On the perturbation of bimodal systems

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Abstract

Given a bimodal system defined by the equations

\[
\begin{align*}
\dot{x}(t) &= A_1x(t) + Bu(t) \quad \text{if } c^tx(t) \leq 0 \\
\dot{x}(t) &= A_2x(t) + Bu(t) \quad \text{if } c^tx(t) \geq 0
\end{align*}
\]

where \(B \in \mathcal{M}_{n,m}\) and \(A_i \in \mathcal{M}_n, i = 1, 2\), are such that \(A_1, A_2\) coincide on the hyperplane \(V = \ker c^t\). We consider in the set of matrices defining the above systems the simultaneous feedback equivalence defined by \(([A_1, B], [A_2, B]) \sim ([A'_1, B'], [A'_2, B'])\) if

\[
[A'_iB'] = S^{-1}[A_iB][\begin{bmatrix} S & 0 \\ R & T \end{bmatrix}] \quad i = 1, 2 \text{ with } S(V) = V
\]

This equivalent relation corresponds to the action of a Lie group. Under this action we obtain, in the case \(m \leq 1\), the semiuniversal deformation, following Arnold’s technique. Then the problem of structural stability is studied.

1 Preliminaries

(1.1) From now on, \(\mathcal{M}_{n,m}\) denotes the set of \(n \times m\) complex matrices. We write \(\mathcal{M}_{n,n} = \mathcal{M}_n\). If \(A \in \mathcal{M}_{n,m}\), \(A^*\) (resp. \(A^t\)) denotes conjugate transpose of \(A\), (resp. transpose of \(A\)) and \(\text{tr } A\) the trace of \(A\).

(1.2) In [3] the following reduced form is obtained under the above equivalent relation: Let \(J\) be a \(h \times h\) complex Jordan matrix and \(N\) be the \(l \times l\) standard nilpotent matrix. Then any pair \(((A_1, b), (A_2, b))\) with \(A_i \in \mathcal{M}_n, b \in \mathcal{M}_{n,1}\) and \(A_1|V = A_2|V, V = \ker (0, ..., 0, 1)^t\), is equivalent to a pair \(((A_{10}, b_0), (A_{20}, b_0))\) where

\[
A_{10} = \begin{pmatrix}
J & 0 & \alpha^1 \\
0 & N & 0 \\
0 & \alpha_1 & 0
\end{pmatrix}, \text{ with } \alpha_1 = (0, ..., 0, 1), \alpha^1 = (\alpha_1^1, ..., \alpha_h^1)^t
\]

\[
A_{20} = \begin{pmatrix}
J & 0 & \beta_1^1 \\
0 & N & \beta_2^1 \\
0 & \alpha_1 & \beta
\end{pmatrix}, b_0 = \begin{pmatrix} 0 \\ p \\ \epsilon_0 \end{pmatrix}
\]

with

\[
\beta_1^1 = (\beta_{11}^1, ..., \beta_{1h}^1)^t, \beta_2^1 = (\beta_{21}^1, ..., \beta_{2l}^1)^t, p = (0, ..., 0, 1)^t.
\]
We shall say that this pair is in Kronecker reduced form. (1.3) Let \( M = \{((A_1 b), (A_2 b)); A_1|\mathcal{V} = A_2|\mathcal{V}\} \) and

\[
\mathcal{G} = \begin{pmatrix}
    S & O \\
    f & t
\end{pmatrix}; S \in \text{Gl}(n), S(\mathcal{V}) = \mathcal{V}, t \neq 0.
\]

Notice that \( S \) has the form \( S = \begin{pmatrix} S_{11} & s^1 \\ 0 & s \end{pmatrix} \), so that \( \mathcal{G} \) can be identified with an open set of \( \mathbb{C}^{n^2+2} \).

We consider in \( M \) the hermitian product defined by

\[
<(\begin{pmatrix} A_1 b \\ \mathcal{V} \end{pmatrix}, (\begin{pmatrix} A_2 b \\ \mathcal{V} \end{pmatrix})) = \text{tr}\left(\begin{pmatrix} A_1^* \\ b^* \\
    A_2^* \\ b'^* 
\end{pmatrix}\right)
\]

and the action of \( \mathcal{G} \) on \( M \) defined by

\[
(\mathcal{S} \mathcal{O} t) * (\begin{pmatrix} A_1 b \\ \mathcal{V} \end{pmatrix}, (\begin{pmatrix} A_2 b \\ \mathcal{V} \end{pmatrix})) = (S(A_1, b), (S(A_2, b)) \begin{pmatrix} A_1^* \\ b^* \\
    A_2^* \\ b'^* 
\end{pmatrix}
\]

We fix a pair \(((A_{10}, b_0), (A_{20}, b_0)) \in M \) and let \( \phi : \mathcal{G} \rightarrow M \) be the map defined by

\[
\phi(S) = S * ((A_{10}, b_0), (A_{20}, b_0))
\]

with \( S = \begin{pmatrix} S & O \\
    f & t
\end{pmatrix} \).

Let \( \mathcal{A}_0 = ((A_{10}, b_0), (A_{20}, b_0)) \) and denote \( \mathcal{O}_0 = \{S \ast \mathcal{A}_0; S \in \mathcal{G}\} \). We know that the orbit \( \mathcal{O}_0 \) is a locally closed submanifold of \( M \) (see for example [2]). Then if we denote \( \mathcal{B} = (T_{\mathcal{A}_0}\mathcal{O}_0)^\perp \) and \( \mathcal{I} \) the unit element in \( \mathcal{G} \), we have the following theorem due to Arnold ([1]; see also [4]).

**Theorem 1** The linear variety \( \mathcal{A}_0 + \mathcal{B} \) has the following universal property. Let \( \psi : \mathcal{B} \rightarrow M \) defined by \( \psi(\chi) = \mathcal{A}_0 + \chi \). Then for any differentiable map \( \varphi : \mathbb{C}^N \rightarrow M \) such that \( \varphi(0) = \mathcal{A}_0 \), there exist a neighborhood \( U \) of 0 in \( \mathbb{C}^N \) a differentiable map \( \eta : U \rightarrow \mathcal{B} \) such that \( \eta(0) = 0 \) and a differentiable map \( \xi : U \rightarrow \mathcal{G} \) with \( \chi(0) = \mathcal{I} \) such that \( \varphi(\mu) = \xi(\mu) \ast \psi(\eta(\mu)) \).

The linear variety \( \mathcal{A}_0 + \mathcal{B} \) has the minimum dimension having this universal property. It is called a miniversal deformation of \( \mathcal{A}_0 \).

Finally we recall that \( \mathcal{A}_0 \) is said to be structural stable if it is an interior point of its orbit. Equivalently, if \( \mathcal{B} = 0 \).

## 2 Construction of a miniversal deformation

As we have said, in order to obtain a miniversal deformation of \( \mathcal{A}_0 \) we have to compute \((T_{\mathcal{A}_0}\mathcal{O}_0)^\perp\).

Let \( \mathcal{I} \) be the unit element in \( \mathcal{G} \) and \( \mathcal{P} = \begin{pmatrix} P & O \\
    p_1 & q
\end{pmatrix} \in T_{\mathcal{G}} \) with \( P = \begin{pmatrix} p_{11} & p^1 \\
    0 & p
\end{pmatrix} \).

Then we have the following lemma.
Lemma 1

\[
d\phi_\mathcal{I}(P) = \langle[P, A_{10}] + b_0p_1, b_0q + Pb_0, ([P, A_{20}] + b_0p_1, b_0q + Pb_0)\rangle.
\]

Since \(T_{\mathcal{A}_0}\mathcal{O}_0 = Imd\phi_\mathcal{I}\) one has that \((A_1, b), (A_2, b)\) \(\in (T_{\mathcal{A}_0}\mathcal{O}_0)^\perp\) if and only if

\[
<\langle[P, A_{10}] + b_0p_1, b_0q + Pb_0, ([P, A_{20}] + b_0p_1, b_0q + Pb_0)\rangle, ((A_1, b), (A_2, b))> = 0
\]

for every \(P \in T_\mathcal{G}\).

Let \(P\) be as above and introduce the following notation:

\[
A_{10} = \begin{pmatrix}
A_{11} & \alpha_1^1 \\
\alpha_1 & \alpha
\end{pmatrix}, A_{20} = \begin{pmatrix}
A_{11} & \beta_1^1 \\
\beta_1 & \beta
\end{pmatrix}, b_0 = \begin{pmatrix}
b_1^1 \\
\epsilon_0
\end{pmatrix}
\]

and

\[
A_1 = \begin{pmatrix}
B_{11} & \delta_1^1 \\
\delta_1 & \delta
\end{pmatrix}, A_2 = \begin{pmatrix}
B_{11} & \gamma_1^1 \\
\gamma_1 & \gamma
\end{pmatrix}, b = \begin{pmatrix}
b_1^1 \\
\epsilon
\end{pmatrix}.
\]

Then, we have the following result.

Theorem 2 A miniversal deformation of \(\mathcal{A}_0\) is given by the linear variety \(\mathcal{A}_0 + ((A_1, b), (A_2, b))\), where \(A_1, A_2\) and \(b\) are any solution of the following system:

(i) \(2[A_{11}, B_{11}^*] + \alpha^1\delta^{1*} - 2\delta^*_1\alpha_1 + \beta^1\gamma^{1*} + b_0^1b^{1*} = 0\)

(ii) \(2\alpha_1B_{11}^* + \alpha\delta^{1*} - (\delta^{1*} + \gamma^{1*})A_{11} - \bar{\delta}\alpha_1 + \beta\gamma^{1*} - \bar{\gamma}\alpha_1 + \epsilon_0b^{1*} = 0\)

(iii) \(2\alpha_1\delta^*_1 - \delta^{1*}\alpha^1 - \gamma^{1*}\beta^1 + \epsilon_0\bar{\epsilon} = 0\)

(iv) \(B_{11}^*b_0^1 + \delta^*_1\epsilon_0 = 0\)

(v) \((\delta^{1*} + \gamma^{1*})b_0^1 + (\bar{\delta} + \bar{\gamma})\epsilon_0 = 0\)

(vi) \(tr(b_0^1b^{1*}) + \epsilon_0\bar{\epsilon} = 0\)

Since the number of unknowns is \(n^2 + 2n\) and the number of equations is \(n^2 - n + 4\), we have the following result.

Proposition 1 There is no pair structural stable in \(\mathcal{M}\).

Remark 1 If \(A_{10}, A_{20}\) and \(b_0\) are real matrices, we can substitute the symbol \(*\) for the symbol \(t\), corresponding to the transpose matrix.

If the pair \(((A_1, b), (A_2, b))\) is in Kronecker reduced form, the above equations take a simplified form allowing in many cases the obtention of an explicit solution of a miniversal deformation. In fact, we have in this case,

\[
A_{10} = \begin{pmatrix}
J & 0 & \alpha^1 \\
0 & N & 0 \\
0 & \alpha_1 & 0
\end{pmatrix}, \alpha_1 = (0, \ldots, 0, 1), \alpha^1 = (\alpha^1_1, \ldots, \alpha^1_h) = d(\gamma)^f
\]
\[
A_{20} = \begin{pmatrix}
J & 0 & \beta_1^* \\
0 & N & \beta_2^* \\
0 & \alpha_1 & \beta
\end{pmatrix},
b_0 = \begin{pmatrix}
0 \\
p \\
\epsilon_0
\end{pmatrix},
p = (0, \ldots, 0)^t.
\]

Then if accordingly with the above notation we write
\[
A_1 = \begin{pmatrix}
B_{11} & B_{12} & \delta_1^* \\
B_{21} & B_{22} & \delta_2^* \\
\delta_{11} & \delta_{12} & \delta
\end{pmatrix},
A_2 = \begin{pmatrix}
B_{11} & B_{12} & \gamma_1^* \\
B_{21} & B_{22} & \gamma_2^* \\
\delta_{11} & \delta_{12} & \gamma
\end{pmatrix},
b = \begin{pmatrix}
\beta_1^* \\
\beta_2^* \\
\epsilon
\end{pmatrix},
\]

the following proposition follows.

**Proposition 2** \(((A_1, b), (A_2, b)) \in (T_{A_0}O_0)^\perp\) if and only if

(i) \[2[J, B_{11}] + \alpha_1^1 \delta_1^* + \beta_1^1 \gamma_1^* = 0\]

(ii) \[2(JB_{21} - B_{21}J) + \alpha_1^1 \delta_2^* + 2 \delta_1^* \alpha_1 + \beta_1^1 \gamma_2^* = 0\]

(iii) \[2(NB_{12} - B_{12}J) + \beta_2^1 \gamma_1^* + \rho \beta_1^* = 0\]

(iv) \[2[N, B_{22}] - 2 \delta_1^* \alpha_1 + \beta_1^1 \gamma_1^* + \rho \beta_2^* = 0\]

(v) \[-(\delta_1^* + \gamma_1^*)J + \beta \gamma_1^* + \epsilon_0 \beta_1^* + 2 \alpha_1 B_{12} = 0\]

(vi) \[2 \alpha_1 B_{22} - (\delta_1^* + \gamma_2^*)N - \delta_1^* \alpha_1 + \beta \gamma_2^* + \epsilon_0 \beta_2^* = 0\]

(vii) \[2 \alpha_1 \delta_1^* - \delta_1^* \alpha_1 - \gamma_1^* \beta_1^* - \gamma_2^* \beta_2^* + \epsilon_0 \epsilon = 0\]

(viii) \[B_{21}^* p + \delta_1^* \epsilon_0 = 0\]

(ix) \[B_{22}^* p + \delta_1^* \epsilon_0 = 0\]

(x) \[(\delta_2^* + \gamma_2^*) p + (\delta + \gamma) \epsilon_0 = 0\]

(xi) \[tr \left( \begin{pmatrix} 0 & \rho \beta_1^* \\
0 & \rho \beta_2^*
\end{pmatrix} \right) + \epsilon_0 \epsilon = 0\]

Notice that \(l = 0\) implies \(\epsilon_0 = 1\), \(\alpha_1 = 0\) and \(l > 0\) implies \(\epsilon_0 = 0\), so that we have

**Corollary 1** The above equations reduced to:

(1) If \(l = 0\):

(i) \[2[J, B_{11}^*] + \alpha_1^1 \delta_1^* + \beta_1^1 \gamma_1^* = 0\]

(ii) \[-(\delta_1^* + \gamma_1^*)J + \beta \gamma_1^* + \beta_1^1 \gamma_1^* = 0\]

(iii) \[-\delta_1^* \alpha_1 - \gamma_1^* \beta_1^* = 0\]

(iv) \[\delta_{11} = 0\]

(v) \[\delta + \gamma = 0\]

(2) If \(h = 0\):
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(i) \[ 2[N, B_{22}^*] - 2\delta_{12}^*\alpha_1 + \beta_2^1\gamma_{12}^* + p\delta_{12}^1 = 0 \]

(ii) \[ 2\alpha_1 B_{22}^* - (\delta_2^1 + \gamma_2^1)N - \delta\alpha_1 + \beta\gamma_2^1 - \gamma\alpha_1 = 0 \]

(iii) \[ 2\alpha_1\delta_{12}^* - \gamma_1\beta_{12}^* = 0 \]

(iv) \[ B_{22}^* p = 0 \]

(v) \[ (\delta_2^1 + \gamma_2^1)p = 0 \]

(vi) \[ trp\delta_{12}^1 = 0 \]

3 The case \( n = 3 \)

If \( n = 3 \) (and of course if \( n = 2 \)) the above equations can be solved easily. We limit ourselves to give in the following three examples the dimension of the corresponding orbit. We denote this orbit by \( O_1, O_2, O_3 \), respectively.

(1) \( J = (\lambda), N = (0) \), so that \( \alpha_1 = 1, \alpha_1^1 = 1, p = 1, \epsilon_0 = 0 \). Then \( dimO_1 = 9 \).

(2) \( J = (0), N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \), so that \( \alpha_1 = (0, 1), \alpha_1^1 = 0, p = (0, 1)^t, \epsilon_0 = 0 \). Then \( dimO_2 = 11 \).

(3) \( J = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \) so that \( \alpha_1 = (0, 0), \alpha_1^1 = (1, 0)^t, \alpha = 0, \epsilon_0 = 1 \). Then \( dimO_3 = 11 \).

Notice that, according Proposition 1, any of these pairs is structurally stable.

References


