum.

To see the difference between the role of infinite $\oplus_i$ in $\underrightarrow{\boldsymbol{lin}}$ and in $\underleftarrow{\boldsymbol{lin}}$: in $\underleftarrow{\boldsymbol{lin}}$ there is a diagonal map $\boldsymbol{grpd}^S \to \oplus_i \boldsymbol{grpd}^S = \boldsymbol{grpd}^{\sum_i S}$ given by sending $X \to S$ to $\sum_i X \to \sum_i S$. This makes sense for a finite map $X \to S$, since the infinite sum of copies of that map is still finite, but it does not make sense in $\underrightarrow{\boldsymbol{lin}}$ since $\sum_i X$ is not finite. On the other hand, $\underrightarrow{\boldsymbol{lin}}$ sports a codiagonal $\oplus_i \boldsymbol{grpd}_{/S} = \boldsymbol{grpd}_{/\sum_i S} \to \boldsymbol{grpd}_{/S}$ given by sending $\alpha \to \sum_i S$ to the composite $\alpha \to \sum_i S \to S$, where the second map is the codiagonal for the infinite sum of ∞-groupoids. Composing with $\sum_i S \to S$ obviously does not alter the finiteness of α, so there is no problem with this construction. In contrast this construction does not work in $\underleftarrow{\boldsymbol{lin}}$: even if $X \to \sum_i S$ is finite, $X \to \sum_i S \to S$ will generally not be so.

6.8. Summability. In algebraic combinatorics, profinite notions are often expressed in terms of notions of summability. We briefly digress to examine our constructions from this point of view.

For B a locally finite ∞-groupoid, a B-indexed family $g : E \to B \times I$ (as in 2.1) is called *summable* if the composite $E \to B \times I \to I$ is a finite map. The condition implies that in fact the members of the family were already finite maps. Indeed,

with reference to the diagram

$$
\begin{array}{ccccc}
E_{b,i} & \longrightarrow & E_i & \longrightarrow & E \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\{b\} \times \{i\} & \longrightarrow & B \times \{i\} & \longrightarrow & B \times I \\
& & \downarrow & \lrcorner & \downarrow \\
& & \{i\} & \longrightarrow & I
\end{array}
$$

summability implies (by Lemma 3.14) that each E_i is finite, and therefore (by Lemma 3.16 since B is locally finite) we also conclude that each $E_{b,i}$ is finite, which is precisely to say that the members $g_b : E_b \to I$ are finite maps (cf. 3.14 again). It thus makes sense to interpret the family as a family of objects in $\mathbf{Grpd}_{/I}^{\mathrm{rel.fin.}}$. And finally we can say that a summable family is a family $g : E \to B \times I$ of finite maps $g_b : E_b \to I$, whose homotopy sum $p_!(g)$ is again a finite map. If I is finite, then the only summable families are the finite families (i.e. $E \to B \times I$ with E finite). A family $g : E \to B \times I$, given equivalently as a functor

$$
F : B \to \mathbf{grpd}^I,
$$

is summable if and only if it is a cofiltered limit of diagrams $F_\alpha : B \to \mathbf{grpd}^\alpha$ (with finite α and full $\alpha \subset I$).

It is easy to check that a map $q : M \to T$ (between locally finite ∞-groupoids) is finite if and only if for every finite map $f : X \to M$ we have that also $q_!(f)$ is finite. Hence we find

Lemma 6.9. *A span $I \xleftarrow{p} M \xrightarrow{q} J$ preserves summable families if and only if q is finite.*

7. Duality

7.1. The perfect pairing. We have a perfect pairing

$$
\begin{array}{rcl}
\mathbf{grpd}_{/S} \times \mathbf{grpd}^S & \longrightarrow & \mathbf{grpd} \\
(p, f) & \longmapsto & f(p)
\end{array}
$$

given by evaluation. In terms of spans, write the map-with-finite-total-space $p : \alpha \to S$ as a finite span $1 \leftarrow \alpha \xrightarrow{p} S$, and write the presheaf $f : S \to \mathbf{grpd}$ as the finite span $S \xleftarrow{f} F \to 1$, where F is the total space of the Grothendieck construction of f. (In other words, the functor F on S corresponds to a linear functor on $\mathbf{grpd}_{/S}$; write the representing span.) Then the evaluation is given by composing these two spans, and hence amounts just to taking the pullback of p and f.

The statements mean: for each $p : \alpha \to S$ in $\mathbf{grpd}_{/S}$, the map

$$
\begin{array}{rcl}
\mathbf{grpd}^S & \longrightarrow & \mathbf{grpd} \\
f & \longmapsto & f(p)
\end{array}
$$

is prolinear. The resulting functor

$$\mathbf{grpd}_{/S} \longrightarrow \underleftarrow{\mathbf{Lin}}(\mathbf{grpd}^S, \mathbf{grpd})$$
$$p \longmapsto \quad (f \mapsto f(p))$$

is an equivalence of ∞-categories (cf. Proposition 6.5).

Conversely, for each $f : S \to \mathbf{grpd}$ in $\mathbf{grpd}^S$, the map

$$\mathbf{grpd}_{/S} \longrightarrow \mathbf{grpd}$$
$$p \longmapsto f(p)$$

is linear. The resulting functor

$$\mathbf{grpd}^S \longrightarrow \underrightarrow{\mathbf{Lin}}(\mathbf{grpd}_{/S}, \mathbf{grpd})$$
$$f \longmapsto \quad (p \mapsto f(p))$$

is an equivalence of ∞-categories (cf. Proposition 6.5).

7.2. Remark. By the equivalences of Proposition 4.2, we also get the perfect pairing

$$\mathbf{Grpd}_{/S}^{\mathrm{rel.fin.}} \times \mathbf{grpd}_{\mathrm{fin.sup.}}^S \longrightarrow \mathbf{grpd}$$
$$(p, f) \longmapsto f(p).$$

7.3. Bases. Both $\mathbf{grpd}_{/S}$ and $\mathbf{grpd}^S$ feature a canonical basis, actually an essentially unique basis. The basis elements in $\mathbf{grpd}_{/S}$ are the names $\ulcorner s \urcorner : 1 \to S$: every object $p : X \to S$ in $\mathbf{grpd}_{/S}$ can be written as a finite homotopy linear combination

$$p = \int^{s \in S} |X_s| \ulcorner s \urcorner.$$

Similarly, in $\mathbf{grpd}^S$, the representables $h^t := \mathrm{Map}(t, -)$ form a basis: every presheaf on S is a colimit, and in fact a homotopy sum, of such representables. These bases are dual to each other, except for a normalisation: if $p = \ulcorner s \urcorner$ and $f = h^t = \mathrm{Map}(t, -)$, then they pair to

$$\mathrm{Map}(t, s) \simeq \begin{cases} \Omega(S, s) & \text{if } t \simeq s \\ 0 & \text{else.} \end{cases}$$

The fact that we obtain the loop space $\Omega(S, s)$ instead of 1 is actually a feature: we shall see below that on taking cardinality we obtain the canonical pairing

$$\mathbb{Q}_S \times \mathbb{Q}^S \longrightarrow \mathbb{Q}$$
$$(\delta_i, \delta^j) \longmapsto \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

8. Cardinality as a functor

The goal is that each slice $\boldsymbol{grpd}_{/S}$ and each finite-presheaf category $\boldsymbol{grpd}^S$ should have a notion of homotopy cardinality, with values in the vector space $\mathbb{Q}_{\pi_0 S}$ and the profinite-dimensional vector space $\mathbb{Q}^{\pi_0 S}$, respectively. The idea of Baez, Hoffnung and Walker [4] is to achieve this by a global assignment, which specialises to every slice ∞-category to define a relative cardinality, a cardinality of families, by the observation that families are special cases of spans, just as vectors can be identified with linear maps from the ground field. In our setting this amounts to functors $\underrightarrow{\boldsymbol{lin}} \to \boldsymbol{Vect}$ and $\underleftarrow{\boldsymbol{lin}} \to \underleftarrow{\boldsymbol{vect}}$.

8.1. Definition of cardinality. We define *meta cardinality* by the assignment

$$\| \quad \| : \underrightarrow{\boldsymbol{lin}} \to \boldsymbol{Vect}$$

defined on objects by

$$\|\boldsymbol{grpd}_{/T}\| := \mathbb{Q}_{\pi_0 T},$$

and on morphisms by taking a finite-type span $S \xleftarrow{p} M \xrightarrow{q} T$ to the linear map

$$
\begin{aligned}
\mathbb{Q}_{\pi_0 S} &\longrightarrow \mathbb{Q}_{\pi_0 T} \\
\delta_s &\longmapsto \int^t |M_{s,t}|\, \delta_t = \sum_t \left|T_{[t]}\right| |M_{s,t}|\, \delta_t.
\end{aligned}
$$

That is, to the span M we assign the matrix $A_{t,s} := \left|T_{[t]}\right| |M_{s,t}|$, which is column finite since M is of finite type.

Dually we define a meta cardinality

$$\| \quad \| : \underleftarrow{\boldsymbol{lin}} \to \underleftarrow{\boldsymbol{vect}}$$

defined on objects by

$$\|\boldsymbol{grpd}^S\| := \mathbb{Q}^{\pi_0 S},$$

and on morphisms by the *same* assignment of a matrix to a finite span as before.

Proposition 8.2. *The meta cardinality assignments just defined*

$$\| \quad \| : \underrightarrow{\boldsymbol{lin}} \to \boldsymbol{Vect}, \qquad \| \quad \| : \underleftarrow{\boldsymbol{lin}} \to \underleftarrow{\boldsymbol{vect}}$$

are functorial.

Proof. First observe that the functor is well defined on morphisms. Given a finite-type span $S \xleftarrow{p} M \xrightarrow{q} T$ defining the linear functors $L : \boldsymbol{grpd}_{/S} \to \boldsymbol{grpd}_{/T}$ and $L^\vee : \boldsymbol{grpd}^T \to \boldsymbol{grpd}^S$), the linear maps

$$\|L\| : \mathbb{Q}_{\pi_0 S} \longrightarrow \mathbb{Q}_{\pi_0 T}, \qquad \|L^\vee\| : \mathbb{Q}^{\pi_0 T} \longrightarrow \mathbb{Q}^{\pi_0 S}$$

are defined with respect to the given (pro-)bases by the matrix $\|L\|_{t,s} = |M_{s,t}| \left|T_{[t]}\right|$. That is:

$$\|L\| \left(\sum_{s \in \pi_0 S} c_s\, \delta_s \right) = \sum_{s,t} c_s\, |M_{s,t}| \left|T_{[t]}\right|\, \delta_t = \sum_{s \in \pi_0 S} c_s \int^{t \in T} |M_{s,t}|\, \delta_t\,,$$

and

$$\|L^{\vee}\| \left(\sum_{t \in \pi_0 T} c_t \, \delta^t \right) \;=\; \sum_{s,t} c_t \, |M_{s,t}| \, \big|T_{[t]}\big| \, \delta_s$$

In particular, we note

$$\|L^{\vee}\| \, (\delta^t) \;=\; \sum_s |M_{s,t}| \, \big|T_{[t]}\big| \, \delta_s \, .$$

The matrix $|M_{s,t}| \, \big|T_{[t]}\big|$ has finite entries and is column-finite: for each $s \in \pi_0 S$ the fibre M_s is finite so the map $M_s \to T$ is finite by Lemma 3.15, and the fibres $M_{s,t}$ are non-empty for only finitely many $t \in \pi_0 T$. It is clear that equivalent spans define the same matrix, and the identity span $L = (S \leftarrow S \to S)$ gives the identity matrix: $\|L\|_{s_1, s_2} = 0$ if s_1, s_2 are in different components, and $\|L\|_{s,s} = |\Omega(S, s)| \, \big|S_{[s]}\big| = 1$ by Lemma 3.10. It remains to show that composition of spans corresponds to matrix product: for $L = (S \leftarrow M \to T)$, $L' = (T \leftarrow N \to U)$ we have

$$|(M \times_T N)_{s,u}| = \int^{t \in T} |M_{s,t} \times N_{t,u}| = \sum_{t \in \pi_0 T} |M_{s,t}| \, \big|T_{[t]}\big| \, |N_{t,u}|$$

and so $\|L'L\|_{u,s} = \displaystyle\sum_{t \in \pi_0 T} |M_{s,t}| \, \big|T_{[t]}\big| \, |N_{t,u}| \, \big|U_{[u]}\big| = \sum_{t \in \pi_0 T} \|L'\|_{u,t} \|L\|_{t,s}.$ $\qquad\square$

8.3. Cardinality of families. As a consequence of Proposition 8.2 we obtain for each locally finite ∞-groupoid T a notion of cardinality of T-indexed families. Let T be a locally finite ∞-groupoid and define the functor

$$|\;| : \mathbf{grpd}_{/T} \longrightarrow \big\|\mathbf{grpd}_{/T}\big\| = \mathbb{Q}_{\pi_0 T}, \qquad |x| \;:=\; \|L_x\| \, (\delta_1) \, .$$

Here $x : X \to T$ is an object of $\mathbf{grpd}_{/T}$ and $\|L_x\| : \mathbb{Q}_{\pi_0 1} \to \mathbb{Q}_{\pi_0 T}$ is induced by the linear functor L_x defined by the finite span $1 \leftarrow X \xrightarrow{x} T$. By the definition of $\|L\|$ in Proposition 8.2, we can write

$$|x| \;=\; \sum_{t \in \pi_0 T} |X_t| \, |T_{[t]}| \, \delta_t \;=\; \int^{t \in T} |X_t| \, \delta_t$$

Lemma 8.4. *Let T be a locally finite ∞-groupoid.*

  (1) *If T is connected, with $t \in T$, and $x : X \to T$ in $\mathbf{grpd}_{/T}$, then*

$$|x| \;=\; |X| \, \delta_t \;\in\; \mathbb{Q}_{\pi_0 T}.$$

  (2) *The cardinality of $\ulcorner t \urcorner : 1 \to T$ in $\mathbf{grpd}_{/T}$ is the basis vector δ_t.*

Proof. (1) By definition, $|x| \;=\; |X_t| \, |T| \, \delta_t$, and by Lemma 3.4, this is $|X| \, \delta_t$
(2) The fibre of $\ulcorner t \urcorner$ over t' is empty except when t, t' are in the same component, so we reduce to the case of connected T and apply (1). $\qquad\square$

8.5. Cardinality of presheaves. We also obtain a notion of cardinality of presheaves: for each S, define

$$|\;| : \mathbf{grpd}^S \longrightarrow \big\|\mathbf{grpd}^S\big\| = \mathbb{Q}^{\pi_0 S}, \qquad |f| \;:=\; \|L_f\|.$$

Here $f : S \to \boldsymbol{grpd}$ is a presheaf, and $L_f : \boldsymbol{grpd}_{/S} \to \boldsymbol{grpd}$ its extension by linearity; L_f is given by the span $S \leftarrow F \to 1$, where $F \to S$ is the Grothendieck construction of f. The meta cardinality of this span is then a linear map $\mathbb{Q}_{\pi_0 S} \to \mathbb{Q}_1$, or equivalently a pro-linear map $\mathbb{Q}^1 \to \mathbb{Q}^{\pi_0 S}$ — in either way interpreted as an element in $\mathbb{Q}^{\pi_0 S}$. In the first viewpoint, the linear map is

$$\begin{aligned} \mathbb{Q}_{\pi_0 S} &\longrightarrow \mathbb{Q}_1 \\ \delta_s &\longmapsto \int^1 |F_s|\, \delta_1 = |F_s|\, \delta_1 \end{aligned}$$

which is precisely the function

$$\begin{aligned} \pi_0 S &\longrightarrow \mathbb{Q} \\ s &\longmapsto |f(s)|. \end{aligned}$$

In the second viewpoint, it is the prolinear map

$$\begin{aligned} \mathbb{Q}^1 &\longrightarrow \mathbb{Q}^{\pi_0 S} \\ \delta_1 &\longmapsto \sum_s |F_s|\, \delta^s \end{aligned}$$

which of course also is the function $s \mapsto |f(s)|$.

In conclusion:

Proposition 8.6. *The cardinality of a presheaf $f : S \to \boldsymbol{grpd}$ is computed point-wise: $|f|$ is the function*

$$\begin{aligned} \pi_0 S &\longrightarrow \mathbb{Q} \\ s &\longmapsto |f(s)|. \end{aligned}$$

In other words, it is obtained by postcomposing with the basic homotopy cardinality.

Example 8.7. The cardinality of the representable functor $h^t : S \to \boldsymbol{grpd}$ is

$$\begin{aligned} \pi_0 S &\longrightarrow \mathbb{Q} \\ s &\longmapsto |\mathrm{Map}(t, s)| = \begin{cases} |\Omega(S, s)| & \text{if } t \simeq s \\ 0 & \text{else.} \end{cases} \end{aligned}$$

8.8. Remark. Note that under the finite fundamental equivalence $\boldsymbol{grpd}^S \simeq \boldsymbol{Grpd}_{/S}^{relfin}$ (4.2), the representable presheaf h^s corresponds to $\ulcorner s \urcorner$, the name of s, which happens to belong also to the subcategory $\boldsymbol{grpd}_{/S} \subset \boldsymbol{Grpd}_{/S}^{relfin}$, but that the cardinality of $h^s \in \boldsymbol{grpd}^S$ *cannot* be identified with the cardinality of $\ulcorner s \urcorner \in \boldsymbol{grpd}_{/S}$. This may seem confusing at first, but it is forced upon us by the choice of normalisation of the functor

$$\| \ \| : \underrightarrow{\boldsymbol{lin}} \to \boldsymbol{Vect}$$

which in turn looks natural since the extra factor $|T_{[t]}|$ comes from an integral. A further feature of this apparent discrepancy is the following.

Proposition 8.9. *Cardinality of the canonical perfect pairing at the ∞-groupoid level (7.1) yields precisely the perfect pairing on the vector-space level.*

Proof. We take cardinality of the perfect pairing

$$
\begin{aligned}
\boldsymbol{grpd}_{/S} \times \boldsymbol{grpd}^S &\longrightarrow \boldsymbol{grpd} \\
(p, f) &\longmapsto f(p) \\
(\ulcorner s \urcorner, h^t) &\longmapsto
\begin{cases}
\Omega(S, s) & \text{if } t \simeq s \\
0 & \text{else .}
\end{cases}
\end{aligned}
$$

Since the cardinality of $\ulcorner s \urcorner$ is δ_s, while the cardinality of h^t is $|\Omega(S, t)| \, \delta^t$, the cardinality of the pairing becomes

$$
(\delta_s, |\Omega(S, t)| \, \delta^t) \longmapsto
\begin{cases}
|\Omega(S, t)| & \text{if } t \simeq s \\
0 & \text{else ,}
\end{cases}
$$

or equivalently:

$$
(\delta_s, \delta^t) \longmapsto
\begin{cases}
1 & \text{if } t \simeq s \\
0 & \text{else ,}
\end{cases}
$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

8.10. Remarks. The definition of meta cardinality involves a convention, namely for a span $S \leftarrow M \rightarrow T$ to include the factor $|T_{[t]}|$. In fact, as observed by Baez–Hoffnung–Walker [4], other conventions are possible: for any real numbers α_1 and α_2 with $\alpha_1 + \alpha_2 = 1$, it is possible to use the factor

$$
\left| S_{[s]} \right|^{\alpha_1} \left| T_{[t]} \right|^{\alpha_2} .
$$

They use $0+1$ in some cases and $1+0$ in other cases, according to what seems more practical. We think that these choices can be explained by the side of duality on which the constructions take place.

Our convention with the $\left| T_{[t]} \right|$ normalisation yields the 'correct' numbers in all the applications of the theory that motivated us, as exemplified below.

8.11. Incidence coalgebras and incidence algebras of decomposition spaces.
A main motivation for us is the theory of decomposition spaces [11], [12], [13]. A *decomposition space* is a simplicial ∞-groupoid $X : \Delta^{\mathrm{op}} \rightarrow \boldsymbol{Grpd}$ satisfying an exactness condition precisely so as to make the following comultiplication law coassociative, up to coherent homotopy. The natural span

$$
X_1 \xleftarrow{\; d_1 \;} X_2 \xrightarrow{\; (d_2, d_0) \;} X_1 \times X_1
$$

defines a linear functor, the *comultiplication*

$$
\begin{aligned}
\Delta : \boldsymbol{Grpd}_{/X_1} &\longrightarrow \boldsymbol{Grpd}_{/(X_1 \times X_1)} \\
(S \xrightarrow{s} X_1) &\longmapsto (d_2, d_0)_! \circ d_1^*(s).
\end{aligned}
$$

(and similarly the span $X_1 \xleftarrow{s_0} X_0 \rightarrow 1$ defines the counit). This is called the incidence coalgebra of X. If the maps $X_0 \xrightarrow{s_0} X_1 \xleftarrow{d_1} X_2$ are both finite then this coalgebra structure restricts to a coalgebra structure on $\boldsymbol{grpd}_{/X_1}$, which in turn descends to $\mathbb{Q}_{\pi_0 X_1}$ under taking cardinality [12].

An example is given by the fat nerve of $\mathbf{V}$, the category of finite-dimensional vector spaces over a finite field and linear injections. Dür [7] obtained the Hall algebra from this example by a reduction step, identifying two linear injections if their cokernels have the same dimension. The coalgebra can also be obtained directly from a decomposition space, namely the Waldhausen S-construction on $\mathbf{V}$. We check in [14] that the cardinality of this comultiplication gives precisely the classical Hall numbers (with the present convention).

8.12. Zeta functions. For X a decomposition space with $X_0 \xrightarrow{s_0} X_1 \xleftarrow{d_1} X_2$ both finite maps, the dual space of $\boldsymbol{grpd}_{/X_1}$ is $\boldsymbol{grpd}^{X_1}$, underlying the incidence *algebra*. Its multiplication is given by a convolution formula. In here there is a canonical element, the 'constant' linear functor given by the span $X_1 \xleftarrow{=} X_1 \to 1$ (corresponding to the terminal presheaf), which is called the *zeta functor* [12]. By 8.6, the cardinality of the terminal presheaf is the constant function 1. Hence the cardinality of the zeta functor is the classical zeta function in incidence algebras.

8.13. Green functions. The zeta function is the 'sum of everything', with no symmetry factors. A 'sum of everything', but *with* symmetry factors, appeared in our work [9] on the Faà di Bruno and Connes–Kreimer bialgebras, namely in the form of combinatorial Green functions (see also [23]).

The coalgebra in question is then the completion of the finite incidence algebra $\boldsymbol{Grpd}_{/X_1}^{\text{rel.fin.}}$, where X_1 is the groupoid of forests (or more precisely, P-forests for P a polynomial functor [16], [22]). Of course we know that $\boldsymbol{Grpd}_{/X_1}^{\text{rel.fin.}}$ is canonically equivalent to $\boldsymbol{grpd}^{X_1}$, but it is important here to keep track of which side of duality we are on. The Green function, which is in reality a distribution rather than a function, lives on the coalgebra side, and more precisely in the completion. (The fact that the comultiplication extends to the completion is due to the fact that not only $d_1 : X_2 \to X_1$ is finite, but that also $X_2 \to X_1 \times X_1$ is finite (a feature common to all Segal 1-groupoids).)

Our Green function, shown to satisfy the Faà de Bruno formula in $\boldsymbol{Grpd}_{/X_1}^{\text{rel.fin.}}$, is $T \to X_1$, the full inclusion of the groupoid of P-trees T into the groupoid of P-forests. Upon taking cardinality, with the present conventions, we obtain precisely the series

$$G = \sum_{t \in \pi_0 T} \frac{\delta_t}{|\text{Aut}(t)|},$$

the sum of all trees weighted by symmetry factors, which is the usual combinatorial Green function in Quantum Field Theory, modulo the difference between trees and graphs [23]. The important symmetry factors appear correctly because we are on the coalgebra side of the duality.

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Departament de Matemàtiques, Universitat Politècnica de Catalunya, ESEIAAT, Carrer Colom 1, 08222 Terrassa (Barcelona), Spain
  E-mail address: `m.immaculada.galvez@upc.edu`

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain
  E-mail address: `kock@mat.uab.cat`

Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, UK
  E-mail address: `apt12@le.ac.uk`