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Logic as general rationality: a survey

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Abstract: Logic today is urged to confront and solve the problem of reasoning under non-ideal conditions, such as *incomplete information* or *imprecisely formulated statements*, as is the case with *uncertainty*, *approximate* descriptions or linguistic *vagueness*. At the same time, Probability theory has widened its traditional field of analysis (the *expected frequency* of physical phenomena) so as to encompass and analyze general *rational expectations*. Thus, Probability has placed itself in the position of offering Logic a solution for its own long-awaited generalization. The basis for that turns out to be precisely the shared base underlying the two disciplines. This theoretical base predates their common birth, as seen in the early efforts of Bernoulli and Laplace, as well as in Boole's 1854 attempt to formalize the "laws of thought" and then, as he claimed, to "derive Logic and Probability" from them. Once we recover (following Popper's 1938 advice) the underlying *formalism*, we come, by interpreting it in two different directions, back into either *Logic* or *Probability*. The present survey explains the story so far and does the reconstruction work from the logical point of view. The stated aim is to generalize *Logic* so as to cover, as Boole intended, the whole of *rationality*.

Keywords: sentential logic, probability, boolean algebra, logical semantics, probabilistic semantics, probability logic, many-valued logics, supervaluations, uncertainty, rational belief.

INTRODUCTION

This survey could as well be entitled: *"How Logic was once the same as Probability, and then they diverged —and how they may again be formally the same"*, or *"Logic and classical Probability: recovering the lost common ground"*. Before we begin, let us say that the implied desideratum of the title(s) is long overdue. Indeed, that *(a)* standard Logic *can* be generalized, and that *(b)* the natural generalization of Logic *is* —or derives from, or is suggested by— Probability theory seems at present the shared conviction of a number of logicians and probabilists. Thus, to cite a few of the latter, Ramsey wrote (in 1926) that the laws of probability are actually laws of *consistency* (or *rational* behavior), an extension of Formal Logic to cover partial information, and that Probability theory could become the "logic of consistency" which would control and guarantee, as Mathematics does, that our beliefs are not self-contradictory. At about the same time de Finetti concluded that Probability theory is the only possible "logic" to generalize standard Logic. All the same, Patrick Suppes was considering in 1979 that Probability theory is the natural extension of classical deductive inference rules, while, more recently, Glenn Shafer declared that "probability is not really about numbers; it is about the structure of reasoning".

Why the technicalities of Probability theory should be viewed today as a powerful *generalizer* of standard Logic and a suitable formal *unifier* of the two may come as a surprise both to probabilists and logicians. Actually, the idea —that we pursue in our generalization below— is very simple: Probability and Logic are but *two interpretations of a same underlying concept*. This is how the founders of both theories saw it and what Popper later explicitly said (and Kolmogorov claimed he had done). But this old notion, over which notable thinkers like Reichenbach or Carnap agonized after the 1930s, is shared nowadays by a surprisingly exiguous minority of specialists (in both disciplines).

Logic and Probability are overwhelmingly seen today as two completely disparate fields, with a very few, if any, points of contact. Logic deals with *reasoning* and *truth*, Probability with inference on poor data. They seem to have nothing in common. Though they both start with a set $\mathcal{B}$ of Boolean-structured objects (respectively *sentences* and *events* —or, confusingly, "propositions") and though they assign them values in a simple number system containing the one and the zero (here logicians prefer 'truth' and 'falsity', though), at this point the similarity apparently ends, for the two valuations are perceived to be very different: the probabilistic $P : \mathcal{B} \to [0, 1]$ obeys a set of axioms set forth by Kolmogorov in the 1930s (that do not actually follow from any particularly "probabilistic" rules but rather reduce it to a simple 'measure' —in the technical sense— of the "event" objects), while the *logical* valuation is felt to be of a quite different nature and regulated by a semantics set forth by Tarski, also in the thirties, and buttressed by elaborate, specialized logical considerations. Moreover, either field not only has a different type of problem to solve, it also has a different, incompatible set of base concepts and interpretations to work on. And the diverging traditions have bred different strokes of unrelated practitioners and two methodologies that are seen by the mainstream mathematician as far distant (even lying at opposite fields of Mathematics, i.e. *real* vs. *discrete*).

However, this is not how things were seen in the first stages of the modern theories of Probability and Logic. Their founders, notably Bernoulli and Laplace, or Boole and Peirce, dithered a lot on what might "probability" or "truth" mean, and often tended to explain one through the other in incipient, half-baked intersecting intuitions, as can be readily seen by browsing into the original literature. Later developments, as well as the progressively firmer foundations and the more specialized and mutually deviating interpretations that either field painstakingly acquired, created an increasing gap between Probability and Logic, in which both contenders apparently never found a ground or occasion to reconcile into one unified approach (which, as we suggest below, is not only desirable but feasible and even natural).

Why striving to recover an encompassing view should be interesting at all may be not obvious at first, but there are strong reasons for it. First, because, as we said, both theories are two differing interpretations of the same idea. Second, because both probabilists and logicians have recently been hard pressed against the limits of their own disciplines when confronting new challenges with consecrated methods. Prominent challenges include: (1) for probabilists, how to clarify the ultimate meaning, and the practical import, of the apparently obvious idea of "probability" (doubters here include names as Keynes [1921], Ramsey [1926] or de Finetti [1931], and the questions raised prolong well to this day into widely-discussed conundrums as the status of *subjective probability*, *rational belief* or *bayesianism*); or (2) for logicians, how to validate reasoning under *uncertainty* or with *incomplete* or *approximate* information (a problem that eventually gave rise, also around the 1930s, to non-standard formalisms such as the *many-valued logics* of Lukasiewicz [1920] (and [1930], with Tarski) or Kleene [1938], or the attempts at defining a *probability logic* by Reichenbach [1935a,b], discussed by Carnap [1950].

The material below is structured in two parts: the first is a short survey explaining why Logic and classical Probability were once the same thing —and gave the (common) pioneers

(Bernoulli, Hume, Laplace, Boole) lots of cross-supporting arguments— and why they soon diverged to the point of being considered unrelated. The second part –considerably longer– is a summary of how we locate Logic firmly in the Logic/Probability common heritage; it is based on former work by the author (Sales [1982a,b, 92,94,96]) and the starting point is Popper's 1938 suggestion (see Popper [1959]) to set forth a *unique* algebraic *uninterpreted* formalism as the common source from which, through distinct interpretations, both Logic *and* Probability can be formally derived as particular instances. The common formal idea we advance is, as will be later explained, that we can postulate an *additive valuation* (in e.g. [0,1]) of the elements of a given abstract Boolean structure $\mathcal{B}$ —that is later interpreted by Probability as a (set-theoretic) *event*, and by Logic as a (non-set) *sentence*. Our generalization proceeds from this point on as an exclusively *logical* reading of the common uninterpreted formalism. (The development is satisfactory also in a second, non-formal sense, since it can be seen as a vindication and reconstruction of the pioneers' historical common source of insights.)

A. THE PROBABILITY/LOGIC INTERFACE

A.1. THE VIEW FROM PROBABILITY

a. Classical Probability

Around the year 1700, Jakob Bernoulli tried to define the *probability* of a phenomenon as a non-evident and non-subjective something that fortunately had effects: the observable frequency, or relative number of cases in which the phenomenon manifested itself. This number was supposed fixed and objective. So, measuring frequencies was the way to estimate "probability". Conversely, knowing the probability of a phenomenon allowed to predict its expected frequency. This is, in essence, Bernoulli's theorem. It is the first clear, albeit implicit, definition of probability. It is also the first instance of a duality that is present since in Probability theory: probability P —a supposedly *objective* property of phenomena— is conceived simultaneously as (1) the ratio of positive cases (call it P_c) and (2) the number we have (call it P_b, b for 'belief') to estimate P_c. The first is assumedly an objective reality, the second an inevitably subjective entity that depends on our past history of observations (a paradox that is the common theme of many reflections, like those of e.g. de Finetti). Obviously, the fewer our interactions, the more subjective our P_b estimate is. The idea is that P_b "aproximates" P_c, and the aim is getting $P_b = P_c$ (in some limit situation).

The Rev. Bayes developed Bernoulli's idea of P_b converging to P_c through observational updates and came up with his celebrated formula (posthumously revealed in 1763) to compute P. Hailed by observational scientists for more than a century, it is now the heart of a debate about what is this Bayes-computed probability. Called "a priori" probability, anti-Bayesians contend it is nothing more than simple, non-objective belief based on a hypothetical view of our ignorance.

Laplace, in 1774 and later, defined probability as P_c, the ratio of favorable *cases*, all assumed having "the same probability". The obvious circularity raised some eyebrows in the 1920s, but Laplace's has been the standard and successful definition since, at least for non-sophisticated applications. Note that it places probability clearly on the *frequency* side, and cavalierly dismisses any subjective-sounding *belief* content, perhaps the reason for its long-standing success. Note, too, that *cases* are a logical notion, since they can be defined —and were by Laplace himself— as the *true* instances of a proposition. Thus, Laplace's [1774,1820] probability can be seen as an early generalization of Logic, particularly of the concept of validity (resp. consistency), now interpretable as "true in all (resp. some) cases". Some years before this, Hume had already implied too that probability was a generalization of logical inference by considering that, given a proposition obtained from some conditions or premises, its probability was the proportion of premises (or premise extensions) in which the proposition was satisfied.

The classical probabilists thus conceived probability, more than as something associated with indeterminism or uncertainty, as a measure of our knowledge of phenomena in the presence of incomplete information (or, dually, as a measure of our partial ignorance). This concept was considered objective because, though not directly measurable, (1) it referred to a supposedly objective situation indirectly parameterizable through its observable effects, and (2) it was manipulable through rules of an objective calculus. So, after Laplace, Probability theory came to be dominated by the probability-as-frequency view. This was convenient as it was objective and "scientific" and adequately eschewed the estimation or "belief" problem.

It lasted until the 1920s, when von Mises, dissatisfied with the classical solution, formalized (in 1928) the estimation or approximation problem by postulating a "sample space" Ω, defining frequency in it and computing probability as the result of some limit process, in which the number of observations tended to infinity. Since this was no ordinary limit and the process not quite satisfactory, Kolmogorov [1933] came up with the now universally accepted solution: probability is just the *measure* of an *event* (an event being a set of outcomes); this measure is taken in the mathematical sense, i.e. as a countably additive valuation (though the need for countable additivity has been challenged by many, notably de Finetti [1970] or Popper [1959]). An interesting thing to note: Kolmogorov declared that his formalization was "neutral" in the sense that it was abstract (and thus previous to any interpretation); in his words, probability had to be formal, pure mathematics, merely ruled by axioms. Nevertheless he also declared that his measured entities (in theory merely the members of a σ-algebra over Ω) *are* actually sets. And though he added that this was irrelevant, Popper protested (in 1955) that it *is* relevant in some important cases, and noted that a truly abstract formalization must admit *any* interpretations, including those in which the measured entities are not sets. (But, we add, this is precisely the case of Logic, where we have non-set entities, namely *sentences* from a *language*, not *events* from a *sample space*.) Popper [1959] offered an axiomatic alternative first suggested in 1938, fully developed in 1955, and now fashionable (under the guise of "probabilistic semantics" or "Popper functions").

b. "Subjective probability", "Probability logic" and "Logical probability"

Following Keynes's [1921] lead, Ramsey [1926] was the first to consider that the belief side of probability, already present in Bernoulli or the Bayes' formula, was the core of the concept, since the "true" value of probability was beyond our reach and the best we could do is approximate it through a careful, consistent, rational procedure; so he defined probability as *belief* and defined this as a number obtained on the basis of *consistency* considerations about the belief-holder's *rational* behavior (as deducible from betting protocols); the rationality-induced consistency insured that the number, though inevitably "subjective", was nevertheless the most objective measure one could obtain. In a similar spirit, and in same years, Bruno de Finetti [1931,37] approached probability as an inherently non-objective concept. He summarized it in his well-known slogan "probability does not exist" (i.e. objectively, at least "not more than the cosmic ether"), and nothing beyond consistency assures its imagined objectivity. According to him, probability is merely what we expect on the basis of past experience and the assumed consistency of what we do. As a number, probability (which de Finetti [1937,70] constructed formally on the basis of a vector space of rational expectations), is "objective" as far as the procedure to obtain it obeys coherent assumptions.

The "subjective probability" thesis of Ramsey/de Finetti has found continuation till now in the work of Jeffreys [1939], Koopman [1940], Savage [1954] or Jeffrey [1965], to name a few, all of which reject the epithet "subjective"; they prefer to be called simply probabilists and at most admit that the probability they deal with is a (non-subjective) partial or rational belief, i.e. the value we assign propositions in absence of complete information.

On the other hand, in a series of studies beginning in 1932 Hans Reichenbach [1935a,b], a physicist with an interest in foundations, interpreted probability as a logic. The logic (*probability logic* he called it) was not truth-functional, but he could subsume all classical tautologies as particular cases of propositions p that had unit probability (i.e. $|p| = 1$). He obtained formulas for the value (probability) of the connectives which are basically like the ones we obtain below; he says that e.g. $|p \vee q|$ is a function of $|p|$ and $|q|$ *plus* a third parameter k he calls Kopplungsgrad or 'degree of coupling' (defined roughly as the relative size of the intersection of overlapping areas or classes to which a measure is applied that coincides with the conditional probability). This is equivalent to what we obtain in our generalization below, but note that Reichenbach never moves out of probability and events: he always speaks of probability in its standard meaning and only in a translation of senses he says he can interpret the probability of an event sequence —a sequence of binary truth values— as its (non-binary) "truth". His world is clearly that of Probability, and what he obtains is a Logic only in the sense that he speaks of truth, albeit probability by another name. Moreover, though his contemporary critics (including Tarski [1935a]) argued against the construction, they did not because of the subsidiary role of truth in it (as a surrogate for a probability of one) but on the arguable ground that a proper logic ought to be truth-functional (see Urquhart's [1986] comment below as to the contrary).

(Work on 'probability logic' by Kemeny *et al.*(see Leblanc [1960]), Bacchus [1990], Halpern [1990] or Fagin et al. [1990] is not mentioned here because what these authors deal with is not what is understood by that name in the the classical tradition. Instead, they apply the standard treatment of any first-order theory *inside* Logic, i.e. augmenting ordinary first-order logic with a number of specific axioms that syntactically describe the real numbers (or an ordered field) and the probability operations on them, so that the Probability laws can be derived formally as theorems.)

Another indirect view of Logic-as-valuation was the one adopted by Rudolf Carnap [1950,62], starting in the 1940s. The Logic/Probability link began in his case by trying to justify probability logically. In his view (that he called *logical probability* and surmised as theoretical ground on which to base a "logic of induction" to which Popper came to be fiercely opposed), the probability of an event was the proportion or, more generally, the measure ("ratio of ranges of propositions") of the intervening circumstances (described as logical sentences) concurring in the event. For this measure he said he was inspired by a definition of Wajsberg —which was inspired in turn by proposition *5.15 of Wittgenstein's [1922] Tractatus (ultimately Bolzano-inspired, see below). Carnap hesitated and changed his approach often along the 1950s; for instance, notably, he came to value sentences instead of events, but came back to events later, shortly before giving up the whole scheme. Wittgenstein and Wajsberg's extensional rendition and Carnap's use of them is, like Reichenbach's implicit grounding of probability on rather obscure "overlapping classes", strongly reminiscent of the Stone representation we obtain below out of our general, non-probabilistic truth valuation of logical sentences. It is interesting to note that Carnap's logically-described components of events correspond rather precisely to what Laplace had called the (positive i.e. true) "cases" concurring in an event. They are also almost interchangeable with Boole's *cases* (his "conceivable sets of circumstances") underlying a logical proposition (or with equivalent descriptions by McColl and Peirce and Wittgenstein, see below).

A.2. THE VIEW FROM LOGIC

a. The pioneer logicians

The laplacian idea of having "cases" (a logical concept, we noted) and then measuring the proportion of the true ones seem to have been floating all over. Laplace's uninfluential contemporary, the Austrian philosopher Bolzano, had this to say about first-order propositions and truth: propositions, he says, have an associated "degree of validity", a number in [0,1] which equals "the proportion of true 'variants'" (Bolzano's "variants" are our term substitutions).

And then Boole [1854], when just after studying classes he sets out to analyze *propositions* (in 1847), conceives them by means of an alternative interpretation of his elective symbol x (already introduced for classes) and says it now stands for the *cases* (defined informally as "conceivable sets of circumstances") —out of a given hypothetical "universe" (a De Morgan's idea)— in which the proposition is *true*. In the last chapters of his 1854 book (significantly entitled "An investigation of the laws of thought, on which are founded the mathematical theories of logic and probabilities") Boole even likens the product $x \cdot y$ of two propositions (i.e. the conjunction value, actually) to the *probability* of simultaneously having them both and the (value of) the sum $x + y$ to the *probability* of having either (provided they are mutually exclusive).

An equivalent idea is present in MacColl's [1906] partially published reflections (started before 1897), where he says that propositions are generally "variable", meaning they are sometimes the case, depending on their (basically probabilistic) modality. Peirce [1902], in an unpublished work that deliberately follows MacColl's steps, sets out to distinguish "necessary" from "contingent" propositions, most being the latter sort, characterized by their (probabilistic) occurrence.

In a similar vein Wittgenstein [1922] considers a little later that propositions reflect —and are basically decomposable into— "states of affairs" (an idea borrowed from Leibniz). That those states of affairs (reminiscent of Laplace's or Boole's cases) are in some way a measurable universe whose proportions gave information on the truth of the composite propositions is obvious from the Tractatus (and is the inspiration of the extensional view of Wajsberg and Carnap mentioned above).

b. "Multi-valued logic"

While some probabilists (from Ramsey to Reichenbach) agonized in the 1930s over their base concepts, there was intense soul searching also in the logicians' camp. The main new idea came from Lukasiewicz in 1930 (down from antecedents since 1918) when he postulated a logic in which "truth" values could take any value from the infinite real [0,1] interval (see Łukasiewicz & Tarski [1930]). To compute the value of composite propositions in his "many-valued logic" he obtained (truth-functional) formulas which are exactly the ones we obtain below except for the fact that they presuppose full *compatibility* (a concept we explain below) among *all* propositions, no matter some are the negation of others. This was not perceived as a problem at the time but *it was*, as some fellow logicians pointed out to him at the 1938 Zurich workshop (see Łukasiewicz [1938]). They considered that either we assign $p \wedge \neg p$ the value $\min(|p|,\ 1 - |p|)$ given by the formulas (thus blatantly contradicting the basic logical *law of non-contradiction*) or else we assign it the –correct– zero value (meaning *falsity*, so in accordance with ordinary Logic) but then we arbitrarily disobey the postulated truth-functionality. As this predicament found no decisive solution, many-valued logic has continued to this day —along with unexpected offshoots like "fuzzy logic"— consecrating truth-functionality as a rigid principle and thus putting itself *out* of mainstream Logic, of which is a weak incompatible variety (unless we add, as suggested by van Fraassen [1968], some super-valuation mechanism to it). At the end of a detailed survey of multi-valued logic, Alasdair Urquhart [1986] comments that it is hardly surprising that those systems have remained logical toys or curiosities since "there seems to be a fundamental error [truth functionality] at the root". Some modern cultivators wonder whether is it possible to combine the two best-known formalisms, Probability and Logic, in any way (but Lee [1972] admits that "we do not seem how to do this"); others, like Hamacher [1976], Zimmerman [1977] or Trillas *et al.* [1982], in asking what are the correct truth-value formulas for the fuzzy calculi, hesitate among a variety of candidates, while still another, Minker (with Aronson *et al.* [1980]), would like to know the "truth bounds" of many-valued conclusions obtained from premises.

A.3. THE VIEW FROM THE OUTSIDE

a. Mathematics

Alfred Tarski straightforwardly supposed (in Tarski & Horn [1948]) he had simply a *Boolean algebra* and then set out to analyze thoroughly all possible *measures* in it. So did Gaifman [1962] —and Scott (with Krauss,[1965])— who extended this analysis to first-order logical formulas; these were assigned (additive) values, that were called 'measures' by Gaifman (and 'probabilities' by Scott). True to mathematicians' fashion (i.e. approaching topics in uninterpreted, "abstract" formalisms), they did not understand 'probability' as other people do; they just used the word as a synonym for *normalized measure* (a *measure* being a σ-additive valuation on the positive reals). In this sense, their "probability" is a blanket term for any common generalization —such as the one we attempt here— for the two (heavily interpreted) fields of Logic and Probability.

Also in this line, J. Loṡ [1962] explored general "probability" valuations of logical *sentences* and came up with a (reasonably unsurprising) *representation theorem* of probabilities on a (set-theoretical) space of *models* (or *interpretations*) in the logical sense. Loṡ's line has been consistently followed by Fenstad since 1965. (Fenstad's papers [1967,68,80,81] have been a source of inspiration for our generalization below.)

b. Philosophy

Philosophers have also been exploring the common material. A few representative examples are Hintikka and Suppes (both presenting first results in 1965), Stalnaker (in 1968-70), Lewis (in 1972) or Popper (e.g. in 1987, with Miller). The first (Jaakko Hintikka [1968]), inspired by Bar-Hillel's (and Carnap's,[1952]) information-measure ideas, was suggesting in 1965 (with Pietarinen,[1968]) various formulas to parameterize the information contained in sentences. Also in 1965, the second (Patrick Suppes [1968]) did a circumscribed analysis of the Modus Ponens logical inference rule from a generalized perspective (what he called 'probabilistic inference') in which he got formulas fully consistent with the ones we obtain below. Another philosopher traditionally preoccupied with logic/probability differences (especially those centered on the conditional/conditioning operation), Robert Stalnaker [1970], revealed some fine points (among which our "$A \rightarrow B$" $\neq$ "$B|A$" conceptual and practical distinction). His work and David Lewis's [1976] have done much to clarify and distinguish concepts shared by logicians and probabilists.

But, prior to these 1960s efforts, the single philosopher to do this most explicitly is surely Popper [1959], in lucid but little known pioneering work. He did not only see (in 1938) that the two concepts were different interpretations of a (yet to be written) formalism —Kolmogorov [1933] also saw this— but he designed one in a very simple and intuitive way by defining a valuation in [0,1] on pairs (a, b) of sentences (of a very elementary language) that was directly constructible by users (i.e. reasoners and probability-estimators alike) and that gave way naturally to a Boolean structure with the usual properties (including measurability). Whatever sense the user gave the valuation ("probability", "truth likelihood", "truth content" or simply "truth") it was the user's concern. Popper later used his own formalism (and the derived Booleanity assumption) to deduce properties of his 'truth content' measure and so emit (with David Miller,[1987]) a post-mortem indictment against Carnap's [1950,62] "inductive logic".

c. "Fuzzy Logic"

From a logical point of view, Fuzzy Logic (under development since 1965) can be considered as an "interpreted" variety of Łukasiewicz & Tarski's [1930] infinite-valued logic. ("Interpreted" because it adds to many-valued logic an extensional interpretation of predicates in terms of non-standard sets.) Thus, it was already mentioned in a former section, where we considered it as an (unexpected) offshoot of the many-valued logic family, and we dedicated it some short comments. Nevertheless, the overgrown "fuzzy" tradition, now largely applications-oriented, has its own self-contained rules and momentum and is *not* exactly logic nor probability. Nor, it claims, has barely anything to do with them, with which it is pretendedly "orthogonal", only devoted to linguistically-motivated imprecision (i.e. *vagueness*).

One may doubt the claim by fuzzy theorists that the issues they currently discuss have no *logical* bearing. On the contrary, they seem fully relevant for logical discussion, so we have dedicated an appendix to comment rather expansively on fuzzy 'logic', as well as to mention an early generalization of Logic that arose inside the fuzzy tradition (by B.R. Gaines [1978]).

d. Artificial Intelligence

Since the moment the first "expert systems" dealt with uncertain information (the obvious cases are Mycin and Prospector), AI as a discipline got involved too in the Probability/Logic dilemma about what is the ultimate nature of "truth" measures of sentences presented to the expert system user (see Shortliffe [1976]). Leaving aside Mycin's "uncertainty factors" (later revealed to be actually measuring belief *change*, see Heckermann [1986]), the typical measures are Prospector's "*probability* assignments" (see Duda *et al.* [1976]), that are considered unproblematic and intuitive (to the user, who can easily estimate them), and are combined according to Bayes' formula as though they were really what their chosen name implied. Whatever the true status (probabilistic, or logical) of the calculus, the formulas on offer happen to become corollaries of our generalized calculus below (where, unlike in AI's rather *ad hoc* formalisms, nothing is assumed about whether the measures are actually "probabilities" or "truths" or something else).

Nils Nilsson's [1986] stated goal in his 'probabilistic logic' paper (orally anticipated in 1983) was to rationalize past work in the Prospector expert system project (1976-80) and give it a formal background by propounding a 'probabilistic entailment' that would do for this formalism what the Modus Ponens rule (A , $A \rightarrow B \ \vdash B$) does for ordinary Logic. He obtained the well-known bounds for the probability of B (see e.g. our formula (8) below) by Venn diagram techniques, that he extended to the study of convex hulls in a "probability space" of "possible worlds" (the latter terms are both familiar terminology to de Finetti and Loś readers). As Nilsson [1993] acknowledged later, his method is similar to work by Good [1950] and Smith [1961] (not to mention de Finetti [1937,70]), authors of whom he was unaware at the time. His goal is, in fact, shared by many since the first 1980s —including the present author and others mentioned in previous and later paragraphs. (The Nilsson effort is briefly discussed in the next section.)

The Artificial Intelligence context has continued to breed practical motivation for the Logic/Probability demarcation. A series of special conferences (*'Uncertainty in A.I.'*) has been called (beginning in 1986, see Kanal & Lemmer [1986]) and given useful insights into the differences and similarity of the once-separate fields, including expert system coefficient analysis by Heckermann [1986], Grosof [1986] and others or the theoretical framework called *belief networks*, developed for the efficient computation of "probabilities" (or whatever they are) by Judea Pearl [1988]. These contributions to general and practical Logic, worthwhile as they are, have the implicit bias that what is actually manipulated is the *probability* of distribution-driven events (rather than the *belief*, *commitment* or *assertiveness* of linguistic sentences), and thus the interpretations are always loaded with unnecessary concessions to probabilistic terminology and methods. So, for instance, Pearl, whose formalism is nominally about "beliefs", is nevertheless overwhelmed with computing probability distributions of *facts* and with assuming simplifying conditions (such as independence or conditioning) to obtain the final value; if this asignment is to be a real "belief", as stated, then presumably the "facts" and their distributional assumptions should be less real and objective than supposed: probably a consistent calculus (consistent in the Ramsey/de Finetti sense) based on possible or estimated (rather than actual) "facts" would suffice —and for this the possible-worlds or the rational-expectations analyses are already at hand (and ready to be usefully supplemented by a practical procedure such as Pearl's).

An unsuspected benefit of Logic-oriented analysis by Artificial Intelligence practitioners has been their growing awareness that a system of premises (what they tend to call "knowledge base") from which predictions are made (or actions are taken) is essentially a set of *beliefs* to which the agent is commited. This is now already clear in classic AI textbooks as Genesereth & Nilsson's [1987], where the distinction is made between inference procedures where the user's *full* commitment must be kept *throughout* the inference process and those where the belief premises are "qualified" (e.g. modally, with a belief operator) or "quantified" (with a "probability" assignment); in the latter cases, it is assumed, the commitment is less than absolute and the conclusion strength, therefore, less than guaranteed —however formally valid the reasoning may be. This approach is welcome, since it implies that however we treat premises in a logical argument —either as commitment-inducing beliefs or as admittedly weak probes— they all take part in the inference process and share with it a common goal: knowing to what extent can we *rely* on conclusions.

e. Probabilistic logic

As mentioned in the previous section, Nilsson [1986] sets out to investigate how Logic would generalize if one were to "assign probabilities instead of truth values to sentences". Though he calls his probabilities *probabilistic truth values* he treats them as real *probabilities* (at least to the extent that Prospector's numerical assignments are). This shows clearly in his subsequent treatment of (sentence) conditioning, that he considers plainly a Bayes process and relates to considerations by authors as Pearl, Heckermann, Grosof or Cheeseman (who explicitly deal with *probability* distributions). Based on entropy considerations by the latter author, Nilsson refines his bounding formulas so as to spot an exact value for the B in his 'probabilistic entailment' (= generalized Modus Ponens) rule —that happens to be the midpoint of the bounding interval (compare that with formulas (9-11) below).

Nilsson's grounding for his 'probabilistic logic' is basically semantic: he exhaustively generates and examines the "possible worlds" inherent in a formula; but the way he then discards some of the worlds —before assigning values to them— amounts to introduce consistency considerations. Akin to this method is what Paass [1988] proposes in a survey: assign basic "subjective probabilities", construct a universe of "relevant propositions" —which turns out to be isomorphic to Shafer's "frame of discernment" (or to our Θ below)— and then evaluate the resulting probability distributions on it. The computation may be done in Dempster-Shafer's terms (see Shafer [1976]) or by other methods: linear programming, stepwise simplification, Pearl's "belief networks" (with interactions) or statistical simulation (see Paass [1988]).

Though they may not use the name (invented by Nilsson), many so-called "probability logics" do not descend from Reichenbach but are really *probabilistic logics* and share Nilsson's conception and aim: finding a logical foundation for the use of [0,1] *probability* assignments to sentences taking part in an inference. Most of them formally derive their technical motivation and analysis from Gaifman [1962] and Scott & Krauss [1968]. The author of one of the first such 'probability logics', Theodore Hailperin [1984] —who also motivates his analysis historically (and also mentions the classics, from Bernoulli to Keynes and beyond)— sets out to generalize "truth" values and Logic in model-theoretical style, through the use of a modified version of the *model* and *consequence* concepts. This is done too by Bolc & Borowik [1992], who base their analysis on Scott & Krauss [1968] and Adams [1966], and by Gerla [1994] in an interesting attempt parallel to ours below.

A.4. BRIDGING THE GAP

a. Attempts at a synthesis

That the need for a generalization of Logic is widely felt, and that the time is now come to try it, is attested by the many surveys —and attempts at Logic/Probability synthesis (like the present one)— that are appearing of late (see for instance Gärdenfors [1988], Paass [1988], Garbolino *et al.* [1991], Bolc & Borowik [1992] or Gerla [1994]). But other such efforts deserve mention: Kyburg's [1993] is an exhaustive survey on logics of "uncertainty" where the author probably respects too much the usual division that insulates the surveyed authors' self-assigned topics, as he divides his survey, too prolixly, in "objectivism", "subjectivism", Nilsson's "probabilistic logic", "belief functions", "measures", "probabilities", "statistical facts", "updating" and "inference", where the very exhaustivity gets in the way of a comprehensive attempt at synthesis. Similarly division-respecting are several 1995 drafts by Friedman & Halpern on plausibility measures, where the probabilistic bias dominates —in the terminology, in the chosen operations, etc.— though apparently the original intention was to widen "the measure" to a general "plausibility" concept. In the case of the swift and consistent work done by Dubois & Prade [1987,93], now a respected tradition, here the drawback to attain wide method-independent generality is the self-imposed limitation to (fuzzy) possibility measures —though some convergence with non-fuzzy approaches may occur in the future (see below on subadditivity). For the sake of completeness, we must add here work in progress by two researchers with a long tradition in trying to bridge the Probability/Logic gap: Richard Jeffrey [1995] and Glenn Shafer [1996].

b. Attempts at finding a meaning for the value

Confronted with the meaning that a "truth value" may have –or may be given– when extended to points in the [0,1] interval, different people have reacted in a number of ways. Here we mention only those who did not surrender to the temptation of subsuming truth value into *probability* (carrying with a it a heavily loaded interpretation of theory). Popper [1972] thought that, in terms of theories rather than sentences, truth value could be made to mean *truth content* (of the theory), degree of *approximation* to truth or, interchangeably, its (appropriately defined) *distance to falsehood*. Haack [1974] saw it could also be interpreted as *partial truth*, roughly defined as the proportion of true components of a sentence or theory, or –equivalently– the "truth" of their conjuntion. (An unwilling distant relative of Haack's is the quantum physicists' interpretation of the "truth" of a probabilistic quantum event sequence, which is similarly defined by reduction –conjunction, actually– to its elementary event components; see e.g. Reichenbach [1935a,b] or Watanabe [1969]). Dana Scott [1973] tried to answer by defining *truth value* as one minus the *error* we commit when ascertaining or deciding it, clearly in analogy with what we do in the observational sciences. Based on ideas advanced in the 1950s by Bar Hillel (with Carnap,[1952]), Johnson-Laird [1975,83] gave too a definition of truth value, albeit indirectly, by positing as a new concept the *informativeness* or "degree of information" of a sentence (a quantity negatively correlated with the "truth-table probability") to see how it evolves through the reasoning process, with an eye more on guiding the process than on controlling the *degree of truth* (or *assertiveness*, or whatever), which is what Logic puts the proper emphasis on.

c. Popper, and Probabilistic Semantics

As repeatedly mentioned above, Popper [1959] (in 1938 and 1955) had decided that Probability and Logic had to be given at last their long-overdue common formalism. Disagreeing with Kolmogorov's [1933] solution (a [0,1]-valuation on sets) because it was already (semi)interpreted —the valued objects were *sets*— and had a bias toward Probability, he proposed instead a totally abstract, uninterpreted system consisting of (1) an elementary algebra of "sentences" (not necessarily Boolean, merely closed by "conjunction" and "negation"), and (2) a [0,1]-valuation $v(a, b)$ on *pairs* of such sentences satisfying very basic and reasonable conditions (see the appendices in Popper [1959]). Such valuations, called "Popper functions", have become a vogue now (under the name of "probabilistic semantics"), following efforts by Harper [1975], Field [1976] or Leblanc [1979,83] to base ordinary (i.e. "unary") probability on it. Great advantages of the Popper formalism are:

— the formulas may be interpreted at will either "logically" or "probabilistically": when in the latter mode, the "sentences" are elementary *events*, the basic operations are *intersection* and *complementation*, and the valuation is just plain *conditional probability* (but with an unsuspected plus: there is no need that the "conditioning" event should be assigned a non-zero (unary) probability)

— there is no need for σ-additivity (as Popper [1959] himself bothers to show), nor is σ-additivity abstract enough for Kolmogorov's [1933] pretendedly neutral formalism to qualify as really neutral (since it is satisfied in an interpretation but discards certain others)

— the resulting quotient algebra *modulo* equi-valuation *is* automatically a *Boolean algebra*, which is not only simple and extraordinarily convenient but, because obtained from very simple assumptions, disarmingly *natural*

— each quotient algebra *class* is interpretable, at will, as an ordinary logical *sentence* or an ordinary probabilistic *event*, and its value turns out to be automatically its *truth value* or, respectively, its (so-called "unary", i.e. ordinary) *probability*

— the formalism being completely abstract (i.e uninterpreted) and the interpretation totally free, the "probability" may be –with equal legitimacy– subjective, objective or whatever; in particular it may be Popper's [1962] *truth content*, or its *probability* (the latter is, according to him, the value we give a theory when nothing is known about its *content*, which correlates *negatively* with it)

— likewise, the "truth value" may be *truth* or simple *belief*; in the present author's view, many other interpetations are equally legitimate, such as "degree of commitment", "assertive value" or any other that gives us information on the *reliability* —subsuming *truth*— of sentences (including premises and conclusion), and that allows us to have *control* over their behavior along a chain of reasoning

— if one does not accept Popper's simple conditions, the same result can provably be obtained by accepting alternative and equally simple conditions set forth –independently– by Cox [1961] (and, lately, also the ones by Woodruff [1995]).

As stated, "truth values" may be interpreted in various legitimate ways. Furthermore, any truly abstract formalism requires that they must. In the following sections —to the end of the article— we consider, in genuine Popperian fashion, several fully *logical* interpretations of the "truth value" concept (*belief*, *assertiveness*, etc.), all motivated by what should be included in any study of Logic: *invariance* (of *truth* —or of what the truth value stands for, be it *approximation* to truth, *reliability* or whatever else) along the whole *reasoning* process (assumed formally *valid*). But, compared to the standard Popper schema, our method proceeds just in the reverse direction: where Popper first defines the two-place probability function and then the unary probability is obtained by taking the quotient, we begin instead by a unary valuation and then we subsidiarily define conditioning (that we prefer to call "truth relativity" or "relative truth") to obtain Popper's basic two-placed function. The process inversion is unimportant, as we could as well have begun by a two-place valuation (of the *assertive value* —or whatever else— of a sentence *relatively* to the others) and then obtain its *absolute* assertive value by taking the quotient. And note that when we proceed in our direction rather than Popper's, evaluating the mutually *relative* position of sentences through the α and σ parameters (see below) amounts to just computing the basic two-place Popper function.

B. STEPS TOWARD A GENERAL LOGIC OF RATIONALITY

B.0. MOTIVATION

Let us advance what is our aim here by starting with an obvious remark. When we argue, we do not always fully assert what we say. We often make half-hearted assertions of sentences we are not sure about, or we even use as assertions sentences we hardly believe to be the case. And yet we proceed by reasoning from such weak premises. If we admit we do, and want to treat this inside Logic, we need to qualify assertions, or, if possible, to quantify their strength, and try to follow and control what effects weak assertions may have in the reasoning process, whether and how they affect its logical validity and how we can tell the strength of the conclusion. All this is indeed a proper *logical* subject (that, however, classical logic never set out to confront).

By tradition, Logic is about *truth*; or, more precisely, it is about truth-preserving manipulations that allow us to validate arguments. *Arguments* are lists of sentences that we note by " $\Gamma \vdash C$ ", where C is called the *conclusion* and Γ is a –possibly empty, or infinite– list of sentences called *premises*. It is no obligation for sentences to be true (or even to have meaning). We merely use them to see whether certain formal manipulations —the *inference rules*— assure us that the prediction embodied by the conclusion C is *true* whenever the premises are. The whole process is dependent on the truth of the premises: if we cannot assure the truth of *all* of them, the whole procedure becomes redundant. This is how ordinary Logic approaches reasoning.

Now, at least two questions arise. First, suppose we are not sure whether some premise applies, yet we want to know the "truth" of C (or what is left of it, or, in other words, the *reliability* we can still attach to C as a prediction). Second, suppose that we deliberately want to *weaken* some true premise to see whether –or up to what point– the conclusion still *holds* (this is : probe the conclusion's dependence on its premises, or, in other words, the argument's "robustness"). By tradition, none of this is approached by Logic (so far). To see how we can generalize Logic to cover weak premises we must get a closer look on how we assign truth to sentences.

In Logic the base material is the *sentence*, say A. (Note A is *not* a set, but merely a member of a given language $\mathcal{L}$.) Once we interpret it we get what we can call a *proposition*. Then, by looking at what is the case, we get a value (a "truth value"). Following Tarski's [1935b] well-known schema

(T) 'A' is true if and only if A is true

the reasoner can verify the sentence (i.e examine the proposition A, obtained by "unquoting" the sentence A) and declare it true whenever the translation A of the object-language sentence A is found true. A is thus assigned the one or *true* value, and we say that A has full credibility and eventually we assert it with the full confidence that truth warrants. If we find A false, we assign it the zero or *false* value and give it null credibility. Note it is the reasoner who is full command of the sentences and the translation process, and thus the only one who can validate their truth. As sentences are used to *assert*, and assertions are defined as "true, believed or merely hypothesized" sentences, it is left to the reasoner who uses them to count actively on the quality of assertions as part of the reasoning process itself (so, for instance, the reasoner usually qualifies the conclusions, conditioned on the strength the original assertions carried).

Now suppose we are reasoning in Physics and A (a premise) is the positive result of an experiment. If we are sure that A is true, then we are done: we can use it in a reasoning as a true premise and proceed with the assumedly valid argument to obtain the conclusion. But suppose that, as is usually the case in Physics, we have some qualms about the truth of A, so we quantify the error ε of the experiment. Now the "truth" of the assertion 'result is positive' is no longer 1 as before but, say, "$1-\varepsilon$". We then perform the formal —valid— reasoning. The question now is : what *confidence* —as a function $\varphi(\varepsilon)$ of ε— may we have in the conclusion?

Most reasonings are like this. We perform as though premises were really true, often unconvincedly. They are provably true sometimes, but most of the time they are 'assumed true' for the sake of the argument. Now, because the (T) validation process to declare a sentence true is under control of the user (who performs the translation and decides whether the unquoted statement is observed to be the case), so the reasoner is the only one who can qualify the truth with the appropriate provisos, according to the difficulties met in the validation (unquoting) process. It seems only natural to ask this user not only to qualify but to *quantify* (with a number in e.g.[0,1]) what is the degree of credibility (or belief) (s)he assigns it. The user *can* usually do it consistently (this is the "rational" behavior studied by Ramsey [1926]), thereby defining (by de Finetti's [1937] theorem) an *additive* valuation. (S)he can always assign the sentence A the *value* $v(A)$ —or, as we will write hereafter, $[\![A]\!]$—; this value ("truth value" we will call it) may be computed in an unspecified way (by betting preferences, belief networks, simulation, statistical survey, or whatever) and based on *any* preferred interpretation of A, be it standard *probability* of A as an event, or Popper's *truth likelihood* or *truth content* of A as a proposition or theory (that, as Popper found in the 1930s, is negatively correlated with its probability), or its *partial truth* in Haack's [1974] sense, or Shafer's [1976] *belief* (and its dual, *plausibility*), or its *reliability*, or the *credibility* of –or the (user's) *belief* in– A, or whatever (provided the assignment is done consistently). $[\![A]\!]$ represents a rough index of the *confidence* we have in A being the case and –consequently– the force with which we feel we can *assert* A in a particular argument (or the assertiveness we can commit into it). This measure is always possible, provided the user is "rational" in Ramsey's sense (but "non-rational" measures are also possible: they merely give *non*-additive values; see the final section, on subadditivity). What is measured is the user's *belief* in and *commitment* to A: a zero value means that the premise is to be taken as false, 1 means that it is a true (and therefore fully assertable) premise or –more often– that it is to be *assumed true* (and fully endorsed), and $v(A) = 1 - \varepsilon$ ($\varepsilon > 0$) means that we can assert A but with some apprehension or risk (that we assume) ε.

Now, approached in a most general way, the problem to solve is as follows. Suppose we have the reasoning $A_1, A_2, \ldots \vdash B$. Suppose we assign degrees of confidence or assertiveness to the premises. The question is : what will be the effect of those degrees in the confidence or reliability of B ? (We would thus probe the argument's robustness.) And what if we vary our confidence levels in some premises? Can it happen that, though the reasoning may be formally *valid*, the reliability of B turns out to be zero (thus making the argument *unsound*)? Or does B maintain its "truth" (or reliability = 1) though all the premises get null confidence themselves (thus making B's truth independent from the premises)? This analysis, like the physicist's $\varphi(\varepsilon)$ estimation problem above, *is* a legitimate logician's concern. (It is what we proceed to develop in our *proof theory* below.)

We illustrate this with an example. This is the well-known sorites about bald men: "If a man with i hairs is not bald then a man with $i-1$ hairs is still not bald. Suppose a man has n hairs. Therefore, a man with 0 hairs is still not bald". Formally:

$$\begin{array}{l} A_i \rightarrow A_{i-1} \quad (i : 1, \ldots, n) \\ A_n \\ \hline A_0 \end{array}$$

This is a paradox because the reasoning is formally correct (it consists of merely n applications of the Modus Ponens rule), the $n+1$ premises are deemed flawless, but the conclusion is outright false (or, more precisely, a *contradictio in terminis*). Usually, it is the length of the argument that is put to blame. There is, however, a more concrete and satisfactory answer we can offer. The n premises $A_i \rightarrow A_{i-1}$ cannot obviously be asserted with the same assurance whatever the index value. That's why the argument fails: for low values of i the premises simply cannot be asserted, even if the rest can, so we can *never* have all premises asserted, and the reasoning is formally valid but vacuously so.

Formally, what happens is that the value $[\![A_i \rightarrow A_{i-1}]\!]$ decreases with i, so that when i is n (or even, say, around $n/2$ or $n/3$) it is 1 or very near 1, but when i approaches, say, $n/10$ —and surely when it becomes zero— the value of $A_i \rightarrow A_{i-1}$ (= the predisposition we have to assert it —or the willingness to assume the risk) comes down to an exceedingly low number. According to a simple proof theory (that we describe below), the conclusion A_0 has the same truth value, at best, as that lowest of numbers (and, thus, the reasoner would be willing to assert the conclusion just no more than he or she would willing to assert $A_1 \rightarrow A_0$).

Note that, though we use words such as 'belief' or 'commitment' as surrogates for truth, *truth* itself is not merely reducible to belief (or probability): compare de Finetti's well-known position ("probability tells us only what to expect, not what will actually be the case") with the more recent comments of Cohen [1990] (a probabilist), who admits that when we say that something is true with probability, say, .26, "this result tells us nothing about the truth" of the predicted fact or the postulated hypothesis, which will be –is, actually— true or not regardless of our (expectation-inducing) probability computations. This is not to say that Logic should continue to treat truth exclusively, as it now does. On the contrary, we contend that Logic, as it becomes a *general theory of rationality*, should center on a new object of study: the *assertive value* of sentences (that subsumes *truth* and which we will hereafter call –somewhat misleadingly– *truth value*), because this is what is really manipulated in arguments, and because this concept may let us analyze them in full generality, be it through weak premises, strong conclusions or argument robustness.

B.1. "TRUTH" VALUATIONS OVER SENTENCES

We assume we have a set $\mathcal{L}$ of sentences that form a Boolean algebra (with respect to the three connectives and two special sentences $\perp$ and $\top$). Now we have the whole Proof Theory of Sentential Logic by identifying the " $\vdash$ " order defined by the Boolean algebra with the deductive consequence relation. Thus the algebra of sentences we started with automatically becomes the Lindenbaum-Tarski algebra of all sentences modulo the interderivability relation " $\dashv\vdash$ " given by the $\vdash$ order (i.e. $A \dashv\vdash B$ iff $A = B$). We then assume that all sentences are valued in $[0,1]$, which we do in the standard way of a *normalized measure* $v : \mathcal{L} \longrightarrow [0,1] : A \mapsto [\![A]\!]$, by just requiring that $\top$ gets a value of 1 (1 is the only 'designated value' we consider) and that the valuation v is *additive* (i.e. $[\![A \vee B]\!] = [\![A]\!] + [\![B]\!] - [\![A \wedge B]\!]$). So we now have also the whole Model Theory of Sentential Logic.

This "truth" valuation is merely a (finitely aditive) *probability* in all technical senses, but here A is a *sentence* (in a language $\mathcal{L}$), not an *event* (in a sample space Ω). $[\![A]\!] = 1$, $[\![A]\!] = 0$ and $[\![A]\!] = 1/2$ here just mean –respectively– *truth* (or, more precisely, that "A is taken as truth"), *falsity* and *undecided belief* (when expressly *asserting* the A *sentence*); this is to be contrasted with (respectively) probabilistic "certainty", zero-probability or balanced odds (when evaluating the *uncertain* outcome of A as an event). We do not require that the valuations —even when interpreted fully as "truth" valuations— to be "extensional" or "truth-functional" as done in many-valued logics. As for the Booleanity of the sentences, either this is assumed (which is undemanding) or it derives from the "minimal algebra" of sentences suggested by Popper [1959] —or from provably equivalent simple assumptions (e.g. by Cox [1961] or Woodruff [1995]). Also, the additive character of the valuation amounts to having a 'rational' (Ramsey [1926]) or 'coherent' (De Finetti [1937,70]) *belief*, a concept so akin to 'strength of assertion' in Logic as to be all but exchangeable.

From the Booleanity of $\mathcal{L}$ and the above properties of the v valuation we immediately obtain:

$$[\![\neg A]\!] = 1 - [\![A]\!] \qquad (1)$$

$$[\![A \wedge B]\!] \leq \min([\![A]\!], [\![B]\!]) \qquad (2)$$

$$[\![A \vee B]\!] \geq \max([\![A]\!], [\![B]\!])$$

$$[\![A \rightarrow B]\!] = 1 - [\![A]\!] + [\![A \wedge B]\!]$$

$$[\![A \leftrightarrow B]\!] = 1 - [\![A \vee B]\!] + [\![A \wedge B]\!]$$

B.2. SENTENCES AS SET EXTENSIONS, AND TRUTH AS MEASURE

Any Boolean Algebra has a *representation* on a set structure (a field of sets) as Stone proved long ago in a famous theorem (see, for example, Koppelberg *et al.* [1989]). Thus, given the Boolean sentence algebra $\mathcal{L}$, there exist both a set Θ (whatever the meaning we give to its elements θ) *and* a 'representation' function that can be characterized as an isomorphism of $\mathcal{L}$ into the Boolean subalgebra $\mathcal{B}$ of clopens in $\mathcal{P}(\Theta)$, i.e.

$$\rho : \mathcal{L} \longleftrightarrow \mathcal{B} : A \mapsto \mathbf{A} \quad (\mathcal{B} \subset \mathcal{P}(\Theta), \mathbf{A} \subset \Theta)$$

We call the members of Θ *possible worlds*, or *cases* (as Laplace [1774] or Boole [1854]) or *possibilities* (Shafer [1976]) or even *observers*, *states*, etc. Θ is the *universe of discourse* or *reference frame* (the set of *possible worlds*). It coincides with Fenstad's [1968] *model space* (where the θs are *interpretations* in the standard logic sense).

We can establish a general, one-to-one correspondence between the two worlds (the language world $\mathcal{L}$ and the referential universe Θ, both made up of "propositions") and their constituent parts, thus:

$$\begin{array}{ccc}
\mathcal{L} & \Longleftrightarrow & \mathcal{B} \\
A\ (A \in \mathcal{L}) & \Longleftrightarrow & \mathbf{A}\ (\mathbf{A} \subset \Theta) \\
A \wedge B & \Longleftrightarrow & \mathbf{A} \cap \mathbf{B} \\
A \vee B & \Longleftrightarrow & \mathbf{A} \cup \mathbf{B} \\
\neg A & \Longleftrightarrow & \mathbf{A}^c \\
\top & \Longleftrightarrow & \Theta \\
\bot & \Longleftrightarrow & \emptyset \\
A \vdash B & \Longleftrightarrow & \mathbf{A} \subset \mathbf{B}
\end{array}$$

If $\mathcal{L}$ has a finite number of generators, then it has 2^n atoms a and the two bijective correspondences $\mathcal{L} \Longleftrightarrow \mathcal{P}(\Theta)$ and $a \Longleftrightarrow \{\theta\}$ also hold.

The valuation v and the representation isomorphism ρ induce a [0,1]-valued measure μ in $\mathcal{B} \subset \mathcal{P}(\Theta)$, in such a way that $\mu = v \circ \rho^{-1}$, i.e. $\mu(\mathbf{A}) = [\![A]\!]$.

Intuitively, the measure $\mu(\{\theta\})$ of each individual θ in a *finite* Θ universe is the relevance or the degree of realizability of the given possible world. The μ measure corresponds to the weighing function λ in Fenstad's [1968] *model space*. As it is known, μ (or λ) is not only additive but –by the compactness property– countably so; thus μ is eligible as a standard "probability" measure (in the technical sense).

B.3. CONNECTIVES AND SENTENTIAL STRUCTURE

If we want to compute the truth value of composite sentences, the task is easy. For the negation connective, the formula is given by (1) above. For the binary sentences composed of A and B we have the formulas below, where we observe that, besides $[\![A]\!]$ and $[\![B]\!]$, we now need a third parameter that we note " α_{AB} " and that we call "*compatibility* between A and B"; its value is defined by $\alpha_{AB} =_{df} 1 - \beta_{AB}$, where:

$$\beta_{AB} = \frac{\min\ ([\![A]\!],[\![B]\!])\ -\ [\![A \wedge B]\!]}{\min\ (\ [\![A]\!],\ [\![B]\!],\ 1-[\![A]\!],\ 1-[\![B]\!]\ )} \ .$$

We call β_{AB} the "*degree of incompatibility* of sentences A and B" and we rename the denominator by calling it "Δ_{AB}". Then, the formulas for the connectives are:

$$\boxed{\begin{aligned}
[\![A \wedge B]\!] &= \min\ ([\![A]\!],[\![B]\!])\ -\ \beta_{AB}\cdot\Delta_{AB} \\
[\![A \vee B]\!] &= \max\ ([\![A]\!],[\![B]\!])\ +\ \beta_{AB}\cdot\Delta_{AB} \\
[\![A \rightarrow B]\!] &= \min\ (1, 1-[\![A]\!]+[\![B]\!])\ -\ \beta_{AB}\cdot\Delta_{AB} \\
[\![A \leftrightarrow B]\!] &= 1\ -\ \mid [\![A]\!]-[\![B]\!] \mid\ -\ 2\cdot\beta_{AB}\cdot\Delta_{AB}
\end{aligned}}$$

So, by knowing a single value (either of $[\![A \wedge B]\!]$, $[\![A \vee B]\!]$, $[\![A \to B]\!]$, $[\![A \leftrightarrow B]\!]$, α_{AB} or β_{AB} —or $[\![A|B]\!]$ or $[\![B|A]\!]$, see below) we can compute, via α_{AB} (or β_{AB}), the other seven. The parameter α_{AB} (which is a modern version of Reichenbach's [1935b] "Kopplungsgrad") acts as an indicator or measure of the "relative position" of $\mathbf{A}$ and $\mathbf{B}$ inside Θ, while "Δ_{AB}" is a quadruple minimum that only depends on the values of $[\![A]\!]$ and $[\![B]\!]$. Note that if we suppose that $\beta_{AB} = 0$ for *all* A and B then the above formulas are

$$\begin{aligned}
[\![A \wedge B]\!] &= \min\,([\![A]\!], [\![B]\!]) \\
[\![A \vee B]\!] &= \max\,([\![A]\!], [\![B]\!]) \\
[\![A \to B]\!] &= \min\,(1, 1 - [\![A]\!] + [\![B]\!]) \\
[\![A \leftrightarrow B]\!] &= 1 - |[\![A]\!] - [\![B]\!]|
\end{aligned}$$

and coincide with those given in ordinary *many-valued logics*. Instead, if $\beta_{AB} = 1$, the connectives are

$$\begin{aligned}
[\![A \wedge B]\!] &= \max\,(0, [\![A]\!] + [\![B]\!] - 1) \\
[\![A \vee B]\!] &= \min\,(1, [\![A]\!] + [\![B]\!]) \\
[\![A \to B]\!] &= \max\,(1 - [\![A]\!], [\![B]\!]) \\
[\![A \leftrightarrow B]\!] &= |[\![A]\!] + [\![B]\!] - 1|
\end{aligned}$$

and coincide with those given in *threshold logics*.

B.4. RELATIVE TRUTH

Now suppose we want to express the conjunction value as a product:

$$[\![A \wedge B]\!] = [\![A]\!] \cdot \tau$$

With the current $\mathcal{L}/\mathcal{P}(\Theta)$ representation in mind, we obtain:

$$\tau = \frac{\mu(\mathbf{A} \cap \mathbf{B})}{\mu(\mathbf{A})} \tag{3}$$

We define τ as the *relative truth* " $[\![B|A]\!]$ " (i.e. the "truth of B relative to A"). Though this definition exactly parallels that of conditional probability, the account we give leaves out any probabilistic interpretation of the concept and retrieves it for exclusively logical contexts.

In particular, if $[\![B|A]\!] = [\![B]\!]$ then we say that A and B are *independent.* In that case, the conjunction can be expressed as the product:

$$[\![A \wedge B]\!] = [\![A]\!] \cdot [\![B]\!]$$

In any other case we say that A and B are mutually *dependent* and speak of the *relative truth* of one with respect to the other. Note the dependence goes both ways and the two situations are symmetric. We have (assuming $[\![A]\!] \neq 0$):

$$[\![A]\!] \cdot [\![B|A]\!] = [\![B]\!] \cdot [\![A|B]\!] = [\![A \wedge B]\!] \quad \text{(a logical \textit{``Bayes formula''})}$$

$$[\![B|A]\!] = 1 - \frac{1 - [\![A \to B]\!]}{[\![A]\!]} \quad .$$

From the latter, note that, in general,

$$[\![B|A]\!] \neq [\![A \to B]\!]$$

Particularly, we have *always*

$$[\![B|A]\!] \;<\; [\![A \rightarrow B]\!]$$

except when either $[\![A]\!] = 1$ or $[\![A \rightarrow B]\!] = 1$, in which cases (and they are the *only* ones) $[\![B|A]\!] = [\![A \rightarrow B]\!]$. (These facts have been repeatedly noticed by many people, notably by Reichenbach [1935b], Popper [1959], Stalnaker [1970] or Lewis [1976].)

When two sentences A and B are *independent* then (and this is a necessary and sufficient condition for that to happen):

$$\begin{aligned} \alpha_{AB} &= \max\,([\![A]\!], [\![B]\!]) \quad\;\; \text{if } [\![A]\!] + [\![B]\!] \leq 1 \\ &= \max\,([\![\neg A]\!], [\![\neg B]\!]) \;\; \text{if } [\![A]\!] + [\![B]\!] \geq 1 \end{aligned}$$

and then the connectives obey the formulas

$$\begin{aligned} [\![A \wedge B]\!] &= [\![A]\!] \cdot [\![B]\!] \\ [\![A \vee B]\!] &= [\![A]\!] + [\![B]\!] - [\![A]\!] \cdot [\![B]\!] \\ [\![A \rightarrow B]\!] &= 1 - [\![A]\!] + [\![A]\!] \cdot [\![B]\!] \;. \end{aligned}$$

The statement '$A \rightarrow B$' can have, among other readings, one logical ("A is sufficient for B" or "B is necessary for A"), another (loosely) "causal" ("A occurs and B follows"). But because $A \rightarrow B$ is valued in [0,1], its value $[\![A \rightarrow B]\!]$ (and the values $[\![B|A]\!]$ and $[\![A|B]\!]$) now mean only *degrees*, and so $B \rightarrow A$ may be —and usually is— read "evidentially" ("B is evidence for A"). Within such a frame of mind,

— $[\![B|A]\!]$ (or "$\sigma_{A(B)}$") could be termed "degree of *sufficiency* or *causality*" of A (or "*causal support* for B"), to be read as "degree in which A is sufficient for B" or "degree in which A is a cause of B". In view of (3), it is roughly a measure of how much of **A** is contained in **B**.

— $[\![A|B]\!]$ (or "$\nu_{A(B)}$") could be termed "degree of *necessity*" or "*evidence*" of A (or "*evidential support* for A", to be read as "degree in which A is necessary for B" or "degree in which B is evidence (= support of hypothesis) for A (=the hypothesis)". With (3) in mind, it can be seen as how much of **B** overlaps with **A**.

Such measures may be directly estimated by experts, normally by interpreting the θs frequentially, in terms of *cases*, like Boole [1854], *possibilities* (Shafer [1976]), *elementary events* in Θ, or possible *interpretations*; at any rate, they may be statistically-based or simply imagined, presumably on the basis of past experience or sheer plausibility. Thus, $\sigma_{A(B)}$ in a causal reading of "$A \rightarrow B$" would be determined by answering the question: "How many times (proportionally) —experience shows— A occurs and B follows?" For $\nu_{A(B)}$, the question would be: "How many times effect B occurs and A has occurred previously?" (Similarly for the evidential reading of "$A \rightarrow B$".) Once σ and ν have been guessed, they may be adjusted (via the

$$\frac{\sigma_{A(B)}}{\nu_{A(B)}} = \frac{[\![B]\!]}{[\![A]\!]}$$

relation) and then lead —by straightforward computation— to $[\![A \rightarrow B]\!]$, $[\![B \rightarrow A]\!]$ and α_{AB}, which allows one to compute all other values for connectives and also to get a picture of the structural relations linking A and B. This process of eliciting the σ value for every $\langle A, B\rangle$ sentence pair is closely equivalent to computing Popper's *binary probability* function.

Note first that a similar computing process takes place in "Bayesian reasoning", and in the "approximate reasoning" methods as implemented in marketable expert systems, though none of them satisfactorily explains on what *logical* grounds are the procedures justified, nor can they avoid supplying a (nominally) *probabilistic* account of them. Such an account would be here clearly misplaced, since there is usually neither (a) a sample space of *events*, but a language of *sentences* (i.e. linguistic descriptions —not necessarily of any "event"), nor (b) a measure based on *uncertainty* and *outcomes* (the topics Probability is supposed to deal with) but rather simple *beliefs* or, at most, mere *a priori* estimates, nor (c) an adequate updatable statistical or probabilistic basis to compute the values of the "events" (and their ongoing, dynamical change).

Note also that any reasoning can proceed here in both directions (from A to B and from B to A), because both conditionals $A \to B$ and $B \to A$ claim a non-zero value and thus a "causal" top-down reasoning can be complemented by an "evidential" bottom-up reasoning on the same set of given sentences (as also happens in Pearl's [1988] belief networks).

B.5. THE GEOMETRY OF LOGIC: DISTANCE, TRUTH LIKELIHOOD, INFORMATION CONTENT AND ENTROPY IN $\mathcal{L}$

The fact that we have:

$$[\![A \leftrightarrow B]\!] = 1 - ([\![A \vee B]\!] - [\![A \wedge B]\!])$$

strongly suggests using $1 - [\![A \leftrightarrow B]\!] = [\![A \vee B]\!] - [\![A \wedge B]\!]$ as a measure of the *distance* $\overline{AB}$ (under a given valuation v). So we do. (We remark that all definitions we give here of distance and related concepts are not only applicable to sentences but to *theories* as well, because for a general lattice $\mathcal{L}$ the lattice $\hat{\mathcal{L}}$ of theories derived from each sentence in $\mathcal{L}$ is isomorphic to $\mathcal{L}$.)

[*Definition 1*:] *Distance* (or *Boolean distance*) between two sentences or theories A and B is:

$$d(A,B) =_{df} 1 - [\![A \leftrightarrow B]\!] \;=\; [\![A \vee B]\!] - [\![A \wedge B]\!] \;=\; |[\![A]\!] - [\![B]\!]| + 2 \cdot \beta_{AB} \cdot \Delta_{AB}$$

[*Definition 2*:] *Compatible distance* between two sentences or theories A and B is:

$$d^{+}(A,B) =_{df} |[\![A]\!] - [\![B]\!]| \;=\; [\![A]\!] + [\![B]\!] - 2 \min([\![A]\!],[\![B]\!])$$

We now define a *truth likelihood* value for A —approximating Popper's [1972] (and Miller's [1978]) *truth likelihood* or *verisimilitude* measure— by making it to equal the distance from A to falsehood, i.e. $d(A,\bot)$. We obtain, immediately:

$$d(A,\bot) = d(\top,\bot) - d(\top,A) = 1 - d(A,\top) = 1 - d(A \triangle \top,\bot) = 1 - d(\neg A,\bot) = 1 - [\![\neg A]\!] = [\![A]\!]$$

So here we have a further interpretation of our "truth values" $[\![A]\!]$ in terms of Popper's [1972] *truth likelihood* or *verisimilitude*. We might as well consider $[\![A]\!]$ as a rough measure of *partial truth* or *truth content* of A. In a similar vein, we may recall that Scott [1973] suggested the "truth value" $[\![A]\!]$ of many-valued logics could be interpreted as *one* (meaning *truth*) less the *error* of A (or rather of a measure settling the truth of A) or the *inexactness* of A (as a theory); in this framework, it comes out that, in our terms, $[\![A]\!] = 1 - \varepsilon_A$ and $\varepsilon_A = 1 - [\![A]\!] = d(A,\top)$.

We further observe that, for any sentential letters P and Q, any uniform truth valuation yields $[\![P]\!] = [\![\neg P]\!] = .50$, $[\![P \wedge Q]\!] = .25$ and $[\![P \vee Q]\!] = .75$, which is like saying that, if all letters are equiprobable, the given values are the probability of the given sentence being true (a number that Johnson-Laird [1975,83] calls, appropriately, "truth-table probability"). This value's complement to one is reasonably made to correspond to the amount of information — in a loose sense— we have when the sentence is true. This is precisely what Johnson-Laird [1975,83] defines as "degree of information", "semantical information" or *informativeness* $I(A)$ of a sentence A. (Viewed in our terms, this *information content* $I(A)$ equals $1 - [\![A]\!]$, or $I(A) = [\![\neg A]\!] = d(A,\top) = \varepsilon_A$.) The concept, based on Bar-Hillel's (and Carnap's,[1952]) ideas, was originally designed to model the reasoning process, assumed to be driven by an increase both of informativeness and parsimony. What is interesting is that the informativeness of composite sentences is computed by combining them according to non-truth-functional rules which yield values that coincide with those predicted by our formulas.

A new measure we can define, which can also be used as an *entropy* (in the sense of De Luca & Termini [1972]), is this one:

[*Definition*:] *Imprecision* (or, perhaps, "fuzziness") of a sentence (or theory) A is the value for A of the function

$$f : \mathcal{L} \longrightarrow [0,1] \quad \text{such that} \quad f(A) = 1 - d^{+}(A, \neg A)$$

It is immediate that:

$$f(A) = 2 \min ([\![A]\!], 1 - [\![A]\!]) \ ,$$

which is equivalent to saying that the imprecision of a sentence A equals twice the error we make when we evaluate on A the truth of the law of non-contradiction or of the excluded middle by considering there really *is* maximum compatibility between A and $\neg A$. (Actually, there is *null* compatibility, as α_{AB} is provably *zero* when B is $\neg A$.) In connection with this measure, we note in passing that:

– *classical* (two-valued) *logic* is the special case of ours in which *all* sentences in $\mathcal{L}$ have *zero* imprecision.

– ordinary *multi-valued logics* —like Lukasiewicz-Tarski's L_∞— are the ones where at least *one* sentence in $\mathcal{L}$ has non-zero imprecision. This measure being —as it is— an error function, imprecision is here just the degree in which these logics fail to distinguish contradictions (their lack of "resolving power") .

B.6. ELEMENTARY PROOF THEORY FOR GENERAL ASSERTIONS

Once we have valued sentences in $[0,1]$ we now need a Proof Theory that in the most natural way extends standard logic so as to treat imprecise statements or weak assertions, and measure and control whatever effect they may have on reasoning (as well as to explain some results in approximate reasoning methods from Artificial Intelligence).

To begin with, suppose a valid argument, noted $\Gamma \vdash B$ (where Γ are the premises, or a finite subset of them). Classical logic declares it *valid* if B is derivable from Γ in an appropriate deduction calculus. By the completeness property, this amounts to assert the truth of B whenever the premises in Γ are true. Now, as we said, the ultimate judge of the truth of the premises is the reasoner. It is the reasoner who decides that each premise used is true (or to be considered true). To justify such a decision, the reasoner applies a truth criterion such as Tarski's [1935b] (T) schema. Thus the reasoner declares A true when assured that what A describes is precisely the case. If the reasoner is not sure of the result of his/her validation or does not want to commit him/herself to it, then the reasoner may choose not to make a full assertion by claiming that A's verification does not yield an obvious result. In that case, the reasoner may rather easily "qualify" the assertion by assigning numbers in $[0,1]$ such as $v(A)$ —that we noted "$[\![A]\!]$"— or $\varepsilon(A)$ [$= 1 - v(A)$] meaning that the reasoner believes or is willing to assert A to the degree $v(A)$ or assume it with a risk or estimated error of $\varepsilon(A)$.

The proof theory we now sketch is a slightly extended version of the standard one. Here we understand by *proof theory* the usual syntactical deduction procedures *plus* the computation of numerical coefficients that we must perform alongside the standard deductive process. We do that because a final value of zero for the conclusion would invalidate the whole argument as thoroughly as though the reasoning were formally —syntactically— invalid. As always, any formally valid *argument* will have, by definition, the following *sequent* form:

$$\Gamma \vdash B$$

where B is the conclusion and Γ stands for the list —or, rather, the conjunction— of the premises (or, by the compactness property, of a finite number of them). We have, elementarily:

$$\Gamma \vdash B \quad \Rightarrow \quad [\![\Gamma]\!] \leq [\![B]\!]. \tag{4}$$

We henceforth assume that we have a *valid argument* (so $\Gamma \vdash B$ will always hold), and that all premises are non-zero (i.e. $\forall i\ [\![A_i]\!] > 0$). We distinguish four possible cases:

1) $[\![\Gamma]\!] = 0$ (i.e. the premises are –materially– *inconsistent*). Here by (4) $[\![B]\!]$ can be anywhere between 0 and 1; this value is in principle undetermined, and uncontrollably so.

2) $[\![B]\!] = 0$. This entails, by (4), $[\![\Gamma]\!] = 0$ and we are in a special instance of the previous case. The reasoning is formally valid, the premises are not asserted, and the conclusion is false.

3) $[\![\Gamma]\!] \in (0, 1)$ (i.e. the premises are consistent). Then, by (4), $[\![B]\!] > 0$. We have a formally valid argument, we risk assessing the premises (though with some apprehension) and get a conclusion which can be effectively asserted though by assuming a –bounded– risk. This will be the case we will set to explore below.

4) $[\![\Gamma]\!] = 1$. This condition means that $[\![A_1]\!] = \ldots = [\![A_n]\!] = 1$ and, by (4), $[\![B]\!] = 1$. So the premises are *all* asserted —with no risk incurred— and the conclusion *holds* inconditionally (remember $\Gamma \vdash B$ is formally valid). This is the classical case studied by ordinary two-valued Logic.

We are interested in examining *case 3* above, i.e. formally valid reasoning *plus* assertable premises (though not risk-free assertions) *plus* assertable conclusion (but at some measurable cost). Cases 3 and 4 characterize in a most general way all *sound* reasoning. Case 2 characterizes *unsound* arguments (since in this case having a formally valid argument $\Gamma \vdash B$ does not preclude getting an irrelevant conclusion ($[\![B]\!] = 0$). As this case is the one to avoid, we have:

[*Definition:*] *Unsoundness* of a valid argument $\Gamma \vdash B$ is having $[\![B]\!] = 0$ though the premises are themselves non-zero. So:

[*Definition:*] *Soundness* of a valid argument $\Gamma \vdash B$ is having $[\![B]\!] > 0$ whenever the premises are non-zero.

With that in mind, we can now turn to the basic inference rule, the Modus Ponens (MP). From a strictly logic point of view, this rule is

$$\begin{array}{ll} A & m \\ A \to B & n \\ \hline B & p \end{array} \tag{5}$$

where m, n and p stand for the strength or force (or "truth value") we are willing to assign each assertion; so, in our terms, m, n and p are just our $[\![A]\!]$, $[\![A \to B]\!]$ and $[\![B]\!]$. They are numbers in [0,1] that take part in a (numerical) computation which parallels and runs along the logical, purely syntactical deduction process. This is well understood and currently exploited by reasoning systems in Artificial Intelligence that must rely on numerical evaluations —given by users— that amount to *credibility* assignments (or "certainty factors"), *belief* coefficients, or even —rather confusingly— *probabilities* (often just *a priori* probability estimates); this is the case of successful *expert systems* such as Prospector or Mycin. The trouble with such systems is that they tend to view Modus Ponens as a probability rule (this is made explicit in systems of the Prospector type, see Duda *et al.* [1976]). They use it to present the MP rule in this way:

$$\begin{array}{l} A(m) \\ A \rightarrow B(\sigma) \\ \hline B(p) \end{array} \qquad (6)$$

where m and p are the 'probability' (a rather loose term here) of A and B, and "$A \rightarrow B(\sigma)$" means that "whenever A happens, B happens with probability σ". Here σ turns out to be just $v(B|A)$ or "$[\![B|A]\!]$", the "relative truth" of B given A; this concept, modelled on a close analogy —*ceteris paribus*— with that of (probabilistic) conditioning, is what we have defined above (in (3)) and called "degree of sufficiency" σ of A —or of necessity of B—, assumed easily elicitable by experts. So it is just natural, and immediate, to compute the p value thus:

$$p \geq \sigma \cdot m$$

or, in our notation,

$$[\![B]\!] \geq [\![B|A]\!] \cdot [\![A]\!]$$

which is just another version of formula (2).

The problem is that what we have, from our purely logical, probability-rid standpoint, is (5), not (6), and in (5) n is not $[\![B|A]\!]$ but $[\![A \rightarrow B]\!]$. Recall that $[\![B|A]\!]$ and $[\![A \rightarrow B]\!]$ not only do *not* coincide but mean different things (as repeatedly noticed by logicians, and as explained above). Indeed, $[\![A \rightarrow B]\!]$ is the value ("truth" we may call it, or "truth minus risk") we assign to the (logical) assertion $A \rightarrow B$. Instead, $[\![B|A]\!]$ is a relative measure linking materially, *factually*, A and B (or, better still, the **A** and **B** sets), with no concern whether a true *logical* relation between them exists; we might even have $[\![B|A]\!] < [\![B]\!]$, thereby indicating there exists an *anti*correlation (thus rather contradicting any —logical or other— reasonable kind of relationship between A and B). So we turn back to our (5) rule; note that $m + n \geq 1$ (this *always* holds) and that $[\![B|A]\!]$ can be obtained from $[\![A \rightarrow B]\!]$, or vice versa: $[\![A \rightarrow B]\!]$ from $[\![B|A]\!]$ through

$$[\![A \rightarrow B]\!] \;=\; 1 - [\![A]\!] \cdot (1 - [\![B|A]\!]) \qquad (7)$$

(which is useful, since $[\![B|A]\!]$ is directly obtainable from experts).

The following theorem states the *soundness* condition for the MP rule.

The Modus Ponens rule

$$\begin{array}{ll} A & m \\ A \rightarrow B & n \\ \hline B & p \end{array} \qquad \text{(we assume } m \text{ and } n \text{ are both non-zero)}$$

is *sound* (and thus $[\![B]\!] \neq 0$) if one of these four equivalent conditions hold:

1) $m + n > 1$

2) $[\![B|A]\!] > 0$

3) $[\![A \wedge B]\!] > 0$

4) Either $[\![A]\!] + [\![B]\!] > 1$ (and thus $[\![B]\!] > 1 - m$) or both A and B are compatible ($\alpha_{AB} > 0$) and not binary-valued.

In both sound and unsound cases we have the following easily computable bounds for the value $[\![B]\!]$ of the MP conclusion (Sales [1992,96]):

$$[\![A]\!] + [\![A \rightarrow B]\!] - 1 \quad \leq \quad [\![B]\!] \quad \leq \quad [\![A \rightarrow B]\!] \tag{8}$$

or equivalently, in shorter notation:

$$m + n - 1 \ \leq \ p \ \leq \ n \ .$$

(Such bounds have been discovered again and again by quite diverse authors; see e.g. Genesereth & Nilsson [1987]). The lower bound —which equals $[\![A \wedge B]\!]$ — is reached when $\alpha_{AB} = 1$ and $[\![A]\!] \geq [\![B]\!]$, while the upper bound is reached when $\alpha_{AB} = 0$ and $[\![A]\!] + [\![B]\!] \geq 1$. Naturally we usually know neither $[\![B]\!]$ nor α_{AB} beforehand, so we don't know whether the actual value $[\![B]\!]$ reaches either bound or not, nor which is it; we can merely locate $[\![B]\!]$ inside the $[m + n - 1, n]$ interval.

But this interval can be narrowed. Since a very reasonable constraint a conditional $A \rightarrow B$ may be expected to fulfill is that A and B be (assumedly) non-independent –and not binary– and *positively* correlated (i.e.: $[\![A \wedge B]\!] > [\![A]\!] \cdot [\![B]\!]$), so we have:

$$[\![A]\!] + [\![A \rightarrow B]\!] - 1 \quad \leq \quad [\![B]\!] \quad < \quad [\![B|A]\!]. \tag{9}$$

Here, if A and B are fully or strongly compatible, $[\![B]\!]$ will be nearer the lower bound. Thus, we can only increase our $[\![B]\!]$ if we are assured that A and B are independent (in the sense that $[\![A \wedge B]\!]$ equals $[\![A]\!] \cdot [\![B]\!]$, see above): we then obtain the highest value $[\![B]\!] = [\![B|A]\!]$. On the other hand, the more we confide instead in a strong logical relation between A and B, the more we should lean towards the low value given by

$$[\![B]\!] = [\![A \wedge B]\!] = m + n - 1. \tag{10}$$

Under very reasonable elementary hypotheses (like this one: $[\![A \rightarrow B]\!] > [\![A \rightarrow \neg B]\!]$ or, equivalently, $\sigma_{A(B)} > 1/2$, see Sales [1996]), we easily get these bounds for $[\![B]\!]$: $[\![A]\!]/2 \ < \ [\![B]\!] \ \leq \ [\![A]\!]$.

Now, if what we want is not an interval, however narrowed, but a precise value for $[\![B]\!]$ we should favor the lower value, the one given by (10) above. There are lots of reasons (some mentioned in Sales [1996]) for this choice of value in the absence of more relevant information.

But if we want not merely a pair of bounds —or a favored lower bound— for the conclusion B of an MP but the *exact* value $[\![B]\!]$, the obvious candidate formula for this follows easily: suppose we are given not only $[\![B|A]\!]$ but also $[\![A|B]\!]$ (that we note by σ and ν) and we assume them estimated by experts. We then formulate MP as

$$\begin{array}{l} A(m) \\ A \rightarrow B(\sigma, \nu) \\ \hline B(p) \end{array}$$

which is exactly (6) except that the conditional has prompted evaluation of relative truths of A and B in both directions. The value is computable at once from the above definition of $[\![A|B]\!]$:

$$[\![B]\!] \ = \ \frac{[\![B|A]\!] \cdot [\![A]\!]}{[\![A|B]\!]} \qquad \text{or} \qquad p \ = \ \frac{\sigma \cdot m}{\nu}.$$

Note that the above logical formula coincides formally with Bayes's theorem —whence the adjective ("Bayesian") for any calculus that uses it— except that it deals not with hard-to-compute probabilities of events but with beliefs (or assertion strengths) of sentences. Note also that this value is the one that some *approximate reasoning* systems (e.g. Prospector) unqualifiedly assign to $[\![B]\!]$ supposedly on purely probabilistic grounds —and falsely assuming that $[\![B|A]\!]$ is the same as $[\![A \to B]\!]$—; see, for instance, the Genesereth & Nilsson [1987] text, where the logical equation above is said to be Bayes's formula.

If we wanted the MP presented in the more traditional *logical* way (5), first we would directly estimate the truth value $[\![A \to B]\!]$ of the conditional, or compute it from σ through (7) —or both, and use each estimate as a cross-check on the other—, so we would now have, along with the expert guess of ν:

$$
\begin{array}{ll}
A & m \\
A \to B & n\ (\nu) \\
\hline
B & p
\end{array}
$$

(where $n = 1 - m \cdot (1-\sigma)$), and so

$$[\![B]\!] \; = \; \frac{[\![A \wedge B]\!]}{[\![A|B]\!]} \; = \; \frac{m+n-1}{\nu} \tag{11}$$

that naturally fits the (9) bounds (when ν runs along from 1 to $[\![A]\!]$).

B.7. THREE COROLLARIES AND ONE EXTENSION

A few remarks can be made on some apparent advantages of the "logic-as-truth-valuation" approach that, following the advice of Popper *et alii*, we have advocated and described above. First, we mention three well-known fields that have been traditionally perceived as separate but that now automatically become special cases of a single formulation. Then, in subsection *b*, we hint at an obvious extension of the approach.

a. The special cases

1) *Classical* (two-valued) *logic* is the special case of our general logic in which *every* sentence is *binary* (i.e. $\forall A \in \mathcal{L} \quad [\![A]\!] \in \{0,1\}$) or, equivalently, in which *every* sentence has *zero* imprecision (i.e. $\forall A \in \mathcal{L} \quad f(A) = 0$).

2) For *three-valued logics* first note that classical examples, especially Kleene's system of strong connectives [1938] and Łukasiewicz's [1920] L_3, give the following tables for the values of connectives (where U stands for "undetermined"):

$\wedge$	$0 \quad U \quad 1$		$\vee$	$0 \quad U \quad 1$		$\to$	$0 \quad U \quad 1$
0	$0 \quad 0 \quad 0$		0	$0 \quad U \quad 1$		0	$1 \quad 1 \quad 1$
U	$0 \quad X \quad U$		U	$U \quad Y \quad 1$		U	$U \quad Z \quad 1$
1	$0 \quad U \quad 1$		1	$1 \quad 1 \quad 1$		1	$0 \quad U \quad 1$

with $X = Y = U$ in both Kleene's and Łukasiewicz's tables, and $Z = U$ in Kleene's (but $Z = 1$ in Łukasiewicz's).

Now, if we abbreviate the "$[\![A]\!] \in (0,1)$" of our general logic by "$[\![A]\!] = U$" (as in Kleene or Łukasiewicz) the values given in the above tables coincide exactly with those that would have been computed by our formulas, *except* that X, Y and Z would remain undetermined until we knew α_{AB}. In general, our values would match Kleene's, but in certain cases they would yield differing results:

1) If $[\![A]\!] + [\![B]\!] \leq 1$ and A and B are *incompatible* (typically because $\mathbf{A} \cap \mathbf{B} = \emptyset$) then $X = 0$.

2) If $[\![A]\!] + [\![B]\!] \geq 1$ and A and B are *incompatible* (typically because $\mathbf{A} \cup \mathbf{B} = \Theta$) then $Y = 1$.

3) If $[\![A]\!] \leq [\![B]\!]$ and A and B are *compatible* (typically because $\mathbf{A} \subset \mathbf{B}$) then $Z = 1$.

Note that in the particular case in which B is $\neg A$ we have *always* $[\![A \wedge \neg A]\!] = 0$ and $[\![A \vee \neg A]\!] = 1$ for any valuation, so that, for instance, the three classical Aristotelian principles ($\vdash A \rightarrow A$, $\vdash \neg(A \wedge \neg A)$ and $\vdash A \vee \neg A$), which do not hold in these logics (except that the first one does in L_3), do now hold in ours. These three results are perfectly classical and in full agreement with what is to be expected from a (Boolean) logic.

3) *Łukasiewicz & Tarski*'s [1930] L_∞ logic, as ours, generalizes classical (two-valued) logic in the sense that it allows the members of the sentential lattice to take values in [0,1] other than 0 or 1. Both systems of logic include classical two-valued logic as a special case. Nevertheless, while L_∞ renounces *Booleanity*, we renounce *functionality* (though not quite, since each connective is actually truth-functional in *three* arguments: $[\![A]\!]$, $[\![B]\!]$ and a third parameter such as, e.g., α_{AB}). In fact, if the $\{0,1\}$ truth set is extended to [0,1] those two properties of classical logic cannot be both maintained, and one must sacrificed; we find much easier to justify logically, and more convenient, the sacrifice of truth-functionality.

Our general logic admits L_∞ as a special case since, indeed, L_∞ behaves exactly as ours would do if *no* sentence in $\mathcal{L}$ could be recognized as a *negation* of some other and, then, it would be assigned systematically the maximum compatibility ($\alpha = 1$) connective formulas. Naturally that would give an error in the values of composite sentences involving non-fully-compatible subsentences, but it would also restore the lost truth-functionality of two-valued logic. L_∞ amounts —from our perspective— to viewing *all* sentences as having *always* maximum mutual compatibility. That means that L_∞ conceives *all* sentences as *nested* (i.e. for every A and A', either $\mathbf{A} \subset \mathbf{A}'$ or $\mathbf{A}' \subset \mathbf{A}$) (Sales [1994]). Such a picture is strongly reminiscent of Shafer's [1976] description of conditions present in what he calls *consonant* valuations, and which entails the fiction of a *total*, linear order $\vdash$ in $\mathcal{L}$ —and $\subset$ in $\mathcal{P}(\Theta)$— (that means a coherent, negationless universe) ...which is probably the best assumption we can make when information on sentences is lacking and negations are not involved —or cannot be identified as such. (Such an option is as legitimate as that of assuming, in the absence of information on sentences, that these are independent.) In L_∞ obvious negations may be a problem, but it can be solved by applying error-correcting *supervaluations* (van Fraassen [1968]) —that our logic supplies automatically. And note that, in particular, the error incurred in by L_∞ when failing to distinguish between a sentence and its negation —thus not being able to recognize a contradiction— is just the quantity we called *imprecision* (or curiously, for the historical record, what Black called *vagueness* and defined formally as we did with imprecision, see Sales [1982b]).

b. Ignorance as subadditivity

If we now suppose that $\mathcal{L}$ is still in a Boolean algebra but the v valuation we impose on $\mathcal{L}$ is *subadditive*, i.e.:

$$\{\![A \wedge B]\!\} + \{\![A \vee B]\!\} \geq \{\![A]\!\} + \{\![B]\!\} \quad \textit{(Subadditivity)}$$

(where we note explicitly by the $\{\![\ ]\!\}$ brackets that v is subadditive), then v can be characterized as a *lower probability* (see Good [1962]) or as a *belief* $Bel(A)$ (in Shafer's [1976] sense). If the inequality sign were inverted, the valuation would become an *upper probability* or (Shafer's) *plausibility* $Pl(A)$. The defective value —the part of the value $v(A)$ attributed neither to A nor to $\neg A$— can be expressed as:

$$\partial(A) \;=\; 1-(\{\!\![A]\!\!\}+\{\!\![\neg A]\!\!\}) \;=\; (1-\{\!\![\neg A]\!\!\})-\{\!\![A]\!\!\} \;=\; Pl(A)-Bel(A)$$

In the finite $\mathcal{L}$ case we know (Shafer [1976]) that a subadditive valuation like $v:\mathcal{L}\longrightarrow[0,1]:A\mapsto\{\!\![A]\!\!\}$ —or, better, the measure $\mu:\mathcal{P}(\Theta)\longrightarrow[0,1]:\mathbf{A}\mapsto\mu(\mathbf{A})$ induced on $\mathcal{P}(\Theta)$ by that valuation— defines a function $m:\mathcal{P}(\Theta)\longrightarrow[0,1]$ (called "*basic assignment*" by Shafer) that satisfies:

1. $m(\vee)=0$
2. $\sum_{\mathbf{A}\subset\Theta}\; m(\mathbf{A})=1$

so that the $\mu(\mathbf{A})$ ($=\{\!\![A]\!\!\}$) values are computed from this measure through

$$\{\!\![A]\!\!\}=\textstyle\sum_{\mathbf{B}\subset\mathbf{A}}\; m(\mathbf{B})$$

and, conversely, the $m(\mathbf{A})$ values can be obtained from $\{\!\![A]\!\!\}$ ($=\mu(\mathbf{A})$) through

$$m(\mathbf{A})=\textstyle\sum_{\mathbf{B}\subset\mathbf{A}}\;(-1)^{|\mathbf{A}-\mathbf{B}|}\,\mu(\mathbf{B})\qquad\text{for any } \mathbf{A}\subset\Theta\ .$$

$Bel(A)$ and $Pl(A)$ happen to coincide with the traditional concept (in Measure Theory) of *inner measure* (P_*) and *outer measure* (P^*), so that the following chain of equivalences has a transparent meaning (notice Shafer calls $Bel(\neg A)$ "degree of doubt of A"):

$$Pl(A)=P^*(\mathbf{A})=\textstyle\sum_{\mathbf{B}\cap\mathbf{A}\neq\phi}m(\mathbf{B})=\sum_{\mathbf{B}\subset\Theta}m(\mathbf{B})-\sum_{\mathbf{B}\subset\mathbf{A}^c}m(\mathbf{B})=1-P_*(\mathbf{A}^c)=1-Bel(\neg A)$$

We know, also, that an *additive* valuation μ is just a basic assignment m such that $m(\mathbf{A})=0$ for all $\mathbf{A}\subset\Theta$ *except* for the singletons $\{\theta\}$ of Θ. This is what Shafer calls '*Bayesian* belief'.

Subadditive belief derives from "non-rational" valuations of evidence by a reasoner (in the Ramsey/de Finetti sense). It can model and explain situations like this classic result: Confronted with the question "Should the Government allow public speeches against democracy?", one user assented 25% of the time. Substituting the word "prohibit" for "allow" elicited a 54% of assenting responses. Since both words are antonyms (the contrary of prohibiting is allowing), it is clear that this user had an unattributed gap left between those two complementary concepts, thus:

$$\{\!\![A]\!\!\}+\{\!\![\neg A]\!\!\}=.25+.54\leq 1$$

which reveals that sentence A (=speeches allowed) was being valued *subadditively*, and also that "allow" and "fail to prohibit" are here analyzable, in Shafer's terms, as $Bel(A)$ and $Pl(A)$, respectively, with values .25 (for belief) and .46 (for plausibility).

Ignorance (or, rather *total ignorance*) is the particular instance of subadditive valuation in which all non-*true* sentences get the zero value, i.e.

$$\{\!\![A]\!\!\}\;=1\quad\text{iff } A=\top\ ,\qquad \{\!\![A]\!\!\}\;=0\text{ otherwise}$$

(this is what Shafer calls '*vacuous* belief function'). It is just the particular instance of valuation in which all non-*true* sentences get the zero value: indeed, we have, for a given A , $\{\!\![A]\!\!\}\;=\;\{\!\![\neg A]\!\!\}\;=\;0$. In a strict parallel with *total ignorance*, subadditive valuations can also adequately formalize *total certainty* (meaning that $\{\!\![A]\!\!\}=1$ while at the same time $\{\!\![B]\!\!\}=0$ for all $B\vdash A$).

Subadditivity enables us to analyze other interesting situations related to Logic. For instance, suppose we have a subadditive valuation assigning A the value $\mu(\mathbf{A})=\{\!\![A]\!\!\}=Bel(A)$, and that a set $\mathbf{A}'$ (not necessarily a subset of $\mathbf{A}$) can be found in $\mathcal{P}(\Theta)$ such that there is an *additive* valuation which assigns $\mathbf{A}'$ precisely the same value. We denote $\mathbf{A}'$ by "$\Box\mathbf{A}$" (where $\Box$ is a set-operator) and $A' = \delta(\mathbf{A}')$ by "$\Box A$". So we have:

$$Bel(A) \;=\; \{\!\![A]\!\!\} \;=\; [\![\Box A]\!]$$

The "$\Box$" is here a linguistic operator that acts on a sentence A and transforms it into another whose value is the belief (= subadditive truth-value) one can assign non-additively to A, so it seems proper to interpret "$\Box$" syntactically as "it is believed that", and "$\Box_\alpha$" or "$[\alpha]$" as "α (= the name of a subject or agent) believes that". Further, we would have

$$Pl(A) \;=\; 1 - Bel(\neg A) \;=\; 1 - \{\!\![\neg A]\!\!\} \;=\; 1 - [\![\Box\neg A]\!] \;=\; [\![\neg\Box\neg A]\!] \;=\; [\![\Diamond A]\!]$$

where we have defined a new operator "$\Diamond$" as an abbreviation for "$\neg\Box\neg$" to be interpreted as syntactically as "it is *plausible* that" or "α admits as credible" (so that "$\Diamond_\alpha A$" —or $\langle\alpha\rangle A$— would read "α finds that A can be believed"), because α just does not believe the contrary.

Dubois & Prade [1987] speak rather of "necessity" (or "degree of knowledge") and write $Nec(A) =_{df} \{\!\![A]\!\!\}$ $(= [\![\Box A]\!])$, and "possibility" (or "degree of admissibility") $Poss(A) =_{df} 1 - \{\!\![\neg A]\!\!\}$ $(= [\![\Diamond A]\!])$; naturally, $Nec(A) \leq [\![A]\!] \leq Poss(A)$ for any A (and also:

$$Nec(A) + Poss(\neg A) \;=\; Poss(A) + Nec(\neg A) \;=\; 1 \;).$$

If necessity (or knowledge) of A and $\neg A$ are totally incompatible (in the sense that $Nec(\neg A) = 0$ whenever $Nec(A) > 0$) then $Nec(A \vee B) = \max(Nec(A), Nec(B))$ and $Poss(A \wedge B) = \min(Poss(A), Poss(B))$.

It is interesting to notice that a subadditive valuation may be superimposed on a sentential lattice without breaking its Boolean character, so that $A \wedge \neg A = \bot$ (bivalence) and $A \vee \neg A = \top$ (excluded middle) still hold, while at the same time $\{\!\![A]\!\!\} + \{\!\![\neg A]\!\!\} \leq 1$. This slightly paradoxical fact may explain that many often-encountered situations —where mere subadditivity ($\{\!\![A]\!\!\} + \{\!\![\neg A]\!\!\} \leq 1$) was probably the case— have been analyzed historically as invalidating the law of the *excluded middle*, because it was felt that there was a "third possibility" between A and $\neg A$ making for the unattributed value $1 - \{\!\![A]\!\!\} - \{\!\![\neg A]\!\!\}$, covered neither by A nor $\neg A$. If our analysis is correct, such situations are analyzable in terms of incomplete valuations, but this does not imply the breaking of bivalence of any Boolean algebra (except, naturally, for intuitionistic logic, where the algebra is explicitly non-Boolean).

Subadditivity valuations on a Boolean algebra allow also analysis not only of the concept of *ignorance* and *certainty* (as we sketched above) but of the paradoxes of Quantum Logic as well. These arise, according to the 'Quantum Logic' proponents (e.g. Reichenbach in 1944), in explaining why the distributivity fails in this logic, following the standard interpretation of certain experimental results where:

$$p(a) \cdot p(b|a) + p(\bar{a}) \cdot p(b|\bar{a}) < p(b)$$

a relationship we write in this way:

$$\{\!\![A]\!\!\} \cdot \{\!\![B|A]\!\!\} + \{\!\![\neg A]\!\!\} \cdot \{\!\![B|\neg A]\!\!\} < \{\!\![B]\!\!\} \;. \tag{12}$$

What is odd is that this inequality is normally interpreted by quantum logicians (e.g. Watanabe) as meaning:

$$(A \wedge B) \vee (\neg A \wedge B) \neq B \tag{13}$$

which obviously signals the breaking of distributivity. However, the even-handed transcription of the value-version (12) into the algebraic one (13) is clearly abusive: (13) is much stronger than

(12), since it is equivalent to requiring that (12) hold for *all* conceivable valuations. Actually, in our notation, the reading of the experimental results translates immediately into (12), not (13). From there we conclude that:

$$\{\![A \wedge B]\!\} + \{\![\neg A \wedge B]\!\} < \{\![B]\!\}$$

which is in clear violation of additivity, but not of distributivity. So we need not consider that quantum phenomena occur in a non-Boolean algebra (orthomodular lattices are the preferred alternatives) because a *subadditively*-valued Boolean lattice surely would do for most quantum-logic applications.

Finally we mention that subadditive valuations can as well satisfactorily model how a scientific explanation frame (i.e. the appropriate lattice of theories that cover any observed or predictable true fact) is dynamically replaced by another once the first can no longer account for observed facts: as the valuation of explanations turns subadditive —reflecting that they no longer cover all predictable facts— one is naturally forced to replace the original sentential structure of theories by a new one (still a Boolean lattice) provided with a new —now again additive— valuation on it that restores the balance; the augmented lattice generators are the new vocabulary, and the elementary components of the new structure are the required new explanatory elements (atomic theories) for the presently observed facts. Such a simple valuation-revision and lattice-replacement mechanism may serve to illustrate the basic dynamics of theory change in scientific explanation (see Sales [1982b]).

APPENDIX: ON FUZZY LOGIC

a. The "Fuzzy Logic" tradition

In a joint reflection with Richard Bellman in 1964 Lotfi Zadeh, then a process control engineer, considered the nonsense of painstakingly computing numerical predictions in control theory contexts where the situation is complex enough to render them meaningless. So he proposed instead to rely confidently on broad –and inherently *vague*– linguistic descriptions like 'high' (for a temperature) or 'open' (for a valve) rather than on misleadingly precise values. He then went to suggest (in Zadeh [1965]) a non-standard extensional interpretation of Predicate Calculus. Though first advanced by Karl Menger (in a 1951 note to the French Académie entitled *Ensembles flous et fonctions aléatoires*), the idea was nevertheless original and simple: an atomic predicate sentence like 'the temperature is high' is assigned truth values in [0,1] reflecting the applicability of the sentence to circumstances (the actual temperatures); those values then define a "set", a "fuzzy" set, by considering them to be the values of a generalized *characteristic* or *set membership* function, in a way that is strictly parallel to the standard procedure for defining predicate extensions (e.g. of 'prime number') by equating the $\{0,1\}$ truth values with the characteristic function of the set (i.e 'the prime numbers') so that if for instance $[\![\text{ Prime } (3)]\!] = 1$ then $\chi_{\,\mathbf{Primes}}(3) = 1$, i.e. $3 \in \mathbf{Primes}$ (and so now, accordingly, if e.g. $[\![\text{ High } (170)]\!] = 0.8$ then $\chi_{\,\mathbf{High\_temps}}(170) = 0.8$ or $170 \in_{0.8}$ **High_temps**).

Some snags soon arose to question the utter simplicity of the scheme, doubts such as: should set inclusion (defined as $\mathbf{A} \subset \mathbf{B}$ iff $\forall x\ \chi_{\mathbf{A}}(x) \leq \chi_{\mathbf{B}}(x)$) fail if a single point x does not satisfy the relation? or, more fundamentally: what are the appropriate formulas for the connectives? Zadeh had first proposed the usual Łukasiewicz connective formulas (the *min* and the *max* for the $\wedge$ and $\vee$) but then (in 1970, again with R. Bellman) considered that these were "non-interactive" and to be preferred only in the absence of more relevant information, so he offered further formulas he called "soft" or "interactive" (the product and sum-minus-product).

Bellman and Gierz [1973] showed that under certain pre-established conditions (notably, *truth-functionality*) the only possible connectives were the *min* and the *max*. For a (standard) logician this is an unfortunate result, since the *min* formula when applied to a (0,1)-valued sentence A and its negation $\neg A$ yields always a non-zero value ($[\![A \wedge \neg A]\!] = \min([\![A]\!], 1 - [\![A]\!]) \neq 0$), so explicitly *negating* the classical law of non-contradiction (never questioned before by any Logic) and thus placing Fuzzy Logic outside the standard logic mainstream, for which $\neg(A \wedge \neg A)$ is always guaranteed theoremhood (and so, necessarily, $[\![A \wedge \neg A]\!] = 0$ for any valuation — provided 1 is the only designated value, as we assume). Another unfortunate by-product, this one algebraic in character, is that the neatness and simplicity of the Boolean algebra structure —preservable only by sacrificing truth-functionality— are irrecoverably lost.

Note that *(a)* Boolean structure had to be sacrificed in Fuzzy Logic just by technical reasons (the *min* formula), not by a deliberate or methodological bias, and that *(b)* the choice of connectives was historically motivated, in fuzzy-set theory, by pragmatic reasons (prediction accuracy in applications) rather than by logical method: thus, many formulas were tried and discussed (see e.g Rodder [1975] and Zimmerman [1977]). (Interestingly, while theoreticians stuck to Lukasiewicz's *min*, practitioners —e.g. Mamdani [1977]— preferred the product, perhaps recognizing that in complex or poorly-known systems the best policy is to suppose sentences *independent* –in our sense, see above–; whence the product.) Anyway, after a brief flurry of discussion in the 1970s about the right connectives, fuzzy-set theory proceeded from then onwards *non-compatibly* by emphasizing that its basic theme is linguistic *vagueness*, not *logic* or *uncertainty*, and that the [0,1]-values are grounded on a (possibly non-additive) valuation called "possibility", for which a complete subtheory has been elaborated since.

The initially intuitive "fuzzy logic" approach has since become an independent growth industry. From a logical point of view, some foundational points are arguable: (1) the apparently undisputable *truth-functionality* requirement, already present in Lukasiewicz, imposes a radical departure from (ordinary) logic: the sentences *cannot* form any longer a Boolean structure, and traditional logic principles are gone forever; (2) the theory's justification for using (0,1)-values (values that reflect *imprecision*, since $[\![A]\!] \in (0,1)$ implies $f(A) > 0$) is strictly linguistic: the cause of imprecision is attributed solely to the *vagueness* of language and explicitly excludes any non-linguistic component or dimension such as *uncertainty* (considered non-intersectingly to be the domain of Probability theory) or simply *approximation*.

The emphasis fuzzy theorists place on *vagueness* and its "orthogonality" with respect to other causes of unattributed truth value is understandable considering the basic tenets of the theory but questionable from a non-partisan stand.

First: in all cases we finally obtain a number, and this has a transparent function in reasoning: keeping track of our confidence in what we say. In the vagueness case the value we assign is clearly the *degree of applicability* of the sentence to the circumstances at hand (and this is seen by executing the (T) schema); in the uncertainty case it is our "belief" or its "probability" (in some more or less standard sense) what emerges from the (T) evaluation procedure. In either case (*vagueness* or *uncertainty*), what we try to capture and measure is the *degree of approximation* (to truth, or to full reliability) that we can confidently assign the sentence, and what we have in both cases is *imprecision* (difficulties in ascertaining the $\{0,1\}$ truth value and also, consequently, doubts about committing ourselves to it along a whole inference process). Plausibly, it is easier to assign values consistently (in the [0,1] interval) to the sentence, regardless of where or why imprecision arose in the first place, because what we are mainly interested in is the *degree of confidence* we attach to this piece of information we are manipulating through the (hopefully truth-preserving) inferences.

Second: the two dimensions of imprecision, *linguistic* and *epistemic* (for vagueness and uncertainty, resp.), are not so separable as claimed, either conceptually or practically. (We do not consider here occasional claims –often made by Zadeh– that vagueness is *in things* –even in truth–, i.e. it is not linguistic, but ontological). As seen through application of the Tarski (T) schema, when the agent is uncapable of making up his/her own mind as to the truth of the (unquoted) sentence, the *cause* for the under-attribution and the origin of the ε residual value need not be considered, only the resulting *confidence* matters. The non-attribution of "normal" (i.e. $\{0,1\}$) truth values may originate in language (i.e. the sentence is *vague*) or in imperfect verification conditions (i.e. the sentence, even being linguistically precise, is nevertheless *uncertain* due to identification or measurement difficulties or other causes), but spotting the source is mostly an academic exercise: consider, for example, the sentence 'this is probable', which is imprecise in either or both senses, or the historically motivating illustration by Bellman/Zadeh (the 'high temperature' process-control case), where a discrimination of origins, either linguistic (the expression is vague) or epistemic (we don't know what is the precise case, or we have difficulties in identifying it) is indifferent or pointless.

Moreover, not only such boundary examples cast doubts on the claimed vagueness/uncertainty orthogonality; even the *linguistic* character of fuzziness (or of "possibility") is arguable. First, because a vague sentence, that to the utterer may mean just that the expression lacks straightforward *applicability* to actual fact, to the hearer –if not involved in the described situation– it may be *epistemic* information about what to expect (for instance, hearing a temperature is "high" may set the unknowing hearer into a –correspondingly imprecise– alert state; (s)he then even may usefully turn the "membership" or "possibility" number into an *a priori*-probability or belief estimate). Second, because the number we assign to approximate membership in a class, which reflects the *applicability* of the sentence to the observed situation (a semantic quantity shared by the speakers of the language), is *constructed* and continually adjusted by the language speakers on the base of their experience of past cases, in a process which is the same as that of constructing all other $[\![A]\!]$ assignments, however called ("truths", "degrees", "applicabilities", "probabilities", "beliefs", "approximation" or whatever). The process, described above, consists in setting a universe Θ (here the application instances of the sentences in the language) and then considering cases $\theta \in \rho(A)$ (here the θs are application instances –*utterances*– of A) from past experience; the weighed result is $[\![A]\!] = \mu(\rho(A))$ (here the *applicability* of the –vague– sentence A, i.e. its "fuzziness"), and the relationship between A and all other sentences can also be user-evaluated through the compatibilities α or the sufficiency/causality degrees σ introduced in the text (in a way that otherwise amounts to Popper's 'binary probability' evaluation process).

This process of *constructing* values may be considered a further instance of our general approach to rationality based on the universe Θ of *cases*, and the resulting number $[\![A]\!]$ may be used in general reasoning as all other "truth values" are. We thus assure compatibility inside a shared formalism from which Logic, Probability and (e.g.) fuzzy reasoning or belief theories can be derived directly without costly or unnatural translations. To do this we merely need that the different interpretations (including the "fuzzy" one) obey the same laws and have the same formal components: (1) a common sentential language equipped with a *Boolean* structure (easy to justify and convenient for preserving the commonly accepted laws of logic), (2) *coherence* in attributing the values (regardless of the meaning we give them) to assure *additivity*, and (3) a predisposition to sacrifice truth-functionality when required. If the notion of fuzziness, however justified, could be conceived in this way, then the original 1964 Bellman/Zadeh proposal could be subsumed and solved in a very natural way, and we probably could do without a separate, costly and incompatible additional formalism. (In other words, we probably wouldn't need "fuzzy logic".)

b. Gaines's 'Standard Uncertainty Logic'

In an interesting effort, born inside the "fuzzy" tradition, to construct a common formalization by discriminating between algebraic *structures* and truth *valuations* on them —somewhat paralleling and anticipating the aim of our present development— Gaines [1978] set out to define what he called 'Standard Uncertainty Logic', that covered and formalized two known subcases. This logic postulates (a) a proposition lattice with an algebraic structure that is initially assumed *Boolean* but —by technical reasons— finally admitted to be merely distributive, and (b) a finitely *additive* valuation of the lattice on the [0,1] interval. Gaines then distinguishes two special cases of his logic: (1) what he calls 'probability logic' and defines to be the particular case of his logic where the law of the excluded middle holds, and (2) what he terms 'fuzzy logic', defined (in one among several alternative characterizations) to be his logic when that law does not hold. Apparently, Gaines's encompassing logic should correspond to our general logic above (that covers classical logic as a special case, just as Gaines's logic becomes his 'probability logic' when "all propositions are binary"), but closer examination reveals that Gaines's confusingly called 'probability logic' turns out to be actually coextensional with *classical* logic, while his 'fuzzy logic' is, simply, *Lukasiewicz's* L_∞. This fact is spelled out by the property he mentions of propositional equivalence (here in our notation):

$$[\![A \leftrightarrow B]\!] = \min\,(1 - [\![A]\!] + [\![B]\!], 1 - [\![B]\!] + [\![A]\!])$$

in which the right hand is clearly $1 - |[\![A]\!] - [\![B]\!]|$. This is an expression that is deduced from Gaines's postulates *only* if, necessarily, either $A \vdash B$ or $B \vdash A$ (note this either/or condition —exactly corresponding to our $\alpha_{AB} = 1$ full-compatibility situation— is explicitly mentioned by Gaines as a characteristic property of his 'fuzzy logic'). So Gaines makes the implicit assumption that the base lattice is *linearly* ordered (perhaps induced to it by the $\leq$ symbol used for the –partial– propositional order in the lattice). A confirmation for this comes from the fact that classical logic principles —that hold, by definition, in his 'probability logic'— do not hold in this one. No wonder, then, the base lattice cannot be but merely distributive.

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