SOME NUMERICAL INVARIANTS OF LOCAL RINGS

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Abstract. Let $R$ be a formal power series ring over a field of characteristic zero and $I \subseteq R$ be any ideal. The aim of this work is to introduce some numerical invariants of the local rings $R/I$ by using the theory of algebraic $\mathcal{D}$-modules. More precisely, we will prove that the multiplicities of the characteristic cycle of the local cohomology modules $H^n_I(R)$ and $H^n_{R_p}(R_p)$, where $p \subseteq R$ is any prime ideal that contains $I$, are invariants of $R/I$.

1. Introduction

Let $(R, m, k)$ be a regular local ring of dimension $n$ containing the field $k$, and $A$ a local ring which admits a surjective ring homomorphism $\pi : R \longrightarrow A$. Set $I = \text{Ker} \pi$. G. Lyubeznik [10] defines a new set of numerical invariants of $A$ by means of the Bass numbers $\lambda_{p,i}(A) := \mu_p(m, H^n_{\pi^{-1}}(R)) := \dim_k \text{Ext}^p_R(k, H^n_{\pi^{-1}}(R))$. This invariant depends only on $A$, $i$ and $p$, but neither on $R$ nor on $\pi$. Completion does not change $\lambda_{p,i}(A)$ so one can assume $R = k[[x_1, \ldots, x_n]]$, with $x_1, \ldots, x_n$ independent variables.

Lyubeznik numbers can be described as the multiplicities of the characteristic cycle of the local cohomology modules $H^0_R(H^n_{\pi^{-1}}(R))$. The aim of this work is to prove that the multiplicities of the characteristic cycle of the local cohomology modules $H^0_I(R)$ and $H^0_{R_p}(R_p)$, where $p \subseteq R$ is any prime ideal that contains $I$, are also invariants of $R/I$. Among these invariants we may find the Bass numbers $\mu_p(p, H^n_I(R)) := \dim_k(k(p), H^n_{R_p}(R_p))$.

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2. The characteristic cycle

In the sequel, $\mathcal{D}$ will denote the ring of differential operators corresponding to the formal power series ring $R = k[[x_1, \ldots, x_n]]$, where $k$ is a field of characteristic zero and $x_1, \ldots, x_n$ are independent variables. For details we refer to [5], [6]. The ring $\mathcal{D}$ has a natural increasing filtration given by the order such that the corresponding associated graded ring $gr(\mathcal{D})$ is isomorphic to the polynomial ring $R[\xi_1, \ldots, \xi_n]$.

Let $M$ be a finitely generated $\mathcal{D}$-module equipped with a good filtration, i.e. an increasing sequence of finitely generated $R$-submodules such that the associated graded module $gr(M)$ is a finitely generated $gr(\mathcal{D})$-module. The characteristic ideal of $M$ is the ideal in $gr(\mathcal{D}) = R[\xi_1, \ldots, \xi_n]$ given by $J(M) := \text{rad}(\text{Ann}_{gr(\mathcal{D})}(gr(M)))$. One may prove that $J(M)$ is independent of the good
filtration on $M$. The characteristic variety of $M$ is the closed algebraic set given by:

$$C(M) := V(J(M)) \subseteq \text{Spec} \left( gr(D) \right) = \text{Spec} \left( R[\xi_1, \ldots, \xi_n] \right).$$

The characteristic variety allows us to describe the support of a finitely generated $\mathcal{D}$-module as $R$-module. Let $\pi : \text{Spec} \left( R[\xi_1, \ldots, \xi_n] \right) \rightarrow \text{Spec} \left( R \right)$ be the map defined by $\pi(x, \xi) = x$. Then $\text{Supp}_R(M) = \pi(C(M))$.

The characteristic cycle of $M$ is defined as:

$$CC(M) = \sum m_i V_i$$

where the sum is taken over all the irreducible components $V_i = V(q_i)$ of the characteristic variety $C(M)$, where $q_i \in \text{Spec} \left( gr(D) \right)$ and $m_i$ is the multiplicity of the module $gr(M)_{q_i}$. Notice that the contraction of $q_i$ to $R$ is a prime ideal so the variety $\pi(V_i)$ is irreducible.

2.1. Bass numbers and characteristic cycle. Let $p \in \text{Spec} \left( R \right)$ be a prime ideal. The Bass numbers $\mu_p(H^p_I(R^n-i)(R))$ of the local cohomology modules $H^p_I(R^n-i)(R)$, where $I \subseteq R$ is any ideal, can be described as the multiplicities of the characteristic cycle of $H^p_I(R^n-i)(R)$. Namely we have:

**Proposition 2.1.** Let $I \subseteq R$ be an ideal, $p \subseteq R$ a prime ideal and

$$CC(H^p_I(R^n-i)(R))) = \sum \lambda_{p, i, \alpha} V_{\alpha}$$

be the characteristic cycle of the local cohomology module $H^p_I(R^n-i)(R))$. Then, the Bass numbers with respect to $p$ of $H^p_I(R^n-i)(R)$ are

$$\mu_p(p, H^p_I(R^n-i)(R))) = \lambda_{p, i, \alpha},$$

where $\pi(V_{\alpha})$ is the subvariety of $X = \text{Spec} \left( R \right)$ defined by $p$.

**Proof.** Let $\widehat{R}_p$ be the completion with respect to the maximal ideal $p\widehat{R}_p$ of the localization $R_p$. Notice that $\widehat{R}_p$ is a formal power series ring of dimension $ht p$. Since Bass numbers are invariant by completion we have:

$$\mu_p(p, H^p_I(R^n-i)(R))) = \mu_p(p\widehat{R}_p, H^p_{\widehat{R}_p}(\widehat{R}_p)) = \mu_0(p\widehat{R}_p, H^p_{p\widehat{R}_p}(H^n_{\widehat{R}_p}(\widehat{R}_p))),$$

where the last assertion follows from [10, Lemma 1.4]. By using [10, Theorem 3.4] we have:

$$H^p_{p\widehat{R}_p}(H^n_{\widehat{R}_p}(\widehat{R}_p)) = E_{\widehat{R}_p}(\widehat{R}_p/p\widehat{R}_p)^{\mu_0(p\widehat{R}_p, H^p_{p\widehat{R}_p}(H^n_{\widehat{R}_p}(\widehat{R}_p)))}.$$ 

So, its characteristic cycle is:

$$CC(H^p_{p\widehat{R}_p}(H^n_{\widehat{R}_p}(\widehat{R}_p))) = \mu_p(p, H^p_I(R^n-i)(R))) \cdot V'_{\alpha},$$

where $\pi(V'_{\alpha})$ is the subvariety of $X' = \text{Spec} \widehat{R}_p$ defined by the ideal $p\widehat{R}_p$. Notice that we have used the following fact (see [10] and [1] for details):

$$CC(H^p_{p\widehat{R}_p}(\widehat{R}_p)) = CC(E_{\widehat{R}_p}(\widehat{R}_p/p\widehat{R}_p)) = V'_{\alpha}.$$ 

Finally, by using the flatness of the morphism $R \rightarrow \widehat{R}_p$, this characteristic cycle can be obtained from the characteristic cycle of $H^p_I(R^n-i)(R))$. Namely, if

$$CC(H^p_I(R^n-i)(R))) = \sum \lambda_{p, i, \alpha} V_{\alpha}.$$
is the characteristic cycle of the module $H^p_{\mathcal{R}}(H^{n-i}_R(R))$, then we have

$$CC(H^p_{\mathcal{R}}(H^{n-i}_R(R))) = \lambda_{p,i,o} V_{\alpha}.$$ 

\[\square\]

2.2. **Inverse and direct image.** Some geometrical operations as the direct image have a key role in the theory of $\mathcal{D}$-modules. Our aim in this section is to give a brief survey of these operations in the particular case of the injection of $\mathbb{A}_k^n$ in $\mathbb{A}_k^{n+1}$. The main references we will use in this section are [6] and [11].

Let $\mathcal{D}_{n+1}$ and $\mathcal{D}_n$ be the rings of differential operators corresponding to $R = k[[x_1, \ldots, x_n, t]]$ and $R = k[[x_1, \ldots, x_n]]$ respectively. Let $M$ be a $\mathcal{D}_n$-module. The direct image corresponding to the injection is the $\mathcal{D}_{n+1}$-module $i_+(M)$ defined as

$$i_+(M) = k[\partial_t] \widehat{\otimes}_k M = M[\partial_t].$$

The characteristic variety of $i_+(M)$ can be computed from the characteristic variety of $M$. Namely, we have:

$$C(i_+(M)) = \{ (x, 0, \xi, \tau) \mid (x, \xi) \in C(M) \} \subseteq \text{Spec} (R[t][\xi_1, \ldots, \xi_n, \tau]),$$

where we have considered $C(M) \subseteq \text{Spec} (R[\xi_1, \ldots, \xi_n])$.

The direct image of local cohomology modules can be easily described. The following result is stated in the way we will use in our work.

**Lemma 2.2.** Let $p \subseteq R$ be a prime ideal that contains an ideal $I \subseteq R$. The direct image of the local cohomology module $H^p_\mathcal{D}(H^{n-i}_R(R))$ is:

$$i_+(H^p_\mathcal{D}(H^{n-i}_R(R))) = H^1(I)(H^p_{\mathcal{D}R}(H^{n-i}_{\mathcal{R}}(R))).$$

**Proof.** Let $\mathcal{D}_I$ be the ring of differential operators corresponding to the formal power series ring $k[[t]]$. For simplicity we will denote the local cohomology modules $H^p_\mathcal{D}(H^{n-i}_R(R))$ and $H^p_{\mathcal{D}R}(H^{n-i}_{\mathcal{R}}(R))$ by $N$ and $N'$ respectively. Then we have:

$$H^1(I)(N') = H^1(I)(N \otimes_k k[[t]]) = H^1(I)(k[[t]]) \otimes_k N = (\mathcal{D}_I / \mathcal{D}_I \cdot (t)) \otimes_k N = i_+(N).$$

\[\square\]

**Remark 2.3.** In general, let $I_1, \ldots, I_s$ be a set of ideals of $R$. Then, the direct image of the local cohomology module $H^{i_1}_{I_1}(\cdots (H^{i_s}_{I_s}(R))\cdots)$ is:

$$i_+(H^{i_1}_{I_1}(\cdots (H^{i_s}_{I_s}(R))\cdots)) = H^1(I)(H^{i_1}_{I_1,R}(\cdots (H^{i_s}_{I_s,R}(R))\cdots)).$$

### 3. Multiplicities of the characteristic cycle.

Let $A$ be a ring that admits a presentation $A \cong R/I$ for a given ideal $I \subseteq R = k[[x_1, \ldots, x_n]]$. Recall that we have $\text{Spec}(A) = \{ p \in \text{Spec}(R) \mid I \subseteq p \}$. Throughout this section, a prime ideal of $A$ will also mean the corresponding prime ideal of $R$ that contains $I$.

Let $R/I$ and $R'/I'$ be two different presentations of the local ring $A$. Then, for any prime ideal of $A$, we will denote $p \in \text{Spec}(R')$ the prime ideal that corresponds to $p \in \text{Spec}(R)$ by the isomorphism $\text{Spec}(A) \cong \text{Spec}(R'/I')$. 


Theorem 3.1. Let $A$ be a local ring which admits a surjective ring homomorphism $\pi : R \to A$, where $R = k[[x_1, \ldots, x_n]]$ is the formal power series ring. Set $I = \ker \pi$, let $p \subseteq A$ be a prime ideal and let
\[
CC(H_p^i(H_I^{n-i}(R))) = \sum \lambda_{p, i, \alpha} V_{\alpha},
\]
be the characteristic cycle of the local cohomology modules $H_p^i(H_I^{n-i}(R))$. Then the multiplicities $\lambda_{p, i, \alpha}$ depend only on $A$, $p$, $i$ and $\alpha$ but neither on $R$ nor on $\pi$.

The proof of the theorem is inspired in the proof of [10, Theorem 4.1], but here we must be careful with the behavior of the characteristic cycle so instead of [10, Lemma 4.3] we will use the following:

Lemma 3.2. Let $g : R' \to R$ be a surjective ring homomorphism, where $R'$ is a formal power series ring of dimension $n'$. Set $I' = \ker g \pi$ and let
\[
CC(H_p^i(H_I^{n-i}(R'))) = \sum \lambda_{p, i, \alpha} V_{\alpha},
\]
be the characteristic cycle of the local cohomology modules $H_p^i(H_I^{n-i}(R'))$. Then, the characteristic cycle of $H_p^i(H_{I'}^{n-i}(R'))$ is
\[
CC(H_p^i(H_{I'}^{n-i}(R'))) = \sum \lambda_{p, i, \alpha} V_{\alpha},
\]
where $\pi(V_{\alpha})$ is the subvariety of $X' = \text{Spec } R'$ defined by the defining ideal of $\pi(V_{\alpha})$ contracted to $R'$.

Proof. $R$ is regular so $\text{Ker } g$ is generated by $n' - n$ elements that form part of a minimal system of generators of the maximal ideal $m' \subseteq R'$. By induction on $n' - n$ we are reduced to the case $n' - n = 1$, so $\text{Ker } g$ is generated by one element $f \in m' \setminus m'^2$. By Cohen’s structure theorem $R' = k[[x_1, \ldots, x_n, t]]$ where we assume $f = t$ by a change of variables. We identify $R$ with the subring $k[[x_1, \ldots, x_n]]$ of $R'$. In particular we have to consider $I' = IR' + (t)$ and $p' = pR' + (t)$.

By using Lemma 2.2 and the degeneration of the Grothendieck’s spectral sequence $E_2^{p, q} = H^p_{t}(H^q_{I'}(M)) \Rightarrow H^{p+q}_{t + (I')}(M)$ we have:

\[
i_+(H_p^i(H_I^{n-i}(R))) = H^1_{t}(H_p^0(H_I^{n-i}(R'))) = H^0_{pR'}(H^1_{t}(H_I^{n-i}(R'))) = \]
\[
= H^0_{pR'}(H_I^{n+1-i}(R')) = H^0_{pR'}(H_I^{n-i}(R')) = \]
\[
= H^0_{pR' + (t)}(H_{I'}^{n-i}(R')) = H^0_{p'}(H_{I'}^{n-i}(R')), \]

where the second last assertion comes from the fact that $H_{I'}^{n-i}(R')$ is a $(t)$-torsion module. Then we are done by the results in Section 2.2.

Now we continue the proof of Theorem 3.1.

Proof. Let $\pi' : R' \to A$ and $\pi'' : R'' \to A$ be surjections with $R' = k[[y_1, \ldots, y_{n'}]]$ and $R'' = k[[z_1, \ldots, z_{n''}]]$. Let $I' = \ker \pi'$ and let $I'' = \ker \pi''$. Let $R'' = R' \otimes_k R''$ be the external tensor product, $\pi''' = \pi' \otimes_k \pi'' : R' \otimes_k R'' \to A$ and $I''' = \ker \pi'''$.

By Lemma 3.2, if the characteristic cycle of $H_p^i(H_{I'}^{n-i}(R'))$ is
\[
CC(H_p^i(H_{I'}^{n-i}(R'))) = \sum \lambda_{p, i, \alpha} V_{\alpha},
\]

then the characteristic cycle of $H_{\nu}^{n}(H_{\nu}^{n} \cap R^{m})$ is

$$CC(H_{\nu}^{n}(H_{\nu}^{n} \cap R^{m})) = \sum \lambda_{p,i,\alpha}^{n} V_{\alpha},$$

where $\pi(V_{\alpha})$ is the subvariety of $X^{m} = \text{Spec } R^{m}$ defined by the defining ideal of $\pi(V_{\alpha})$ contracted to $R^{m}$.

By Lemma 3.2, if the characteristic cycle of $H_{\nu}^{n}(H_{\nu}^{n} \cap R^{m})$ is

$$CC(H_{\nu}^{n}(H_{\nu}^{n} \cap R^{m})) = \sum \lambda_{p,i,\alpha}^{n} V_{\alpha},$$

then the characteristic cycle of $H_{\nu}^{n}(H_{\nu}^{n} \cap R^{m})$ is

$$CC(H_{\nu}^{n}(H_{\nu}^{n} \cap R^{m})) = \sum \lambda_{p,i,\alpha}^{n} V_{\alpha},$$

where $\pi(V_{\alpha})$ is the subvariety of $X^{m} = \text{Spec } R^{m}$ defined by the defining ideal of $\pi(V_{\alpha})$ contracted to $R^{m}$.

In particular we have $\lambda_{p,i,\alpha}^{n} = \lambda_{p,i,\alpha}^{n}$ for all $p$, $i$ and $\alpha$.

\[\square\]

Remark 3.3. With the same arguments one may prove that the multiplicities of the characteristic cycle of the local cohomology modules $H_{I}^{n}(H_{I}^{n} \cap R)$, where $I_{1}, \ldots, I_{s}$ is a set of ideals of $R$ containing the ideal $I = I_{s}$, are also invariants of $R/I$.

Since Bass numbers $\mu_{p}(p, H_{I}^{n-i}(R))$ are multiplicities of the characteristic cycle of $H_{p}^{n}(H_{I}^{n-i}(R))$, we recover Lyubeznik’s result:

Corollary 3.4. Let $A$ be a ring which admits a surjective ring homomorphism $\pi : R \to A$, where $R = k[[x_{1}, \ldots, x_{n}]]$ is the formal power series ring. Set $I = \ker \pi$ and let $p \subseteq A$ be a prime ideal. The Bass numbers $\mu_{p}(p, H_{I}^{n-i}(R))$ depend only on $A$, $p$, $i$ and $\pi$ but neither on $R$ nor on $\pi$.

When $p$ is the zero ideal, we obtain the invariance with respect to $R/I$ of the multiplicities of the characteristic cycle of $H_{I}^{n-i}(R)$.

Corollary 3.5. Let $A$ be a local ring which admits a surjective ring homomorphism $\pi : R \to A$, where $R = k[[x_{1}, \ldots, x_{n}]]$ is the formal power series ring. Set $I = \ker \pi$ and let

$$CC(H_{I}^{n-i}(R)) = \sum m_{i,\alpha} V_{\alpha},$$

be the characteristic cycle of the local cohomology modules $H_{I}^{n-i}(R)$. Then the multiplicities $m_{i,\alpha}$ depend only on $A$, $i$ and $\alpha$ but neither on $R$ nor on $\pi$.

Collecting these multiplicities by the dimension of the corresponding irreducible varieties we define the following invariants:

Definition 3.6. Let $I \subseteq R$ be an ideal. If $CC(H_{I}^{n-i}(R)) = \sum m_{i,\alpha} V_{\alpha}$ is the characteristic cycle of the local cohomology modules $H_{I}^{n-i}(R)$ then we define:

$$\gamma_{p,i}(R/I) := \left\{ \sum m_{i,\alpha} \mid \dim(\pi(V_{\alpha})) = p \right\}.$$
One may prove that these invariants have the same properties as Lyubeznik numbers (see [10, Section 4]). Namely, let $d = \dim(R/I)$ then $\gamma_{p,i}(R/I) = 0$ if $i > d$, $\gamma_{p,i}(R/I) = 0$ if $p > i$ and $\gamma_{d,i}(R/I) \neq 0$. In particular we can collect them in a triangular matrix that we will denote by $\Gamma(R/I)$. We point out that these invariants are finer than the Lyubeznik numbers.

**Example 3.7.** Let $R = k[[x_1, x_2, x_3, x_4, x_5]]$. Consider the ideals:

- $I_1 = (x_1, x_2, x_3) \cap (x_3, x_4, x_5)$.
- $I_2 = (x_1, x_2, x_3) \cap (x_3, x_4, x_5) \cap (x_1, x_2, x_3, x_4)$.

The characteristic cycle of the corresponding local cohomology modules can be computed by means of [1, Theorem 3.8]. Collecting the multiplicities we obtain the triangular matrices:

$$
\Gamma(R/I_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ \end{pmatrix} \quad \Gamma(R/I_2) = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ \end{pmatrix}
$$

Computing the Lyubeznik numbers (see [1, Theorem 4.4]), we obtain the triangular matrix:

$$
\Lambda(R/I_1) = \Lambda(R/I_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ \end{pmatrix}
$$

We have to point out that the quotient ring $R/I_1$ is Buchsbaum but $R/I_2$ is not.

**Remark 3.8.** In order to compute the Lyubeznik numbers $\lambda_{p,i}(R/I)$ for a given ideal $I \subseteq R$ and arbitrary $i, p$ we have to refer to U. Walther’s algorithm [12]. When $I$ is a squarefree monomial ideal, a description of these invariants is given in [1] and [14]. Some other particular computations may also be found in [7], [8],[9] and [13]. The multiplicities of the characteristic cycle of $H^i_p(H^{\nu-1}_I(R))$, where $I$ is a squarefree monomial ideal and $p$ is any homogeneous prime ideal, have been computed in [2].

When $I$ is a squarefree monomial ideal (resp. the defining ideal of an arrangement of linear varieties), the multiplicities of the characteristic cycle of $H^{\nu-1}_I(R)$ have been computed in [1] (resp. [3]).

**References**


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