DECAY OF SOLUTIONS FOR A MIXTURE OF THERMOELASTIC SOLIDS WITH DIFFERENT TEMPERATURES

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Abstract. We study a system modeling thermomechanical deformations for mixtures of thermoelastic solids with two different temperatures, that is, when each component of the mixture has its own temperature. In particular, we investigate the asymptotic behavior of the related solutions. We prove the exponential stability of solutions for a generic class of materials. In case of the coupling matrix $\mathbf{B}$ being singular, we find that in general the corresponding semigroup is not exponentially stable. In this case we obtain that the corresponding solution decays polynomially as $t^{-1/2}$ in case of Neumann boundary condition. Additionally, we show that the rate of decay is optimal. For Dirichlet boundary condition, we prove that the rate of decay is $t^{-1/6}$. Finally, we demonstrate the impossibility of time-localization of solutions in case that two coefficients (related with the thermal conductivity constants) agree.

1. Introduction

Under the theory of non-classical elastic solids we understand certain generalizations of the classical theory of elasticity. The most known non-classical elastic solids are the elastic solids with voids, micropolar elastic solids, nonsimple elastic solids and the mixtures of elastic solids. Micropolar elastic solids have first been introduced by the Cosserat brothers at the begining of the last century and they were recovered, analyzed and extended by Eringen and many other researchers in the second part of the past century. For an overview on these so called microcontinuum theories we refer, e.g., to [10,11,21]. In the same period, the theories concerning the nonsimple materials, materials with voids and mixtures of material were established. It is worth recalling here the book of Ieşan [15] where several of these theories are analyzed. This manuscript is concerned with one of these theories: the mixtures of elastic solids.

Thermoelastic mixtures of solids have deserved a big interest in the last decades (see, e.g., [4,5,7,8,12,13,29,30]). Qualitative properties of solutions to the problems defining this kind of materials have been the scope of many investigations. Several results concerning existence, uniqueness, continuous dependence and asymptotic stability can be found in the literature [1–3,16,19,26,27]. In this paper, we study the decay of solutions in case of a one-dimensional rod composed by a mixture of two thermoelastic solids with two different temperatures. We will prove the exponential stability in a generic case, however, we cannot expect that the solutions can identically vanish after a finite time and we will see the impossibility of localization for various scenarios. In several situations, the decay is not so fast and we will prove the polynomial decay for these situations. It is worth recalling that studying the rate of decay of the solutions for several non-classical theories has been the goal of many articles in this last decade [18,22–24]. Thus, the present paper aims to be a new contribution in this line.

For a rod composed by a mixture of two interacting continua occupying the interval $(0,\ell)$ the displacements of each component of typical particles at time t are denoted, by u and w, respectively, where $u=u(x,t):(0,\ell)\times(0,T)\to\mathbb{R}$ and $w=w(y,t):(0,\ell)\times(0,T)\to\mathbb{R}$, with $T>0$. We assume that the particles under consideration are in the same position at time $t=0$, so that $x=y$.

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We also assume the existences of two different temperatures (see [14]), in each point x and at time t, given by $\theta_i = \theta_i(x,t) : (0,\ell) \times (0,T) \to \mathbb{R}$, $i = 1,2$. We denote by ρ_i, $i = 1,2$ the mass density of each constituent at time $t = 0$. We introduce $\mathcal{T}$ and $\mathcal{S}$ as the partial stresses associated with these two constituents, P the internal diffusive force, $\Xi^{(i)}$, $i = 1,2$, the entropy densities, $Q^{(i)}$, $i = 1,2$, the heat flux vector and $\mathfrak{T}_0$ is the absolute temperature in the reference configuration. In the absence of body forces, the system consists of the following equations:

- equations of motion

$$\rho_1\, u_{tt} = \mathcal{T}_x - P, \qquad \rho_2\, w_{tt} = \mathcal{S}_x + P, \tag{1.1}$$

- energy equations

$$\rho_1\, \mathfrak{T}_0\, \Xi^{(1)}{}_t = Q^{(1)}_x + W^{(1)} + G, \qquad \rho_2 \mathfrak{T}_0\, \Xi^{(2)}{}_t = Q^{(2)}_x + W^{(2)} - G, \tag{1.2}$$

- constitutive equations

$$\mathcal{T} = a_{11}u_x + a_{12}w_x - \beta_1\theta_1 - \beta_2\theta_2, \qquad \mathcal{S} = a_{12}u_x + a_{22}w_x - \gamma_1\theta_1 - \gamma_2\theta_2, \tag{1.3}$$

$$P = \alpha(u - w), \qquad G = -a(\theta_1 - \theta_2), \tag{1.4}$$

$$\rho_1\, \Xi^{(1)} = \beta_1 u_x + \beta_2 w_x + M_1^{(1)}\theta_1 + M_2^{(1)}\theta_2 + T_0^{-1}\rho_2\kappa_1^{(2)}(\theta_1 - \theta_2), \tag{1.5}$$

$$\rho_2\, \Xi^{(2)} = \gamma_1 u_x + \gamma_2 w_x + M_2^{(1)}\theta_1 + M_2^{(2)}\theta_2 + T_0^{-1}\rho_1\kappa_2^{(1)}(\theta_1 - \theta_2), \tag{1.6}$$

$$Q^{(1)} = K_{11}\theta_{1,x} + K_{12}\theta_{2,x}, \qquad Q^{(2)} = K_{21}\theta_{1,x} + K_{22}\theta_{2,x}. \tag{1.7}$$

Functions $W^{(i)}$ are given by

$$W^{(1)} = \rho_1\kappa_2^{(1)}\theta_{2,t} - \rho_2\kappa_1^{(2)}\theta_{1,t}, \qquad W^{(2)} = \rho_2\kappa_1^{(2)}\theta_{1,t} - \rho_1\kappa_2^{(1)}\theta_{2,t}. \tag{1.8}$$

If we denote

$$b_1 = T_0 M_1^{(1)} + 2\rho_2\kappa_1^{(2)}, \qquad b_2 = T_0 M_2^{(1)} - \rho_1\kappa_2^{(1)} - \rho_2\kappa_1^{(2)}, \qquad b_3 = M_2^{(2)} + 2\rho_2\kappa_2^{(1)},$$

and substitute constitutive equations (1.3)–(1.8) into dynamical equations (1.1)–(1.2), we obtain the following evolution system

$$\begin{array}{ll}
\rho_1 u_{tt} - a_{11}u_{xx} - a_{12}w_{xx} + \alpha(u-w) + \beta_1\theta_{1,x} + \beta_2\theta_{2,x} = 0 & \text{in } (0,\ell)\times(0,T),\\
\rho_2 w_{tt} - a_{12}u_{xx} - a_{22}w_{xx} - \alpha(u-w) + \gamma_1\theta_{1,x} + \gamma_2\theta_{2,x} = 0 & \text{in } (0,\ell)\times(0,T),\\
b_1\theta_{1,t} + b_2\theta_{2,t} - K_{11}\theta_{1,xx} - K_{12}\theta_{2,xx} + \beta_1 u_{xt} + \beta_2 w_{xt} + a(\theta_1 - \theta_2) = 0 & \text{in } (0,\ell)\times(0,T),\\
b_2\theta_{1,t} + b_3\theta_{2,t} - K_{21}\theta_{1,xx} - K_{22}\theta_{2,xx} + \gamma_1 u_{xt} + \gamma_2 w_{xt} - a(\theta_1 - \theta_2) = 0 & \text{in } (0,\ell)\times(0,T),
\end{array}$$

where, for the sake of simplicity, we assume that $\mathfrak{T}_0 = 1$. Therefore the corresponding evolution system can be written as

$$\mathbf{R}_1 U_{tt} - \mathbf{A}U_{xx} + \alpha\mathbf{N}U + \mathbf{B}\Upsilon_x = 0, \quad \text{in } (0,\ell)\times(0,T), \tag{1.9}$$

$$\mathbf{R}_2 \Upsilon_t - \mathbf{K}\Upsilon_{xx} + a\mathbf{N}\Upsilon + \mathbf{B}U_{xt} = 0, \quad \text{in } (0,\ell)\times(0,T). \tag{1.10}$$

Here

$$U = \begin{pmatrix} u \\ w \end{pmatrix}, \quad \Upsilon = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},$$

$$\mathbf{A} = (a_{ij})_{2\times 2}, \quad \mathbf{R}_1 = (\delta_{ij}\rho_i)_{2\times 2}, \quad \mathbf{R}_2 = (b_i)_{2\times 2},$$

are symmetric matrices, δ_{ij} is the usual Kroneker's delta, $\mathbf{K} = (K_{ij})_{2\times 2}$, $\mathbf{N} = ((-1)^{i+j})_{2\times 2}$ and $\mathbf{B} = \begin{pmatrix} \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$. In general $\mathbf{B}$ is neither symmetrical nor positive definite. We supplement our system with the initial conditions

$$U(x,0) = \begin{pmatrix} u_0 \\ w_0 \end{pmatrix}, \quad U_t(x,0) = \begin{pmatrix} u_1 \\ w_1 \end{pmatrix}, \quad \Upsilon(x,0) = \begin{pmatrix} \theta_{10} \\ \theta_{20} \end{pmatrix}, \tag{1.11}$$

and boundary conditions

$$U(0,t) = U(\ell,t) = \Upsilon(0,t) = \Upsilon(\ell,t) = 0. \tag{1.12}$$

Alternatively, we can also consider the Dirichlet–Neumann boundary conditions, namely

$$U(0,t) = U(\ell,t) = \Upsilon_x(0,t) = \Upsilon_x(\ell,t) = 0^1. \tag{1.13}$$

We state the general assumptions we impose in the paper:

(H.1) The mass densities are positive numbers, i.e., $\rho_1 > 0$ and $\rho_2 > 0$.
(H.2) The matrices $\mathbf{A}, \mathbf{R}_2$ and $\mathbf{K}$ are positive definite.
(H.3) The constitutive parameters α and a are positive, i. e., $\alpha > 0$ and $a > 0$.

The system of field equations is composed of four equations. We will show that the coupling is generically so strong that the thermal dissipation brings the whole system to an exponential decay. We will say that the decay of the solutions is exponential if they are exponentially stable and, if they are not, we will say that the decay of the solutions is slow.

Concerning the terminology, we do not distinguish between the *stability of our problem* and *its associated solution*.

In two recent papers [2, 20], the authors proved that, under suitable conditions on the coefficients of the problem, the solution decays exponentially in the case of one temperature. Here, we want to continue this line of study and to improve the result under the following point of view. When two different temperatures are considered, the coupling is so strong that the exponential decay is guaranteed whenever the vectors which define the coupling are linearly independent. Consequently, we find a sufficient condition to guarantee that the imaginary axis is contained in the resolvent, and then the exponential decay of solutions. Some other situations, where the exponential decay of solutions holds, are presented.

In linear thermoelasticity, the asymptotic behavior of solutions as $t \to +\infty$ has been studied by many authors. We refer, e.g., to the book of Liu and Zheng [17] for a general survey on those topics. For the system (1.9)–(1.12) we can not expect that its solution always decays exponentially. For instance, in case that $\beta_1 + \beta_2 = \gamma_1 + \gamma_2 = 0$ and $\rho_2(a_{11} + a_{12}) = \rho_1(a_{12} + a_{22})$, we can obtain solutions of the form $u = w$ and $\theta_1 = \theta_2 = 0$. These solutions are undamped and do not decay to zero. When $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0$, the mechanical and thermal parts are not coupled and the displacements do not decay. These are very particular cases, but we will see that there are some other cases where the solutions decay, but the decay is not so fast to be controlled by an exponential function. By applying also some results obtained recently (see, e.g., [6]), polynomial decay will be proved in these cases.

Our main aim is to show that the C_0-semigroup (and thus the solution $\mathbf{U}(t)$) associated with system (1.9)–(1.12) is exponentially stable if suitable conditions on the coefficients are satisfied.

This paper is organized as follows. In Section 2, we establish the well-posedness of system (1.9)–(1.12). The exponential stability in a generic case is proved in Section 3. Later in Section 4 we present several cases where the decay is not exponential, and for these cases in Section 5 we obtain the polynomial decay. Finally, Section 6 is devoted to proving the impossibility of localization of the solutions when we assume that $K_{12} = K_{21}$.

[1] We point out that this boundary condition is not sound from a thermomechanical point of view. Therefore, our viewpoint with respect this boundary condition is mainly mathematical.

2. Existence and uniqueness of solutions

The aim of this section is to prove existence and uniqueness of solutions for problem (1.9)–(1.12) or (1.9)–(1.11), (1.13).

We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and the norm defined on $L^2(0, \ell)$, respectively. In general, for a Banach space X, we let $\| \cdot \|_X$ be the usual norm defined on X. Moreover, for the sake of simplicity, here and in that follows we will employ the same symbols C and c for different constants, even in the same formula.

In case of Dirichlet thermal boundary conditions, let us consider the vectorial space

$$\mathcal{H} = [H_0^1(0, \ell)]^2 \times [L^2(0, \ell)]^2 \times [L^2(0, \ell)]^2.$$

In case of Neumann thermal boundary conditions we take

$$\mathcal{H}_* = [H_0^1(0, \ell)]^2 \times [L^2(0, \ell)]^2 \times [L_*^2(0, \ell)]^2,$$

where

$$L_*^2(0, \ell) = \left\{ f \in L^2(0, \ell) : \quad \int_0^\ell f(x)\, dx = 0 \right\}, \qquad H_*^1(0, \ell) = H^1(0, \ell) \cap L_*^2(0, \ell).$$

Here and in that follows we denote by $M^\top$ the transpose of a matrix M. Putting $Y = U_t$, for any couple of vectors $\mathbf{U} = (U, Y, \Upsilon)^\top$, $\mathbf{U}^* = (U^*, Y^*, \Upsilon^*)^\top$ in the phase space $\mathcal{H}$ (or $\mathcal{H}_*$) we define the inner product

$$\langle \mathbf{U}, \mathbf{U}^* \rangle_{\mathcal{H}} = \int_0^\ell \Big(U_x^\top \mathbf{A} U_x^* + Y^\top \mathbf{R}_1 Y^* + \alpha U^\top \mathbf{N} U^* + \Upsilon^\top \mathbf{R}_2 \Upsilon^* \Big)\, dx.$$

It constitutes a squared norm

$$\|\mathbf{U}\|_{\mathcal{H}}^2 = \int_0^\ell \Big(U_x^\top \mathbf{A} U_x + Y^\top \mathbf{R}_1 Y + \alpha U^\top \mathbf{N} U + \Upsilon^\top \mathbf{R}_2 \Upsilon \Big)\, dx.$$

Together with the above defined inner product, the phase space is a Hilbert space. In particular, there exist two positive constants c_0 and c_1 such that inequality

$$c_0 |||\mathbf{U}||| \le \|\mathbf{U}\|_{\mathcal{H}} \le c_1 |||\mathbf{U}|||$$

is satisfied, with

$$|||\mathbf{U}|||^2 = \|u_x\|^2 + \|w_x\|^2 + \|v\|^2 + \|\eta\|^2 + \|\theta_1\|^2 + \|\theta_2\|^2.$$

We now consider the matrix operator

$$\mathcal{A} = \begin{pmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{R}_1^{-1} \mathbf{A} \partial_x^2 - \alpha \mathbf{R}_1^{-1} \mathbf{N} & 0 & -\mathbf{R}_1^{-1} \mathbf{B} \partial_x \\ \mathbf{0} & -\mathbf{R}_2^{-1} \mathbf{B} \partial_x & \mathbf{R}_2^{-1} \mathbf{K} \partial_x^2 - a \mathbf{R}_2^{-1} \mathbf{N} \end{pmatrix},$$

where here $\mathbf{I}$ and $\mathbf{0}$ are the 2×2 identity matrix and the 2×2 zero matrix, respectively. Symbols ∂_x and ∂_x^2 denote the first and second–order partial derivatives with respect to the spatial variable x.

Under boundary conditions (1.12) the domain of operator $\mathcal{A}$ is

$$D(\mathcal{A}) = [H_0^1(0, \ell) \cap H^2(0, \ell)]^2 \times [H_0^1(0, \ell)]^2 \times [H_0^1(0, \ell) \cap H^2(0, \ell)]^2.$$

In case of Dirichlet–Neumann boundary condition (1.13) we have

$$D(\mathcal{A}) = [H_0^1(0, \ell) \cap H^2(0, \ell)]^2 \times [H_0^1(0, \ell)]^2 \times [H_*^2(0, \ell)]^2,$$

where $H_*^2(0, \ell) = \big\{ f \in H^2(0, \ell) : \quad f_x(0) = f_x(\ell) = 0 \big\} \cap L_*^2(0, \ell)$.

The domain $D(\mathcal{A})$ is dense in the Hilbert space $\mathcal{H}$. Our initial-boundary value problem (1.9)–(1.12) or (1.9)–(1.11), (1.13) can be rewritten as the following initial abstract form

$$\frac{d}{dt}\mathbf{U}(t) = \mathcal{A}\mathbf{U}(t), \qquad \mathbf{U}(0) = \mathbf{U}_0,$$

where $\mathbf{U}_0 = (U(x,0), U_t(x,0), \Upsilon(x,0))^\top$ and according to (1.11).

Lemma 2.1. *Under hypotheses (H.1)–(H.3), the operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions denoted by $S(t) = e^{\mathcal{A}t}$, $t \geq 0$.*

Proof. It is enough to show that $\mathcal{A}$ is a dissipative operator and $0 \in \rho(\mathcal{A})$ (see, [17], pag. 3, Theorem 1.2.4).
In fact, because of (H.2)–(H.3),

$$\mathrm{Re}\langle \mathcal{A}\mathbf{U}, \mathbf{U}\rangle_{\mathcal{H}} = -\,\mathrm{Re}\int_0^\ell \left(\Upsilon_x^\top \mathbf{K}\Upsilon_x + a\Upsilon^\top \mathbf{N}\Upsilon\right) dx \leq 0. \tag{2.1}$$

Therefore, the operator $\mathcal{A}$ is dissipative. We now prove that for $\lambda = 0$, the resolvent system

$$\begin{aligned} i\lambda U - Y &= F_U, & (2.2)\\ i\lambda \mathbf{R}_1 Y - \mathbf{A}U_{xx} + \alpha\mathbf{N}U + \mathbf{B}\Upsilon_x &= \mathbf{R}_1 F_Y, & (2.3)\\ i\lambda \mathbf{R}_2 \Upsilon - \mathbf{K}\Upsilon_{xx} + a\mathbf{N}\Upsilon + \mathbf{B}Y_x &= \mathbf{R}_2 F_\Upsilon. & (2.4)\end{aligned}$$

has a unique solution $\mathbf{U} = (U, V, \Upsilon)^\top$ in $D(\mathcal{A})$. In fact, from (2.2) and (2.4) ($\lambda = 0$) we get

$$\mathbf{K}\Upsilon_{xx} - a\mathbf{N}\Upsilon = -\mathbf{R}_2 F_\Upsilon - \mathbf{B}F_{U,x}, \quad \text{in } [L^2(0,\ell)]^2 \text{ (or } [L^2_*(0,\ell)]^2),$$

which is a well-posed second-order Dirichlet (or Neumann) elliptic PDE for Υ. With Υ known, we have that (2.3) is a well-posed problem for U. Therefore, there exists only one solution to the $\mathcal{A}\mathbf{U} = \mathcal{F}$. Since

$$\|\mathbf{U}\|_{\mathcal{H}} \leq C\|\mathbf{F}\|_{\mathcal{H}},$$

with $C > 0$, we conclude that $0 \in \varrho(\mathcal{A})$. □

Consequently, we establish the following result.

Proposition 2.2. *For any $\mathbf{U}_0 \in \mathcal{H}$ (or $\mathcal{H}_*$), there exists a unique solution $\mathbf{U} = (u, w, u_t, w_t, \theta_1, \theta_2)$ of system (1.9)–(1.12) (or (1.9)–(1.11), (1.13)) satisfying*

$$\begin{aligned} &u, w \in C(0,T; H^1_0(0,\ell)) \cap C^1(0,T; L^2(0,\ell)),\\ &\theta_1, \theta_2 \in C(0,T; L^2(0,\ell)) \cap L^2(0,T; H^1(0,\ell)),\end{aligned}$$

(or

$$\begin{aligned} &u, w \in C(0,T; H^1_0(0,\ell)) \cap C^1(0,T; L^2(0,\ell)),\\ &\theta_1, \theta_2 \in C(0,T; L^2(0,\ell)) \cap L^2(0,T; H^1_*(0,\ell))\big).\end{aligned}$$

3. A generic case. Exponential stability

The asymptotic behavior of solutions is determined by the coupling between the conservative and the dissipative parts of the system, that is, the parameters $\beta_1, \beta_2, \gamma_1, \gamma_2$. A generic assumption is to assume that the vectors (β_1, β_2) and (γ_1, γ_2) are linearly independent. The main aim of this section is to verify that this is a sufficient condition to guarantee the exponential stability of the solutions. We also prove the exponential stability in case when these vectors are linearly dependent.

Because of the result due to J. Prüss [25], it is well-known that the exponential stability depends on the uniform estimate of the resolvent operator over the imaginary axes. This result is contained in the following Theorem.

Theorem 3.1. *Let $S(t) = e^{\mathcal{A}t}$, $t \geq 0$, be a C_0-semigroup of contractions on a Hilbert space. Then $(S(t))_{t\geq 0}$ is exponentially stable if and only if $i\mathbb{R} \subset \rho(\mathcal{A})$ and*

$$\overline{\lim_{|\lambda|\to\infty}} ||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} < \infty. \tag{3.1}$$

Note that taking an inner product in $\mathcal{H}$ with $\mathbf{U}$ and using (2.1), we get

$$\mathrm{Re} \int_0^\ell \left(\Upsilon_x^\top \mathbf{K} \Upsilon_x + a \Upsilon^\top \mathbf{N} \Upsilon \right) dx \leq \|\mathbf{U}\|_{\mathcal{H}} \|\mathbf{F}\|_{\mathcal{H}}. \tag{3.2}$$

Let us remind that the one-dimensional thermoelastic model is exponentially stable. From the mathematical point of view, it means that a one-dimensional hyperbolic equation coupled to one parabolic equation results in a two by two system which is exponentially stable. Can we extend this result? That is to say, if we have two hyperbolic systems coupled with two parabolic system, does the resulting four-by-four system is exponentially stable? It is natural to expect that the answer will depend on the coupling terms which are determined by the matrix $\mathbf{B}$.

We have three possibilities, if $\mathbf{B} = 0$, the system is conservative. If $\mathrm{Rank}(\mathbf{B}) = 2$, we prove that the system is exponentially stable. Finally, the interesting case is when the $\mathrm{Rank}(\mathbf{B}) = 1$, that is, $\mathbf{B}$ can be written as

$$\mathbf{B} = \begin{pmatrix} \beta_1 & \beta_2 \\ \tau\beta_1 & \tau\beta_2 \end{pmatrix} = \begin{pmatrix} \overrightarrow{\beta} \\ \tau\overrightarrow{\beta} \end{pmatrix} \in \mathbb{R}^{2\times 2}, \quad \overrightarrow{\vartheta} = \begin{pmatrix} 1 \\ \tau \end{pmatrix} \in \mathbb{R}^2 \tag{3.3}$$

for some real number τ. In what follows, $\overrightarrow{\vartheta} \cdot Z$ denotes the scalar product of the vector $\overrightarrow{\vartheta}$ and the column vector Z.

Because of the structure of the system, we have that $\overrightarrow{\beta} \cdot Y$ is bounded in sense of Lemma 3.4 below. To show the exponential stability, we need to find another direction denoted as $\overrightarrow{\zeta}$ such that $\overrightarrow{\zeta}$ and $\overrightarrow{\beta}$ is a basis of $\mathbb{R}^2$ and $\overrightarrow{\zeta} \cdot Y$ is bounded. The next Lemma plays an important role in the sequel.

Lemma 3.2. *Let $\overrightarrow{\beta}$ and $\overrightarrow{\zeta}$ be two linearly independent vectors. Assume that for every $\epsilon > 0$ there exists a constant C_ϵ such that the inequality*

$$\left(\|\overrightarrow{\beta} \cdot Y\|^2 + \|\overrightarrow{\zeta} \cdot Y\|^2 \right) \leq \epsilon \|\mathbf{U}\|^2_{\mathcal{H}} + C_\epsilon |\lambda|^p \|\mathbf{F}\|^2_{\mathcal{H}},$$

or

$$\left(\|\overrightarrow{\beta} \cdot U_x\|^2 + \|\overrightarrow{\zeta} \cdot U_x\|^2 \right) \leq \epsilon \|\mathbf{U}\|^2_{\mathcal{H}} + C_\epsilon |\lambda|^p \|\mathbf{F}\|^2_{\mathcal{H}},$$

holds for every $\lambda \in \mathbb{R}$ and $p \in [0, \infty)$. Then we have that

$$\|\mathbf{U}\|^2_{\mathcal{H}} \leq c_\epsilon |\lambda|^p \|\mathbf{F}\|^2_{\mathcal{H}}.$$

Proof. Let us denote

$$\overrightarrow{\beta} = (\beta_1, \beta_2), \quad \text{and} \quad \overrightarrow{\zeta} = (\zeta_1, \zeta_2).$$

By the assumption, there exist two bounded functions $\mathfrak{F}$ and $\mathfrak{G}$ such that $\overrightarrow{\beta} \cdot Y = \beta_1 y_1 + \beta_2 y_2 = \mathfrak{F}$ and $\overrightarrow{\zeta} \cdot Y = \zeta_1 y_1 + \zeta_2 y_2 = \mathfrak{G}$, which implies that

$$y_1 = \frac{\mathfrak{F}\zeta_2 - \mathfrak{G}\beta_2}{\beta_1\zeta_2 - \beta_2\zeta_1}, \qquad y_2 = -\frac{\mathfrak{F}\zeta_1 - \mathfrak{G}\beta_1}{\beta_1\zeta_2 - \beta_2\zeta_1}.$$

Therefore, each component of Y is bounded. In fact, we can deduce that the inequality

$$\|Y\|^2 \leq \epsilon \|\mathbf{U}\|^2_{\mathcal{H}} + C_\epsilon |\lambda|^p \|\mathbf{F}\|^2_{\mathcal{H}} \tag{3.4}$$

holds for every $\lambda \in \mathbb{R}$. Finally, multiplying by $\overline{U}$ Equation (2.3) and using (2.2), we get

$$\int_0^\ell \left(U_x^\top \mathbf{A} U_x + \alpha U^\top \mathbf{N} U \right) dx = \int_0^\ell Y^\top \mathbf{R}_1 Y \, dx - \int_0^\ell U^\top \mathbf{B} \Upsilon_x \, dx + \int_0^\ell U^\top \mathbf{R}_1 F_U \, dx + \int_0^\ell F_Y^\top \mathbf{R}_1 Y \, dx.$$

Recalling the definition of the norm of $\|\mathbf{U}\|_{\mathcal{H}}$, we get that the above identity implies that

$$\|\mathbf{U}\|_{\mathcal{H}}^2 = 2\int_0^\ell Y^\top \mathbf{R}_1 Y\,dx - \int_0^\ell U^\top \mathbf{B}\Upsilon_x\,dx + \int_0^\ell \Upsilon^\top \mathbf{R}_2 \Upsilon\,dx + \int_0^\ell U^\top \mathbf{R}_1 F_U\,dx + \int_0^\ell F_Y^\top \mathbf{R}_1 Y\,dx.$$

Using Poincaré's inequality and the estimate in Equation (3.2), we get

$$\begin{aligned}\|\mathbf{U}\|_{\mathcal{H}}^2 &\leq C\|Y\|^2 + \frac{\epsilon}{4}\|\mathbf{U}\|_{\mathcal{H}}^2 + c_\epsilon\|\Upsilon_x\|^2 + C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}\\ &\leq C\|Y\|^2 + \frac{\epsilon}{2}\|\mathbf{U}\|_{\mathcal{H}}^2 + c_\epsilon\|\mathbf{F}\|_{\mathcal{H}}^2.\end{aligned}$$

In view of the estimate (3.4), we have that $\|\mathbf{U}\|_{\mathcal{H}}^2 \leq c_\epsilon|\lambda|^p\|\mathbf{F}\|_{\mathcal{H}}^2$ for ϵ small and $|\lambda|$ large. Then the first part of our claim follows. Finally, multiplying equation (2.3) by $\overrightarrow{\zeta}\mathbf{R}_1^{-1}$, we get

$$i\lambda\overrightarrow{\zeta}\cdot Y - \overrightarrow{\zeta}\cdot\mathbf{R}_1^{-1}\mathbf{A}U_{xx} + \alpha\overrightarrow{\zeta}\cdot\mathbf{R}_1^{-1}\mathbf{N}U + \overrightarrow{\zeta}\cdot\mathbf{R}_1^{-1}\mathbf{B}\Upsilon_x = \overrightarrow{\zeta}\cdot F_Y.$$

Multiplying the above equation by $\overrightarrow{\zeta}\cdot U$ and using Poincaré's inequality, we get

$$\begin{aligned}\|\overrightarrow{\zeta}\cdot Y\|^2 &= -\langle\overrightarrow{\zeta}\cdot\mathbf{R}_1^{-1}\mathbf{A}U_x\,,\ \overrightarrow{\zeta}\cdot U_x\rangle - \langle\alpha\overrightarrow{\zeta}\cdot\mathbf{R}_1^{-1}\mathbf{N}U + \overrightarrow{\zeta}\cdot\mathbf{R}_1^{-1}\mathbf{B}\Upsilon_x\,,\ \overrightarrow{\zeta}\cdot U\rangle\\ &\quad +\langle\overrightarrow{\zeta}\cdot F_Y\,,\ \overrightarrow{\zeta}\cdot U\rangle - \langle\overrightarrow{\zeta}\cdot Y\,,\ \overrightarrow{\zeta}\cdot F_U\rangle\\ &\leq c_\epsilon\|\overrightarrow{\zeta}\cdot U_x\|_{\mathcal{H}}^2 + \epsilon\|\mathbf{U}_x\|_{\mathcal{H}}^2 + C_\epsilon\|\mathbf{F}_x\|_{\mathcal{H}}^2.\end{aligned}$$

So we have that $\overrightarrow{\zeta}\cdot Y$ is bounded, and using the first part of this Lemma, our claim follows. □

From now on we will consider the dissipative directions,

$$\overrightarrow{N} = (1,-1),\quad \overrightarrow{\beta} = (\beta_1,\beta_2),\quad \overrightarrow{\beta}^\perp = (\beta_2,-\beta_1).$$

We assume that the vectors $\overrightarrow{N}$ and $\overrightarrow{\beta}$ are linearly independent. Our goal is to get estimates for U and Y in the direction of $\overrightarrow{N}$, $\overrightarrow{\beta}$. To this end, we introduce the functions V, W, Ψ, Φ and Θ given by

$$V = \overrightarrow{\beta}\cdot U,\quad W = \overrightarrow{N}\cdot U,\quad \Psi = \overrightarrow{\beta}\cdot Y,\quad \Phi = \overrightarrow{N}\cdot Y,\quad \Theta = \overrightarrow{\beta}\cdot\Upsilon. \tag{3.5}$$

From Equation (2.3) we have that

$$i\lambda\mathbf{R}_1 Y - \mathbf{A}U_{xx} + \alpha\overrightarrow{N}W + \overrightarrow{\vartheta}\Theta_x = \mathbf{R}_1 F_U, \tag{3.6}$$

where $\overrightarrow{\vartheta} = (1,\tau)$ is given in (3.3). Therefore,

$$i\lambda Y - \mathbf{R}_1^{-1}\mathbf{A}U_{xx} + \alpha\mathbf{R}_1^{-1}\overrightarrow{N}W + \mathbf{R}_1^{-1}\overrightarrow{\vartheta}\Theta_x = F_U. \tag{3.7}$$

Taking the inner product with $\overrightarrow{\beta}$, we get

$$i\lambda\overrightarrow{\beta}\cdot Y - \overrightarrow{\beta}\cdot\mathbf{R}_1^{-1}\mathbf{A}U_{xx} + \underbrace{\alpha\overrightarrow{\beta}\cdot\mathbf{R}_1^{-1}\overrightarrow{N}}_{:=\alpha\chi}W + \underbrace{\overrightarrow{\beta}\cdot\mathbf{R}_1^{-1}\overrightarrow{\vartheta}}_{:=\sigma^*}\Theta_x = \overrightarrow{\beta}\cdot F_U, \tag{3.8}$$

where

$$\chi := \frac{\beta_1}{\rho_1} - \frac{\beta_2}{\rho_2} \quad\text{and}\quad \sigma^* := \frac{\beta_1}{\rho_1} + \frac{\beta_2\tau}{\rho_2}. \tag{3.9}$$

This number will be important to describe the asymptotic behavior of corresponding semigroup.

Lemma 3.3. *Let U satisfy homogeneous Dirichlet boundary conditions. With notations as above, we find*

$$|U_x(0)|^2+|U_x(\ell)|^2\le C\left(\|U_x\|^2+\|Y\|^2\right)+c\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}.$$

In case that $\operatorname{Rank}(\mathbf{B})=1$ *and* $\overrightarrow{\beta}$, $\overrightarrow{\beta}\mathbf{R}_1^{-1}\mathbf{A}$ *are linearly dependent, we have*

$$|\overrightarrow{\beta}\cdot U_x(0)|^2+|\overrightarrow{\beta}\cdot U_x(\ell)|^2\le C\left(\|\overrightarrow{\beta}\cdot Y\|^2+\|W\|^2\right)+c\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}, \tag{3.10}$$

$$\|\overrightarrow{\beta}\cdot U_x\|^2\le c_\delta\|\overrightarrow{\beta}\cdot Y\|^2+\frac{\delta}{|\lambda|^2}\|W\|^2+\frac{c}{|\lambda|}\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}. \tag{3.11}$$

Moreover

$$|\Upsilon_x(\ell)|^2\le c|\lambda|\left(\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+\|\overrightarrow{\beta}\cdot U_x\|^2\right)+\frac{C}{|\lambda|}\|\mathbf{F}\|^2_{\mathcal{H}}. \tag{3.12}$$

Proof. Multiplying Equation (2.3) by $(x-\ell/2)\overline{U_x}$, integrating by parts and taking the real part, our first claim follows.

If $\overrightarrow{\beta}$, $\overrightarrow{\beta}\mathbf{R}_1^{-1}\mathbf{A}$ are linearly dependent, we can write $\overrightarrow{\beta}\cdot\mathbf{R}_1^{-1}\mathbf{A}Z=\gamma_0\overrightarrow{\beta}\cdot Z$, for any column vector Z. Hence, from (3.8) , we have

$$i\lambda\overrightarrow{\beta}\cdot Y-\gamma_0\overrightarrow{\beta}\cdot U_{xx}+\alpha\chi W+\sigma^*\Theta_x=\overrightarrow{\beta}\cdot F_U, \tag{3.13}$$

Multiplying Equation (3.13) by $(x-\ell/2)\overrightarrow{\beta}\cdot\overline{U_x}$, integrating by parts and taking the real part, we get

$$|\overrightarrow{\beta}\cdot U_x(0)|^2+|\overrightarrow{\beta}\cdot U_x(\ell)|^2\le C\left(\|\overrightarrow{\beta}\cdot U_x\|^2+\|\overrightarrow{\beta}\cdot Y\|^2\right)+\|W\|^2+c\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}. \tag{3.14}$$

Multiplying Equation (3.13) by $\overline{\overrightarrow{\beta}\cdot U}$, we obtain

$$\|\overrightarrow{\beta}\cdot U_x\|^2\le c\|\overrightarrow{\beta}\cdot Y\|^2+\frac{c}{|\lambda|^2}\|W\|^2+c\|\overrightarrow{\beta}\cdot U\|\|\mathbf{F}\|_{\mathcal{H}}+\frac{c}{|\lambda|}\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}. \tag{3.15}$$

Here, we used (2.2) and

$$\begin{aligned}\operatorname{Re}\int_0^\ell W\overline{\overrightarrow{\beta}\cdot U}\,dx &= \operatorname{Re}\frac{1}{i\lambda}\int_0^\ell W\left(\overline{\overrightarrow{\beta}\cdot Y+\overrightarrow{\beta}\cdot F_U}\right)dx\\ &\le \frac{c}{|\lambda|^2}\|W\|^2+\frac{c}{|\lambda|^2}\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+c\|\overrightarrow{\beta}\cdot Y\|^2,\end{aligned}$$

$$\begin{aligned}\operatorname{Re}\int_0^\ell \Theta_x\overline{\overrightarrow{\beta}\cdot U}\,dx &= \operatorname{Re}\frac{1}{i\lambda}\int_0^\ell \Theta_x\left(\overline{\overrightarrow{\beta}\cdot Y+\overrightarrow{\beta}\cdot F_U}\right)dx\\ &= \operatorname{Re}\frac{1}{i\lambda}\int_0^\ell\Theta_x\overline{\overrightarrow{\beta}\cdot Y}\,dx-\operatorname{Re}\frac{1}{i\lambda}\int_0^\ell\Theta\overline{\overrightarrow{\beta}\cdot F_{U,x}}\,dx\\ &\le \frac{c}{|\lambda|}\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+c\|\overrightarrow{\beta}\cdot Y\|^2.\end{aligned}$$

From inequalities (3.14) and (3.15), our claim follows for λ large enough. Using (2.4), we get

$$\|\Upsilon_{xx}\| \le c|\lambda|\left(\|\Upsilon\|+\|\overrightarrow{\beta}\cdot U_x\|\right)+C\|\mathbf{F}\|_{\mathcal{H}}, \tag{3.16}$$

Since

$$|\Upsilon_x(\ell)|\le C\|\Upsilon\|_{H^{3/2}}\le C\|\Upsilon_x\|^{1/2}\|\Upsilon_{xx}\|^{1/2},$$

we see

$$|\Upsilon_x(\ell)|^2\le c|\lambda|\left(\|\Upsilon_x\|\|\Upsilon\|+\|\Upsilon_x\|\|\overrightarrow{\beta}\cdot U_x\|\right)+C\|\mathbf{F}\|_{\mathcal{H}}\|\Upsilon\|_{H^1}. \tag{3.17}$$

Therefore, we obtain

$$|\Upsilon_x(\ell)|^2 \leq c|\lambda| \left(\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}} + \|\mathbf{F}\|_{\mathcal{H}}^{1/2}\|\mathbf{U}\|_{\mathcal{H}}^{1/2}\|\overrightarrow{\beta}\cdot U_x\| \right) + C\frac{\|\mathbf{F}\|_{\mathcal{H}}}{|\lambda|^{1/2}}(|\lambda|^{1/2}\|\mathbf{F}\|_{\mathcal{H}}^{1/2}\|\mathbf{U}\|_{\mathcal{H}}^{1/2}).$$

Exploiting the arithmetic-geometric mean inequality, we obtain the estimate. □

Lemma 3.4. *With the above notations, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$\|\mathbf{B}Y\|^2 \ \leq \epsilon\|\mathbf{U}\|_{\mathcal{H}}^2 + C_\epsilon\|\mathbf{F}\|_{\mathcal{H}}^2. \tag{3.18}$$

In case that $\mathrm{Rank}(\mathbf{B}) = 1$ *and* $\overrightarrow{\beta}, \quad \mathbf{R}_1^{-1}\overrightarrow{\beta}\mathbf{A}$ *are linearly dependent vectors, with the Neumann boundary condition for* Υ *, we get*

$$\|\overrightarrow{\beta}\cdot Y\|^2 \leq \frac{\epsilon}{|\lambda|^2}\|\mathbf{U}\|_{\mathcal{H}}^2 + C_\epsilon|\lambda|^2\|\mathbf{F}\|_{\mathcal{H}}^2. \tag{3.19}$$

Instead, for the Dirichlet boundary condition for Υ*, we have*

$$\|\overrightarrow{\beta}\cdot Y\|^2 \leq \frac{\epsilon}{|\lambda|^4}\|\mathbf{U}\|_{\mathcal{H}}^2 + C_\epsilon|\lambda|^8\|\mathbf{F}\|_{\mathcal{H}}^2. \tag{3.20}$$

Proof. Multiplying Equation (2.4) by $\displaystyle\int_0^x \overline{\mathbf{B}Y}ds$, we get

$$\begin{aligned}
\|\mathbf{B}Y\|^2 \quad &= \quad \int_0^\ell (i\lambda\mathbf{R}_2\Upsilon - \mathbf{K}\Upsilon_{xx} + a\mathbf{N}\Upsilon - \mathbf{R}_2F_\Upsilon)\left(\int_0^x \overline{\mathbf{B}Y}ds\right)dx \\
&= \quad \int_0^\ell i\lambda\mathbf{R}_2\Upsilon\left(\int_0^x \overline{\mathbf{B}Y}ds\right)dx - \int_0^\ell \mathbf{K}\Upsilon_{xx}\left(\int_0^x \overline{\mathbf{B}Y}ds\right)dx \\
&\quad + \int_0^\ell a\mathbf{N}\Upsilon\left(\int_0^x \overline{\mathbf{B}Y}ds\right)dx - \int_0^\ell \mathbf{R}_2F_\Upsilon\left(\int_0^x \overline{\mathbf{B}Y}ds\right)dx \\
&= \quad \underbrace{\int_0^\ell i\lambda\mathbf{R}_2\Upsilon\left(\int_0^x \overline{\mathbf{B}Y}ds\right)dx}_{:=J} - \underbrace{\mathbf{K}\Upsilon_x(\ell)\int_0^\ell \overline{\mathbf{B}Y}\,dx}_{:=J_1} + S,
\end{aligned} \tag{3.21}$$

where

$$S = \int_0^\ell \mathbf{K}\Upsilon_x\overline{\mathbf{B}Y}\,dx + \int_0^\ell a\mathbf{N}\Upsilon\left(\int_0^x \overline{\mathbf{B}Y}ds\right)dx - \int_0^\ell \mathbf{R}_2F_\Upsilon\left(\int_0^x \overline{\mathbf{B}Y}ds\right)dx,$$

and

$$|S| \leq c\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + \frac{1}{2}\|\mathbf{B}Y\|^2$$

by virtue of Equation (3.2). Note that from (2.3)

$$\begin{aligned}
J &= \int_0^\ell i\lambda \mathbf{R}_2\Upsilon\left(\int_0^x \overline{\mathbf{B}Y}ds\right)dx\\
&= -\int_0^\ell \mathbf{R}_2\Upsilon\left(\mathbf{B}\mathbf{R}_1^{-1}\int_0^x \overline{i\lambda\mathbf{R}_1 Y}ds\right)dx\\
&= -\int_0^\ell \mathbf{R}_2\Upsilon\mathbf{B}\mathbf{R}_1^{-1}\int_0^x \left(\overline{\mathbf{A}U_{ss}-\alpha\mathbf{N}U-\mathbf{B}\Upsilon_s+\mathbf{R}_1F_Y}\right)ds\,dx\\
&= -\int_0^\ell \mathbf{R}_2\Upsilon\mathbf{B}\mathbf{R}_1^{-1}\overline{\mathbf{A}U_x}\,dx+\int_0^\ell \mathbf{R}_2\Upsilon\,dx\ \ \mathbf{B}\mathbf{R}_1^{-1}\overline{\mathbf{A}U_x(0)}\\
&\quad +\int_0^\ell \mathbf{R}_2\Upsilon\mathbf{B}\left(\int_0^x \alpha\mathbf{R}_1^{-1}\mathbf{N}\overline{U}ds+\mathbf{R}_1^{-1}\mathbf{B}\overline{\Upsilon_s}-\overline{F_Y}ds\right)dx. \qquad (3.22)
\end{aligned}$$

Using Young's inequality and Lemma 3.3, we get

$$J\leq \epsilon\|\mathbf{U}\|_{\mathcal{H}}^2+C_\epsilon\|\mathbf{F}\|_{\mathcal{H}}^2$$

Denote

$$R=\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+c\|\mathbf{F}\|_{\mathcal{H}}^2$$

Lemma 3.3 implies that there exists $c>0$ such that

$$|\Upsilon_x(\ell)|^2\leq c|\lambda|\|\overrightarrow{\beta}\cdot U_x\|^2+c|\lambda|R.$$

On the other hand, using (2.3) and Lemma 3.3, we get

$$\begin{aligned}
\left|\int_0^\ell \mathbf{B}Y\,dx\right| &\leq \frac{C}{|\lambda|}\left(\|\Upsilon_x\|+|U_x(0)|+|U_x(\ell)|+\|W\|+\|\mathbf{F}\|_{\mathcal{H}}\right)\\
&\leq \frac{C}{|\lambda|}\left(\|\mathbf{U}\|_{\mathcal{H}}+\|\mathbf{F}\|_{\mathcal{H}}\right).
\end{aligned}$$

From the two above inequalities, we obtain that J_1 defined in equality (3.21) is estimated as

$$|J_1|\leq \frac{c}{|\lambda|^{1/2}}\left(\|\mathbf{U}\|_{\mathcal{H}}^2+\|\mathbf{F}\|_{\mathcal{H}}^2\right).$$

Therefore, the right-hand side to Equation (3.21) can be estimated as

$$\|\mathbf{B}Y\|^2\leq \epsilon\|\mathbf{U}\|_{\mathcal{H}}^2+c_\epsilon\|\mathbf{F}\|_{\mathcal{H}}^2,$$

for λ large enough. Therefore, the first part of this Lemma follows.

Finally, let us suppose that Rank$(\mathbf{B})=1$ and $\overrightarrow{\beta}$, $\overrightarrow{\beta}\mathbf{R}_1^{-1}\mathbf{A}$ are linearly dependent. In this case, we have that $\overrightarrow{\beta}\cdot\mathbf{R}_1^{-1}\mathbf{A}Z=\gamma_0\overrightarrow{\beta}\cdot Z$, for any column vector Z. For

$$\mathbf{B}=\begin{pmatrix}\overrightarrow{\beta}\\ \tau\overrightarrow{\beta}\end{pmatrix},\quad \text{we further have}\quad \mathbf{B}Y=\begin{pmatrix}\overrightarrow{\beta}\cdot Y\\ \tau\overrightarrow{\beta}\cdot Y\end{pmatrix}. \qquad (3.23)$$

Therefore, from (3.22) we can write

$$|J| \leq c\|\Upsilon_x\|\|\Psi\|+c\|\Upsilon\|\|W\|+R.$$

for some positive constant c. Using that $i\lambda W-\Phi=F_W$ with $F_W=\vec{N}\cdot F_U$, we get that

$$|J| \leq c_\epsilon\|\Upsilon_x\|^2+\epsilon\|\Psi\|^2+\frac{c}{|\lambda|}\|\Upsilon\|\|\Phi\|+R. \qquad (3.24)$$

An application of Young's inequality to inequality (3.24) yields

$$|J| \quad \leq \quad c_\epsilon|\lambda|^8\|\mathbf{F}\|_{\mathcal{H}}^2+\epsilon\|\overrightarrow{\beta}\cdot Y\|^2+\frac{\epsilon}{|\lambda|^4}\|\mathbf{U}\|_{\mathcal{H}}^2.$$

We also get from inequality (3.24)

$$|J| \quad \leq \quad c_\epsilon|\lambda|^2\|\mathbf{F}\|_{\mathcal{H}}^2+\epsilon\|\overrightarrow{\beta}\cdot Y\|^2+\frac{\epsilon}{|\lambda|^2}\|\mathbf{U}\|_{\mathcal{H}}^2.$$

Therefore, our conclusion follows in case of Neumann boundary condition. Now we consider the Dirichlet boundary condition, that is our next task is to estimate J_1. Because of (3.23), we have

$$|J_1|\leq c|\Upsilon_x(\ell)|\left|\int_0^\ell\overrightarrow{\beta}\cdot Y\;ds\right|.$$

From (3.13) we get

$$|\lambda|\left|\int_0^\ell\overrightarrow{\beta}\cdot Y\;ds\right|\leq c|\overrightarrow{\beta}\cdot U_x(0)|+c|\overrightarrow{\beta}\cdot U_x(\ell)|+c\|W\|+R.$$

Using the second part of Lemma 3.3, we get

$$|\lambda|\left|\int_0^\ell\overrightarrow{\beta}\cdot Y\;ds\right|\leq C\left(\|\overrightarrow{\beta}\cdot Y\|+\|W\|\right)+R.$$

Therefore, we obtain

$$|\lambda|\left|\int_0^\ell\overrightarrow{\beta}\cdot Y\;ds\right|\leq C\left(\|\overrightarrow{\beta}\cdot Y\|+\frac{1}{|\lambda|}\|\mathbf{U}\|_{\mathcal{H}}\right)+R. \tag{3.25}$$

for some positive constant C. Concluding from (3.17)

$$|\Upsilon_x(\ell)|^2\leq c|\lambda|\|\Upsilon_x\|\|\overrightarrow{\beta}\cdot U_x\|+c|\lambda|\,R$$

and using inequality (3.11), we get

$$|\Upsilon_x(\ell)|\leq c|\lambda|^{1/2}\|\Upsilon_x\|^{1/2}\|\overrightarrow{\beta}\cdot Y\|^{1/2}+c\|W\|^{1/2}\|\Upsilon_x\|^{1/2}+c|\lambda|^{1/2}\,R^{1/2}.$$

It can be written as

$$|\Upsilon_x(\ell)|\leq c|\lambda|^{1/2}\|\Upsilon_x\|^{1/2}\|\overrightarrow{\beta}\cdot Y\|^{1/2}+\frac{c}{|\lambda|^{1/2}}\|\mathbf{U}\|_{\mathcal{H}}^{1/2}\|\Upsilon_x\|^{1/2}+c|\lambda|^{1/2}\,R^{1/2}.$$

From (3.25) and the last estimate, we get

$$\begin{aligned}|J_1| \quad \leq \quad & \frac{c}{|\lambda|^{1/2}}\|\Upsilon_x\|^{1/2}\|\Psi\|^{3/2}+\frac{c}{|\lambda|^{1/2}}\|\mathbf{U}\|_{\mathcal{H}}^{1/2}\|\Upsilon_x\|^{1/2}\|\Psi\|+c|\lambda|^{1/2}\,R^{1/2}\|\Psi\|\\ &+\frac{c}{|\lambda|^{1/2}}\|\Upsilon_x\|^{1/2}\|\overrightarrow{\beta}\cdot Y\|^{1/2}\|\mathbf{U}\|_{\mathcal{H}}+\frac{c}{|\lambda|^{3/2}}\|\mathbf{U}\|_{\mathcal{H}}^{1/2}\|\Upsilon_x\|^{1/2}\|\mathbf{U}\|_{\mathcal{H}}+\frac{c}{|\lambda|^{1/2}}\,R^{1/2}\|\mathbf{U}\|_{\mathcal{H}}\\ &+\frac{c}{|\lambda|^{1/2}}\|\Upsilon_x\|^{1/2}\|\overrightarrow{\beta}\cdot Y\|^{1/2}R^{1/2}+\frac{c}{|\lambda|^{3/2}}\|\mathbf{U}\|_{\mathcal{H}}^{1/2}\|\Upsilon_x\|^{1/2}R^{1/2}+\frac{c}{|\lambda|^{1/2}}\,R.\end{aligned}$$

Using several times Young's inequality, we find

$$|J_1|\leq\epsilon\|\Psi\|^2+\frac{\epsilon}{|\lambda|^4}\|\mathbf{U}\|_{\mathcal{H}}^2+c_\epsilon|\lambda|^8\|\mathbf{F}\|_{\mathcal{H}}^2$$

Plugging J and J_1 into (3.21), we obtain

$$\|\mathbf{B}Y\|^2 \quad \leq \quad \frac{\epsilon}{|\lambda|^4}\|\mathbf{U}\|_{\mathcal{H}}^2+c_\epsilon|\lambda|^8\|\mathbf{F}\|_{\mathcal{H}}^2.$$

$\square$

Since $\overrightarrow{\beta}$ and $\overrightarrow{N}$ are linearly independent, we can write

$$\overrightarrow{\beta}\mathbf{R}_1^{-1}\mathbf{A} = \gamma_0\overrightarrow{\beta} + \gamma_1\overrightarrow{N}, \tag{3.26}$$

$$\overrightarrow{N}\mathbf{A}^{-1}\mathbf{R}_1 = \tau_0\overrightarrow{\beta} + \tau_1\overrightarrow{N}. \tag{3.27}$$

So we have that $\overrightarrow{\beta}$ and $\overrightarrow{\beta}\mathbf{R}_1^{-1}\mathbf{A}$ are linearly independent if and only if $\gamma_1 \neq 0$. Moreover, it is not difficult to see that the spectrum of $\mathbf{R}^{-1}\mathbf{A}$ is given by

$$\sigma(\mathbf{R}^{-1}\mathbf{A}) = \left\{\lambda_0^+,\ \lambda_0^-\right\},$$

where $\lambda_0^\pm$ are positive eigenvalues, given as

$$\begin{aligned}\lambda_0^\pm &= \frac{a_{11}\rho_1^{-1} + a_{22}\rho_2^{-1} \pm \sqrt{\left(a_{11}\rho_1^{-1} + a_{22}\rho_2^{-1}\right)^2 - 4\rho_1^{-1}\rho_2^{-1}\left(a_{11}a_{22} - a_{12}^2\right)}}{2}\\ &= \frac{a_{11}\rho_1^{-1} + a_{22}\rho_2^{-1} \pm \sqrt{\left(a_{11}\rho_1^{-1} - a_{22}\rho_2^{-1}\right)^2 + 4\rho_1^{-1}\rho_2^{-1}a_{12}^2}}{2}.\end{aligned}$$

Note that

$$\lambda_0^+ = \lambda_0^- \quad \Leftrightarrow \quad a_{12} = 0 \quad \text{and} \quad a_{11}\rho_1^{-1} - a_{22}\rho_2^{-1} = 0.$$

That is, we have eigenvalues of algebraic multiplicity 2 if and only if $\mathbf{R}^{-1}\mathbf{A}$ is diagonal.

Lemma 3.5. *$i\mathbb{R} \subset \varrho(\mathcal{A})$ if one of the following conditions holds.*

(1) The rank of matrix $\mathbf{B}$ is 2,

(2) The rank of matrix $\mathbf{B}$ is 1, the vectors $\overrightarrow{\beta}$, $\overrightarrow{\beta}\mathbf{R}_1^{-1}\mathbf{A}$ are linearly independent and

- **(i):** *$\gamma_1\chi > 0$, or*
- **(ii):** *$\tau_1 \leq 0$, or*
- **(iii):** *$\gamma_1\chi < 0$ and $-\frac{\alpha}{\gamma_1}\chi \neq \left(\frac{k\pi}{l}\right)^2$ for any $k \in \mathbb{N}$.*

(3) Rank$(\mathbf{B}) = 1$, *the vectors $\overrightarrow{\beta}$, $\overrightarrow{\beta}\mathbf{R}_1^{-1}\mathbf{A}$ are linearly dependent ($\gamma_1 = 0$), and $\chi \neq 0$.*

Proof. Note that the domain of $\mathcal{A}$ has a compact embedding into the phase space. So to prove that the imaginary axes is contained in the resolvent set, it is enough to show that there is no imaginary eigenvalues. In fact, let us suppose that $\mathbf{U}$ is an eigenvector with an imaginary eigenvalue, then we have

$$i\lambda\mathbf{U} = \mathcal{A}\mathbf{U}.$$

Taking the inner product with $\mathbf{U}$ and considering the real part as well as using inequality (2.1), we conclude that $\boldsymbol{\Upsilon} = 0$. If the rank of $\mathbf{B}$ is equal to 2, then $Y_x = 0$, which implies that $U = Y = 0$, that is, $\mathbf{U} = 0$ which is a contradiction. Therefore, our first condition holds. Let us suppose that the rank of $\mathbf{B}$ equals to 1. This implies that $\overrightarrow{\beta} \cdot Y_x = 0$, so we have that $\overrightarrow{\beta} \cdot Y = \overrightarrow{\beta} \cdot U = 0$. From (3.7) we get that

$$i\lambda\overrightarrow{\beta} \cdot Y - \overrightarrow{\beta} \cdot \mathbf{R}_1^{-1}\mathbf{A}U_{xx} + \alpha\chi W = 0.$$

So, using (3.26) and that $\overrightarrow{\beta} \cdot Y = \overrightarrow{\beta} \cdot U = 0$, and recalling the definition of χ, we have that the above equation implies

$$-\gamma_1 W_{xx} + \alpha\chi W = 0. \tag{3.28}$$

Therefore, if $\gamma_1\chi > 0$, the only solution of the above equation must be $W = 0$, which implies that $Y = U = 0$. So we conclude that there is no imaginary eigenvalues. Similarly, we have that

$$i\lambda\overrightarrow{N}\cdot\mathbf{A}^{-1}\mathbf{R}_1 Y-\overrightarrow{N}\cdot U_{xx}+\underbrace{\alpha\overrightarrow{N}\cdot\mathbf{A}^{-1}\overrightarrow{N}}_{:=\alpha_N}W=0.$$

Using $i\lambda\Phi=-\lambda^2 W$, we get

$$-\tau_1\lambda^2 W-W_{xx}+\alpha_N W=0.$$

Multiplying by $\overline{W}$ the above equation, we conclude that $W=0$ provided $\tau_1\le 0$ (Note that $\alpha_N>0$). So, condition (ii) holds. If $\gamma_1\chi<0$, from (3.28) the eigenvalues of $-(\cdot)_{xx}$ must be of the form $k^2\pi^2/l^2$ with $k\in\mathbb{N}$. Therefore, we have no eigenvalues if $\alpha\chi\neq-\gamma_1 k^2\pi^2/l^2$ for any $k\in\mathbb{N}$. Finally, if $\gamma_1=0$, from (3.28), we conclude that $W=0$ provided $\chi\neq 0$. Therefore, our claim follows. □

We can now show our first stability result.

Theorem 3.6. *Let us suppose that one of the following assumptions hold true:*

(1) The rank of the matrix $\mathbf{B}$ *is 2,*

(2) The rank of the matrix $\mathbf{B}$ *is 1 and the vectors* $\overrightarrow{\beta}$, $\overrightarrow{\beta}\mathbf{R}_1^{-1}\mathbf{A}$ *are linearly independent and one of the condition (i), (ii), or (iii) of Lemma 3.5 holds.*

(3) The rank of the matrix $\mathbf{B}$ *is 1 and the vectors* $\overrightarrow{\beta}$, $\overrightarrow{\beta}\mathbf{R}_1^{-1}\mathbf{A}$ *are linearly dependent,* $\tau_1=\overrightarrow{N}\mathbf{A}^{-1}\mathbf{R}_1\cdot\overrightarrow{\beta}^{\perp}\le 0$.

Then the operator $\mathcal{A}$ *generates a semigroup which is exponentially stable.*

Proof. If the rank of the matrix $\mathbf{B}$ is 2, the result follows immediately from Lemma 3.2 with $p=0$ and Lemma 3.4.

Let us suppose that case (2) holds. Since the rank of $\mathbf{B}$ is equal to 1, there exists a positive constant c such that $\|\overrightarrow{\beta}\cdot Y\|\le c\|\mathbf{B}Y\|$. Then Lemma 3.4 implies that for any ϵ there exists C_ϵ such that

$$\|\overrightarrow{\beta}\cdot Y\|^2\le\epsilon\|\mathbf{U}\|_{\mathcal{H}}^2+C_\epsilon\|\mathbf{F}\|_{\mathcal{H}}^2. \tag{3.29}$$

Multiplying Equation (3.8) by $\mathbf{R}_1^{-1}\overrightarrow{\beta}\mathbf{A}\cdot\overline{U}$, we get

$$\|\mathbf{R}_1^{-1}\overrightarrow{\beta}\mathbf{A}\cdot U_x\|^2\le c\|\mathbf{U}\|_{\mathcal{H}}\|\overrightarrow{\beta}\cdot Y\|+c\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+\frac{c}{|\lambda|^2}\|\mathbf{U}\|_{\mathcal{H}}^2+c\|\mathbf{F}\|_{\mathcal{H}}^2.$$

Using (3.29) once more time, we conclude

$$\|\mathbf{R}_1^{-1}\overrightarrow{\beta}\mathbf{A}\cdot U_x\|^2\le\epsilon c\|\mathbf{U}\|_{\mathcal{H}}^2+c_\epsilon\|\mathbf{F}\|_{\mathcal{H}}^2, \tag{3.30}$$

for λ large enough. Since $\mathbf{R}_1\overrightarrow{\beta}\mathbf{A}$ and $\overrightarrow{\beta}$ are linearly independent according to Lemma 3.2, we conclude that $\|\mathbf{U}\|_{\mathcal{H}}^2\le C\|\mathbf{F}\|_{\mathcal{H}}^2$, which implies the exponential stability.

Finally, let us assume that condition (3) is satisfied. Multiplying (3.6) by $\mathbf{A}^{-1}$ and then by $\overrightarrow{N}$ as well recalling the definition of W, we get

$$i\lambda\overrightarrow{N}\cdot\mathbf{A}^{-1}\mathbf{R}_1 Y-W_{xx}+\alpha\overrightarrow{N}\cdot\mathbf{A}^{-1}\overrightarrow{N}\cdot W+\overrightarrow{N}\cdot\mathbf{A}^{-1}\vec{\vartheta}\Theta_x=\overrightarrow{N}\cdot\mathbf{A}^{-1}\mathbf{R}_1 F_U.$$

Since $\overrightarrow{\beta}$ and $\overrightarrow{N}$ are linearly independent vectors, we have that there exist two real numbers τ_1 and τ_0 such that

$$\overrightarrow{N}\mathbf{A}^{-1}\mathbf{R}_1=\tau_1\overrightarrow{N}+\tau_0\overrightarrow{\beta},\quad\Rightarrow\quad\overrightarrow{N}\mathbf{A}^{-1}\mathbf{R}_1\cdot Z=\tau_1\overrightarrow{N}\cdot Z+\tau_0\overrightarrow{\beta}\cdot Z, \tag{3.31}$$

for any column vector Z. Therefore, we have

$$i\lambda\tau_1\Phi-W_{xx}+\underbrace{\alpha\overrightarrow{N}\cdot\mathbf{A}^{-1}\overrightarrow{N}}_{:=\alpha_0}W+\underbrace{\overrightarrow{N}\cdot\mathbf{A}^{-1}\vec{\vartheta}}_{:=\sigma_0}\Theta_x+i\lambda\tau_0\Psi=\underbrace{\overrightarrow{N}\cdot\mathbf{A}^{-1}\mathbf{R}_1^{-1}F_U}_{:=F_\Phi}. \tag{3.32}$$

Multiplying equation (3.32) by $\overline{W}$ and using (3.25) we get

$$-\tau_1\|\Phi\|^2+\|W_x\|^2\leq \frac{c}{|\lambda|}\|\Phi\|^2+C\int_0^\ell(|\Theta\overline{W}_x|+|V_x\overline{W}_x|+|F_U\overline{W}|)\,dx+C\|\mathbf{F}\|^2_{\mathcal{H}}.$$

Note that the hypothesis

$$\tau_1(\beta_1+\beta_2)=\overrightarrow{N}\mathbf{A}^{-1}\mathbf{R}_1\cdot(\beta_2,-\beta_1)=\overrightarrow{N}\mathbf{R}^{-1}\mathbf{A}\cdot\overrightarrow{\beta}^{\perp}\leq 0$$

implies that $\tau_1\leq 0$ [2]. Therefore, we have

$$-\tau_1\|\Phi\|^2+\|W_x\|^2\leq C\|V_x\|^2+C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+C\|\mathbf{F}\|^2_{\mathcal{H}}. \tag{3.33}$$

Since $\overrightarrow{\beta}$ and $\overrightarrow{N}$ are linearly independent vectors, using again Lemma 3.2 with $p=0$, we get $\|\mathbf{U}\|^2_{\mathcal{H}}\leq C\|\mathbf{F}\|^2_{\mathcal{H}}$. The proof is now complete. ☐

4. Lack of exponential stability

System (1.9)–(1.11) has two different temperatures. In practice, it will have two dissipative mechanisms only when the rank of $\mathbf{B}$ is equal to 2. Otherwise, the behaviour of the system is similar to the case when only one temperature is present. From now on, we assume that the rank of $\mathbf{B}$ is equal to 1. In this Section, we restrict our attention to the Dirichlet-Neumann case. We will study the necessary and sufficient conditions on the coefficients to furnish polynomial or exponential stability. Our starting point is to show that the system is not exponentially stable in general.

Theorem 4.1. *If* $\mathrm{Rank}(\mathbf{B})=1$*, the vectors* $\overrightarrow{\beta}$*,* $\overrightarrow{\beta}\mathbf{R}_1^{-1}\mathbf{A}$ *are linearly dependent and* $\overrightarrow{N}\cdot\mathbf{R}_1\mathbf{A}\overrightarrow{\beta}^{\perp}>0$*, the system is not exponentially stable.*

Proof. To make the calculations easier, we will assume in this section that the interval length is π. That is, $\ell=\pi$. Let us consider the case when $F_Y=F_\Upsilon=0$. We know that there exists a real number γ_0 such that

$$\overrightarrow{\beta}\cdot\mathbf{R}_1^{-1}\mathbf{A}Z=\gamma_0\overrightarrow{\beta}\cdot Z \tag{4.1}$$

for any column vector Z. Recalling (3.5), system (2.2)–(2.4) can be written as

$$\begin{aligned}
-\lambda^2V-\gamma_0V_{xx}+\alpha_0W+\sigma_0\Theta_x &= \overrightarrow{\beta}\cdot F_U,\\
-\lambda^2W-d_0W_{xx}+\alpha_1W+\sigma_1\Theta_x-d_1V_{xx} &= \overrightarrow{N}\cdot F_U,\\
i\lambda b_1\theta_1+i\lambda b_2\theta_2-K_{11}\theta_{1xx}-K_{12}\theta_{2xx}-i\lambda V_x-a(\theta_2-\theta_1) &= 0,\\
i\lambda b_2\theta_1+i\lambda b_3\theta_2-K_{21}\theta_{1xx}-K_{22}\theta_{2xx}-i\tau\lambda V_x+a(\theta_2-\theta_1) &= 0.
\end{aligned}$$

Taking $\overrightarrow{\beta}\cdot F_U=0$ and $\overrightarrow{N}\cdot F_U=\sin(\mu x)$, we can look for solutions of the form

$$V=A\sin(\mu x),\qquad W=B\sin(\mu x),\qquad \theta_1=D\cos(\mu x),\qquad \theta_2=E\cos(\mu x).$$

Note that $\Theta=\beta_1\theta_1+\beta_2\theta_2$. The above system is equivalent to

$$\begin{aligned}
p_1A+\alpha_0B-\sigma_0\mu C&=0,\\
d_1\mu^2A+p_2B-\sigma_1\mu C&=1,\\
q_1D+q_2E-i\lambda\mu A&=0,\\
q_3D+q_4E-i\lambda\tau\mu A&=0,
\end{aligned}$$

[2] We assume here that $\beta_1+\beta_2$ is positive. However, this is not an extra condition because the case when $\beta_1+\beta_2<0$ can be transformed to $\beta_1+\beta_2>0$ by noting that $(u(\ell-x,t),w(\ell-x,t),\theta_1(\ell-x,t),\theta_2(\ell-x,t))$ is the solution of the system obtained by replacing β_1, γ_1, β_2 and γ_2 with $-\beta_1$, $-\gamma_1$, $-\beta_2$ and $-\gamma_2$ respectively.

where $C = \beta_1 D + \beta_2 E$ and

$$p_1(\lambda) = -\lambda^2 + \gamma_0\mu^2, \quad p_2(\lambda) = -\lambda^2 + d_0\mu^2 + \alpha_1,$$

and

$$q_1 = ib_1\lambda + K_{11}\mu^2 + a, \qquad q_2 = ib_2\lambda + K_{12}\mu^2 - a, \tag{4.2}$$

$$q_3 = ib_2\lambda + K_{21}\mu^2 - a, \qquad q_4 = ib_3\lambda + K_{22}\mu^2 + a. \tag{4.3}$$

From the two last equations, we get

$$D = \frac{q_4 - \tau q_2}{q_1q_4 - q_2q_3} i\lambda\mu A, \qquad E = -\frac{q_3 - \tau q_1}{q_1q_4 - q_2q_3} i\lambda\mu A. \tag{4.4}$$

Therefore,

$$C = \beta_1 D + \beta_2 E = \frac{\beta_1 q_4 - \beta_1\tau q_2 - \beta_2 q_3 + \beta_2\tau q_1}{q_1q_4 - q_2q_3} i\lambda\mu A.$$

We can write the above system as

$$p_1 A + \alpha_0 B - \sigma_0\mu C = 0, \tag{4.5}$$

$$d_1\mu^2 A + p_2 B - \sigma_1\mu C = 1, \tag{4.6}$$

$$-i\lambda\mu A + p_3 C = 0, \tag{4.7}$$

where

$$p_3 = \frac{q_1q_4 - q_2q_3}{\beta_1 q_4 - \beta_1\tau q_2 - \beta_2 q_3 + \beta_2\tau q_1} = O(\mu^2).$$

The matrix associated to system (4.5)–(4.7) is given by

$$\begin{bmatrix} p_1 & \alpha_0 & -\sigma_0\mu \\ d_1\mu^2 & p_2 & -\sigma_1\mu \\ -i\lambda\mu & 0 & p_3 \end{bmatrix}.$$

It follows that

$$B = \frac{p_1p_3 - i\lambda\sigma_0\mu^2}{p_1p_2p_3 - d_1\alpha_0\mu^2 p_3 + i\lambda\mu^2(\alpha_0\sigma_1 - \sigma_0 p_2)}.$$

Now, we take λ such that $p_2(\lambda) = c_0$, that is

$$\lambda^2 = d_0\mu^2 + \alpha_1 - c_0,$$

and we select c_0 to minimize the degree of the following polynomial,

$$\begin{aligned} p_1p_2p_3 - d_1\alpha_0\mu^2 p_3 &= p_3\left(c_0p_1 - d_1\alpha_0\mu^2\right) \\ &= p_3\left[c_0\left(-\lambda^2 + \gamma_0\mu^2\right) - d_1\alpha_0\mu^2\right] \\ &= p_3\left[c_0\left(-d_0\mu^2 - \alpha_1 + c_0 + \gamma_0\mu^2\right) - d_1\alpha_0\mu^2\right] \\ &= p_3\left\{c_0\left[-\left(d_0 - \gamma_0\right)\mu^2 - \alpha_1 + c_0\right] - d_1\alpha_0\mu^2\right\} \\ &= p_3\left[-c_0\left(d_0 - \gamma_0\right)\mu^2 - c_0\alpha_1 + c_0^2 - d_1\alpha_0\mu^2\right]. \end{aligned}$$

Taking c_0 such that $(d_0 - \gamma_0)c_0 = -d_1\alpha_0$, we get

$$p_1p_2p_3 - d_1\alpha_0\mu^2 p_3 = c_2 p_3, \quad \text{where} \quad c_2 = \frac{d_1^2\alpha_0^2}{(d_0 - \gamma_0)^2} + \frac{\alpha_1 d_1\alpha_0}{d_0 - \gamma_0}.$$

Therefore, B can be rewritten as

$$B = \frac{p_1 p_3 - i\lambda\sigma_0\mu^2}{c_2 p_3 + i\lambda\mu^2(\alpha_0\sigma_1 - \sigma_0 p_2)}.$$

In particular, we have $B = O(\mu)$ for λ large. Which implies that

$$\rho_2^{-1}\|U\|_{\mathcal{H}}^2 \geq \lambda^2 \int_0^\pi |B\sin(\mu x)|^2\,dx = O(\lambda^2 B^2) = O(\lambda^4) \text{ at least.} \tag{4.8}$$

So $\|U\|_{\mathcal{H}}^2 \to \infty$ when $\lambda \to \infty$. Therefore, there is no exponential stability.

□

5. Polynomial decay

In this section we prove that solution decays polynomially to zero if $\text{Rank}(\mathbf{B}) = 1$, the vectors $\overrightarrow{\beta}$, $\overrightarrow{\beta}\mathbf{R}_1^{-1}\mathbf{A}$ are linearly dependent and $\tau_1 = \overrightarrow{N} \cdot \mathbf{R}_1^{-1}\mathbf{A}(\beta_2, -\beta_1) > 0$. Here, we assume that $\beta_1 + \beta_2 \neq 0$ and $\overrightarrow{\beta}$, $\overrightarrow{N}$ are linearly independent. Our argument is based on the following theorem (see [6, Theorem 2.4]).

Theorem 5.1. *Let $(S(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with the generator $\mathcal{A}$ such that $i\mathbb{R} \subset \varrho(\mathcal{A})$. Then, for a fixed $\alpha > 0$,*

$$\|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = O\left(|\lambda|^\alpha\right), \quad \lambda \to \infty \qquad \Longleftrightarrow \qquad \|S(t)\mathcal{A}^{-1}\|_{\mathcal{L}(\mathcal{H})} = O\left(t^{-1/\alpha}\right), \quad t \to \infty.$$

The main result of this section is the following theorem.

Theorem 5.2. *Under the hypothesis of Theorem 4.1, the solution $U(t) = e^{\mathcal{A}t}U_0$ decays polynomially as*

$$\|U(t)\|_{\mathcal{H}_D} \leq \frac{c}{t^{1/6}}\|U_0\|_{D(\mathcal{A})}$$

in case of the Dirichlet boundary condition. For the Neumann boundary condition, we have that

$$\|U(t)\|_{\mathcal{H}_N} \leq \frac{c}{t^{1/2}}\|U_0\|_{D(\mathcal{A})}.$$

Provided $\tau \neq \rho_2/\rho_1$, in case of $\mathbf{A}$ being diagonal and $\tau \neq \rho_2\beta_2/\rho_1\beta_1$ otherwise, where τ is defined in (3.3). Moreover, in this later case, the rate of decay is optimal.

Proof. From (3.8) and hypothesis (4.1) we have

$$i\lambda\overrightarrow{\beta} \cdot Y - \gamma_0 V_{xx} + \alpha\chi W + \sigma^*\Theta_x = \overrightarrow{\beta} \cdot F_U, \tag{5.1}$$

where χ and σ^* are defined in (3.9). Multiplying Equation (5.1) by $\overline{W}$, we get

$$\alpha\chi\|W\|_{L^2}^2 = \langle\Psi, \overline{\Phi}\rangle - \gamma_0\langle V_x, \overline{W_x}\rangle - \sigma^*\langle\Theta_x, \overline{W}\rangle + R^*, \tag{5.2}$$

where R^* is such that $|R^*| \leq c\|\mathbf{U}\|_{\mathcal{H}}\,\|\mathbf{F}\|_{\mathcal{H}}$. Multiplying (3.32) by $\overline{V}$, we get that

$$\langle W_x, \overline{V}_x\rangle = \tau_1\langle\Phi, \overline{\Psi}\rangle + \alpha_0\langle W, \overline{V}\rangle + \sigma_0\langle\Theta_x, \overline{V}\rangle + \tau_0\|\Psi\|^2 + \langle F_\Phi, \overline{V}\rangle + \tau_1\langle\Phi, \overline{\overrightarrow{\beta} \cdot F_U}\rangle + \tau_0\langle\Psi, \overline{\overrightarrow{\beta} \cdot F_U}\rangle.$$

Plugging that into (5.2), we get

$$\alpha\chi\|W\|^2 = (1 - \tau_1\gamma_0)\langle\Psi, \overline{\Phi}\rangle - \gamma_0\alpha_0\langle W, \overline{V}\rangle - \gamma_0\sigma_0\langle\Theta_x, \overline{V}\rangle - \sigma^*\langle\Theta_x, \overline{W}\rangle - \tau_0\gamma_0\|\Psi\|^2 + R^*. \tag{5.3}$$

Multiplying with λ^2, we obtain

$$\|\Phi\|^2 \leq c|\lambda|^4\|\Psi\|^2 + c|\lambda|\|\Theta_x\|^2 + c|\lambda|^2\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + c|\lambda|^2\|\mathbf{F}\|_{\mathcal{H}}^2.$$

Using Lemma 3.4, we get

$$\|\Phi\|^2 \leq \epsilon\|\mathbf{U}\|_{\mathcal{H}}^2 + c|\lambda|^{12}\|\mathbf{F}\|_{\mathcal{H}}^2.$$

Analogously, applying Lemma 3.4, we find that $\|\Psi\|^2$ is bounded. Finally, from Lemma 3.2, our claim follows in case of Dirichlet boundary condition.

Now let us consider the case of Neumann boundary condition. In view of the hypothesis, Equation (2.4) can be written as

$$i\lambda\mathbf{R}_2\Upsilon-\mathbf{K}\Upsilon_{xx}+a\mathbf{N}\Upsilon+\overrightarrow{\vartheta}\Psi_x=\mathbf{R}_2F_\Upsilon.$$

Multiplying the above equation by $\mathbf{K}^{-1}$ and taking a scalar product with $\overrightarrow{\beta}$, we get

$$i\lambda\underbrace{\overrightarrow{\beta}\cdot\mathbf{K}^{-1}\mathbf{R}_2\Upsilon}_{:=\overrightarrow{R}_3\cdot\Upsilon}-\Theta_{xx}+\underbrace{a\overrightarrow{\beta}\cdot\mathbf{K}^{-1}\mathbf{N}\Upsilon}_{:=\overrightarrow{R}_4\cdot\Upsilon}+\underbrace{\overrightarrow{\beta}\cdot\mathbf{K}^{-1}\overrightarrow{\vartheta}}_{:=r}\Psi_x=\overrightarrow{\beta}\cdot\mathbf{K}^{-1}\mathbf{R}_2F_\Upsilon.$$

Therefore, we have that

$$i\lambda\overrightarrow{R}_3\cdot\Upsilon-\Theta_{xx}+\overrightarrow{R}_4\cdot\Upsilon+r\Psi_x=F_\Theta.$$

Integrating over the interval $[0,x]$, we obtain

$$i\lambda\int_0^x\overrightarrow{R}_3\cdot\Upsilon ds-\Theta_x+\int_0^x\overrightarrow{R}_4\cdot\Upsilon\,ds+r\Psi=\int_0^x F_\Theta\,ds.$$

Multiplying it by $\overline{\Phi}$, we get

$$r\langle\Psi,\overline{\Phi}\rangle=\underbrace{-\langle i\lambda\int_0^x\left(\overrightarrow{R}_3\Upsilon+\frac{1}{i\lambda}\overrightarrow{R}_4\Upsilon\right)\,ds,\overline{\Phi}\rangle}_{:=J^*}+\langle\Theta_x,\overline{\Phi}\rangle+R^*. \tag{5.4}$$

So we have that

$$\begin{aligned}
J^* &= \langle\int_0^x\left(\overrightarrow{R}_3\cdot\Upsilon+\frac{1}{i\lambda}\overrightarrow{R}_4\cdot\Upsilon\right)\,ds,\overline{i\lambda\Phi}\rangle\\
&= \frac{1}{\tau_1}\langle\int_0^x\left(\overrightarrow{R}_3\cdot\Upsilon+\frac{1}{i\lambda}\overrightarrow{R}_4\cdot\Upsilon\right)\,ds,(\overline{W_{xx}-\alpha_0W-\sigma_0\Theta_x-i\lambda\tau_0\Psi+F_\Phi})\rangle\\
&= \frac{1}{\tau_1}\langle\left(\overrightarrow{R}_3\cdot\Upsilon_x+\frac{1}{i\lambda}\overrightarrow{R}_4\cdot\Upsilon_x\right)\,ds,\overline{W}\rangle-\frac{\alpha_0}{\tau_1}\langle\int_0^x\left(\overrightarrow{R}_3\cdot\Upsilon+\frac{1}{i\lambda}\overrightarrow{R}_4\cdot\Upsilon\right)\,,\overline{W}\rangle\\
&\quad+\frac{\sigma_0}{\tau_1}\langle\left(\overrightarrow{R}_3\cdot\Upsilon+\frac{1}{i\lambda}\overrightarrow{R}_4\cdot\Upsilon\right),\overline{\Theta}\rangle\\
&\quad+\frac{\tau_0}{\tau_1}\langle\int_0^x\left(\overrightarrow{R}_3\cdot\Upsilon+\frac{1}{i\lambda}\overrightarrow{R}_4\cdot\Upsilon\right)\,ds,\overline{i\lambda\Psi}\rangle+R.
\end{aligned}\tag{5.5}$$

Since

$$\begin{aligned}
&= \langle\int_0^x\left(\overrightarrow{R}_3\cdot\Upsilon+\frac{1}{i\lambda}\overrightarrow{R}_4\cdot\Upsilon\right)\,ds,\overline{i\lambda\Psi}\rangle\\
&= \langle\int_0^x\left(\overrightarrow{R}_3\cdot\Upsilon+\frac{1}{i\lambda}\overrightarrow{R}_4\cdot\Upsilon\right)\,ds,(\overline{\gamma_0V_{xx}-\alpha_0W-\sigma_0\Theta_x+F_\Psi})\rangle\\
&= \langle\left(\overrightarrow{R}_3\cdot\Upsilon_x+\frac{1}{i\lambda}\overrightarrow{R}_4\cdot\Upsilon_x\right)\,ds,\overline{\gamma_0V}\rangle\\
&\quad-\alpha_0\langle\int_0^x\left(\overrightarrow{R}_3\cdot\Upsilon+\frac{1}{i\lambda}\overrightarrow{R}_4\cdot\Upsilon\right)\,ds,\overline{W}\rangle\\
&\quad+\sigma_0\langle\left(\overrightarrow{R}_3\cdot\Upsilon+\frac{1}{i\lambda}\overrightarrow{R}_4\cdot\Upsilon\right),\overline{\Theta}\rangle+R,
\end{aligned}$$

we see that

$$\left|\langle\int_0^x [\overrightarrow{R}_3\cdot\Upsilon+\frac{1}{i\lambda}\overrightarrow{R}_4\cdot\Upsilon]\,ds,\overline{i\lambda\Psi}\rangle\right|\leq C\|\Upsilon_x\|\|V\|+C\|\Upsilon_x\|\|W\|+C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}.$$

Plugging this inequality into (5.5), we get

$$|J^*|\leq C\|\Upsilon_x\|\|V\|+C\|\Upsilon_x\|\|W\|+C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}. \tag{5.6}$$

Using (5.4) into (5.3), we find

$$\alpha\chi\|W\|^2=\frac{1-\tau_1\gamma_0}{r}\langle\Theta_x,\overline{\Phi}\rangle+G, \tag{5.7}$$

where

$$G \;=\; \frac{1-\tau_1\gamma_0}{r}J^*-\gamma_0\alpha_0\langle W,\overline{V}\rangle-\gamma_0\sigma_0\langle\Theta_x,\overline{V}\rangle-\sigma^*\langle\Theta_x,\overline{W}\rangle-\tau_0\gamma_0\|\Psi\|^2+R^*.$$

Our next step to estimate W in (5.7), is first to estimate G and $\langle\Theta_x,\overline{\Phi}\rangle$. In fact, using (5.6) we get

$$|G| \;\leq\; C\|\Upsilon_x\|\|V\|+C\|\Upsilon_x\|\|W\|+C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+\frac{C}{|\lambda|^2}\|\Phi\|\|\Psi\|+C\|\Psi\|^2+R^*.$$

Therefore, we have that

$$\begin{aligned}|\lambda|^2|G| \;&\leq\; C|\lambda|\|\Upsilon_x\|\|\Psi\|+C|\lambda|\|\Upsilon_x\|\|\Phi\|+C|\lambda|^2\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+C\|\Phi\|\|\Psi\|\\ &\quad +C|\lambda|^2\|\Psi\|^2+c\|\mathbf{F}\|^2_{\mathcal{H}}\\ &\leq\; C|\lambda|^2\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+\frac{\alpha\chi}{2}\|\Phi\|^2+C|\lambda|^2\|\Psi\|^2+c\|\mathbf{F}\|^2_{\mathcal{H}},\end{aligned}$$

where we used

$$|\lambda|^2\|\Upsilon_x\|\|W\|\leq|\lambda|\|\Upsilon_x\|\|\Phi+F_W\|\leq|\lambda|\|\Upsilon_x\|\|\Phi\|+c|\lambda|^2\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+c\|\mathbf{F}\|^2_{\mathcal{H}}.$$

To estimate $\langle\Theta_x,\overline{\Phi}\rangle$, we consider two cases. First, we suppose that the matrix $\mathbf{R}_1^{-1}\mathbf{A}$ has an eigenvalue of multiplicity 2. This case happens only in case that the matrix $\mathbf{A}$ is diagonal. Therefore, we will have that $\mathbf{R}_1^{-1}\mathbf{A}=\gamma_0\mathbf{I}$. This means that

$$i\lambda Y-\gamma_0U_{xx}+\alpha\mathbf{R}_1^{-1}\overrightarrow{N}W+\mathbf{R}_1^{-1}\vartheta\Theta_x=\mathbf{R}_1^{-1}F_Y.$$

Therefore, multiplying with $\overrightarrow{N}$, we get

$$i\lambda\Phi-\gamma_0W_{xx}+\alpha\overrightarrow{N}\cdot\mathbf{R}_1^{-1}\overrightarrow{N}W+\overrightarrow{N}\cdot\mathbf{R}_1^{-1}\vartheta\Theta_x=\overrightarrow{N}\cdot\mathbf{R}_1^{-1}F_Y$$

Further, multiplying the above expression by $\overline{\Phi}$ and taking the real part, we have that

$$\underbrace{\overrightarrow{N}\cdot\mathbf{R}_1^{-1}\vartheta}_{=\rho_1^{-1}-\tau\rho_2^{-1}}\int_0^l\Theta_x\overline{\Phi}\,dx=R^{**}. \tag{5.8}$$

Note that from hypotheses $\overrightarrow{N}\cdot\mathbf{R}_1^{-1}\vartheta\neq0$, and R^{**} is such that $\|R^{**}\|\leq\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}$. Therefore, in case of $\mathbf{R}_1^{-1}\mathbf{A}=\gamma_0\mathbf{I}$, we have that

$$\left|\langle\Theta_x,\overline{\Phi}\rangle\right|\leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}. \tag{5.9}$$

Therefore, we can assume that $\mathbf{R}_1^{-1}\mathbf{A}\neq\gamma_0\mathbf{I}$. Let us take the imaginary part of identity (5.3) to get

$$(1-\tau_1\gamma_0)\operatorname{Im}\langle\Psi,\overline{\Phi}\rangle=\gamma_0\alpha_0\operatorname{Im}\langle W,\overline{V}\rangle+\gamma_0\sigma_0\operatorname{Im}\langle\Theta_x,\overline{V}\rangle+\sigma^*\operatorname{Im}\langle\Theta_x,\overline{W}\rangle-\operatorname{Im}R^*,$$

Multiplying the above identity by $\lambda\in\mathbb{R}$, we get

$$(1-\tau_1\gamma_0)\operatorname{Im}\lambda\langle\Psi,\overline{\Phi}\rangle=-\gamma_0\alpha_0\operatorname{Re}\langle\Phi,\overline{V}\rangle+\gamma_0\sigma_0\operatorname{Re}\langle\Theta_x,\overline{\Psi}\rangle-\sigma^*\operatorname{Re}\langle\Theta_x,\overline{\Phi}\rangle-\operatorname{Im}R^*,$$

where we used $\operatorname{Im} z = -\operatorname{Re} iz$ and that $\operatorname{Im} z = \operatorname{Re} i\bar{z}$. Therefore, we have

$$\sigma^* \underbrace{\operatorname{Re}\langle \Theta_x, \overline{\Phi}\rangle}_{:=\mathcal{X}} + (1-\tau_1\gamma_0) \underbrace{\operatorname{Im} \lambda(\Psi, \overline{\Phi})}_{:=\mathcal{Y}} = \underbrace{-\gamma_0\alpha_0 \operatorname{Re}\langle \Phi, V\rangle + \gamma_0\sigma_0 \operatorname{Re}\langle \Theta_x, \Psi\rangle - \operatorname{Im} R^*}_{:=H}. \tag{5.10}$$

Multiplying Equation (3.32) by $\overline{\Phi}$ and taking real part, we get

$$\sigma_0 \underbrace{\operatorname{Re}\langle \Theta_x, \overline{\Phi}\rangle}_{:=\mathcal{X}} + \tau_0 \underbrace{\operatorname{Im} \lambda(\Psi, \overline{\Phi})}_{:=\mathcal{Y}} = R^*. \tag{5.11}$$

We can consider system (5.10)–(5.11) as a system in $\mathcal{X}$ and $\mathcal{Y}$, which can be solved in terms of the right-hand side if and only if

$$\sigma^*\tau_0 - \sigma_0 + \sigma_0\tau_1\gamma_0 \neq 0. \tag{5.12}$$

From (3.31) we get

$$\overrightarrow{N}\mathbf{A}^{-1} = \tau_1\overrightarrow{N}\mathbf{R}_1^{-1} + \tau_0\overrightarrow{\beta}\mathbf{R}_1^{-1}.$$

Therefore, we obtain

$$\overrightarrow{N}\mathbf{A}^{-1}\overrightarrow{\vartheta} = \tau_1\overrightarrow{N}\mathbf{R}_1^{-1}\overrightarrow{\vartheta} + \tau_0\overrightarrow{\beta}\mathbf{R}_1^{-1}\overrightarrow{\vartheta}.$$

Recalling the definition of σ_0 and σ^*, we have

$$\sigma_0 = \tau_1\overrightarrow{N}\mathbf{R}_1^{-1}\overrightarrow{\vartheta} + \tau_0\sigma^*.$$

Plugging this into (5.12), we find

$$\sigma_0\gamma_0 \neq \overrightarrow{N}\mathbf{R}_1^{-1}\overrightarrow{\vartheta}, \quad \text{or} \quad \gamma_0\overrightarrow{N}\mathbf{A}^{-1}\overrightarrow{\vartheta} \neq \overrightarrow{N}\mathbf{R}_1^{-1}\overrightarrow{\vartheta}.$$

By contradiction, let us suppose

$$\gamma_0\overrightarrow{N}\mathbf{A}^{-1}\overrightarrow{\vartheta} = \overrightarrow{N}\mathbf{R}_1^{-1}\overrightarrow{\vartheta}.$$

This implies that

$$0 = \overrightarrow{N}\cdot(\gamma_0\mathbf{I} - \mathbf{R}_1^{-1}\mathbf{A})\mathbf{A}^{-1}\overrightarrow{\vartheta}, \tag{5.13}$$

and this means that $(\gamma_0\mathbf{I} - \mathbf{R}_1^{-1}\mathbf{A})\mathbf{A}^{-1}\overrightarrow{\vartheta}$ must be orthogonal to $\overrightarrow{N} = (1,-1)$. In case of $(\gamma_0\mathbf{I} - \mathbf{R}_1^{-1}\mathbf{A})\mathbf{A}^{-1}\overrightarrow{\vartheta} = 0$, we have two possibilities. First, that $\gamma_0\mathbf{I} - \mathbf{R}_1^{-1}\mathbf{A} = 0$, but for this case we already prove that (5.9) holds. Second, that $\mathbf{A}^{-1}\overrightarrow{\vartheta} = \gamma_3\overrightarrow{\beta}$. This implies that $\mathbf{R}_1^{-1}\overrightarrow{\vartheta} = \gamma_3\mathbf{R}_1^{-1}\mathbf{A}\overrightarrow{\beta} = \gamma_3\gamma_0\overrightarrow{\beta}$. Therefore, we have that

$$(1,\tau) = \overrightarrow{\vartheta} = \gamma_3\gamma_0\mathbf{R}_1\overrightarrow{\beta}, \quad \Rightarrow \quad \tau = \frac{\rho_2\beta_2}{\rho_1\beta_1}.$$

But this is not possible due to our hypotheses on τ.

So, we can assume that $(\gamma_0\mathbf{I} - \mathbf{R}_1^{-1}\mathbf{A})\mathbf{A}^{-1}\overrightarrow{\vartheta} \neq 0$. Therefore, from (5.13) there exist $a \neq 0$ such that

$$(\gamma_0\mathbf{I} - \mathbf{R}_1^{-1}\mathbf{A})\mathbf{A}^{-1}\overrightarrow{\vartheta} = (a, a)$$

Note that $\gamma_0 \in \sigma(\mathbf{R}_1^{-1}\mathbf{A})$ where $\sigma(\mathbf{R}_1^{-1}\mathbf{A})$ denotes the spectrum. The above problem has a solution (infinitely many) if and only if $(a,a) \in [\operatorname{Ker}(\gamma_0\mathbf{I} - \mathbf{R}_1^{-1}\mathbf{A})]^\perp$. But this is not possible, because

$$(\beta_1,\beta_2) = \overrightarrow{\beta} \in \operatorname{Ker}(\gamma_0\mathbf{I} - \mathbf{R}_1^{-1}\mathbf{A}), \quad (\beta_1,\beta_2)\cdot(a,a) = a(\beta_1+\beta_2) \neq 0.$$

Therefore, we have that system (5.11) and (5.12) is a well-posed system, so we have

$$\mathcal{X} = \operatorname{Re}\langle \Theta_x, \overline{\Phi}\rangle = \frac{\tau_0 H - (1-\tau_1\gamma_0)R^*}{\sigma^*\tau_0 - \sigma_0 + \sigma_0\tau_1\gamma_0}.$$

Hence, we can estimate

$$\operatorname{Re}\langle \Theta_x, \overline{\Phi}\rangle \leq C\|\Phi\|\|V\| + C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}} + C\|\Theta_x\|\|\Psi\|,$$

and then we have that

$$|\lambda|^2 \operatorname{Re}\langle \Theta_x, \overline{\Phi}\rangle \le \epsilon \|\Phi\|^2 + C|\lambda|^2 \|\mathbf{U}\|_{\mathcal{H}} \|\mathbf{F}\|_{\mathcal{H}}.$$

Multiplying (5.7) by λ^2, we conclude that

$$\|\Phi\|^2 \le C|\lambda|^2 \|\mathbf{U}\|_{\mathcal{H}} \|\mathbf{F}\|_{\mathcal{H}} + C\|\mathbf{F}\|^2_{\mathcal{H}}.$$

Hence,

$$\|\Phi\|^2 \le \epsilon \|\mathbf{U}\|^2_{\mathcal{H}} + |\lambda|^4 \|\mathbf{F}\|^2_{\mathcal{H}}.$$

Using similar arguments as above, we conclude that

$$\|\mathbf{U}\|_{\mathcal{H}} \le c|\lambda|^2 \|\mathbf{F}\|_{\mathcal{H}},$$

which implies that the solution decays polynomially as $t^{-1/2}$. Using inequality (4.8), we conclude that the rate of decay is optimal. Otherwise, if there exists a better decay rate as $t^{-1/(2-\epsilon)}$, we would conclude that $|\lambda|^{2-\epsilon}\|U\|_{\mathcal{H}}$ must be bounded. But this is contradictory to inequality (4.8). □

6. Impossibility of localization. Case $K_{12} = K_{21}$

The aim of this section is to prove the impossibility of localization of solutions in the particular case that $K_{12} = K_{21}$. To this end it will be sufficient to prove the uniqueness of solutions for the backward in time problem which is determined by the system of equations

$$\begin{aligned}
&\rho_1 u_{tt} - a_{11} u_{xx} - a_{12} w_{xx} + \alpha(u-w) + \beta_1 \theta_{1,x} + \beta_2 \theta_{2,x} = 0 && \text{in } (0,\ell)\times(0,T),\\
&\rho_2 w_{tt} - a_{12} u_{xx} - a_{22} w_{xx} - \alpha(u-w) + \gamma_1 \theta_{1,x} + \gamma_2 \theta_{2,x} = 0 && \text{in } (0,\ell)\times(0,T),\\
&b_1 \theta_{1,t} + b_2 \theta_{2,t} + K_{11}\theta_{1,xx} + K_{12}\theta_{2,xx} - \beta_1 u_{xt} - \beta_2 w_{xt} + a(\theta_2-\theta_1) = 0 && \text{in } (0,\ell)\times(0,T),\\
&b_2 \theta_{1,t} + b_3 \theta_{2,t} + K_{12}\theta_{1,xx} + K_{22}\theta_{2,xx} - \gamma_1 u_{xt} - \gamma_2 w_{xt} - a(\theta_2-\theta_1) = 0 && \text{in } (0,\ell)\times(0,T)
\end{aligned} \tag{6.1}$$

with the initial and boundary conditions posed at (1.11), (1.13). To prove the uniqueness of solutions, it will be sufficient to prove that the only solution for the null initial and boundary conditions is the null solution.

The first relation we need is the energy conservation law, which states that

$$\begin{aligned}
\mathcal{W}_1(t) =& \frac{1}{2}\int_0^\ell \left(U_x^\top \mathbf{A} U_x + Y^\top \mathbf{R}_1 Y + \alpha U^\top \mathbf{N} U + \Upsilon^\top \mathbf{R}_2 \Upsilon \right) dx \\
=& \int_0^t \int_0^\ell \left(\Upsilon_x^\top \mathbf{K} \Upsilon_x + a \Upsilon^\top \mathbf{N} \Upsilon \right) dx\, ds.
\end{aligned} \tag{6.2}$$

Multiply our first equation by u_t, the second by w_t, the third by $-\theta_1$ and the last one by $-\theta_2$, we obtain

$$\begin{aligned}
\mathcal{W}_2(t) =& \frac{1}{2}\int_0^\ell \left(U_x^\top \mathbf{A} U_x + Y^\top \mathbf{R}_1 Y + \alpha U^\top \mathbf{N} U - \Upsilon^\top \mathbf{R}_2 \Upsilon \right) dx \\
=& -\int_0^t \int_0^\ell \left(\Upsilon_x^\top \mathbf{K} \Upsilon_x + a \Upsilon^\top \mathbf{N} \Upsilon \right) dx\, ds - \int_0^t \int_0^\ell 2Y^\top \mathbf{B} \Upsilon_x \, dx\, ds.
\end{aligned} \tag{6.3}$$

The third identity we need follows from the Lagrange identity argument and it could be deduced with the help of [9, 28]. For a fixed t, we consider the identities

$$\frac{\partial}{\partial s}\left[\rho_1 \dot{u}(s)\dot{u}(2t-s)\right] = \rho_1 \ddot{u}(s)\dot{u}(2t-s) - \rho_1 \dot{u}(s)\ddot{u}(2t-s), \tag{6.4}$$

$$\frac{\partial}{\partial s}\left[\rho_2 \dot{w}(s)\dot{w}(2t-s)\right] = \rho_1 \ddot{w}(s)\dot{w}(2t-s) - \rho_1 \dot{w}(s)\ddot{w}(2t-s), \tag{6.5}$$

$$\begin{aligned}\frac{\partial}{\partial s}\left\{\left[b_1\theta_1(s)+b_2\theta_2(s)\right]\theta_1(2t-s)\right\} = b_1 &\left[\dot{\theta}_1(s)\theta_1(2t-s) - \theta_1(s)\dot{\theta}_1(2t-s)\right] \\ &+ b_2\left[\dot{\theta}_2(s)\theta_1(2t-s) - \theta_2(s)\dot{\theta}_1(2t-s)\right],\end{aligned} \tag{6.6}$$

$$\begin{aligned}\frac{\partial}{\partial s}\left\{\left[b_2\theta_1(s)+b_2\theta_3(s)\right]\theta_2(2t-s)\right\} = b_3 &\left[\dot{\theta}_2(s)\theta_2(2t-s) - \theta_2(s)\dot{\theta}_2(2t-s)\right] \\ &+ b_2\left[\dot{\theta}_1(s)\theta_2(2t-s) - \theta_1(s)\dot{\theta}_2(2t-s)\right].\end{aligned} \tag{6.7}$$

In view of the system of equations as well as the null initial and boundary conditions, we obtain

$$\int_0^\ell (Y^\top \mathbf{R}_1 Y - \Upsilon^\top \mathbf{R}_2 \Upsilon)\, dx = \int_0^\ell (U_x^\top \mathbf{A} U_x + \alpha U^\top \mathbf{N} U)\, dx. \tag{6.8}$$

From (6.3) and (6.8) it follows that

$$\begin{aligned}\mathcal{W}_2(t) &= \int_0^\ell (U_x^\top \mathbf{A} U_x + \alpha U^\top \mathbf{N} U)\, dx \\ &= -\int_0^t\int_0^\ell \left(\Upsilon_x^\top \mathbf{K}\Upsilon_x + a\Upsilon^\top \mathbf{N}\Upsilon\right) dx\, ds - \int_0^t\int_0^\ell 2Y^\top \mathbf{B}\Upsilon_x\, dx\, ds.\end{aligned} \tag{6.9}$$

Let ϵ be a small positive constant. We consider $\mathcal{W}_3(t) = \mathcal{W}_2(t) + \epsilon\mathcal{W}_1(t)$. We note that

$$\mathcal{W}_3(t) = \frac{\epsilon}{2}\int_0^\ell (Y^\top \mathbf{R}_1 Y + \alpha U^\top \mathbf{N} U + \Upsilon^\top \mathbf{R}_2 \Upsilon)\, dx + \left(1+\frac{\epsilon}{2}\right)\int_0^\ell (U_x^\top \mathbf{A} U_x + \alpha U^\top \mathbf{N} U)\, dx, \tag{6.10}$$

is a positive function which defines a squared norm on the solutions. Taking into account

$$\mathcal{W}_3(t) = -(1-\epsilon)\int_0^t\int_0^\ell (\Upsilon_x^\top \mathbf{K}\Upsilon_x + a\Upsilon^\top \mathbf{N}\Upsilon)\, dx\, ds - \int_0^t\int_0^\ell 2(Y^\top \mathbf{B}\Upsilon_x)\, dx\, ds,$$

we have

$$\frac{d\mathcal{W}_3}{dt} = -(1-\epsilon)\int_0^\ell (\Upsilon_x^\top \mathbf{K}\Upsilon_x + a\Upsilon^\top \mathbf{N}\Upsilon)\, dx - \int_0^\ell 2(Y^\top \mathbf{B}\Upsilon_x)\, dx.$$

As

$$-\int_0^\ell 2(Y^\top \mathbf{B}\Upsilon_x)\, dx \le K_1\int_0^\ell Y^\top \mathbf{R}_1 Y\, dx + \epsilon_1\int_0^\ell (\Upsilon_x^\top \mathbf{K}\Upsilon_x + a\Upsilon^\top \mathbf{N}\Upsilon)\, dx$$

where ϵ_1 is sufficiently small, with K_1 depending only on the constitutive constants and ϵ_1, we obtain the estimate

$$\frac{d\mathcal{W}_3(t)}{dt} \le C^*\mathcal{W}_3(t),$$

for every $t \ge 0$, where C^* is a computable positive constant. Consequently $\mathcal{W}_3(t) \le \mathcal{W}_3(0)\exp(C^* t)$, for every $t \ge 0$. Since we assume the null initial conditions, we see that $\mathcal{W}_3(t)$ vanishes for every t and the the solution must be the null solution.

Remark 6.1. The analysis proposed in this section can be extended without difficulties to the three-dimensional and for inhomogeneous case, but we did not developed it here to be consistent with the other sections.

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