Single-Pushout Hypergraph Rewriting through Free Completions

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Abstract. Different relationships between single-pushout rewriting of total and partial unary algebras are studied in this paper. In general, given a single-pushout transformation rule r of total algebras, a corresponding single-pushout transformation rule r' of partial algebras can be found such that single-pushout derivations by the application of r are epireflections of single-pushout derivations by the application of r'. But, unlike the case of double-pushout derivations, in this case single-pushout transformation rules of partial algebras become "larger" than corresponding rules of total algebras. Therefore, *ad hoc* methods are developed to compute pushouts of partial homomorphisms through free completions which, in the case of hypergraphs, may lead to a slight improvement in efficiency of pattern-matching and to a moderate improvement in efficiency of single-pushout transformations.

Key words: algebraic graph transformation, single-pushout approach, partial algebras, hypergraphs, free completions

1. Introduction

The algebraic approach to graph transformation models the transformation of total structures (that is, graphs, hypergraphs, algebras) [9]. In the single-pushout form of the algebraic approach, a transformation rule in a category $\mathcal{C}$ is a partial morphism $r : \mathbf{L} \to \mathbf{R}$, and it can be *applied* to an object $\mathbf{G}$ when there exists a total morphism $m : \mathbf{L} \to \mathbf{G}$ and a pushout of r and m in $\mathcal{C}$, where the meaning of "total" and "partial" depends on the context. When such a pushout exists, the object $\mathbf{H}$ is said to be *derived* from $\mathbf{G}$ by the application of rule r through morphism m.

The morphism m is called *occurrence* when it is total. Single-pushout derivations are only of interest in the case of occurrences, since other-

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wise every single-pushout transformation rule could be applied to any object through the empty partial morphism.

More recently, the algebraic approach has been extended to the transformation of partial structures (that is, partial algebras). Single-pushout algebraic transformation has been studied in [2, 5] in different categories of partial morphisms of unary partial algebras which have all pushouts, and necessary and sufficient conditions for the existence of pushouts in a certain category of partial morphisms of arbitrary partial algebras have been given in [11]. Notice, however, that (double-pushout) algebraic transformation of partial graphs had been already dealt with in [10].

In this paper, we study the relationship between single-pushout transformation of hypergraphs and partial unary algebras, aimed at improving efficiency of single-pushout transformations by using "smaller" rules (that is, rules of partial algebras) applied to "smaller" objects (that is, partial algebras).

Unfortunately, one cannot use epireflections with this purpose as the transition tool from partial to total algebras, since for partial morphisms (as they are used in single-pushout transformation) the epireflection (in the category of total algebras) of a partial algebra is, in the categories where it exists, smaller than the original algebra, leading to no improvement in efficiency at all.

But by means of *ad hoc* methods, and similar to the case of double-pushout derivations through free completions dealt with in [7], we establish here that in general we can replace *a priori* single-pushout transformation rules $r : \mathbf{L} \to \mathbf{R}$ of hypergraphs by single-pushout transformation rules $r_0 : \mathbf{L}_0 \to \mathbf{R}_0$ of partial algebras (of a suitable signature Σ_H), in the (informal) sense that the rule r can be applied to a hypergraph $\mathbf{G}$ through a homomorphism $m : \mathbf{L} \to \mathbf{G}$ iff the rule r_0 can be applied (in a suitable category) to $\mathbf{G}$ through the restriction $m_0 : \mathbf{L}_0 \to \mathbf{G}$ of m, and then the derived hypergraph of $\mathbf{G}$ by the application of r through m is the free completion of the derived partial Σ_H-algebra of $\mathbf{G}_0$ by the application of r_0 through $m_0 : \mathbf{L}_0 \to \mathbf{G}_0$, where $\mathbf{G}_0$ is some suitable relative subalgebra of $\mathbf{G}$ containing $m(\mathbf{L}_0)$.

But, unlike the case of double-pushout derivations, there may not exist any significant gain in doing so: for instance, if r is neither total nor empty on the nodes of $\mathbf{L}$ then $\mathbf{L}_0$ must be $\mathbf{L}$, and therefore we have no improvement in the pattern-matching cost.

But we also establish that, given a single-pushout transformation rule $r : \mathbf{L} \to \mathbf{R}$ of hypergraphs and given a homomorphism $m : \mathbf{L} \to \mathbf{G}$ of hypergraphs, their pushout can be computed as the free completion of the pushout of a single-pushout transformation rule $r'_0 : \mathbf{L}'_0 \to \mathbf{R}'_0$ and a homomorphism $m'_0 : \mathbf{L}'_0 \to \mathbf{G}'_0$ of partial Σ_H-algebras (again, in a suitable category). In this case, these $\mathbf{L}'_0$, $\mathbf{R}'_0$ and $\mathbf{G}'_0$ may be significantly smaller than $\mathbf{L}$, $\mathbf{R}$ and $\mathbf{G}$, leading to a certain improvement in efficiency of single-pushout transformation, after pattern-matching has been done. But r_0 cannot be found *a priori*, independently of m, since it does depend on m.

The rest of this paper is organized as follows. Section 2 collects basic notions and results concerning single-pushout algebraic transformation, on the one hand, and free completions of partial algebras and hypergraphs, on the other hand. It also establishes a necessary and sufficient condition for the existence of pushouts in the category of *closed-domain quomorphisms* of unary partial algebras, parallel to the necessary and sufficient condition given in [11] for quomorphisms of arbitrary partial algebras; that necessary and sufficient condition is used later in the paper, although it may also be of interest by itself.

Pushouts of partial morphisms of total algebras are related to epi-reflections of pushouts in categories of partial algebras in Section 3. Such results are carried over to free completions of partial algebras in Section 4. In Section 5, an alternative simplification through free completions is developed which allows to get "smaller" single-pushout transformation rules of hypergraphs *a priori*. Finally, Section 6 summarizes and comments on these results.

There is a convention used throughout this paper that we want to mention here. Unless otherwise stated, we denote algebras (and in particular hypergraphs) by means of boldface capital letters —$\mathbf{A}$, $\mathbf{B}$, etc.—, their respective carriers by the same capital letter, but in italic type —A, B, etc., respectively—, and their carriers of a given sort by the same capital letter in italic type, with the sort as a subscript —A_s, B_t, etc.

2. Preliminaries

2.1. Partial homomorphisms

Throughout this paper we will be interested in pushouts of partial homomorphisms of Γ-algebras.

Definition 1. A signature $\Gamma = (S, \Omega, \eta)$ is called a *graph structure* if all the operations are unary, that is, if $\eta(\Omega) \subseteq S \times S$.

Let now Γ be a graph structure. In this section we recall the main different types of partial homomorphisms of total and partial Γ-algebras used in the single-pushout algebraic transformation; see [2, 5, 6, 9] for detailed definitions. For the convenience of the reader, we recall first the definitions of the different types of subalgebras and (total) homomorphisms of partial Γ-algebras underlying them.

Definition 2. Let $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ and $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ be two arbitrary partial Γ-algebras and let $f : A \to B$ be a mapping of S-sets, that is, a family of mappings $f_s : A_s \to B_s$, $s \in S$. Then

- f is a (*plain*) *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ when it satisfies the following condition for every $\varphi \in \Omega$, say with $\eta(\varphi) = (s, s')$: if $a \in \operatorname{dom}\varphi^{\mathbf{A}}$ then $f_s(a) \in \operatorname{dom}\varphi^{\mathbf{B}}$ and $\varphi^{\mathbf{B}}(f_s(a)) = f_{s'}(\varphi^{\mathbf{A}}(a))$.

- f is a *closed homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ when it is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ and it satisfies the following further condition for every $\varphi \in \Omega$, say with $\eta(\varphi) = (s, s')$: for every $a \in A_s$, if $f_s(a) \in \operatorname{dom}\varphi^{\mathbf{B}}$ then $a \in \operatorname{dom}\varphi^{\mathbf{A}}$.

If $\mathbf{A}$ and $\mathbf{B}$ are total algebras then both notions are equivalent to the usual notion of *homomorphism* of total algebras: for every $\varphi \in \Omega$, say with $\eta(\varphi) = (s, s')$, if $a \in A_s$ then $\varphi^{\mathbf{B}}(f_s(a)) = f_{s'}(\varphi^{\mathbf{A}}(a))$.

We shall denote by Alg_Γ and C-Alg_Γ the categories whose objects are all partial Γ-algebras and whose morphisms are the plain homomorphisms and the closed homomorphisms, respectively. Also, we shall denote by TAlg_Γ the category whose objects are all total Γ-algebras and whose morphisms are the homomorphisms of total algebras: it is a full subcategory of both Alg_Γ and C-Alg_Γ.

Definition 3. Let $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ and $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ be two partial Γ-algebras with $B \subseteq A$, that is, with $B_s \subseteq A_s$ for every $s \in S$. Then

- $\mathbf{B}$ is a *weak subalgebra* of $\mathbf{A}$ (*supported on* B) when it satisfies the following condition for every $\varphi \in \Omega$: if $b \in \operatorname{dom}\varphi^{\mathbf{B}}$ then $b \in \operatorname{dom}\varphi^{\mathbf{A}}$ and $\varphi^{\mathbf{A}}(b) = \varphi^{\mathbf{B}}(b)$.

- $\mathbf{B}$ is a *relative subalgebra* of $\mathbf{A}$ (*supported on* B) when it is a weak subalgebra of $\mathbf{A}$ satisfying the following further condition for every $\varphi \in \Omega$, say with $\eta(\varphi) = (s, s')$: if $b \in B_s \cap \operatorname{dom}\varphi^{\mathbf{A}}$ and $\varphi^{\mathbf{A}}(b) \in B_{s'}$ then $b \in \operatorname{dom}\varphi^{\mathbf{B}}$.

- B is a *closed subset* of $\mathbf{A}$ when it satisfies the following condition for every $\varphi \in \Omega$, say with $\eta(\varphi) = (s, s')$: if $b \in B_s \cap \operatorname{dom}\varphi^{\mathbf{A}}$ then $\varphi^{\mathbf{A}}(b) \in B_{s'}$.

- $\mathbf{B}$ is a *closed subalgebra* of $\mathbf{A}$ (*supported on* B) when it is a relative subalgebra of $\mathbf{A}$ and B is a closed subset of $\mathbf{A}$. That is, when it is a weak subalgebra of $\mathbf{A}$ satisfying the following further condition for every $\varphi \in \Omega$, say with $\eta(\varphi) = (s, s')$: if $b \in B_s \cap \operatorname{dom} \varphi^{\mathbf{A}}$ then $b \in \operatorname{dom} \varphi^{\mathbf{B}}$.

If $\mathbf{A}$ is a total algebra and $\mathbf{B}$ is a closed subalgebra of it, we shall simply say that $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$.

Given a subset B of the universe A of a partial algebra $\mathbf{A}$, there are in principle many weak subalgebras of $\mathbf{A}$ supported on B, but only one relative subalgebra of $\mathbf{A}$ supported on B. And there is a (unique) closed subalgebra of $\mathbf{A}$ supported on B iff B is closed in $\mathbf{A}$. Notice also that if $\mathbf{B}$ is a total Γ-algebra that is a weak or relative subalgebra of some partial Γ-algebra $\mathbf{A}$, then it is a closed subalgebra of it.

Definition 4. Let $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ and $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ be two arbitrary partial Γ-algebras and let $f : A \to B$ be a *partial* mapping of S-sets. Then

- f is a *conformism* from $\mathbf{A}$ to $\mathbf{B}$ when it is a closed homomorphism from some weak subalgebra of $\mathbf{A}$ supported on $\operatorname{Dom} f$ to $\mathbf{B}$.
- f is a *quomorphism* from $\mathbf{A}$ to $\mathbf{B}$ when it is a plain homomorphism from the relative subalgebra of $\mathbf{A}$ supported on $\operatorname{Dom} f$ to $\mathbf{B}$.
- f is a *closed-domain quomorphism*, *cd-quomorphism* for short, from $\mathbf{A}$ to $\mathbf{B}$ when it is a quomorphism with $\operatorname{Dom} f$ a closed subset of $\mathbf{A}$.
- f is a *closed quomorphism*, *c-quomorphism* for short, from $\mathbf{A}$ to $\mathbf{B}$ when it is a quomorphism and a conformism simultaneously: when it is a closed homomorphism from the relative subalgebra of $\mathbf{A}$ supported on $\operatorname{Dom} f$ to $\mathbf{B}$.
- f is a *closed-domain closed quomorphism*, *cdc-quomorphism* for short, from $\mathbf{A}$ to $\mathbf{B}$ when it is a closed quomorphism with $\operatorname{Dom} f$ a closed subset of $\mathbf{A}$.

When $\mathbf{A}$ and $\mathbf{B}$ are total Γ-algebras, f is a *partial homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ if it is a homomorphism from a subalgebra of $\mathbf{A}$ supported on $\operatorname{Dom} f$.

We shall denote by CF-Alg$_\Gamma$, Q-Alg$_\Gamma$, CDQ-Alg$_\Gamma$, CQ-Alg$_\Gamma$, and by CDCQ-Alg$_\Gamma$ the categories whose objects are all partial Γ-algebras and whose morphisms are the conformisms, the quomorphisms, the cd-quomorphisms, the closed quomorphisms, and the cdc-quomorphisms, respectively. Also, we shall denote by P-TAlg$_\Gamma$ the category whose objects are all total Γ-algebras and whose morphisms are the partial homomorphisms.

As we shall see in Section 3, P-TAlg$_\Gamma$ is a full subcategory of CF-Alg$_\Gamma$, CDQ-Alg$_\Gamma$, CQ-Alg$_\Gamma$ and CDCQ-Alg$_\Gamma$, but not of Q-Alg$_\Gamma$.

It is worth mentioning that, when Γ is a graph structure, CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$, CDCQ-Alg$_\Gamma$ and P-TAlg$_\Gamma$ are always cocomplete, while Q-Alg$_\Gamma$ and CDQ-Alg$_\Gamma$ are never cocomplete (see Sect. 2.5 for a deeper study of cd-quomorphisms), unless $\Omega = \emptyset$.

2.2. Hypergraphs

The main object of interest in this paper are hypergraphs. Recall that a hypergraph is a structure $\mathbf{G} = (G_V, G_E, s^{\mathbf{G}}, t^{\mathbf{G}})$ where G_V and G_E are two finite sets (of *nodes* and *hyperarcs*, respectively) and $s^{\mathbf{G}}, t^{\mathbf{G}}$ are mappings $s^{\mathbf{G}} : G_E \to G_V^*$ and $t^{\mathbf{G}} : G_E \to G_V^*$, called the *source* and *target* mappings respectively. (In this definition, X^* stands for the set of all finite words over a set X.)

The graph structure $\Sigma_H = (S_H, \Omega_H, \eta_H)$ with

$$\begin{aligned}
&S_H = \{V\} \cup \{E_{m,n} \mid 0 \le m, n\}\\
&\Omega_H = \{s_i^{m,n} \mid 1 \le i \le m, 0 \le n\} \cup \{t_i^{m,n} \mid 1 \le i \le n, 0 \le m\}\\
&\eta_H(s_i^{m,n}) = \eta_H(t_i^{m,n}) = (E_{m,n}, V)
\end{aligned}$$

allows to understand hypergraphs as finite total Σ_H-algebras, by splitting G_E into $\bigsqcup_{m,n} G_{E_{m,n}}$, where

$$G_{E_{m,n}} = \left\{e \in G_E \mid |s^{\mathbf{G}}(e)| = m, |t^{\mathbf{G}}(e)| = n\right\},\ 0 \le m, n$$

and then, for every $e \in G_{E_{m,n}}$, defining $s_j^{\mathbf{G}}(e)$, $0 \le j \le m$, to be the jth letter of $s^{\mathbf{G}}(e)$, and a similar definition for $t_j^{\mathbf{G}}(e)$, $0 \le j \le n$.

This yields an isomorphism between the full subcategory of TAlg$_{\Sigma_H}$ supported on the class of all finite Σ_H-algebras, and the usual category of hypergraphs; cf. [9, Ex. 3.4].

In this paper, we shall make the abuse of notation of calling *hypergraphs* the finite total Σ_H-algebras, and *partial hypergraphs* the finite partial Σ_H-algebras. In general, we shall understand Σ_H-algebras as sort of possibly partial and infinite hypergraphs, and therefore we shall also make other (consistent) abuses of language and notation. In particular, given a partial Σ_H-algebra $\mathbf{A}$, we shall denote by A_E the disjoint union $\bigsqcup_{p,q} A_{E_{p,q}}$, and given a morphism of Σ_H-algebras $f : \mathbf{A} \to \mathbf{B}$ of any of the kinds introduced in Sect. 2.1, we shall denote by $f_E : A_E \to B_E$ the corresponding mapping $\bigsqcup_{p,q} f_{E_{p,q}}$. Consequently, when it is irrelevant to know the exact sort $E_{p,q}$ of an element $e \in A_E$, we shall omit it. We shall also make the abuse of language, given a partial Σ_H-algebra, of calling *nodes* the elements of A_V and *arcs* the elements of A_E, and calling generically *source* and *target* operations the operations $s_j^{m,n}$ and $t_j^{m,n}$.

Given a partial Σ_H-algebra $\mathbf{A}$ and a node $v \in A_V$, we shall call the *degree* of v, $\deg_{\mathbf{A}} v$ or simply $\deg v$, the number of pairs $(e, \varphi) \in G_E \times \Omega_H$ such that $v = \varphi^{\mathbf{G}}(e)$; when $v = \varphi^{\mathbf{G}}(e)$, we shall say that v is *incident* to e. A node of degree 0 is not incident to any arc and then it is also called *isolated*; a node of degree 1 is obtained exactly once as the result of source or target operations in the algebra; and a node of degree greater than or equal to 2 is obtained at least twice as the result of source or target operations in the algebra, that is, it is incident to two different arcs, or twice to an arc.

2.3. Free completions of hypergraphs

Let $\mathcal{C}$ be any category, and let $\mathcal{K}$ be a full subcategory of $\mathcal{C}$ (which in practice we identify with its class of objects).

Definition 5. Given an object $\mathbf{A}'$ of $\mathcal{C}$, a *$\mathcal{K}$-epireflection* of $\mathbf{A}'$ is an epimorphism (in $\mathcal{C}$) $\ell_{\mathbf{A}'} : \mathbf{A}' \to \mathbf{A}$ with $\mathbf{A} \in \mathcal{K}$ such that for every morphism $f : \mathbf{A}' \to \mathbf{B}$ in $\mathcal{C}$ with $\mathbf{B} \in \mathcal{K}$ there exists one, and only one, morphism $\hat{f} : \mathbf{A} \to \mathbf{B}$ such that $\hat{f} \circ \ell_{\mathbf{A}'} = f$. A full subcategory $\mathcal{K}$ of $\mathcal{C}$ is said to be *epireflective* in $\mathcal{C}$ when every object of $\mathcal{C}$ has a $\mathcal{K}$-epireflection.

Let $\Sigma = (S, \Omega, \eta)$ be an arbitrary signature. It is well known that TAlg_Σ is a full and epireflective subcategory of Alg_Σ [4, Sect. 5.3].

Definition 6. A *free completion* of a partial Σ-algebra $\mathbf{A}'$ is a TAlg_Σ-epireflection $\ell_{\mathbf{A}'} : \mathbf{A}' \to \mathbf{A}$.

A free completion $\ell_{\mathbf{A}'} : \mathbf{A}' \to \mathbf{A}$ of a partial Σ-algebra $\mathbf{A}' = (A', (\varphi^{\mathbf{A}'})_{\varphi\in\Omega})$ is obtained (up to isomorphism) in the following way. Let $\mathbf{T}_\Gamma(A')$ be the free Γ-algebra generated by the set A' and let A^* be its carrier. Define on A^* the following structure of Γ-algebra $\mathbf{A}^* = (A^*, (\varphi^{\mathbf{A}^*})_{\varphi\in\Omega})$: for every $\varphi \in \Omega$, say of arity $\eta(\varphi) = (s_1 \dots s_n, s)$, and for every $\underline{a} \in A^*_{s_1} \times \cdots \times A^*_{s_n}$,

$$\varphi^{\mathbf{A}^*}(\underline{a}) = \begin{cases} \varphi^{\mathbf{A}'}(\underline{a}) & \text{if } \underline{a} \in A'_{s_1} \times \cdots \times A'_{s_n} \\ \varphi^{\mathbf{T}_\Gamma(A')}(\underline{a}) & \text{otherwise} \end{cases}$$

Finally, let $\mathbf{A}$ be the least closed subalgebra of $\mathbf{A}^*$ containing A' and $\ell_{\mathbf{A}'} : \mathbf{A}' \to \mathbf{A}$ the embedding of $\mathbf{A}'$ into $\mathbf{A}$ as a relative subalgebra.

In the case of a partial Σ_H-algebra $\mathbf{A}'$, the previous construction amounts to simply adding to $\mathbf{A}'$ a new node for every element $e \in A'_{E_{p,q}}$

and every operation $\varphi \in \{s_1, \ldots, s_p, t_1, \ldots, t_q\}$ such that $e \notin \operatorname{dom} \varphi^{\mathbf{A}'}$, and the corresponding operation $\varphi^{\mathbf{A}}$ sending e to this new node. Notice that if $\mathbf{A}'$ is a finite Σ_H-algebra then the resulting $\mathbf{A}$ is a hypergraph.

Given a hypergraph or, more in general, a total Σ_H-algebra $\mathbf{G} = (G, (s_i^{m,n\mathbf{G}}, t_i^{m,n\mathbf{G}})_{i,m,n})$, where $G = (G_V, (G_{E_{m,n}})_{m,n})$, the least subset $\widetilde{G} \subseteq G$ that generates $\mathbf{G}$ is described as follows:

$$\begin{aligned} &\widetilde{G}_{E_{m,n}} = G_{E_{m,n}},\ m, n \geq 0 \\ &\widetilde{G}_V = \{v \mid \deg v = 0\} \end{aligned}$$

And given a subset $X \subseteq G$, the least subset $G' \subseteq G$ containing X and such that $\mathbf{G}$ is a free completion of $\mathbf{G}'$ (its relative subalgebra supported on G') is described as follows:

$$\begin{aligned} &G'_{E_{m,n}} = G_{E_{m,n}},\ m, n \geq 0 \\ &G'_V = \{v \mid \deg v = 0\} \cup \{v \mid \deg v \geq 2\} \cup X_V \end{aligned}$$

2.4. Pushouts of cdc-quomorphisms and partial homomorphisms

Since, for every graph structure Γ, P-TAlg$_\Gamma$ is a full and epireflective subcategory of CDCQ-Alg$_\Gamma$ (see Section 3), the corresponding inclusion functor has a left adjoint, and then the pushout of two partial homomorphisms of total Γ-algebras in P-TAlg$_\Gamma$ coincides with their pushout in CDCQ-Alg$_\Gamma$.

We recall in the sequel an explicit construction of the pushout of two cdc-quomorphisms in CDCQ-Alg$_\Gamma$, from [5]. Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two cdc-quomorphisms of partial Γ-algebras. Then their pushout in P-TAlg$_\Gamma$ is obtained as follows:

– One finds

  - the greatest closed subalgebra $\mathbf{K}'$ of $\mathbf{K}$ contained in $\operatorname{Dom} f \cap \operatorname{Dom} g$ and such that $f^{-1}(f(\mathbf{K}')) = \mathbf{K}'$ and $g^{-1}(g(\mathbf{K}')) = \mathbf{K}'$; such $\mathbf{K}'$ is called the *gluing algebra* of f and g;
  - the greatest closed subalgebra $\mathbf{A}'$ of $\mathbf{A}$ contained in $(\mathbf{A} - f(\mathbf{K})) \cup f(\mathbf{K}')$;
  - the greatest closed subalgebra $\mathbf{B}'$ of $\mathbf{B}$ contained in $(\mathbf{B} - g(\mathbf{K})) \cup g(\mathbf{K}')$,

  (Notice that $f(\mathbf{K}') \subseteq \mathbf{A}'$ and $g(\mathbf{K}') \subseteq \mathbf{B}'$.)

– Then, one computes the pushout in C-Alg$_\Gamma$ of the homomorphisms $f|_{\mathbf{K}'} : \mathbf{K}' \to \mathbf{A}'$ and $g|_{\mathbf{K}'} : \mathbf{K}' \to \mathbf{B}'$ obtained by restricting f and g to $\mathbf{K}'$. This pushout is given by the quotient $\mathbf{C} = (\mathbf{A}'+\mathbf{B}')/\theta$, where θ stands for the least equivalence relation on $\mathbf{A}' + \mathbf{B}'$ containing

$$\{(f_s(x), g_s(x)) \mid x \in K'_s\}, \quad s \in S$$

(which in this case turns out to be a closed congruence on $\mathbf{A}'+\mathbf{B}'$), together with the closed homomorphisms $\tilde{g} : \mathbf{A}' \to \mathbf{C}$ and $\tilde{f} : \mathbf{B}' \to \mathbf{C}$ given by the restrictions to $\mathbf{A}'$ and $\mathbf{B}'$ of the quotient homomorphism $\mathbf{A}' + \mathbf{B}' \to \mathbf{C}$.

And then the pushout of f and g in CDCQ-Alg$_\Gamma$ is given by the algebra $\mathbf{C}$ obtained in this way, together with the cdc-quomorphisms $\tilde{g} : \mathbf{A} \to \mathbf{C}$ and $\tilde{f} : \mathbf{B} \to \mathbf{C}$, with domains $\mathbf{A}'$ and $\mathbf{B}'$ respectively, corresponding to the homonymous closed homomorphisms defined on these closed subalgebras.

Notice that, when $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ are finite Σ_H-algebras, the pushout algebra $\mathbf{C}$ is also a finite Σ_H-algebra, i.e., it is a hypergraph.

When it is necessary or convenient to remind the original cdc-quomorphisms, we shall denote the closed subalgebras $\mathbf{K}'$, $\mathbf{A}'$ and $\mathbf{B}'$ appearing in the computation of the pushout of f and g by $\mathbf{K}'(f,g)$, $\mathbf{A}'(f,g)$ and $\mathbf{B}'(f,g)$, respectively.

2.5. Pushouts of cd-quomorphisms

It happens (cf. [3]) that no CDQ-Alg$_\Gamma$, for any non-empty signature Γ (even for Σ_H), has all pushouts. In the monounary (one sort and one unary operation φ) case, the paradigm of this fact is the non-existence of the pushout of $f : \mathbf{A} \to \mathbf{B}$ and itself, where $\mathbf{A}$ is a discrete Γ-algebra with carrier $A = \{a\}$, $\mathbf{B}$ is a partial Γ-algebra with carrier $B = \{b, b'\}$ and operation $\varphi^{\mathbf{B}}(b) = b'$, and f is given by $f(a) = b$: on the one hand, in a pushout both copies of b, and therefore of b', should be identified, but then the cd-quomorphisms $g_1, g_2 : \mathbf{B} \to \mathbf{C}$, with $\mathbf{C}$ a discrete Γ-algebra with carrier $C = \{c_1, c_2\}$, given by $g_1(b') = c_1$, $g_2(b') = c_2$, would not admit a common extension to this pushout. This counterexample is easily generalized to an arbitrary signature.

In Section 5 we will be interested in pushouts of cd-quomorphisms of Σ_H-algebras, and in particular we shall need the following result, which establishes a necessary and sufficient condition for the existence of the pushout of two cd-quomorphisms of partial Σ_H-algebras; we want to point out that this result holds *mutatis mutandi* also for hierarchical graph structures, but it does not hold for more general signatures; for more general results, see [1].

Proposition 7. Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two cd-quomorphisms of partial Σ_H-algebras. Then, f and g have a pushout in CDQ-Alg$_{\Sigma_H}$ iff the following two conditions are satisfied:

- There exists the greatest closed subset K' of $\mathbf{K}$ such that
  - $f^{-1}(f(K')) = K'$, $g^{-1}(g(K')) = K'$,
  - there exists the greatest closed subset A' of $\mathbf{A}$ lying between $f(K')$ and $(A - f(K)) \cup f(K')$,
  - there exists the greatest closed subset B' of $\mathbf{B}$ lying between $g(K')$ and $(B - g(K)) \cup g(K')$;
- The equivalence relation $\theta(K')$ on $A' \sqcup B'$ generated by

$$\{(f_s(x), g_s(x)) \mid x \in K'_s\}, s \in S$$

  is a congruence on $\mathbf{A}' + \mathbf{B}'$ (the disjoint sum of the relative subalgebras of $\mathbf{A}$ and $\mathbf{B}$ supported on A' and B').

And when such K', A' and B' exist, and $\theta(K')$ is a congruence, the pushout of f and g is (up to isomorphism) $(\mathbf{C}, \tilde{g} : \mathbf{A} \to \mathbf{C}, \tilde{f} : \mathbf{B} \to \mathbf{C})$ in CDQ-Alg$_{\Sigma_H}$, where $\mathbf{C} = (\mathbf{A}' + \mathbf{B}')/\theta(K')$ and $\tilde{g} : \mathbf{A} \to \mathbf{C}$ and $\tilde{f} : \mathbf{B} \to \mathbf{C}$ are respectively the cd-quomorphisms given by the restriction to $\mathbf{A}'$ and $\mathbf{B}'$ of the natural homomorphism $\mathbf{A}'+\mathbf{B}' \to (\mathbf{A}'+\mathbf{B}')/\theta(K')$.

(Notice that the existence of A' (respectively, B') is equivalent to the existence of *some* closed subset of $\mathbf{A}$ lying between $f(K')$ and $(A - f(K)) \cup f(K')$ (respectively, of $\mathbf{B}$ lying between $g(K')$ and $(B - g(K)) \cup g(K')$).)

Proof. The "if" implication can be proved by adapting to our (slightly simpler) setting the argument developed in [9, Th. 2.7] for partial homomorphisms: we leave the details to the reader.

As far as the "only if" implication goes, assume that $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ have a pushout $(\mathbf{C}, \tilde{g} : \mathbf{A} \to \mathbf{C}, \tilde{f} : \mathbf{B} \to \mathbf{C})$ in CDQ-Alg$_{\Sigma_H}$.

Let us see first that $K' = \mathrm{Dom}\, \tilde{g} \circ f = \mathrm{Dom}\, \tilde{f} \circ g$ is the greatest closed subset asked for in the first condition. It is clear that $f^{-1}(f(K')) = K'$, $g^{-1}(g(K')) = K'$. Moreover, there exist a closed subset of $\mathbf{A}$ lying between $f(K')$ and $(A - f(K)) \cup f(K')$ and a closed subset of $\mathbf{B}$ lying between $g(K')$ and $(B - g(K)) \cup g(K')$: namely, $\mathrm{Dom}\, \tilde{g}$ and $\mathrm{Dom}\, \tilde{f}$, respectively.

We shall prove that if K_0 is another closed subset of $\mathbf{K}$ satisfying that $f^{-1}(f(K_0)) = K_0$, $g^{-1}(g(K_0)) = K_0$, and that there exist a greatest closed subset A_0 of $\mathbf{A}$ lying between $f(K_0)$ and $(A - f(K)) \cup f(K_0)$

and a greatest closed subset B_0 of $\mathbf{B}$ lying between $g(K_0)$ and $(B - g(K)) \cup g(K_0)$, then $K_0 \subseteq K'$, $A_0 \subseteq \mathrm{Dom}\,\tilde{g}$ and $B_0 \subseteq \mathrm{Dom}\,\tilde{f}$.

Let $\tilde{\theta}(K_0)$ be the least congruence on $\mathbf{A}_0+\mathbf{B}_0$ (the disjoint sum of the closed subalgebras of $\mathbf{A}$ and $\mathbf{B}$ supported on A_0 and B_0, respectively) containing

$$\{(f_s(x), g_s(x)) \mid x \in K_{0s}\}, s \in S_H$$

and let $\mathbf{D}_0$ denote the quotient $(\mathbf{A}_0 + \mathbf{B}_0)/\tilde{\theta}(K_0)$ and $p : \mathbf{A} \to \mathbf{D}_0$, $q : \mathbf{B} \to \mathbf{D}_0$ the cd-quomorphisms with domains, respectively, $\mathbf{A}_0$ and $\mathbf{B}_0$, given by the restrictions to their domains of the quotient mapping $\mathbf{A}_0 + \mathbf{B}_0 \to (\mathbf{A}_0 + \mathbf{B}_0)/\tilde{\theta}(K_0)$.

It turns out that $p \circ f = q \circ g$, because $\mathrm{Dom}\,p \circ f = \mathrm{Dom}\,q \circ g = K_0$, and $p \circ f$ and $q \circ g$ are equal on K_0 by definition. Therefore, there exists a cd-quomorphism $h : \mathbf{C} \to \mathbf{D}_0$ such that $h \circ \tilde{g} = p$ and $h \circ \tilde{f} = q$. But then $h \circ \tilde{g} \circ f = p \circ f$ and therefore $K_0 = \mathrm{Dom}\,p \circ f \subseteq \mathrm{Dom}\,\tilde{g} \circ f = K'$, $A_0 = \mathrm{Dom}\,p \subseteq \mathrm{Dom}\,\tilde{g}$ and $B_0 = \mathrm{Dom}\,q \subseteq \mathrm{Dom}\,\tilde{f}$, as we wanted to prove.

Taking $A' = \mathrm{Dom}\,\tilde{g}$ and $B' = \mathrm{Dom}\,\tilde{f}$, this finishes the proof of the first condition. As far as the second condition goes, notice that from the previous argument we deduce the existence of a unique cd-quomorphism $\mathbf{C} \to (\mathbf{A}' + \mathbf{B}')/\tilde{\theta}(K')$ such that $h \circ \tilde{g} = p$ and $h \circ \tilde{f} = q$, where $\tilde{\theta}(K')$ is the least congruence on $\mathbf{A}' + \mathbf{B}'$ containing $\theta(K')$ and

$$p : \mathbf{A} \to (\mathbf{A}' + \mathbf{B}')/\tilde{\theta}(K'), q : \mathbf{B} \to (\mathbf{A}' + \mathbf{B}')/\tilde{\theta}(K')$$

are the cd-quomorphisms with domains, respectively, $\mathbf{A}'$ and $\mathbf{B}'$, given by the restrictions to their domains of the quotient mapping $\mathbf{A}'+\mathbf{B}' \to (\mathbf{A}' + \mathbf{B}')/\tilde{\theta}(K')$.

On the other hand, since $\tilde{\theta}(K') \subseteq \ker(\tilde{g} + \tilde{f})$, we have that there exists a unique homomorphism $h' : (\mathbf{A}' + \mathbf{B}')/\tilde{\theta}(K') \to \mathbf{C}$ such that $h' \circ \mathrm{nat}_{\tilde{\theta}(K')} = \tilde{g} + \tilde{f}$: i.e., such that $h' \circ q = \tilde{f}$ and $h' \circ p = \tilde{g}$.

A simple argument, using the universal properties of pushouts and of quotients, shows then that h and h' are isomorphisms inverse of each other.

Now, to finish the proof of the direct implication, we must prove that $\theta(K') = \tilde{\theta}(K')$. Since no operation arrives to carriers of sorts $E_{p,q}$, for any p, q, and since $\theta(K') \subseteq \tilde{\theta}(K')$, we actually only have to prove that $\tilde{\theta}(K')_V \subseteq \theta(K')_V$. But if $\mathbf{D}$ is the discrete Σ_H-algebra with $D_E = \emptyset$ and $D_V = (A'_V + B'_V)/\theta(K')_V$, and if $p' : \mathbf{A} \to \mathbf{D}$ and $q' : \mathbf{B} \to \mathbf{D}$ are the cd-quomorphisms that are empty mappings on the arcs and the restriction of the quotient mappings on the sorts, then $p' \circ f = q' \circ g$, and therefore there exists a cd-quomorphism $h' : \mathbf{C} \to \mathbf{D}$ extending p' and q', and in particular a (well-defined) mapping

$$h'_V : (A'_V + B'_V)/\tilde{\theta}(K'_V) \to (A'_V + B'_V)/\theta(K'_V)$$

given by $h'_V([x]_{\tilde{\theta}(K')_V}) = [x]_{\theta(K')_V}$ for every $x \in A'_V + B'_V$. This entails that $\tilde{\theta}(K')_V \subseteq \theta(K')_V$, as we wanted to prove. ∎

As a consequence of Sect. 2.4 and Proposition 7, we have the following result.

Corollary 8. Two cdc-quomorphisms have always a pushout in CDQ-Alg_{Σ_H}, and it is equal to their pushout in CDCQ-Alg_{Σ_H}.

We shall also need the following generalization of Proposition 7. For every $S_0 \subseteq S_H$, we shall call a S_0-*cd-quomorphism* to a cd-quomorphism $f : \mathbf{A} \to \mathbf{B}$ that is totally defined on all carriers of $\mathbf{A}$ of sorts in S_0, i.e., such that $\mathrm{Dom}\, f_s = A_s$ for every $s \in S_0$. For every such S_0, let CDQ-$\mathrm{Alg}_{\Sigma_H}(S_0)$ denote the category of S_0-cd-quomorphisms of partial Σ_H-algebras: notice that CDQ-$\mathrm{Alg}_{\Sigma_H}(\emptyset) =$ CDQ-Alg_{Σ_H} and CDQ-$\mathrm{Alg}_{\Sigma_H}(S_H) = \mathrm{TAlg}_{\Sigma_H}$.

Proposition 9. Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two S_0-cd-quomorphisms of partial Σ_H-algebras. Then, f and g have a pushout in CDQ-$\mathrm{Alg}_{\Sigma_H}(S_0)$ iff the following two conditions are satisfied:

- There exists the greatest closed subset K' of $\mathbf{K}$ such that
  - $K'_s = K_s$ for every $s \in S_0$,
  - $f^{-1}(f(K')) = K'$, $g^{-1}(g(K')) = K'$,
  - there exists the greatest closed subset A' of $\mathbf{A}$ lying between $f(K')$ and $(A - f(K)) \cup f(K')$ and such that $A'_s = A_s$ for every $s \in S_0$,
  - there exists the greatest closed subset B' of $\mathbf{B}$ lying between $g(K')$ and $(B - g(K)) \cup g(K')$ and such that $B'_s = B_s$ for every $s \in S_0$;
- The equivalence relation $\theta(K')$ on $A' \sqcup B'$ generated by

  $$\{(f_s(x), g_s(x)) \mid x \in K'_s\}, s \in S$$

  satisfies the following condition: If $(e, e') \in \theta(K')_{E_{p,q}}$ with $E_{p,q} \notin S_0$, and $\varphi \in \{s_1, \ldots, s_p, t_1, \ldots, t_q\}$ then $(\varphi^{\mathbf{A}'+\mathbf{B}'}(e), \varphi^{\mathbf{A}'+\mathbf{B}'}(e')) \in \theta(K')_V$ (i.e., when $\theta(K')$ is a congruence on $\mathbf{A}' + \mathbf{B}'$ for the sorts not belonging to S_0).

And when such K', A' and B' exist, and $\theta(K')$ satisfies the previous condition, the pushout of f and g is (up to isomorphism) $(\mathbf{C}, \tilde{g} : \mathbf{A} \to \mathbf{C}, \tilde{f} : \mathbf{B} \to \mathbf{C})$ in CDQ-$\mathrm{Alg}_{\Sigma_H}(S_0)$, where $\mathbf{C} = (\mathbf{A}' + \mathbf{B}')/\tilde{\theta}(K')'$, $\tilde{\theta}(K')$

is the least congruence on $\mathbf{A}' + \mathbf{B}'$ generated by $\theta(K')$, and $\tilde{g} : \mathbf{A} \to \mathbf{C}$ and $\tilde{f} : \mathbf{B} \to \mathbf{C}$ are respectively the cd-quomorphisms given by the restriction to $\mathbf{A}'$ and $\mathbf{B}'$ of the natural homomorphism $\mathbf{A}' + \mathbf{B}' \to (\mathbf{A}' + \mathbf{B}')/\tilde{\theta}(K')$.

The proof is similar to the one for Proposition 7, and we leave the details to the reader.

(Notice that the existence of A' (respectively, B') is equivalent to the existence of *some* closed subset of $\mathbf{A}$ lying between $f(K')$ and $(A - f(K)) \cup f(K')$ and such that $A'_s = A_s$ for every $s \in S_0$ (respectively, of $\mathbf{B}$ lying between $g(K')$ and $(B - g(K)) \cup g(K')$ and such that $B'_s = B_s$ for every $s \in S_0$).)

It is important to notice that when two S_0-cd-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ have a pushout in CDQ-$\mathrm{Alg}_{\Sigma_H}(S_0)$ then it is given by the pushout of $f|_{K'} : \mathbf{K}' \to \mathbf{A}'$ and $g|_{K'} : \mathbf{K}' \to \mathbf{B}'$ in Alg_{Σ_H}: taking $S_0 = \emptyset$, this observation is also valid for pushouts in CDQ-Alg_{Σ_H}.

The following notation will be useful in Section 5. For any cd-quomorphism r, let $S(r) = \{s \in S_H \mid r_s \text{ total}\}$. Then, we shall denote CDQ-$\mathrm{Alg}_{\Sigma_H}(S(r))$ simply by CDQ-$\mathrm{Alg}_{\Sigma_H}(r)$.

3. There is no gain in computing pushouts of partial homomorphisms through epireflections

The following result is known from [7].

Theorem 10. Let $\mathcal{C}$ be a category and let $\mathcal{K}$ be a full and epireflective subcategory of $\mathcal{C}$. Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two morphisms in $\mathcal{K}$. Let $\mathbf{K}', \mathbf{A}', \mathbf{B}'$ be objects of $\mathcal{C}$ such that $\ell_{\mathbf{A}'} : \mathbf{A}' \to \mathbf{A}$ and $\ell_{\mathbf{B}'} : \mathbf{B}' \to \mathbf{B}$ are $\mathcal{K}$-epireflections and there exist an epimorphism $\ell : \mathbf{K}' \to \mathbf{K}$ and two morphisms $f' : \mathbf{K}' \to \mathbf{A}'$ and $g' : \mathbf{K}' \to \mathbf{B}'$ such that $f \circ \ell = \ell_{\mathbf{A}'} \circ f'$ and $g \circ \ell = \ell_{\mathbf{B}'} \circ g'$.

Let $(\mathbf{H}', \tilde{g}' : \mathbf{A}' \to \mathbf{H}', \tilde{f}' : \mathbf{B}' \to \mathbf{H}')$ be a pushout of $f' : \mathbf{K}' \to \mathbf{A}'$ and $g' : \mathbf{K}' \to \mathbf{B}'$ in $\mathcal{C}$, let $\ell_{\mathbf{H}'} : \mathbf{H}' \to \mathbf{H}$ be a $\mathcal{K}$-epireflection, and let $\tilde{g} : \mathbf{A} \to \mathbf{H}$ and $\tilde{f} : \mathbf{B} \to \mathbf{H}$ be respectively the unique morphisms such that $\tilde{g} \circ \ell_{\mathbf{A}'} = \ell_{\mathbf{H}'} \circ \tilde{g}'$ and $\tilde{f} \circ \ell_{\mathbf{B}'} = \ell_{\mathbf{H}'} \circ \tilde{f}'$.

Then $(\mathbf{H}, \tilde{g} : \mathbf{A} \to \mathbf{H}, \tilde{f} : \mathbf{B} \to \mathbf{H})$ is a pushout of $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ in $\mathcal{K}$.

This result can then be applied to compute the pushout of two morphisms in $\mathcal{K}$ as the epireflection of the pushout of two suitable morphisms in $\mathcal{C}$. The result has some interest, from the point of view of efficiency, when epireflections increase the size of the objects, as it was the case for instance with free completions of partial algebras, in the context of [7]. But, unfortunately, this is not the case for partial homomorphisms.

Let Γ denote in the sequel any graph structure.

Notice first of all that P-TAlg$_\Gamma$ is not a full subcategory of Q-Alg$_\Gamma$ since, in general, a relative subalgebra of a total algebra need not be total; take then a total algebra $\mathbf{A}$, a non-total relative subalgebra $\mathbf{B}$ of it, and consider the identity mapping on $\mathbf{B}$ as a partial mapping $\mathbf{A} \to \mathbf{A}$. It is a quomorphism, but not a partial homomorphism.

With the different categories of conformisms of partial Γ-algebras introduced in the previous section, we have more positive (but useless) results.

Proposition 11. P-TAlg$_\Gamma$ is a full and epireflective subcategory of CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$, and CDCQ-Alg$_\Gamma$.

Proof. If $\mathbf{A}$ is a partial Γ-algebra, $\mathbf{B}$ is a total Γ-algebra and $f : \mathbf{A} \to \mathbf{B}$ is a closed homomorphism, then $\mathbf{A}$ must be total. This implies that if $f : \mathbf{A} \to \mathbf{B}$ is a conformism and $\mathbf{B}$ is a total Γ-algebra, then the weak subalgebra of $\mathbf{A}$ supported on Dom f, out from which f is a closed homomorphism, is a total algebra, and therefore a closed subalgebra of $\mathbf{A}$.

Taking $\mathbf{A}$ also a total Γ-algebra, this entails that f is a partial homomorphism from $\mathbf{A}$ to $\mathbf{B}$. Thus, P-TAlg$_\Gamma$ is a full subcategory of CF-Alg$_\Gamma$, and since P-TAlg$_\Gamma$ is contained in CQ-Alg$_\Gamma$ and CDCQ-Alg$_\Gamma$, which are subcategories of CF-Alg$_\Gamma$, it is also a full subcategory of them.

Let now $\mathbf{A}'$ be a partial Γ-algebra and let $\mathbf{A}$ be the largest weak subalgebra of $\mathbf{A}'$ that is total. It always exists, as the union of all weak subalgebras of $\mathbf{A}'$ that are total, and it is actually a closed subalgebra of $\mathbf{A}'$. Let $\ell_{\mathbf{A}'} : \mathbf{A}' \to \mathbf{A}$ be the partial homomorphism given by the identity over $\mathbf{A}$: it is a cdc-quomorphism (and therefore, also a c-quomorphism and a conformism). Moreover, it is an epimorphism, since it is surjective.

If $f : \mathbf{A}' \to \mathbf{B}$ is a conformism from $\mathbf{A}'$ to a total Γ-algebra $\mathbf{B}$ then, by the discussion at the beginning of this proof, its domain is a total (weak) subalgebra of $\mathbf{A}'$, and therefore it is contained in $\mathbf{A}$. Thus, f can be regarded as a conformism $\widehat{f} : \mathbf{A} \to \mathbf{B}$. And such $\widehat{f}$ is clearly the only conformism $\mathbf{A} \to \mathbf{B}$ such that $\widehat{f} \circ \ell_{\mathbf{A}'} = f$. Therefore, P-TAlg$_\Gamma$ is an epireflective subcategory of CF-Alg$_\Gamma$.

The same argument can be carried over for c-quomorphisms and cdc-quomorphisms, showing that P-TAlg$_\Gamma$ is also epireflective in CQ-Alg$_\Gamma$, and CDCQ-Alg$_\Gamma$. ∎

The proof of the previous result shows that, contrary to the case of total homomorphisms, P-TAlg$_\Gamma$-epireflections in categories of conformisms of partial Γ-algebras are “smaller” than the original partial algebras.

Remark 12. The first part of the proof of the previous proposition shows that P-TAlg$_\Gamma$ is a full subcategory of CDQ-Alg$_\Gamma$. But it turns out that P-TAlg$_\Gamma$ is not epireflective in CDQ-Alg$_\Gamma$: actually, no strictly partial Γ-algebra has a P-TAlg$_\Gamma$-reflection in CDQ-Alg$_\Gamma$.

Indeed, let $\mathbf{A}'$ be a partial Γ-algebra, let $\ell_{\mathbf{A}'} : \mathbf{A}' \to \mathbf{A}$ be its free completion, and assume that $\mathbf{A}'$ has a P-TAlg$_\Gamma$-reflection $\gamma_{\mathbf{A}'} : \mathbf{A}' \to \mathbf{A}''$ in CDQ-Alg$_\Gamma$. Since $\ell_{\mathbf{A}'}$ is a homomorphism, and therefore a cd-quomorphism, there exists a cd-quomorphism $\widetilde{\ell} : \mathbf{A}'' \to \mathbf{A}$ such that $\widetilde{\ell} \circ \gamma_{\mathbf{A}'} = \ell_{\mathbf{A}'}$. This implies that $\gamma_{\mathbf{A}'}$ is a homomorphism, and then there exists a homomorphism $\widehat{\gamma} : \mathbf{A} \to \mathbf{A}''$ such that $\gamma_{\mathbf{A}'} = \widehat{\gamma} \circ \ell_{\mathbf{A}'}$.

Now, the universal properties of P-TAlg$_\Gamma$-epireflections and of free completions show that $\widehat{\gamma}$ and $\widetilde{\ell}$ are inverses of each other.

We have then proved that if $\mathbf{A}'$ has a P-TAlg$_\Gamma$-reflection in CDQ-Alg$_\Gamma$ then it must its free completion $\ell_{\mathbf{A}'} : \mathbf{A}' \to \mathbf{A}$. But if $\mathbf{A}'$ is non total then this is not its P-TAlg$_\Gamma$-reflection in CDQ-Alg$_\Gamma$, because the empty mapping, understood as a cd-quomorphism $\emptyset : \mathbf{A}' \to \mathbf{A}$, admits (at least) two cd-quomorphisms $\mathbf{A} \to \mathbf{A}$ extending it: namely, the empty mapping itself, and the identity on the complementary of $\mathbf{A}'$, which is a non-empty closed subalgebra of $\mathbf{A}$. ∎

The (negative) results in this section motivate the development, for the sake of efficiency, of *ad hoc* methods to compute pushouts of partial homomorphisms through free completions.

4. Computing single-pushout derivations *a posteriori* through free completions

Let Γ denote a graph structure. The goal of this section is to show that, given a single-pushout transformation rule $r : \mathbf{L} \to \mathbf{R}$ in P-TAlg$_\Gamma$ (i.e., a partial homomorphism of total Γ-algebras) and an occurrence $m : \mathbf{L} \to \mathbf{G}$ (i.e., a homomorphism of total Γ-algebras), a single-pushout transformation rule $r_0 : \mathbf{L}_0 \to \mathbf{R}_0$ in Alg$_\Gamma$ can be extracted from r such that the pushout algebra $\mathbf{H}$ of r and m in P-TAlg$_\Gamma$ is the free completion of the pushout algebra $\mathbf{H}_0$ of r_0 and $m_0 : \mathbf{L}_0 \to \mathbf{G}_0$ in Alg$_\Gamma$, where $\mathbf{G}_0$ is a suitable relative subalgebra of $\mathbf{G}$ containing $m(\mathbf{L}_0)$ and m_0 is the corresponding restriction of m. Since $\mathbf{L}_0$, $\mathbf{R}_0$ and $\mathbf{G}_0$ may be significantly smaller than $\mathbf{L}$, $\mathbf{R}$ and $\mathbf{G}$, depending among other factors on the inner freedom of these total algebras, this means that, after pattern-matching, the derived algebra can be obtained in simpler computations.

The following result has been first stated, without proof, in [7].

Theorem 13. Let Γ be a graph structure. Let $r : \mathbf{L} \to \mathbf{R}$ and $m : \mathbf{L} \to \mathbf{G}$ be two partial homomorphisms of total Γ-algebras. Let $L' \subseteq L$ be the greatest closed subset of $\mathbf{L}$ such that $r^{-1}(r(L')) = L'$ and $m^{-1}(m(L')) = L'$, and let $\mathbf{L}'$ be the subalgebra of $\mathbf{L}$ supported on L'. Let $R' \subseteq R$ be the greatest closed subset of $\mathbf{R}$ contained in $(R - r(L)) \cup r(L')$, and let $\mathbf{R}'$ be the subalgebra of $\mathbf{R}$ supported on R'. Let also $G' \subseteq G$ be the greatest closed subset of $\mathbf{G}$ contained in $(G - m(L)) \cup m(L')$, and let $\mathbf{G}'$ be the subalgebra of $\mathbf{G}$ supported on G'.

Let $\mathbf{L}_0$ be a relative subalgebra of $\mathbf{L}'$ supported on a subset of L' that generates $\mathbf{L}'$. Let $\mathbf{R}_0$ be a relative subalgebra of $\mathbf{R}'$ containing $r(L_0)$ and freely completed by $\mathbf{R}'$, and let r_0 be the restriction of r to $\mathbf{L}_0$ and $\mathbf{R}_0$. Let also $\mathbf{G}_0$ be a relative subalgebra of $\mathbf{G}'$ containing $m(L_0)$ and freely completed by $\mathbf{G}'$, and let m_0 be the restriction of m to $\mathbf{L}_0$ and $\mathbf{G}_0$. Then the pushout algebra $\mathbf{H}$ of r and m in P-TAlg$_\Gamma$ is the free completion of the pushout algebra $\mathbf{H}_0$ of r_0 and m_0 in Alg$_\Gamma$.

Proof. Consider the commutative diagram formed by the objects and morphisms described in the statement, extended by the embedding morphisms $\ell : \mathbf{L}' \to \mathbf{L}$, $\ell_{\mathbf{R}'} : \mathbf{R}' \to \mathbf{R}$ and $\ell_{\mathbf{G}'} : \mathbf{G}' \to \mathbf{G}$, by the free completion $\ell_{\mathbf{H}_0} : \mathbf{H}_0 \to \mathbf{H}$, and by the morphisms defined below.

$$
\begin{tikzcd}
\mathbf{L} \arrow[rrrrr, "r"] \arrow[ddddd, "m"'] & & & & & \mathbf{R} \arrow[ddddd, bend left, "p"] \arrow[ddddl, bend left, "\tilde{m}'"'] \\
& \mathbf{L}' \arrow[ul, "\ell"'] \arrow[rrr, "r'"] \arrow[ddd, "m'"'] & & & \mathbf{R}' \arrow[ur, "\ell_{\mathbf{R}'}"] \arrow[ddd, "\tilde{m}'"] & \\
& & \mathbf{L}_0 \arrow[ul] \arrow[r, "r_0"] \arrow[d, "m_0"'] & \mathbf{R}_0 \arrow[ur] \arrow[d, "\tilde{m}_0"] & & \\
& & \mathbf{G}_0 \arrow[r, "\tilde{r}_0"'] \arrow[dl] & \mathbf{H}_0 \arrow[dr, "\ell_{\mathbf{H}_0}"] & & \\
& \mathbf{G}' \arrow[rrr, "\tilde{r}'"'] \arrow[dl, "\ell_{\mathbf{G}'}"'] & & & \mathbf{H} \arrow[dr, dotted, "r"] & \\
\mathbf{G} \arrow[urrrr, bend right=10, "\tilde{r}'"'] \arrow[rrrrr, bend right=10, "q"'] & & & & & \mathbf{H}'
\end{tikzcd}
$$

Let $r' : \mathbf{L}' \to \mathbf{R}'$ and $m' : \mathbf{L}' \to \mathbf{G}'$ be the restrictions of r and m to $\mathbf{L}'$. By construction, they are homomorphisms of total Γ-algebras.

Let $\tilde{m}_0 : \mathbf{R}_0 \to \mathbf{H}_0$ and $\tilde{r}_0 : \mathbf{G}_0 \to \mathbf{H}_0$ be the respective pushout morphisms, and let $\tilde{m}' : \mathbf{R}' \to \mathbf{H}$ and $\tilde{r}' : \mathbf{G}' \to \mathbf{H}$ be respectively the unique homomorphisms of total Γ-algebras extending $\ell_{\mathbf{H}_0} \circ \tilde{r}_0 : \mathbf{G}_0 \to \mathbf{H}$ and $\ell_{\mathbf{H}_0} \circ \tilde{m}_0 : \mathbf{R}_0 \to \mathbf{H}$. By [7, Cor. 7], $(\mathbf{H}, \tilde{m}' : \mathbf{R}' \to \mathbf{H}, \tilde{r}' : \mathbf{G}' \to \mathbf{H})$ is a pushout of $r' : \mathbf{L}' \to \mathbf{R}'$ and $m' : \mathbf{L}' \to \mathbf{G}'$ in TAlg_Γ.

Let also $\tilde{m}' : \mathbf{R} \to \mathbf{H}$ and $\tilde{r}' : \mathbf{G} \to \mathbf{H}$ be the same homomorphisms $\tilde{m}' : \mathbf{R}' \to \mathbf{H}$ and $\tilde{r}' : \mathbf{G}' \to \mathbf{H}$ but considered as partial homomorphisms with source algebras $\mathbf{R}$ and $\mathbf{G}$, respectively. A simple diagram chasing shows that $(\tilde{m}' \circ r) \circ \ell = (\tilde{r}' \circ m) \circ \ell$. Since ℓ is an epimorphism, this implies that $\tilde{m}' \circ r = \tilde{r}' \circ m$.

Now, let $\mathbf{H}'$ be a total Γ-algebra and let $p : \mathbf{R} \to \mathbf{H}'$ and $q : \mathbf{G} \to \mathbf{H}'$ be two partial homomorphisms such that $(p \circ \ell_{\mathbf{R}'}) \circ r' = (q \circ \ell_{\mathbf{G}'}) \circ m'$. Then, there exists a unique partial homomorphism $r : \mathbf{H} \to \mathbf{H}'$ such that $r \circ \tilde{m}' = p \circ \ell_{\mathbf{R}'}$ and $r \circ \tilde{r}' = q \circ \ell_{\mathbf{G}'}$, that is, such that $r \circ \tilde{m}' = p$ and $r \circ \tilde{r}' = q$. Therefore, $(\mathbf{H}, \tilde{m}' : \mathbf{R} \to \mathbf{H}, \tilde{r}' : \mathbf{G} \to \mathbf{H})$ is a pushout of $r' : \mathbf{L}' \to \mathbf{R}'$ and $m' : \mathbf{L}' \to \mathbf{G}'$ in P-TAlg_Γ. ∎

If, in the proof of the previous Proposition, we take Γ to be a *hierarchical* graph structure [9, Def. B.1], as for instance Σ_H, then we can take $\mathbf{L}_0$ as the relative subalgebra of $\mathbf{L}'$ supported on the least subset of L' that generates $\mathbf{L}'$, $\mathbf{R}_0$ as the least relative subalgebra of $\mathbf{R}'$ containing $r(L_0)$ and freely completed by $\mathbf{R}'$, and $\mathbf{G}_0$ as the least relative subalgebra of $\mathbf{G}'$ containing $m(L_0)$ and freely completed by $\mathbf{G}'$. In the case of hypergraphs, these “least” objects are described in Sect. 2.2.

5. Simplifying *a priori* single-pushout transformation rules

In the previous section, the rule r_0 depended on r and m, and therefore it could not be extracted from r *a priori*, without a prior knowledge of the occurrence m, as it was done for double-pushout production rules in [7]. In this section we show that if we want to extract r_0 *a priori*, then $\mathbf{L}_0$ and $\mathbf{R}_0$ may not be significantly smaller than $\mathbf{L}$ and $\mathbf{R}$.

More specifically, in this section we address the following (informally stated) problem:

> Given a partial homomorphism of hypergraphs $r : \mathbf{L} \to \mathbf{R}$, find a rule $r_0 : \mathbf{L}_0 \to \mathbf{R}_0$, with $\mathbf{L}_0$ and $\mathbf{R}_0$ suitable relative subalgebras of $\mathbf{L}$ and $\mathbf{R}$, respectively, and r_0 the corresponding restriction of r, such that

– The rule r can be applied to a hypergraph $\mathbf{G}$ through a homomorphism $m : \mathbf{L} \to \mathbf{G}$ iff r_0 can be applied to $\mathbf{G}$ through the restriction $m_0 : \mathbf{L}_0 \to \mathbf{G}$ of m; and
– The derived hypergraph of $\mathbf{G}$ by the application of r through m can be obtained in an explicit way from the derived partial Σ_H-algebra of $\mathbf{G}_0$ by the application of r_0 through $m_0 : \mathbf{L}_0 \to \mathbf{G}_0$, where $\mathbf{G}_0$ is some suitable relative subalgebra of $\mathbf{G}$ containing $m(\mathbf{L}_0)$.

To give a formal content to this problem, notice that we talk in it about rules $r_0 : \mathbf{L}_0 \to \mathbf{R}_0$ and pushouts of these rules and homomorphisms $m_0 : \mathbf{L}_0 \to \mathbf{G}_0$. We need first of all to fix the category where these pushouts will be computed. Since the restriction of a partial homomorphism of total algebras to a relative subalgebra of the source algebra is always a cd-quomorphism, but in general it need not be a cdc-quomorphism, and since m_0 will also be in general only a homomorphism, the natural category where we can consider both r_0 and m_0 is some CDQ-$\mathrm{Alg}_{\Sigma_H}(S_0)$ (the category of cdc-quomorphisms of partial Σ_H-algebras that are total on the sorts of S_0; see Sect. 2.5) containing r_0. Notice that it is not a big question that the category where to work in depends on r_0, since in all these categories pushouts are computed exactly in the same way. But we still have the drawback that, in general, the pushout of r_0 and m_0 need not exist herein. Therefore, as a side requirement we must impose that *it always exists*; this is were Propositions 7 and 9 enter into scene.

We shall require, as a first condition in the problem, that $\mathbf{L}$ be a free completion (as Σ_H-algebra) of $\mathbf{L}_0$, since then every homomorphism $m_0 : \mathbf{L}_0 \to \mathbf{G}$ can be extended in a unique way to a homomorphism $m : \mathbf{L} \to \mathbf{G}$. Recall, then, that $\mathbf{L}_0$ will be obtained from $\mathbf{L}$ by deleting some nodes of degree 1.

Consistently with this interpretation of the first condition, we shall require as a second condition that the pushout of r and m in P-TAlg_{Σ_H} be the free completion of the pushout of r_0 and $m_0 : \mathbf{L}_0 \to \mathbf{G}_0$ in some CDQ-$\mathrm{Alg}_{\Sigma_H}(S_0)$. To do that, we shall use Theorem 10. To be able to apply it, and with the notations of Section 2, we need that $\mathbf{L}'_0(r_0, m_0)$ generates $\mathbf{L}'(r, m)$, and in particular that every isolated node of $\mathbf{L}'(r, m)$ belongs to $\mathbf{L}_0$, for every homomorphism $m : \mathbf{L} \to \mathbf{G}$. We also need to require that $\mathbf{R}'(r, m)$ be a free completion of $\mathbf{R}'_0(r_0, m_0)$, for every homomorphism $m : \mathbf{L} \to \mathbf{G}$. In particular, $\mathbf{R}$ must be a free completion of $\mathbf{R}_0$ (taking $m = \mathrm{Id}_{\mathbf{L}} : \mathbf{L} \to \mathbf{L}$).

Summarizing, we can re-state our original problem as follows.

Given a partial homomorphism of hypergraphs $r : \mathbf{L} \to \mathbf{R}$, find a cd-quomorphism of partial Σ_H-algebras $r_0 : \mathbf{L}_0 \to \mathbf{R}_0$ such that:

- $\mathbf{L}_0$ is a relative subalgebra of $\mathbf{L}$ such that $\mathbf{L}$ is its free completion, also containing all nodes that become isolated in some gluing algebra $\mathbf{L}'(r,m)$;
- $\mathbf{R}_0$ is a relative subalgebra of $\mathbf{R}$, containing $r(\mathbf{L}_0)$ and such that $\mathbf{R}$ is its free completion, and $r_0 : \mathbf{L}_0 \to \mathbf{R}_0$ is the corresponding restriction of r;
- There always exists the pushout of $r_0 : \mathbf{L}_0 \to \mathbf{R}_0$ and any homomorphism of partial Σ_H-algebras $m_0 : \mathbf{L}_0 \to \mathbf{G}_0$ in some CDQ-$\mathrm{Alg}_{\Sigma_H}(S_0)$;
- For every homomorphism of hypergraphs $m : \mathbf{L} \to \mathbf{G}$, there exists a relative subalgebra $\mathbf{G}_0$ of $\mathbf{G}$ containing $m(\mathbf{L}_0)$ such that, if we denote by $m_0 : \mathbf{L}_0 \to \mathbf{G}_0$ the corresponding restriction of m, $\mathbf{R}'(r,m)$ is the free completion of $\mathbf{R}'_0(r_0,m_0)$ and $\mathbf{G}'(r,m)$ is the free completion of $\mathbf{G}'_0(r_0,m_0)$.

And then, by the description of the pushout of two S_0-cd-quomorphisms given in Sect. 2.5, we have that the pushout hypergraph of r and m will be indeed the free completion of the pushout algebra of r_0 and m_0 in CDQ-$\mathrm{Alg}_{\Sigma_H}(S_0)$.

In this section we show that there is no spectacular solution to this problem. We shall distinguish two cases, with different solutions: when r_V is not total and when r_V is total.

5.1. The case when r_V is partial

Let $r : \mathbf{L} \to \mathbf{R}$ be in the sequel a fixed partial homomorphism of hypergraphs, with $\mathrm{Dom}\, r_V \neq L_V$. Recall that $\mathbf{L}_0$ must be obtained from $\mathbf{L}$ by deleting some nodes of degree 1 and nothing else.

Remark 14. Let v_0 be a node of degree 1 belonging to $\mathrm{Dom}\, r_V$ such that the only arc e to which it is incident does not belong to $\mathrm{Dom}\, r_E$. Then it is an isolated node in $\mathbf{Dom} r$, which is the gluing hypergraph of r and $\mathrm{Id}_{\mathbf{L}}$. Thus, v_0 must belong to $\mathbf{L}_0$.

Remark 15. Let v_0 be a node of degree 1 belonging to $\mathrm{Dom}\, r_V$, and let $e_0 \in \mathrm{Dom}\, r_E$ be the only arc to which it is incident. We shall distinguish two cases:

- Assume that there is some other node v_1 incident to e_0, and let $v \in L_V - \mathrm{Dom}\, r_V$. Let θ be the least congruence on $\mathbf{L}$ identifying v_1 and v_0, and let $m : \mathbf{L} \to \mathbf{L}/\theta$ be the correspoding quotient homomorphism. Then $v_0 \in L'(r,m)_V$ but $e_0 \notin L'(r,m)_E$, and therefore v_0 will be an isolated node in $\mathbf{L}'(r,m)$. Thus, v_0 must belong to $\mathbf{L}_0$.

– Assume now that v_0 is the only node incident to e_0: to fix ideas, assume that $e_0 \in L_{E_{1,0}}$ and $v_0 = s_1^{\mathbf{L}}(e_0)$. As before, let $v \in L_V - \mathrm{Dom}\, r_V$.

  Let $\{e_0, e_1, \ldots, e_k\} = r_E^{-1}(r_E(e_0))$. Since r is a partial homomorphism of hypergraphs, $e_1, \ldots, e_k \in E_{1,0}$. Let $v_i = s_1^{\mathbf{L}}(e_i)$ for $i = 1, \ldots, k$, let θ be the least congruence on $\mathbf{L}$ identifying $v_0, v_1, \ldots, v_k$ and v, and let $m : \mathbf{L} \to \mathbf{L}/\theta$ be the corresponding quotient homomorphism.

  A simple computation shows that

$$
\begin{aligned}
L'(r,m)_V &= \mathrm{Dom}\, r_V - \{v_0, \ldots, v_k\} \\
L'(r,m)_E &= \mathrm{Dom}\, r_E - \{e_0, \ldots, e_k\}
\end{aligned}
$$

  and, being $\mathbf{R}'(r,m) = (\mathbf{R} - r(\mathbf{L})) \cup r(\mathbf{L}')$, we have in particular that $r_E(e_0) \notin R'_E$.

  Let now $\mathbf{L}_0$ be obtained from $\mathbf{L}$ by deleting at least v_0 (we may delete other nodes of degree 1, even $v_1, \ldots, v_k$), and let $\mathbf{R}_0$ be any partial subhypergraph of $\mathbf{R}$ containing $r(\mathbf{L}_0)$ and such that $\mathbf{R}$ is its free completion. Let $r_0 : \mathbf{L}_0 \to \mathbf{R}_0$ be the corresponding restriction. Let now $m_0 : \mathbf{L}_0 \to \mathbf{G}_0$ be the restriction of m to $\mathbf{L}_0$, considered with target algebra some relative subalgebra $\mathbf{G}_0$ of $\mathbf{G}$ containing $m(\mathbf{L}_0)$ and such that $\mathbf{G}$ is its free completion.

  Notice that, at least, $\mathbf{G}_0$ is the quotient of $\mathbf{L}_0$ by the least equivalence relation on $\mathbf{L}_0$ identifying v with the nodes among $v_1, \ldots, v_k$ still remaining in $\mathbf{L}_0$, and m_0 is the corresponding quotient mapping.

  But then $e_0 \in L'_0(r_0, m_0)_E$, and therefore $r_E(e_0) \in R'_0(r_0, m_0)_E$. And in this case $\mathbf{R}'(r,m)$ cannot be the free completion of $\mathbf{R}'_0(r_0, m_0)$, because it even does not contain it.

  This shows that v_0 (and, for that matter, $v_1, \ldots, v_k$) must also belong to $\mathbf{L}_0$.

Remark 16. Let $v_0 \in L_V$ be a node of degree 1 not belonging to $\mathrm{Dom}\, r_V$, and assume $\mathrm{Dom}\, r_V \neq \emptyset$. Let $v_1 \in \mathrm{Dom}\, r_V$, let $\{v_1, \ldots, v_k\} = r_V^{-1}(r_V(v_1))$ and let θ be the least congruence on $\mathbf{L}$ identifying $v_0, v_1, \ldots, v_k$: it is simply the equivalence relation identifying these nodes. Let $m : \mathbf{L} \to \mathbf{L}/\theta$ be the quotient homomorphism. Arguing as in the second case of the previous remark, we have that $r_V(v_1) \notin R'(r,m)_V$.

Let now $\mathbf{L}_0$ be the relative subalgebra of $\mathbf{L}$ obtained by removing (at least) v_0; by the previous Remarks, we can assume that $v_1, \ldots, v_k \in \mathbf{L}_0$ (those v_i not of degree 1 must belong to $\mathbf{L}_0$ by definition, and those of degree 1 cannot be removed by the previous Remarks).

Let $\mathbf{R}_0$ be any relative subalgebra of $\mathbf{R}$ containing $r(\mathbf{L}_0) = r(\mathbf{L})$ and such that $\mathbf{R}$ is a free completion of $\mathbf{R}_0$, and let $r_0 : \mathbf{L}_0 \to \mathbf{R}_0$ be the corresponding restriction of r; in particular, $r_V(v_1) \in R_{0V}$. Let moreover $\mathbf{G}_0$ be any relative subalgebra of $\mathbf{G}$ containing $m(\mathbf{L}_0)$ and such that $\mathbf{G}$ is its free completion, and let $m_0 : \mathbf{L}_0 \to \mathbf{G}_0$ be the restriction of m. The effect of m_0 is simply the identification of $v_1, \ldots, v_k$. Since all these nodes are in the domain of r, we have that $\mathbf{L}'_0(r_0, m_0) = \mathrm{Dom}\, r$ (the subhypergraph of $\mathbf{L}_0$ supported on $\mathrm{Dom}\, r$) and then $\mathbf{R}'_0(r_0, m_0) = \mathbf{R}_0$. And then again, $\mathbf{R}'$ does not contain $\mathbf{R}_0$.

This shows that $\mathbf{L}_0$ must contain any node of degree 1 not in $\mathrm{Dom}\, r_V$.

Summarizing these remarks, we have that if $\emptyset \neq \mathrm{Dom}\, r_V \neq L_V$ then $\mathbf{L}_0$ must be $\mathbf{L}$ itself: we cannot remove any node from it. But then we have the following result.

Proposition 17. Given a partial homomorphism of hypergraphs $r : \mathbf{L} \to \mathbf{R}$, let $\mathbf{R}_0$ be the least relative subalgebra of $\mathbf{R}$ containing $r(\mathbf{L})$ and such that $\mathbf{R}$ is its free completion, and let $r_0 : \mathbf{L} \to \mathbf{R}_0$ be r understood with target algebra $\mathbf{R}_0$.

Let now $m : \mathbf{L} \to \mathbf{G}$ be any homomorphism of hypergraphs, let $\mathbf{G}_0$ be the least relative subalgebra of $\mathbf{G}$ containing $m(\mathbf{L})$ and such that $\mathbf{G}$ is its free completion, and let $m_0 : \mathbf{L} \to \mathbf{G}_0$ be m understood with target algebra $\mathbf{G}_0$.

Then r_0 and m_0 have a pushout in CDQ-Alg$_{\Sigma_H}$, and its free completion yields the pushout of r and m in P-TAlg$_{\Sigma_H}$.

Proof. Notice that r_0 is a cdc-quomorphism and m_0 is a closed homomorphism: therefore, by Corollary 8 in § 2.5, they have always a pushout in CDQ-Alg$_{\Sigma_H}$, and it is also their pushout in CDCQ-Alg$_{\Sigma_H}$.

Notice also that $\mathbf{L}'(r, m) = \mathbf{L}'(r_0, m_0)$; let us call it simply $\mathbf{L}'$. To prove the second part of the statement, it is enough to prove that $\mathbf{R}' = \mathbf{R}'(r, m)$ is a free completion of $\mathbf{R}'_0 = \mathbf{R}'_0(r_0, m_0)$ and $\mathbf{G}' = \mathbf{G}'(r, m)$ is a free completion of $\mathbf{G}'_0 = \mathbf{G}'_0(r_0, m_0)$, and then apply Theorem 10.

Let us prove then that $\mathbf{R}'$ is a free completion of $\mathbf{R}'_0$. In our case, $\mathbf{R}'$ is the greatest closed subalgebra of $\mathbf{R}$ contained in $\mathbf{R} - (r(\mathbf{L}) - r(\mathbf{L}'))$ and $\mathbf{R}'_0$ is the greatest closed subalgebra of $\mathbf{R}_0$ contained in $\mathbf{R}_0 - (r(\mathbf{L}) - r(\mathbf{L}'))$; notice moreover that $R_{0E} = R_E$. Therefore, $R'_V = R_V - (r_V(L_V) - r(L'_V))$ and R'_E consists of those arcs in $R_E - (r_E(L_E) - r_E(L'_E))$ with all their incident nodes in R'_V, and a similar description (replacing R_V by R_{0V}) for $\mathbf{R}'_0$.

Since $R_{0V} \subseteq R_V$, we have that $\mathbf{R}'_0 \subseteq \mathbf{R}'$, and in particular $R'_{0E} \subseteq R'_E$. On the other hand, if $e \notin R'_{0E}$ then it is incident to some node $v \in r_V(L_V) - r_V(L'_V)$ and therefore $e \notin R'_E$ either. Therefore $R'_E = R'_{0E}$.

Let now $v \in R'_V$. If $v \notin R'_{0V} = (R_{0V} - r_V(L_V)) \cup r(L'_V)$ then v has degree 1, say $v = s_j^{\mathbf{R}}(e)$, and $v \notin r_V(L_V)$. Then $e \in R_E - r_E(L_E)$ (because otherwise $v \in r_V(L_V)$, since $r(V)$ is closed in $\mathbf{R}$) and therefore $e \in R'_E = R'_{0E}$. This shows that $\mathbf{R}'_0$ generates $\mathbf{R}'$, and therefore that $\mathbf{R}'$ is a free completion of $\mathbf{R}'_0$. ∎

There is still one case to consider: when $\mathrm{Dom}\, r_V = \emptyset$. In this case $\mathrm{Dom}\, r_E \subseteq L_{E_{0,0}}$ (because $\mathrm{Dom}\, r$ is a closed subset of $\mathbf{L}$) and then we can remove from $\mathbf{L}$ all nodes of degree 1 without any trouble.

Proposition 18. Let $r : \mathbf{L} \to \mathbf{R}$ be a partial homomorphism of hypergraphs with $\mathrm{Dom}\, r_V = \emptyset$, and therefore $\mathrm{Dom}\, r_E \subseteq L_{E_{0,0}}$. Let $\mathbf{L}_0$ be the least relative subalgebra of $\mathbf{L}$ such that $\mathbf{L}$ is its free completion, let $\mathbf{R}_0$ be the least relative subalgebra of $\mathbf{R}$ containing $r(\mathbf{L}_0) = r(\mathbf{L})$ and such that $\mathbf{R}$ is its free completion, and let $r_0 : \mathbf{L}_0 \to \mathbf{R}_0$ be r understood with source algebra $\mathbf{L}_0$ and target algebra $\mathbf{R}_0$.

Let now $m : \mathbf{L} \to \mathbf{G}$ be any homomorphism of hypergraphs, let $\mathbf{G}_0$ be the least relative subalgebra of $\mathbf{G}$ containing $m(\mathbf{L}_0)$ and such that $\mathbf{G}$ is its free completion, and let $m_0 : \mathbf{L}_0 \to \mathbf{G}_0$ be the restriction of m to $\mathbf{L}_0$ understood with target algebra $\mathbf{G}_0$.

Then r_0 and m_0 have a pushout in CDQ-Alg_{Σ_H}, and its free completion is the pushout of r and m in P-TAlg_{Σ_H}.

Proof. In this case, $\mathbf{L}'(r,m)_V = \mathbf{L}'(r_0,m_0) = \emptyset$ and $\mathbf{L}'(r,m)_E = \mathbf{L}'(r_0,m_0)_E \subseteq L_{E_{0,0}}$, and then it is clear that $\mathbf{R}'(r,m)$ is a free completion of $\mathbf{R}'_0(r_0,m_0)$ and $\mathbf{G}'(r,m)$ is a free completion of $\mathbf{G}'_0(r_0,m_0)$. Indeed, notice that $R'(r,m)_E = R'_0(r_0,m_0)_E$ and $R'(r,m)_V = R_V$ and $R_0(r_0,m_0)_V = R_{0V}$.

Then Theorem 10 applies. ∎

5.2. The case when r_V is total

Let $r : \mathbf{L} \to \mathbf{R}$ be in the sequel a fixed partial homomorphism of hypergraphs with r_V total; as before, $\mathbf{L}_0$ must be obtained from $\mathbf{L}$ by deleting some nodes of degree 1 and nothing else.

Let v_0 be a node of degree 1 and let $e_0 \in L_E$, say $e_0 \in L_{E_{p,q}}$, be the only arc to which it is incident; the case when $e_0 \notin \mathrm{Dom}\, r_E$ has been considered in Remark 14, and therefore we can assume that $e_0 \in \mathrm{Dom}\, r_E$.

Assume that there exists some $e' \in L_{E_{p,q}}$ not belonging to $\mathrm{Dom}\, r_E$. Then, let θ be the least congruence on $\mathbf{L}$ identifying e_0 and e', and let $m : \mathbf{L} \to \mathbf{L}/\theta$ be the correspoding quotient homomorphism. Then

$L'(r,m)_V = L_V$, but $e_0 \notin L'(r,m)_E$ and therefore v_0 is an isolated node in $\mathbf{L}'(r,m)$. Thus, in this case, v_0 must belong to $\mathbf{L}_0$.

But if r is totally defined on $L_{E_{p,q}}$ then both e_0 and v_0 will always belong to the gluing algebra $\mathbf{L}'(r,m)$, and therefore v_0 will never be an isolated node therein.

So, if r_V is total then we can only remove from $\mathbf{L}$ those nodes of degree 1 incident to arcs belonging to some layer $L_{E_{p,q}}$ where r_E is total: i.e., with the notations introduced at the end of Sect. 2.5, to arcs $e \in L_{E_{p,q}}$ with $E_{p,q} \in S(r)$. And in this case we have the following result.

Proposition 19. Let $r : \mathbf{L} \to \mathbf{R}$ be a partial homomorphism of hypergraphs with $\mathrm{Dom}\, r_V = L_V$. Let $\mathbf{L}_0$ be the relative subalgebra of $\mathbf{L}$ obtained by removing from $\mathbf{L}$ those nodes of degree 1 incident to arcs belonging to layers $L_{E_{p,q}}$ of L_E where r_E is total. Let $\mathbf{R}_0$ be the least relative subalgebra of $\mathbf{R}$ containing $r(\mathbf{L}_0)$ and such that $\mathbf{R}$ is its free completion, and let $r_0 : \mathbf{L}_0 \to \mathbf{R}_0$ be the corresponding restriction of r.

Let now $m : \mathbf{L} \to \mathbf{G}$ be any homomorphism of hypergraphs, let $\mathbf{G}_0$ be the least relative subalgebra of $\mathbf{G}$ containing $m(\mathbf{L}_0)$ and such that $\mathbf{G}$ is its free completion, and let $m_0 : \mathbf{L}_0 \to \mathbf{G}_0$ be the corresponding restriction of m.

Then r_0 and m_0 have a pushout in CDQ-$\mathrm{Alg}_{\Sigma_H}(r)$, and its free completion is the pushout of r and m in P-TAlg_{Σ_H}.

Proof. We must prove first that r_0 and m_0 have a pushout in CDQ-$\mathrm{Alg}_{\Sigma_H}(r)$.

Let L'_0 be the greatest subset of $\mathbf{L}_0$ such that $r_0^{-1}(r_0(L'_0)) = L'_0$ and $m_0^{-1}(m_0(L'_0)) = L'_0$; since $\mathrm{Dom}\, r_{0V} = L_{0V}$, we have that $L'_{0V} = L_{0V}$ and therefore L'_0 is a closed subset of $\mathbf{L}_0$. Moreover, if $E_{p,q} \in S(r)$ (i.e., if $r_{E_{p,q}}$ is total) then $L'_{0E_{p,q}} = L_{0E_{p,q}} = L_{E_{p,q}}$.

Let now $R'_0 = (R_0 - r_0(L_0)) \cup r_0(L'_0)$ and $G'_0 = (G_0 - m_0(L_0)) \cup m_0(L'_0)$; as before, these are closed subsets of $\mathbf{R}_0$ and $\mathbf{G}_0$, respectively, because $R'_{0V} = R_{0V}$ and $G'_{0V} = G_{0V}$ (since $L'_{0V} = L_{0V}$). And it also happens that if $E_{p,q} \in S(r)$ then $R'_{0E_{p,q}} = R_{0E_{p,q}}$ and $G'_{0E_{p,q}} = G_{0E_{p,q}}$.

This shows that the first condition in Proposition 9 is satisfied by r_0 and m_0. Assume now that $(e_1, e_2) \in \theta(L'_0)_{E_{p,q}}$ with $E_{p,q} \notin S(r)$ and let $v_1 = \varphi^{\mathbf{R}'_0 + \mathbf{G}'_0}(e_1)$ $v_2 = \varphi^{\mathbf{R}'_0 + \mathbf{G}'_0}(e_2)$ for some source or target operation φ with source sort $E_{p,q}$. If $e_1 = e_2$ then $v_1 = v_2 \in \theta(L'_0)_V$. And if $e_1 \neq e_2$ then $e_1, e_2 \in (r_0 + m_0)_{E_{p,q}}(L'_{0E_{p,q}})$; to fix ideas, say $e_1 = r_{0E_{p,q}}(e'_1)$ and $e_2 = m_{0E_{p,q}}(e'_2)$ (the other cases are solved in the same way). But then, since $E_{p,q} \notin S(r)$, we have that $\varphi^{\mathbf{L}'_0}(e'_1)$ and $\varphi^{\mathbf{L}'_0}(e'_2)$ are defined. This implies that if $e'_1 = e'_2$ then $(v_1, v_2) = (r_{0V}(\varphi^{\mathbf{L}'_0}(e'_1)), m_{0V}(\varphi^{\mathbf{L}'_0}(e'_1))) \in \theta(L'_0)_V$. And then, a simple and typical induction on the length of the chain that, by transitivity, makes (e_1, e_2) to belong to $\theta(L'_0)_{E_{p,q}}$, entails that $(v_1, v_2) \in \theta(L'_0)_V$.

Therefore, by Proposition 9, r_0 and m_0 have a pushout in CDQ-$\mathrm{Alg}_{\Sigma_H}(r)$, and it is obtained as the quotient of $\mathbf{R}'_0 + \mathbf{G}'_0$ by the least congruence that identifies the images by r_0 and m_0 of the same elements of $\mathbf{L}'_0$: i.e., as the pushout in Alg_{Σ_H} of $r_0 : \mathbf{L}'_0 \to \mathbf{R}'_0$ and $m_0 : \mathbf{L}'_0 \to \mathbf{G}'_0$.

Let now $\mathbf{L}' = \mathbf{L}'(r, m)$, $\mathbf{R}' = \mathbf{R}'(r, m)$, $\mathbf{G}' = \mathbf{G}'(r, m)$, $\mathbf{L}'_0 = \mathbf{L}'_0(r_0, m_0)$, $\mathbf{R}'_0 = \mathbf{R}'_0(r_0, m_0)$ and $\mathbf{G}'_0 = \mathbf{G}'_0(r_0, m_0)$.

Since $L'_{0E_{p,q}} = L_{E_{p,q}} = L'_{E_{p,q}}$ for every $E_{p,q} \in S(r)$, and the only nodes in $L'_V = L_V$ missing in L_{0V} are incident to arcs in $L_{E_{p,q}}$ for some $E_{p,q} \in S(r)$, we conclude that $\mathbf{L}'_0$ generates $\mathbf{L}'$.

And on the other hand, we have that $R'_V = R_V$, $R'_{0V} = R_{0V}$, and $R'_{0E} = R'_E$. Let $v \in R'_V - R'_{0V} = R_V - R_{0V}$: v has degree 1, say $v = s_j^{\mathbf{R}}(e)$ with $e \in R_{E_{p,q}}$, and does not belong to $m_V(L_{0V})$. If $e \in R_0{}'_{E_{p,q}}$ then we are done. If $e \notin R_0{}'_{E_{p,q}} = R'_{E_{p,q}}$ then $e \in r_E(L_{E_{p,q}}) - r_E(L'_{E_{p,q}})$, say $e = r_E(e_0)$. But then $E_{p,q} \notin S(r)$ (because otherwise $L_{E_{p,q}} = L'_{E_{p,q}}$) and therefore $s_j^{\mathbf{L}}(e_0) \in L_{0V}$ and therefore $v \in m_V(L_{0V})$, contradicting the assumption on v.

Therefore $\mathbf{R}'$ is generated by, and in particular is a free completion of, $\mathbf{R}'_0$. By symmetry, $\mathbf{G}'$ is a free completion of $\mathbf{G}'_0$. Then, Theorem 10 applies. ∎

6. Conclusion

In this paper different relationships between single-pushout rewriting of hypergraphs and partial unary algebras, aimed at improving efficiency of single-pushout transformations by using "smaller" rules applied to "smaller" objects, have been studied in detail.

The main result in this paper establishes that (a) the result of the application of a single-pushout transformation rule of total algebras to a total algebra can be obtained as the epireflection of the result of the application of a corresponding single-pushout transformation rule of partial algebras, although unlike the case of double-pushout derivations, in this case single-pushout transformation rules of partial algebras become "larger" than corresponding rules of total algebras, leading then to no improvement in efficiency at all; (b) given a single-pushout transformation rule of hypergraphs, a corresponding single-pushout transformation rule of partial algebras can be extracted *a priori* such that, whenever the rule of hypergraphs can be applied to a hypergraph through an occurrence, the rule of partial algebras can be applied to the same hypergraph through a restriction of the occurrence, and vice versa, where the derived hypergraph is the free completion of the derived partial algebra, although the extracted rule does not become significantly smaller than the original rule, leading then to no substantial improvement in efficiency of pattern-matching; and (c) given a single-pushout transformation rule of hypergraphs and an occurrence morphism of hypergraphs, a "smaller" single-pushout transformation rule of partial algebras can be extracted *a posteriori* such that the pushout of the morphisms of hypergraphs is the free completion of the pushout of the morphisms of partial algebras, although the extracted rule does not become significantly smaller than the original rule either, leading then to no substantial improvement in efficiency of single-pushout transformation after pattern-matching.

The lack of strong results contrasts with the case of double-pushout derivations through free completions [7], and it lies in single-pushout algebraic transformation being a generalization of double-pushout algebraic transformation. In particular, the *gluing condition* that characterizes double-pushout derivations allows a certain *a priori* simplification of transformation rules. Since in single-pushout derivations *clean occurrences* play the role of gluing condition, we expect to report on some improvement in efficiency of single-pushout derivations through free completions in the interesting case of clean occurrences elsewhere.

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