Phase Transitions and Complex Systems

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Abstract

The applicability of concepts from the theory of critical phenomena and phase transitions to complex systems (in relation with the *edge of chaos* hypothesis) is discussed. Different types of phase transitions are defined and their importance in generating fractal structures and optimal information transfer are stressed. Several examples from ecology, computation, evolution and social behavior are discussed. Some general consequences are outlined.

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1 Phase transitions and order parameters

How to define complexity is not a trivial question. Several definitions have been proposed concerning this topic. All of them share the intuitive idea that complexity is neither complete order nor disorder (1-5). But this statement, though fairly intuitive, is far from satisfactory. A quantitative characterization of complexity is necessary. Which "universal" features share apparently different complex systems?. The behavior of physical systems close to critical points comes to our help. It is well known, from the theory of phase transitions, that a given system (eventually made of many sub-systems) can undergo strong qualitative changes in their macroscopic properties if a suitable control parameter is adequately tuned. And also that close to these critical points some key characteristic constants (the so called critical exponents) are the same for very different systems (6). At critical points, fractal structures, complex dynamical patterns and optimal information transfer appear in a spontaneous way. The observation of such properties in those systems which we call "complex" makes plausible the conjecture that complexity tends to appear close to instability points.

In this paper, two main types of theoretical approximations are considered[1] : (i) continuous dynamical systems, eventually including stochastic terms, defined as (1,7-8)

$$\frac{d\mathbf{X}}{dt} = \mathbf{f}(\mathbf{X}) + \sigma(\mathbf{X})\xi \tag{1}$$

where $\mathbf{X} = (x_1, ..., x_n) \in \mathcal{R}^n$, $\mathbf{f}$ is a given nonlinear function and $\sigma(\mathbf{X})\xi$ a noise term (see below), and (ii) cellular automata (CA) models, given by a set of transition rules (4,9):

$$S_{t+1}(\mathbf{r}) = \mathcal{T}\Big[\{c_j\}; \{S_t(\mathbf{r}')\}\Big] \tag{2}$$

where a $d-$dimensional lattice $\Lambda_d(L) = \{\mathbf{k}\}$ $(\mathbf{k} = (k_1, ..., k_d); 1 \leq k_j \leq L)$ is used. For the CA approach, space and time are discrete and, typically, also the number of states: $S \in \Sigma = \{0, 1, ..., s-1\}$. Here $\mathcal{T}$ indicates a set of transition rules which can be deterministic or stochastic. $\{c_j\}$ is a set of parameters and $\mathbf{r}'$ are a given set of nearest neighboring positions of $\mathbf{r}$. Typically, the set of neighboring points, $\mathcal{N}(\mathbf{r})$

[1] Other important approaches, where phase transitions can also be explored, are for example reaction-diffusion models and coupled map lattices

is symmetrically distributed around the site $\mathbf{r}$ and those CA able to simulate real systems often verify some particular constrains.

In this section, we summarize the most relevant phase transitions (PHT). To begin with, we use the approach of equation (1), where spatial degrees of freedom and explicit local properties are neglected[2]. We will assume that these are *nonequilibrium phase transitions*, which are achieved by means of energy or matter inputs into the system.

1.1 First-order transitions

The existence of sudden changes in the behavior of nonlinear systems as a consequence of a continuous change in a given parameter is well known (1,8). An example is given by the one-dimensional nonlinear system:

$$\frac{dx}{dt} = f(x) = x^3 + \mu x + \beta \tag{3}$$

The stationary solutions of (1) are given by $f(x_s) = 0$ and are represented in figure (1) in terms of the parameter space (μ, β), forming a smooth surface with a fold. If a single parameter β is smoothly changed from negative to positive (from i to f in figure (1)), the steady state x_s is also continuously changed but, for a given β_c, a sudden transition takes place. The state variable jumps from the upper branch to the lower one. In the language of catastrophe theory, it corresponds to a "cusp catastrophe" for the x_s solutions. Many physical and biological systems show this type of behavior, known as *first order* phase transitions (1PHT). These are not the type of PHT interesting for our discussion on complex systems: they reflect a transient phenomenon between two different states.

1.2 Second-order transitions

A different type of PHT takes place in the following dynamical system (DS):

$$\frac{dx}{dt} = f(x) = -\mu x - \beta x^3 \tag{4}$$

which can be written as deriving from a scalar potential $\phi(x)$:

$$\frac{dx}{dt} = -\frac{\partial}{\partial x}\phi(x) \tag{5}$$

[2]These are known as mean field theories

where $\phi(x) = \mu x^2/2 + \beta x^4/4$, also known as the ϕ^4 model (8). The stationary solutions of (2) are $x_s^0 = 0$ and $x_s^{\pm} = \pm\sqrt{-\mu/\beta}$. When $\mu, \beta > 0$, the single real solution, x_s^0, is stable (i.e. $\partial_x f(x_s) < 0$). But for $\mu > \mu_c = 0$ a bifurcation takes place, x_s^0 becomes unstable and two new solutions emerge. Any minimal perturbation from x_s^0 will lead to a dynamical evolution towards one of the new solutions. In figure 2a, the bifurcation diagram for x_s is shown: this is a characteristic *second order* PHT. Two stable solutions are available when $\mu < 0$ and the system must choose one of them: we say that *symmetry breaking* takes place (1,8). In figure (2b) a mechanical illustration of this process is also shown.

Fluctuations can be included in this formal approach by means of the so called Fokker-Planck equation (FPE) (7,8). This approach is very important in order to analyse real systems, where fluctuations are inevitable and where probability distributions are the observed data. Starting from a general dynamical system $dx/dt = f(x)$, a stochastic term can be included as:

$$\frac{dx}{dt} = f(x) + \sigma(x)\xi \tag{6}$$

where ξ is a white noise with gaussian distribution and $< \xi >= 0$. It can be shown that the time-dependent probability $f(x,t)$ of finding $x \in (x, x+dx)$ at the time t is given by the FPE:

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big[\sigma^2(x) f\Big] + \frac{\partial f}{\partial x}\Big[G(x) f\Big] \tag{7}$$

where $G(x)$ is a given function, defined from $f(x)$ and the noise term $\sigma(x)$[3]. The stationary solution $f_*(x)$, if it exists, is obtained from

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\Big[\sigma^2(x) f_*\Big] + \frac{\partial f}{\partial x}\Big[G(x) f_*\Big] = 0 \tag{8}$$

and it reads:

$$f_*(x) = \frac{K}{\sigma^2(x)} exp\left\{\int_{x_0}^{x} 2\frac{G(z)}{\sigma^2(z)} dz\right\} \tag{9}$$

K being the normalization constant. Once the distribution $f_*(x)$ is obtained, the moments of the distribution can be calculated.

[3]Two possible $G(x)$ are used, the so called (i) Ito ($G(x) = f(x)$) and (ii) Stratonovich ($G(x) = f(x) + [\partial_x \sigma^2(x)]/4$)

In figure (2a), we show the trajectory of a dynamical system as described by (2) but with a small noisy perturbation of constant strength σ, i.e. we have $dx/dt = f(x) + \sigma\xi$. It can be shown that these small fluctuations are amplified close to the instability point. At μ_c they become very large. This phenomenon is known as *critical slowing down* (6,8) and is a characteristic fingerprint of the onset of a second-order phase transition. These kind of transitions are more interesting for several reasons: and they are linked, in spatially distributed systems, with the appearance of self-similarity and optimal information transfer. Typically, for an arbitrary dynamical system $dx/dt = F_\mu(x)$ which shows symmetry breaking for $\mu > \mu_c$, a macroscopic measure O_μ (the so-called order parameter) can be defined over the system (here μ means a given (set of) parameter(s)) in such a way that this quantity is positive for $\mu > \mu_c$ and vanishes at μ_c. For a second-order phase transition, O_μ grows, close to μ_c (when $\mu > \mu_c$), as $O_\mu \approx (\mu - \mu_c)^{1/2}$ and the relaxation time of fluctuations behaves near μ_c like $\tau \approx |\mu - \mu_c|^{-1}$.

The concept of order parameter has been widely generalized (1,8) as a cornerstone of the theory of nonequilibrium phase transitions. In this sense, when an arbitrary system (from a laser to an ant colony) crosses an instability point μ_c, only just a few variables become relevant and serve as order parameters which define the macroscopic pattern. At the same time, these macroscopic variables govern the behavior of the microscopic parts. This is the core of what we call "emergent properties".

1.3 Noise-induced transitions

A further example of phase transitions is linked with the response of some nonlinear systems to multiplicative external noise i.e. when $\sigma = \sigma(x)$, as in equation (7). These fluctuations, coupled in this system-dependent way, can induce new nonequilibrium transitions, i.e. qualitative changes in the steady state of the system, so the system no longer adjusts its macroscopic behavior to the average properties of the environment, but responds in a more active way. To be specific we treat again a nonlinear system:

$$\frac{dx}{dt} = \alpha - x + \lambda x(1 - x) \tag{10}$$

with $x \in [0, 1]$, i.e. the genetic model (7). In a constant environment (λ constant) no instability occurs, there is not any transition phenomena, and, with $\lambda = 0$ and $\alpha = 0.5$ for example, the steady state is $x^* = 0.5$.

Now if $\lambda_t = \lambda + \sigma\xi_t$, we have a stochastic process where ξ_t fulfill the previous properties. Now, it can be shown that the previous equation can be written similarly as eq.(6), being $\sigma(x) = x(1 - x)$. In figure 3, the stationary density $f_*(x)$ is shown for different intensities of noise. A Noise Induced Transition is observed, where a *qualitative change in the shape of the stationary probability density* is generated. In the case of the genetic model, we can find, for the above mentioned parameters $\lambda = 0$ and $\alpha = 0.5$, a change in the number of extrema of $f_*(x)$: as we increase the noise intensity σ, $f_*(x)$ goes from a maximum in $x^* = 0.5$, i.e. the most probable value of will be the deterministic one, to the appearance of three extrema for $\sigma^2 > 4$, one minimum at $x^* = 0.5$, and two maxima $x_{m\pm}$, so the system displays now bistable behavior. This is a *pure noise induced* phenomenon. It shows the fact that, sometimes, *the knowledge of the average environmental state is insufficient to predict the macroscopic behavior of the system.* In other cases, external noise can lead to even more profound modifications[4].

2 Measuring Complexity. The Ising model

Many examples of second-order or noise-induced phase transitions are known from classical and quantum physics: the phase transitions when magnets are heated up (see below), superconductors which become simple conductors, inflationary models of the early evolution of the universe, etc. In biology, threshold phenomena in cellular dynamics, the collective behavior of insect societies or the large-scale dynamics of brain are just some examples. In all these examples, the system undergoes dramatic changes in its qualitative properties when the critical point is reached. But before to proceed further in our study of phase transitions, let us consider the best known model of them: the Ising model (6), as a reference nonlinear (closed) system formed by many elements with local interactions. Several relevant measures will be defined, with general interest for complex systems.

[4]It can be shown that, similarly as in section 1.2, a *probabilistic potential* $\mathcal{V}(x)$ can be defined in such a way that we can write $f_*(x) = Kexp[-2\mathcal{V}(x)/\sigma^2]$ (see (7))

2.1 Microscopic model

Consider a two-dimensional square lattice of length side L (in one dimension, no phase transition is obtained). The state of each lattice site i is indicated as $S_t(i)$ for each time step t. Time is assumed discrete, and $\Sigma = \{-1, +1\}$. Each "spin" $S_t(i)$ interacts with its four nearest neighbors through an exchange interaction J, which favors parallel alignment (the least-energy configuration).

The dynamics of the 2D Ising model is easily simulated by computer. Each time step, we pick up at random a single spin S_i and its local field h_i, defined as $h_i = J \sum S_j$ (the sum is carried out over nearest neighbors) is computed. Then the i-th spin will change with a probability W_i given by (6):

$$W_i(S_i) = P(S_i \rightarrow -S_i) = \frac{1}{2}\Big[1 - S_i\, tanh\Big(\frac{h_i}{k_B T}\Big)\Big] \tag{11}$$

Here k_B is the Boltzmann constant and T the temperature. The temperature is a very important control parameter: if we start from a system at high temperatures and we cool slowly our system, a sudden change in the behavior of this system takes place at a critical point T_c, the so-called Curie temperature. In order to understand the nature of this change, we can define a new quantity: the global magnetization M given by $M = \sum_i S_i$. In figure (3) we see how this quantity behaves. For $T > T_c$, $M = 0$ as a consequence of the effect of thermal excitations on the interaction of nearest spins. When $T << T_c$, interactions are not perturbed and the spins are aligned (so M becomes maximum). Formally, the magnetization per spin, $m = M/N$ is shown to behave as: $m = 0$ for $T > T_c$ and as

$$m = \left[1 - \Big(sinh(2\beta\epsilon)\Big)^{-4}\right]^{1/8}$$

for $T < T_c$. Close to T_c, (for $T < T_c$) m behaves as $m \approx |\tau|^{\alpha}$ where τ is the so called reduced temperature, i.e. $\tau = (T - T_c)/T_c$. The exponent α is named a *critical exponent*. Generally, the critical exponent λ associated with a function $F(t)$ is defined as $\lambda = \lim_{t \to 0}[log|F(t)|/log|t|]$ i. e. $F(t) \approx |t|^{\lambda}$.

2.2 Correlations and percolation

At the critical point, something happens. What?. The behavior at the critical point T_c can be visualized by means of the simulation model. In figure (4) we see three examples of this system for three different points, for (a) $T = 1.2\ T_c$, (b) $T = T_c$ and (c) $T = 0.95\ T_c$. For temperatures greater than the Curie temperature, entropy dominates over energy and the distribution of spins is essentially random (fig. 4a). We say that the so called *correlation length* ξ is small. It expresses the typical distance over which the behavior of a microscopic variable is correlated with (influenced by) the behavior of another. Close to T_c, ξ scales as $\xi \approx |\tau|^{-\nu}$ being ν another critical exponent. Below the critical point the model is said to exhibit long-range order: the exchange interaction works over thermal perturbations (fig. 4c). Now let us move towards the critical point. If we observe this state (fig. 4b) we see that both types of spins are equally represented but their distribution appears to be clustered in a non-trivial way. In fact, if an enough large system is seen, we can appreciate the existence of a self-similar pattern (i.e., a fractal object). There is no upper cut-off length and ordered structures exist on every length scale. In fact this is the microscopic physics underlying a critical phase transition: fluctuations on all scales of length are meaningful.

The correlation length ξ also represents the typical distance of a connected cluster of spins. For a fixed ξ, fluctuations of blocks of all sizes up to size ξ can be found. When moving towards the critical point, ξ diverges to infinity, and so does the size of the maximum connected cluster. This means that there exists a connected path on this cluster that spreads from side to side of the system. When this happens, the system is said to percolate. There are many different types of percolation (9), depending on the geometry of the system and on the existence of correlations in the formation of connected sites. The simplest example of percolation consists in randomly filling the sites of a square lattice with some probability p. If p is increased from 0 to a critical value p_c, the properties of a 2^{nd} order phase transition are recovered: one finds divergency of the correlation length and of the size of the clusters (until, at $p = p_c$, a percolation cluster that connects both sides of the system appears), among others. At the critical point, the system displays fractal geometry (the size of the clusters scales following a power-law that helps to characterize the transition, too), and fluctuations of all sizes can be found. At or close to a critical point the amount of information that might be transferred throughout the system is maximal.

Percolating systems are likely to be very robusts against external perturbations, provided that each and every different size is already considered in the system, both in space and in time.

2.3 Entropy and Mutual information functions

Let us define two additional measures, which will be used in the next sections. Here we consider a given system formed by many subsystems (as in the Ising model). Let $\{x_i\}$ with $i = 1, 2, ..., m$, a given set of values $x_i \in \Sigma$, where $\sharp(\Sigma) = S$. Here $\{x_i\}$ can be a time series obtained from one of the subsystems or the complete set of current states, or a combination of both. And let us assume that we can define two sets of probabilities $\{P_\alpha\}$ and $\{P_{\alpha\beta}\}$. Here P_α will be the probability of observing the $\alpha \in \Sigma$ state, and $\{P_{\alpha\beta}\}$ the probability of observing both $\alpha, \beta \in \Sigma$ states. This probabilities would be obtained by averaging over time or averaging, at a given time step, over the elements of the system, or both.

Then the Boltzmann (Shannon) entropy is given by:

$$H = -\sum_{\alpha\in\Sigma} P_\alpha \, log[P_\alpha] \tag{12}$$

and the mutual information by (10,11):

$$M = -\sum_{\alpha\in\Sigma} \left\{ \sum_{\beta\in\Sigma} P_{\alpha\beta} \, log\left[\frac{P_{\alpha\beta}}{P_\alpha P_\beta}\right] \right\} \tag{13}$$

The first quantity give us a measure of disorder, and it is fitted by $0 \leq H \leq \log(S)$, which correspond to the simplest state ($P_\alpha = 1; P_{\beta\neq\alpha} = 0$) and the uniform distribution ($P_\alpha = 1/S$, $\forall\alpha \in \Sigma$). If we consider the Ising model, this can be observed for the entropy per site. It is given by $H = -\{p_+ \, log(p_+) + (1-p_+) \, log(1-p_+)\}$, and is $H = 0$ at low temperatures (when $m \to m_+$) and $H = log(2)$ at high temperatures. Using the same example, let us now consider two neighboring spins, and let us calculate the previous probabilities by averaging over time many steps. At low temperatures, the system is very ordered, and we have to expect $P_{++} \to 1$ or $P_{--} \to 1$, and so $M \to 0$. At the other extreme, both elements will behave randomly and nearly independently, and so we have to expect $P_{\alpha\beta} = P_\alpha P_\beta$, and again $M \to 0$. At intermediate values, when correlations emerge close to the critical point, we expect a maximum in M [5] , as it is shown in figure 5.

[5] When $\sharp\Sigma > 2$, this situation can be more complicated, as shown by W.Li (11)

3 Cellular Automata and Phase Transitions

Though the Ising model is a very good representation of a wide set of equilibrium physical systems, we can ask ourselves if other (now *open*) open complex systems can be equally well represented, at least in their qualitative properties. Cellular automata (CA) models have been widely used as models of complex phenomena (9). They also exhibit phase transitions (see (12) and references therein).

3.1 Deterministic cellular automata (DCA)

Invented in 1948 by the hungarian mathematicians Von Neumann and Ulam, these dynamical systems are defined in a general way by equation (2), these models, exhibit an enormous variety of spatiotemporal patterns, but they can be classified into four main qualitative categories (figure 6a-d): (a) Class I: evolution leads to a homogeneous state (steady attractor, fixed point); (b) Class II: evolution leads to a set of localized stable and/or periodic structures (periodic attractors); (c) Class III: evolution leads to chaotic patterns (chaotic attractor) and (d) Class IV: evolution leads to complex localized structures, with very long transients. These automata are the only ones able to perform universal computation (figure 6d).

This scheme was revised by means of the Langton's λ parameter (4). Following the notation of section 1, and using $\mathcal{S} = \sharp(\Sigma)$ and $\kappa = \sharp(\mathcal{N})$, the Langton's parameter is defined, for a given CA, as:

$$\lambda = \frac{\mathcal{S}^\kappa - n_q}{\mathcal{S}^\kappa} \tag{14}$$

Here S^{2r+1} is the number of all the possible transitions for a specific rule-table, and n_q is the subset of such transitions (in the rule-table) leading to the so called *quiescent state* (usually 0). This parameter gives a rough quantitative characterization of the CA complexity. Using λ, we can order the automata rules following the sequence: $Class \quad I \rightarrow Class \quad II \rightarrow Class \quad IV \rightarrow Class \quad III$ and so the complexity of CA rules is ordered. It was shown that information grows up to a maximum at a "critical" λ_c, *where class-IV automata are observed.* This fact suggested to some authors that computation in natural systems should emerge close to phase transition points (3,4) (see also Haken (7,14) for a detailed discussion on physical systems). The $H - M$ diagram shows us how complexity (M) and disorder (H) are related: information needs some degree of order to be stored (low entropy) but also some degree of disorder if transmission and manipulation are necessary (7,15). The critical point, where the maximum M is obtained, shows the compromise between both tendencies. This conjecture is known as *Computation at the edge of chaos* (see, however, (14)).

3.2 Stochastic cellular automata (SCA)

As it happens with continuous, noise-driven systems (see section 1), CA models incorporating noise (the so called stochastic cellular automata, (15)) can generate unexpected patterns of behavior. A study of the simplest one-dimensional DCA ($\mathcal{S} = 2, \kappa = 3$) shows that they do not exhibit class-IV behavior[6]. But some SCA, which are classified as class-I if no randomness is used, belong to class-IV for a suitable noise level.

Stochasticity can be introduced in several ways. As an example, if a one-dimensional CA is used, with $\Sigma = \{0, 1\}$ and copupling with nearest neighbors. Then the i-th automaton takes the value $\mu \in \Sigma$ according to a conditional probability: $P[S_{t+1}(i) = \mu | S_t(i-1), S_t(i), S_t(i+1)]$. It can be shown that, by means of a percolation phenomenon (the so called directed percolation (9)) complex patterns emerge at critical points, with statistical properties reminiscent of class-IV DCA. In other cases, a small amount of noise (something unavoidable in real systems) can modify the long-term behavior of a class-II CA in such a way that the transient structures (which can look as class-IV propagating structures) are stabilized by means of noise.

4 Self-organized Criticality

As we have seen, complex patterns involving the appearance of fractal structures are observed at critical points. Fractal forms are widespread in nature, from mountain landscapes to river networks or clusters of galaxies. Other self-similar patterns are observed in the time evolution of many systems which exhibit the so-called $1/f$ noise. But the origin of such patterns has been a matter of debate for a long time. Free-scale behavior seems to be linked with complexity. How does it emerge?.

[6]Except perhaps rule 193

4.1 The sandpile automaton

In 1987 Per Bak, Chen Tang and Kurt Wiesenfield (16) suggested a new theory which they called "self-organized criticality" (SOC). They suggested that a wide variety of natural dissipative systems spontaneously evolve towards the critical point. No tuning of some external parameter is necessary and in this sense these phenomena should be "parameter-free". The best known example of a physical system exhibiting SOC (at least in some cases) is the sandpile (fig. 7a). When we build the pile of sand by adding a few grains at a time, the slope grows until a maximum value is reached. Once this situation is obtained, the addition of new grains very often generates small avalanches. Larger avalanches are less common but also occur. From time to time, a very large avalanche takes place. These events generate a $1/f$ dynamics (i.e. self-similar fluctuations in time) and we have reached a critical state: the distribution of avalanches is in fact a power law.

Bak et al. (16) suggested that a simple cellular automata model (the sandpile automaton, SA) is able to reproduce the observed behavior. Let us consider a 2D grid where the "state" of each point is given by the local "slope" $z(i, j)$. There are two essential rules defining the automaton: (i) Addition of a unit: we randomly choose a grid point and add a "grain": $z(i, j) \to z(i, j) + 1$ and (ii) Toppling: if the local slope is larger than a critical value K (ussually K=4 with Neumann neighbourhood) then some units are transferred to nearest positions: $z(i, j) \to z(i, j) - 4$ and

$$z(i \pm 1, j \pm 1) \to z(i \pm 1, j \pm 1) + 1$$

Power laws are a consequence of the self-organization process: once the pile reaches the critical slope, the model exhibits the general features of the sandpile. In particular, if we study how many avalanches of each size occur, after the addition of a single "grain" to the pile, we find that the distribution of events where a total of s sites topple obeys a power law (fig. 7b): $N(s) \approx s^{-\tau}$. Thus, if one waits enough, one will certainly see events that are as large as one has the patience to wait for. As Bak has pointed out, avalanches are an unavoidable and intrinsic part of sandpile dynamics. On the other hand, the spatial distribution of with critical slope K shows also fractal behavior. This result strongly suggests a non-trivial connection between temporal and spatial self-similarity (17).

4.2 SOC and universality

The sandpile model (and other models of real systems which exhibit SOC) is a metaphor of the real sandpile, but a powerful one. The key word for this consistence in *universality*. What this means (and it is particularly true for critical systems) is that important features of large scale phenomena are roughly insensitive to the particular details of the models, and are shared by very (apparently) different systems. In this sense, it has been suggested that earthquakes or even turbulence would be associated with SOC phenomena (see (17) and references therein) belonging to different universality classes. Actually, this is the only known mechanism by means of which an explanation for self-organized complexity is obtained. Whether or not SOC is a universal mechanism of "complexity generation" will be a matter of future research.

5 Examples: models and reality

Now let us proceed with a study of several examples of complex systems exhibiting critical properties. Together with these examples, some theoretical approaches to phase transitions are also introduced.

5.1 Kauffman Networks and Genome Stability

Random boolean networks (also known as Kauffman networks, KN) were first proposed by Kauffman as simple (yet reasonable) models of genetic systems (18). Since their definition in 1962, they have become one of the most celebrated models of complex systems. In these networks, a set of N binary elements is used. The state of each element at a given time step t, is given by $S_i(t) \in \{0, 1\}$, $(i = 1, ..., N)$. The dynamical state of $S_i(t)$ is updated (synchronously) by means of a boolean function Λ_i. Each element receives inputs from exactly K elements, and we have a dynamical system defined from:

$$S_i(t+1) = \Lambda_i \left[\, S_{i_1}(t), S_{i_2}(t), ..., S_{i_K}(t) \, \right] \tag{15}$$

$(i = 1, 2, ..., N)$. The critical point K_c was analitically determined by Derrida and his colleages (19) in a set of remarkable theoretical studies, using the so called *annealed* approximation. The dynamical and structural properties of the critical point can be well understood by means of a very interesting example: the problem of cell differentiation from the global activity of a complex genomic system.

All cells in a multicellular organism share the same genome formed by N genes (and some amount of selfish DNA). If we assume (reasonably) that genes are essentially on-off elements, a given differentiated cell (a neuron, kidney cell, etc) is characterized by a subset of active genes. The number of different cell types is not arbitrary, and much less than N. Why?. The existence of a critical point in KN might be the answer. The key assumption of Kauffman's theory is that *a cell type is an attractor*, more precisely, a state cycle attractor. Starting from different initial conditions (IC), a KN evolve towards a more or less complex periodic pattern, the system's attractor. Several IC can reach identical attractors, forming a *basin of attraction*. The global properties of these attractors, i.e. their length, number (n_A) and stanility against small perturbations, are controlled by the connectivity K, and are summarized in table I.

At $K_c = 2$, nets *crystallize spontaneous order*. KN with critical connectivity exhibit unexpected and powerful collective order. Both the expected length of state cycles and the number of attractors are $n_A = O(\sqrt{N})$. This implies that in a genome of 10000 elements with $2^{10000} = 10^{3000}$ combinations of activities, only 100 asymptotic patterns (different cell types) are observed. Another property of critical KN is that (as cells) each attractor is stable to most minimal perturbations. We have then an example of phase transition where the properties of real genomes and cells are found to appear. Even if a distribution of input connectivities is used, instead of a fixed K, the previous results are obtained (20). A very important consequence of these results is that the observed properties involved in genetic regulatory systems would be the result of intrinsic dynamical processes, and not just the action of natural selection.

5.2 Social Insects and swarms

Social insects are a paradigmatic example of the collective properties that might emerge from a set of simple individual entities. The appearance of social behavior is a quantum leap in evolution: the insect societies have found their place in all ecosystems (21). In spite that, typically, social insects in isolation have a rather simple behavior, the colony as a whole is able to perform very complex tasks, including computation. Insect societies share common features with neural structures like the brain. Both share some general properties: (i) They are formed by a more or less large number of elements (neurons or, say, ants) which interact locally; (ii) Single elements display a reduced number of behavioral patterns; (iii) When we look at the system as a whole, macroscopic patterns of activity emerge as a result of local interactions among elements, not reducible to the behavior of single units; (iv) The dynamics of these systems is robust against noise, failure or even the removal of single units.

The remarkable studies of Deneubourg and his colleages of the Bruxells school (22) early showed that several patterns of activity in social insects were linked with self-organization processes. These phenomena are sometimes the result of symmetry-breaking processes (22,23) as those analysed in section 1.2.

As an example of computation at the edge of chaos in ant colonies, let us consider the colony oscillations of the *Leptothorax* ants (24,25). It has been shown that these colonies show a striking pattern of global short-time periodic oscillations of activity. The number of active individuals moves up and down in a roughly periodical way. It is also known that individual ants are not periodic, but chaotic (26) and so global behavior is an emergent property. As the density of individuals is increased (this can be experimentally done) from low to high, the coherence of these oscillations becomes more and more clear. In natural conditions, the nest density appears to be at an intermediate value where neither total order nor disorder are allowed (see (27) and references therein).

It has been conjectured (28) that this phenomenon enable ants to perform mutual exclusion (ME). ME is an important general problem of concurrent programming, an instance of which is resource allocation. We say that activity A_1 of process P_1 and activity A_2 of process P_2 must exclude each other if the execution of A_1 may not overlap the execution of A_2 In our case, mutual exclusion is attained with the combined effects of autosynchronization and the spatial arrangement of brood in the nests of *L. acervorum*, which appears to limit the number of workers that can access brood simultaneously (28).

On the other hand, we have introduced clear evidence that the phenomena of autosynchronization is due to a state dependent external noise, in the form of both a spontaneous activation and a density dependent activity propagation, giving raise to a noise induced transition (29). Using the formal approach of fluid neural networks (27), we have shown that, if the entropy and mutual information are calculated (by averaging over time the states of pairs of individuals) a maximum information is reached at a given critical density ρ_c (figure 8). For a biologically plausible set of parameters we have the transition at the same $\rho_c \simeq 0.2$, experimentally observed. These facts point to the possibility of a noise induced computational capability in the colony, for which we have coined the notion of *Noise Induced Computation* (29).

5.3 Macroevolution and Extinction

In the neodarwinist view of evolution, the interest on individuals as the units for selection has been shifted towards genes. In this framework, evolution is then understood as the temporal changes in gene frequencies, driven by natural selection. In the general case, where mutations and selection are considered, the evolution of gene frequencies follows a nonlinear stochatic differential equation for the gene frequency x:

$$\frac{\partial f(x,t)}{\partial t} = -\Phi_\mu(x)\frac{\partial^2 f(x,t)}{\partial^2 x} + \Psi_\beta(x)\frac{\partial f(x,t)}{\partial x} \tag{16}$$

Here $\Phi_\mu(x), \Psi_\beta(x)$ are suitable continuous functions incorporating the specific type of mutation, selection pressures, etc. We also see diffusion and drift terms (the partial derivatives over x). Once the whole set of parameters is known, the steady state solution (i.e. $f_s(x)$ such that $\partial_t f = 0$) gives us the final frequency distribution. Implicitly, it is assumed that these theoretical approaches can be translated to the large-scale events, and so give an explanation for extinction patterns and species diversification.

Some data of the fossil record in fact, suggest that a critical, emergent phenomenon would be behind the overall pattern of macroevolution. One is the frequency distribution of extinction events, which fits a power law (30). Another is the existence of fractal scaling in the structure of taxonomic data (31) and more recently, the study of time fluctuations of some fossil families has shown the existence of 1/f behavior (32).

Some theoretical models have tried to shed light in this problem. One if the Kauffman-Johnsen model, based on random boolean nets (18). The other, more simple, is the Bak-Sneppen (BS) (33) model defined by the following set of rules. We consider a string of N random numbers $x_i \in (0,1)$, which are the "species" of our ecosystem, and the steps are: (a) We choose the minimum x_j and change it by a new random number $x_j \in (0,1)$; (b) The two nearest neighbors are also changed: $x_{j+1} \in (0,1), x_{j-1} \in (0,1)$ (periodic boundaries are assumed) and (c) The previous steps are repeated.

In this model the quantities $\{x_i\}$ play the role of a "barrier height", separating the local fitness maximum from other better maxima. In other words, the barrier height is a measure of how far is the i-th species from its maximum fitness. Then if the fitness x_j is low, it is more likely to find nearby better states. In the BS model, the barriers are the measure of stability: smaller barriers are more unstable and easily changed. In spite of the oversimplified nature of this model, it is able to show punctuated equilibrium (34) in terms of bursts of change into the ecosystem. It also shows power law distributions of several quantities (fig. 9). These properties are in fact shared by other complex systems as economics, and it has been conjectured that punctuated equilibrium is a common property of all complex systems (30)[7]

5.4 Quasispecies and the error catastrophe

Consider now a set of molecules (RNA chains, viruses, etc). We indicate by I_j a given sequence (assumed to be a string of bits of length ν). We can imagine the phase space of our system as a ν-dimensional hypercube. Up to 2^ν different molecules are possible. We asume that replication of molecules takes place together with mutations (i.e. errors). Selective pressures can be introduced in such a way that some sequences survive and others do not. The mutation rates between a given pair $\{I_i, I_k\}$ are indicated as W_{ik} (where $i, k = 1, 2, ..., 2^\nu$). It can be shown (36) that the population $n_i(t)$ of the i-th sequence I_i will evolve in time following the dynamical system:

$$\frac{dn_i(t)}{dt} = W_{ii} n_i(t) + \sum_{k \neq i} W_{ik} n_k(t) + \frac{n_i(t)}{\sum_k n_k(t)} \phi(t) \tag{17}$$

where W_{ik} are the mutation rates from $I_{k \neq i}$ to I_i, and W_{ii} is the net production of I_i through (correct) self-copying. It is defined as $W_{ii} = A_i Q_{ii} - D_i$ where A_i

[7] This observation was first reported by B. Huberman et al. in relation with economics and the so called "computational ecologies" (35)

and D_i gives the rate constants of total (correct and erroneous) replication and degradation, respectively. The factor Q_{ii} is the frequency of correct copying, the so-called quality factor. It can be indicated as $Q_{ii} = q^\nu$, q being the mutation rate per unit. Finally, the last term at the right-hand side stands for the dilution flux that removes molecules in proportion to the amount produced.

Now one of the most important consequences of this model is the appearance a final distribution of sequences (not just one). This distribution is known as a molecular quasi-species. It is formed by a given (dominant) sequence I_m, also known as the master sequence, together with a set of mutants deriving from it. These mutants are grouped around the master sequence in such a way that often their average sequence equals that of the master. Retroviral populations are typically formed by a dynamical quasispecies.

The interesting point is that q plays the role of a temperature [8]. The nonequilibrium Darwinian organizing process has a clear parallel in the existence of order-disorder phase transitions. In our case, very small q values (high fidelity, $q \to 0$) lead to a single copy (a uniform virus population), and very high error values ($q \to 1$) lead to a totally random set of molecules, without any biological identity.

It can be shown that a phase transition point exists for a critical fidelity ϵ_c (or error threshold q_c) given by (36-38):

$$\epsilon_c = 1 - q_c = \frac{Ln(\sigma)}{\nu} \tag{18}$$

In fig. 10, we see how ϵ operates on the stationary distribution of sequences for $\nu = 50$. We observe a sharp transition, known as the "error catastrophe" at $\epsilon_c \approx 0.045$. At the left-hand side the master sequence dominates but once reached (when $\epsilon > \epsilon_c$) it becomes a small fraction inside a random population.

How is ϵ_c related with viruses?. For a retrovirus, very small ϵ means to be very sensitive to the immune response. The cells of the immune system recognize very specifically foreign molecules. Once detected, it is eliminated. Viruses will survive only if new mutants are generated fast enough. But obviously ϵ should be less than ϵ_c or the new sequences will be not functional. Where are real viruses organized?. Experimental evidence shows clearly that retroviruses are typically self-organized

[8] It has been shown by Leuthausser that the Eigen's evolution model is equivalent to a two-dimensional Ising model, see (37)

close to the error catastrophe. (39). In this sense, RNA viruses explore their "RNA world" (38) by moving towards the edge of disorder. In this way, the broad spectrum of mutants makes evolutionary optimization faster. The virus population is less likely caught in local fitness optima.

5.5 Diversity and Rainforest Dynamics

Complex systems find their brightest example in the tropical rainforest. The great variety of species, even in local areas, is legendary. Since the early times of ecological theory, the causal explanation for this phenomenon was that the species composition of such ecosystems is maintained near an equilibrium state. But other analyses soon suggested a different explanation: that high diversity is maintained only in a nonequilibrium state (40). Here "disturbance" means an external one. However, the system can be maintained far-from equilibrium by means of its internal dynamics. In a recent study (41-42) it was shown that the spatial distribution of trees in a rainforest (the Barro Colorado rainforest, in Panama) shows (multi-) fractal properties. Specifically, we analysed the spatial distribution of those places where the size of the canopy is less than 10 meters (usually, the canopy reaches 50-60 meters high). These low canopy points are the result of recent treefall. Treefall and gap formation are known to contribute, very importantly, to the generation and maintenance of diversity in rainforests. The size distribution of gap sizes, $N(G)$, was also shown to be a power-law (fig. 11). $N(G) \approx G^{-\psi}$ with $\psi \approx 2$. These results strongly suggest that BCI (and may be other rainforests) might be an ecosystem poised at a critical state.

With only these quantities at hand, the evidence of criticality is appealing but may be not enough conclusive. Now again theoretical models comes to our help. A CA model of forest dynamics, the *Forest Game (FG)* (41,42) can be used in order to analyse the previous conjecture. The model is defined on a $L \times L$ lattice where trees of size $S_i \in [0, S_c]$ are defined. A simple set of rules are (asynchronously) applied. We take a lattice point at random and the following possibilities are allowed: (i) Birth: $0 \xrightarrow{P_b} S_0$; (ii) Death: $S \xrightarrow{P_d} 0$ and $S_c \longrightarrow 0$ and (iii) growth: $S \xrightarrow{\Phi} S + \Delta S$ Here P_d and P_b indicate the death and birth probabilities. When a new tree is born, it has a minimum size $S_0 = 0.1$. The third transition introduces a critical (maximum) size S_c. Finally, the fourth transition enables trees to grow, provided that a given condition is fulfilled. Specifically , the growth term is defined by:

$$\Delta S = \Phi\left[1 - \frac{\gamma}{8}\sum_q S_q\right]$$

where $\Phi(z) = z$ if $z > 0$ and zero otherwise. γ is the strength of the interaction, and the sum is extended over the eight neighbors. Then we study how the dynamics is modified as γ is changed. When very small γ are used, the FG shows coherent behavior: local parts of the forest become higly synchronized. At high γ, the forest becomes frozen with a random pattern of disperse gap points. For large p_d, the forest becomes disordered. But for a wide set of intermediate interaction values, a *complex forest* is observed, where *all* the observed properties of the BCI plot are reproduced. In particular, those properties linked with a system at a critical state: (a) Fractal behavior of gap points (with multifractal spectrum); (b)Power-law distribution of gap sizes (fig. 11) with $\psi \approx 2.0$ and (c) 1/f-fluctuations of biomass.

The agreement of theoretical and field results suggest that rainforests are certainly in a nonequilibrium state, in fact they are on or close to a phase transition point. A direct implication of this result is that high diversity is a consequence of criticality. Again, a very simple model (we only use one "species" of tree, instead of the 186 known species of trees of BCI) leads to a good description of the rainforest, as expected when universality is involved.

6 Summary and prospects

We have reviewed several theoretical problems involving the existence of phase transitions and critical phenomena in natural complex systems. Some others have been studied, as for example the large-scale dynamics of brain activity, as an external stimulus is slowly changed (43), showing clearly those properties expected in phase transition phenomena. A simple theoretical model has been suggested (44) in which the main experimental results were well reproduced.

The previous systems, their nature, parameter dependence (if any) and implications are summarized in table II. In some cases, the existence of a critical boundary is important as an organizing principle: the genome (and the underlying patterns of cell differentiation) or the existence of computational processes in ant colonies are only two examples. Sometimes they are linked with a mechanism of diversity generation (as in RNA viruses and rainforests) and sometimes they introduce sudden changes in the complexity of ecosystems as it happens when landscapes are fragmented below a critical threshold (45).

In relation with modelling real systems, two general comments are in order. The first, that if complexity is linked with critical states, then a simplified model is justified, as far as it is able to reproduce relevant properties of the real situation. The other concerns with constraints: biological systems are often limited in their structural and dynamical properties (46). In this sense not all places in parameter space are available, and sometimes critical states might be not available. And second, if SOC is a robust phenomenon able to drive complex systems towards the critical point, it is assumed that the final state is reached through a parameter-free process. This makes apparently necessary the underlying model to be explicitly parameter-free. Though this can be done, a parameter-dependent model (as it is the forest game) where different types of dynamical states are observed, can nevertheless be used in order to detect criticality in the real world. This is achieved if the parameter-space of the theoretical model has a critical domain where a phase transition takes place. If the predicted properties at this point match those observed in the real system, then the next step will be to look for a mechanism of evolution towards such boundary.

The previous results are only some examples of a rich variety of PHT phenomena occuring in complex systems. They show how simple, nonlinear models are able to capture the essential properties of real systems close to critical points. How general is this situation will be a matter for future research, but clearly, as stated by Prigogine and Stengers (47), "the more complex a system is, the more numerous are the types of fluctuations that threaten its stability". In any case, the interplay between fluctuations -amplified through nonlinearities- and those stabilizing phenomena (as diffusion) find the most creative boundary at the edge of chaos.

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Tables

CONNECTIVITY	CYCLE LENGTH	N ATTRACTORS	STABILITY
$K = N$	$0.5 \times 2^{N/2}$	N/e	low
$K = 1$	$exp(\frac{1}{8}log^2 N)$	$(2/\sqrt{e})^N$	low
$K = 2$	$\sqrt{N}$	$\sqrt{N}$	high

Table I : Properties of Kauffman Nets for different values of K

SYSTEM	MODEL(S)	PARAMETER	PROPERTIES
Ferromagnet	Ising Model	Temperature	Fractals/Power laws
Sand pile	Sand pile CA	SOC	$1/f$-noise
Macroevolution	Kauffman, BS models	SOC	Extinction pattern
Rainforests	Forest Game	Interaction strength	Fractal patterns
RNA viruses	Eigen Model	Mutation rate	Error catastrophe
Ant colonies	Fluid Neural Nets	Density	Maximum I transfer
Brain Dynamics	Jirsa et al.	Behavioral	Critical slowing down
Genome	Kauffman nets	Connectivity	Antichaos

Table II : Phase transitions and complex systems (SOC: self-organized criticality)

Figure Captions

FIGURE 1: First order phase transition. This surface is obtained from the stationary solutions of eq.(2), defined over the (μ, β) parameter space. As a given parameter is varied continuously from an initial value (**i**) to a new one (**f**), the system undergoes a sudden shift from the upper branch to the lower one (see text).

FIGURE 2: Second-order phase transitions. (a) As defined from equation (4), in a 2PHT two new branches appear symmetrically from the old one, which becomes unstable. A symmetry-breaking phenomenon takes place and one of the two possible solutions is choosen, as shown by the stochastic trajectory shown (see text). (b) A mechanical illustration of symmetry-breaking phenomena. A potential $\phi(x)$ can be represented by a surface with two minima. As the ball moves down (starting from a symmetric initial condition) any small fluctuation will decide the final state.

FIGURE 3: Noise-induced phase transition. Here the deterministic dynamical system given by eq.(10) is perturbed by means of a multiplicative noise term (see text). The stationary probability density (as given by eq. (9)) is shown. As the noise level is increased, the single maximum is replaced by two new estrema.

FIGURE 4: Phase transitions and the Ising model. Three snapshots of a 200×200 lattice of spins are shown, for different temperatures T. (a) $T = 1.2T_c$; (b) $T = T_c$ (critical point, Curie temperature); (a) $T = 0.95T_c$.

FIGURE 5: Mutual information M (as defined in section 2.3) for the Ising model. The information transfer betwen two nearest spins has been calculated by using a 20×20 lattice, averaging over 500 time steps, using 50 configurations. A maximum is obtained at the critical point.

FIGURE 6: Patterns of behavior of one-dimensional CA models, as defined by the Wolfram's classification: (a) class I; (b) class II; (c) class III and (d) class IV.

FIGURE 7: (a) The real sandpile; (b) power-law distribution obtained from a computer simulation of a two-dimensional sand pile cellular automaton, as defined in section 4.1

FIGURE 8: Mutual information in a fluid neural network (see section 5.2), calculated for different densities of automata on a 10×10 lattice, and following two

given individuals over 10^4 time steps. A maximum is obtained at a critical density (see text).

FIGURE 9: Power-law behavior of the Bak-Sneppen model of large-scale evolution. The lifetime of "evolutionary avalanches" is shown. It is conjectured that this quantity is related with the life span of species over time.

FIGURE 10: The error catastrophe. By using the model defined by equation (17), we plot the stationary concentrations of the concentrations $x_k = n_k / \sum n_j$ ($k = 1, ..., 50$). As the error rate is increased, a critical boundary is reached where a phase transition takes place. It has been shown that, typically, RNA viruses replicate clos eto such error threshold (see text).

FIGURE 11: Power-law distributions of gap sizes for the Barro Colorado rainforest (see text, dashed line) and for the forest game cellular automaton (continuous line). The last curve has been calculated at the so called "complex forest" domain of parameter space. The agreement between these sets of data seem to suggest that the BCI rainforest would be poised at thje critical state.

References

(1) G. Nicolis and I. Prigogine, Exploring Complexity, Freeman, New York, 1989

(2) P. Grassberger. How to measure self-generated complexity?. Physica A

(3) J. P. Crutchfield and K. Young. Computation at the onset of chaos, in: Complexity, Entropy and the Physics of Information. SFI Studies in the Sciences of Complexity, III. Addison-Wesley (1990)

(4) C. Langton, Computation at the edge of chaos. Physica D42 (1990) 12-32

(5) B. A. Huberman and T. Hogg. Complexity and adaptation. Physica D22 (1986) 376-384

(6) J. J. Binney, N. W. Dowrick, A. J. Fisher and M. E. J. Newman. The theory of critical phenomena. An introduction of the renormalization group. Clarendon Press. Oxford (1993)

(7) W. Horsthemke and R. Lefever, Noise-induced transitions, Springer-Verlag, Berlin, 1984

(8) H. Haken, Synergetics, Springer, Berlin, 1977

(9) D. Stauffer and A. Aharony. Introduction to Percolation Theory (2nd edition). Taylor and Francis, London (1994)

(10) W. Li, Mutual information functions versus correlation functions. J. Stat. Phys. 60 (1990) 823-837

(11) W. Li, "On the relationship between complexity and entropy for Markov chains and regular languages", Complex Systems 5, 1991, 381-399

(12) S. Wolfram, Cellular Automata and Complexity (Collected Papers), Addison-Wesley, Reading, 1994

(13) M. Mitchell, P. Hraber and J.P.Crutchfield. Evolving cellular automata to perform computations. Complex Systems 7 (1993) 89-130

(14) H. Haken. Information and self-organization. Springer-Verlag. Berlin (1989)

(15) P. Rujn. Cellular automata and statistical mechanical models. J. Stat.

Phys. 49 (1987) 139-222

(16) P. Bak, C. Tang and K. Wiesenfeld. Self-organized criticality. Phys. Rev. A 38 (1988) 364-374

(17) P. Bak and M. Creutz. Fractals and selforganized criticality. In: Fractals in Science. A. Bunde and S. Harlin (eds). Springer-Verlag. Berlin (1994)

(18) S. Kauffman, The origins of order, Oxford U. Press, Oxford, 1993

(19) B. Derrida and Y. Pomeau. Random networks of automata: a simple annealed approximation. Europhys. Lett. 1 (1986) 45-52

(20) R. V. Sol and B. Luque. Phase transitions and antichaos in generalized Kauffman networks. Phys. Lett. A196 (1995) 331-334

(21) E. O. Wilson, The Insect Societies, Harvard U. Press, Cambridge, 1971

(22) J.L.Deneubourg, S.Goss, J.M.Pasteels and G. Duerinck. Random behavior, amplification processes and number of participants: how they contribute to the foraging properties of ants. Physica D22 (1986) 176-186

(23) M. Millonas. Swarm field dynamics and functional morphogenesis. Proc. of ECAL 1993. Universite Libre de Bruxelles (1993)

(24) N.R. Franks, S. Bryant, R.Griffiths and L. Hemerik. Synchronization of the behavior within nests of the ant *Leptothorax acervorum*. Bull.Math.Biol. 52 (1990) 597-612.

(25) B. J. Cole. Short-term activity cycles in ants: generation of periodicity by worker interaction. Am. Nat. 137 (1991) 244-259

(26) B. J. Cole. Is animal behavior chaotic?: evidence from the activity of ants. Proc. R. Soc. Lond. B244 (1991) 253-259

(27) R. V. Sol and O. Miramontes, "Information at the edge of chaos in fluid neural networks", Physica D80, 1995, pp. 171-180

(28) N.R.Franks, A.Wilby, B.W.Silverman and C.Tofts. "Self-organizing nest construction in ants". Anim. Behav. 44 (1992) 357-375

(29) J. Delgado and R. V. Solé, "Evidence of a Noise Induced Transition in

Fluid Neural Networks", Research Report LSI 95-10-R (Universitat Politècnica de Catalunya). Submitted to Physica D

(30) P. Bak and M. Paczuski, "Complexity, Contingency and Criticality", Proc. Natl. Acad. Sci. USA, in press, 1995

(31) B. Burlando. "The fractal geometry of evolution". J. Theor. Biol. 163 (1993) 161-172

(32) R. V. Solé and J. Bascompte. Are critical phenomena relevant in large-scale evolution?. Preprint.

(33) P. Bak and K. Sneppen. Punctuated equilibrium in a simple model of evolution. Phys. Rev. Lett. 58 (1993) 381-384

(34) S. J. Gould and N. Eldredge, "Punctuated equilibrium comes of age", Nature 366 (1993) 223-227

(35) J. O. Kephart, T. Hogg and B. A. Huberman. Dynamics of computational ecosystems. Phys. Rev. A40 (1989) 404-421

(36) M. Eigen, J. McCaskill and P. Schuster. "The Molecular Quasispecies". Adv. Chem. Phys. 75 (1989) 149-263

(37) I. Leuthausser. "An exact correspondence between Eigen's evolution model and a two-dimensional Ising system". J. Chem. Phys. 84 (1986) 1884-1885

(38) P. Schuster. How do RNA molecules and viruses explore their RNA worlds?. In: Complexity: Metaphors, Models and Reality. SFI Studies in the Sciences of Complexity, vol. XIX (1994). 382-413

(39) E. Domingo and J.J.Holland. Mutation rates and rapid evolution of RNA viruses. In: The evolutionary biology of viruses. S. S. Morse (ed). Raven Press, Ltd. New York (1994) 161-184

(40) J. H. Connell. "Diversity in tropical rainforests and coral reefs". Science 199 (1978) 1302-1310

(41) R. V. Sol and S. C. Manrubia. "Are rainforests self-organized in a critical state? ". J. Theor. Biol. 173 (1995) 31- 40

(42) R. V. Sol and S. C. Manrubia. "Self-similarity in rainforests: evidence for

a critical state? ". Phys. Rev. E51 (1995)

(43) J. A. S. Kelso, S. L. Bressler, S. Buchanan, G. C. deGuzman, M. Ding, A. Fuchs and T. Holroyd. "A phase transition in human brain and behavior". Phys. Lett. A 169 (1992) 134-144

(44) V. K. Jirsa, R. Friedrich, H. Haken and J. A. S. Kelso. A theoretical model of phase transitions in the human brain. Biol. Cybern. 71 (1994) 27-35

(45) J. Bascompte and R. V. Sol. "Habitat fragmentation and extinction thresholds in spatially explicit models" (preprint)

(46) B. C. Goodwin. "How the leopard changed its spots". Weinfeld and Nicolson. London (1994)

(47) I. Prigogine and I. Stengers. Order out of Chaos. Bantman Books Inc. New York (1984)

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