A Set of Rules for a Constructive Geometric Constraint Solver

Robert Juan-Arinyo and Antoni Soto
Departament de Llenguatges i Sistemes Informàtics
Universitat Politècnica de Catalunya
Av. Diagonal 647, 8a, E–08028 Barcelona

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Abstract

This paper presents the set of constructive rules used by a two-dimensional, variational geometric constraint solver based on a constructive approach. The main goal in defining the rules has been to keep the set of rules as small as possible while being able to yield a useful solver with a large enough domain of solvable constructions.

The rules can be classified according to their functionality as belonging to one of three possible types: creation rules, merging rules, and constructive rules. Creation rules translate the geometric constraints into the appropiate internal representation. Merging rules are used to compute the transitive closure of angle constraints. Construction rules allow to build triangles and they also build some quadrilaterals. The rules can build any triangle defined by a set of three independent relationships given as either distances between two vertices, angles between two sides and altituds. Quadrilateral rules can build three especific, different quadrilaterals.

1 Introduction

Constraints are an important enabling technology for interactive graphics applications. One of such applications where the use of constraints is growing more rapidly is in geometric modelling. The goal is to develop sketching systems based on geometric constraints solvers in which the user defines a rough sketch, annotated with dimensions and constraints, and the system builds an instance that satisfies all constraints.

Several approaches to the geometric constraint solving problem have been reported in the literature. Two main approaches have been addressed in geometric constraints solving: non-linear simultaneous equation solving and constructive approach. In the first approach, constraints are translated to equations. The resulting system of non-linear simultaneous equations is solved using different techniques such as numerical methods, as Newton-Raphson [10] or Relaxation, [2, 8, 15]; symbolic algebraic methods, as Gröbner basis [5] and the Buchberger's algorithm [4] or Wu-Ritt [6, 18] method; or propagation methods [12, 13, 15]. In the constructive approach, constraints are satisfied constructively, by placing geometric elements in some order. It is based on the fact that most configurations on engineering drawings are solvable by using a rather small set of tools like ruler, compass and protractor. Constructive methods are further classified in graph based methods, [3, 7, 11] and rule based mathods, [1, 14, 16, 17]. A thorough review on geometric constraint solving techniques can be found in [3].

This paper describes the set of constructive rules we have selected for the two-dimensional, variational geometric constraint solver based on a constructive approach that our group is building. Both the data representation and the rules used are based on those reported in [16] and [17], but differ from them in some rather important details. The differences are mainly due to the assumption made in [17] that the distance constraint perpendicular to a segment defined by the user has been translated in a preprocess into distance and angle constraints. We will show that this assumption has major drawbacks. As a result, the data representation given in [17] has been extended to include explicitly a new constraint type that represents the perpendicular distance between a point and a segment. This extension is justified in Section 2. Consistently, new rules have been defined to deal with these new constraints. These rules are based on the construction of triangles with ruler and compass collected in [9]. We also include the construction of the quadrilaterals given in [17]. We demonstrate that each of these quadrilateral rules can be expressed as a sequence of creation rules and construction rules where the angles constraint merging rule is not used.

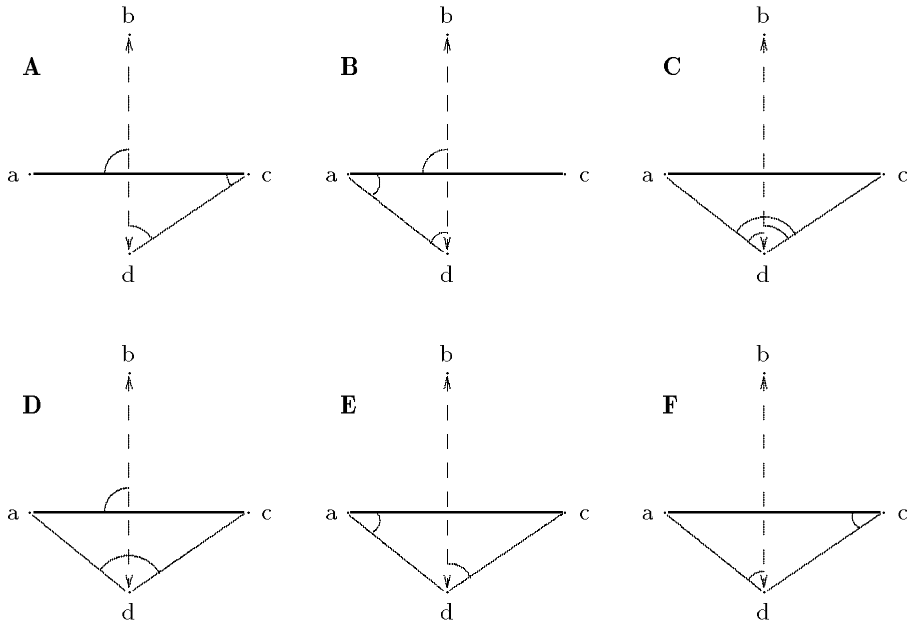


Figure 1: Possible translations of the altitude constraint.

2 Data Representation

We consider two-dimensional geometric models formed from points, segments, circles and arcs. The user can define consraints by giving relations of distances, angles, parallelism, incidence, perpendicularity, tangency, etc. Internally, constraints are represented by relations of distance between two points, angle between two segments, and perpendicular distance from a point to a segment. In what follows, a dimensioning scheme will be a set of points and the set of constraints defined on these points.

The data representation used is based on the representations reported in [14, 16, 17], but differs from them in some significative details. In [17] it is assumed that the distance constraint perpendicular to a segment defined by the user has been translated into distance and angle constraints before calling the solver. Let us show that this assumption has a major drawback.

Let b be the point and (a, c) the segment on which the distance constraint perpendicular to a segment from a point has been defined. This constraint can be represented by means of distance between points and angle between two segments constraints by translating the three points problem into a four points problem. All we have to do is to add an extra point d and two new constraints. The distance between point b and the extra point d must be the initial distance value. The two new constraints will represent the fact that the extra point d must be incident to the line supporting the segment (a, c). Figure 1 shows all the possible translations of the three points problem to the four points problem. In this figure solid lines represent segments, an arc of circle joining two segments represents an angle constraint, and a dashed line represents the former perpendicular distance from a point to a segment. Note that in cases **A**, **B**, and **C** in Figure 1 although only two out of the three marked angles must be generated by the translation process, the third one is also known by the transitive property of the angle constraints.

Figure 2 shows the different dimensioning schemes that can be defined on three points where one constraint is an altitud of the triangle. Constraints are represented in Figure 2 in the same way than in Figure 1 but now a dashed line represents both the perpendicular distance from a point to a segment and distance between two points.

Theorem 2.1. Among all the possible translations of the distance constraint perpendicular to a segment from a point into distance and angle constraints, there is not a unique translation such that allows to solve all the triangles specified by dimensioning schemes where one or more altitudes are included, applying just the set of rules defined in [17].

Proof. We shall prove the theorem by checking each different possible translation of the three points problem into a four points problem.

1. Translation **A**. The dimensioning schemes number **8** and **9** in Figure 2 cannot be solved by translation **A**, but can be solved by translation **B**.

2. Translation **B**. The dimensioning schemes number **7** and **10** in Figure 2 cannot be solved by translation **B**, but can be solved by translation **A**.

3. Translation **C**. The dimensioning schemes number **4**, **5**, **11**, **12**, **13**, **14** and **15** in Figure 2 cannot be solved by translation **C**, but can be solved by translations **A** and **B**.

4. Translation **D**. The dimensioning schemes number **1**, **2**, **3**, **4**, **5**, **9**, **10**, **11**, **12**, **13**, **14** and **15** in Figure 2 cannot be solved by translation **D**, but **1**, **2**, **3**, **4**, **5**, **10**, **11**, **12**, **13**, **14** and **15** can be solved by translation **A** and the scheme number **9** can be solved by translation **B**.

5. Translation **E**. The dimensioning schemes number **1**, **3**, **4**, **5**, **9**, **10**, **11**, **12**, **13**, **14** and **15** in Figure 2 cannot be solved by translation **E**, but **1**, **3**, **4**, **5**, **10**, **11**, **12**, **13**, **14** and **15** can be solved by translation **A** and the scheme number **9** can be solved by translation **B**.

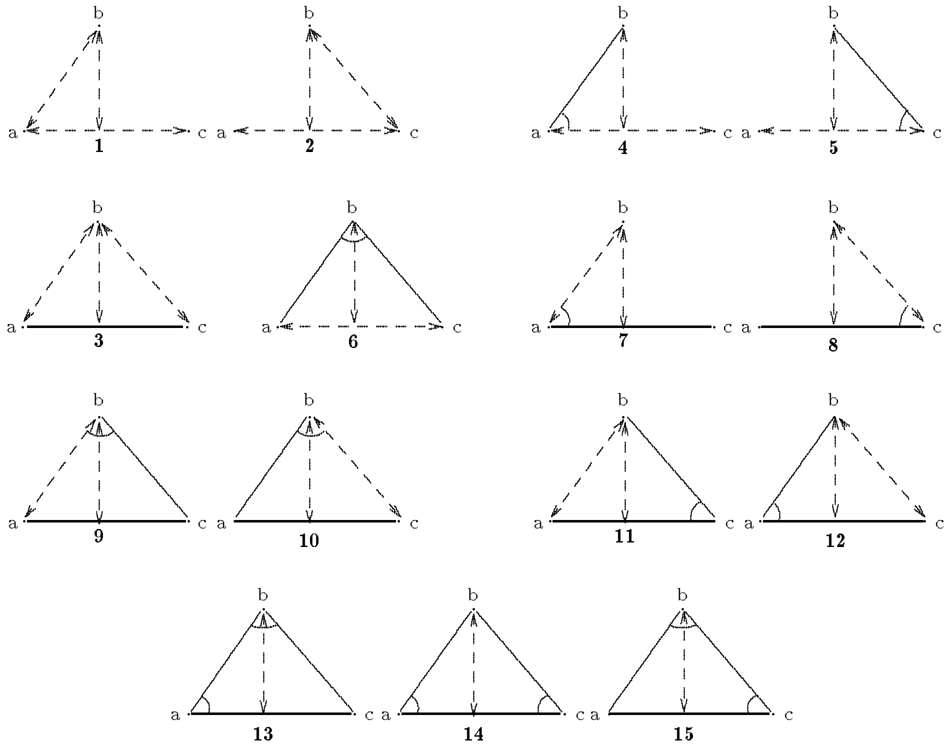


Figure 2: Possible well constrained dimensioning schemes on three points with one altitude constraint.

6. Translation **F**. The dimensioning schemes number **2**, **3**, **4**, **5**, **9**, **10**, **11**, **12**, **13**, **14** and **15** in Figure 2 cannot be solved by translation **E**, but **2**, **3**, **4**, **5**, **10**, **11**, **12**, **13**, **14** and **15** can be solved by translation **A** and the scheme number **9** can be solved by translation **B**.

□

This means that, in the framework of these constructive rules, the constraint translation process is context-dependent, i.e. the dimensioning scheme defined by the user should be recognized in order to translate it to a proper representation. Note that this is equivalent to having available a specific rule for each situation where the distance constraint perpendicular to a segment from a point is used. Since this constraint is so often used, we have included a specific representation for it. Therefore, besades the **CD** sets and the **CA** sets given in [14] and [17], we include the **CH** sets. These constraints sets are defined as follows.

A **CA** set is a pair of oriented segments constrained in angle.

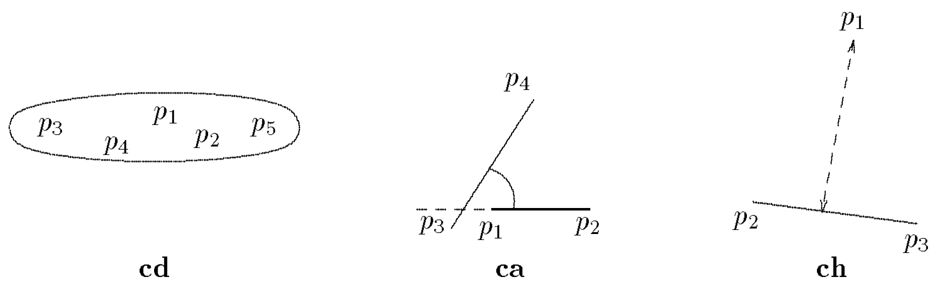


Figure 3: Representation of **CD**, **CA** and **CH** sets

A **CD** set is a set of points with mutually constrained distances. A frame of reference is attached to each **CD** set to which the points in the set are refered to, [14, 17]. It is worth to note that a sketch is solved when all the points in it belong to the same **CD** set.

A **CH** set is a point and a segment constrained by the perpendicular distance from the point to the segment.

Note that the **CA** set definition here is the same than that given in [17], but there is a difference in the role the **CA** sets will play. In [17] **CA** sets are initialy created from angle constraints. Then when two **CA** sets share a segment they are merged, representing then the transitive closure of the angle constraints and the new **CA** set no longer fulfills the definition. Here we will figure out the transitive closure of angle constraints as a preprocess and a **CA** set will always fulfill its definition. Keeping the meaning of the **CA** sets in this way facilitates theoretical reasoning.

In the sequel we will represent graphically a **CD** set by a closed solid line enclosing the points in the set, a **CA** set by an arc joining the two segments in the set, and a **CH** set by a dashed line joining the point an the segment in the set. See Figure 3.

3 Rules

In order to deal with the **CH** sets, we have consistently extended the rules given in [16] and in [17]. According to the functionality of the rules we classify them as creation rules, merging rules or construction rules. Now we shall give a formal definition for each rule.

3.1 Creation Rules

Creation rules create elementary **CD**, **CA** and **CH** sets as an interpretation of the dimensioning scheme defined by the user.

Rule C1

When an angle constraint between two directed segments is defined, a **CA** set is created.

$$\text{angle}((p_1, p_2), (p_3, p_4)) \longrightarrow \text{ca}'$$

where

$$\text{ca}' = \{(p_1, p_2), (p_3, p_4)\}$$

Rule C2

When a distance constraint between two points is given, a **CD** set is created.

$$\text{distance}(p_1, p_2) \longrightarrow \text{cd}'$$

where

$$\text{cd}' = \{p_1, p_2\}$$

Rule C3

Whenever a point, a segment and the perpendicular distance from the point to the segment are given, a **CH** set is created.

$$\text{pdistance}(p_1, (p_2, p_3)) \longrightarrow \text{ch}'$$

where

$$\text{ch}' = \{p_1, (p_2, p_3)\}$$

3.2 Merging Rules

There is only one merging rule. This rule allows to compute the transitive closure of the angle constraints.

Rule M

When two different **CA** sets share one segment a new **CA** set constraining the two non-common segments is created.

$$\{\mathrm{ca}_1, \mathrm{ca}_2\} \longrightarrow \{\mathrm{ca}_1, \mathrm{ca}_2, \mathrm{ca}'\}$$

where

$$\begin{aligned} \mathrm{ca}_1 &= \{(p_1, p_2), (p_3, p_4)\} \\ \mathrm{ca}_2 &= \{(p_1, p_2), (p_5, p_6)\} \\ \mathrm{ca}' &= \{(p_3, p_4), (p_5, p_6)\} \end{aligned}$$

3.3 Construction Rules

Construction rules merge either **CD** sets, **CH** sets or **CA** sets into larger **CD** sets. Basic construction rules merge three constraint sets into a **CD** set. Quadrilateral rules merge five constraint sets into a **CD** set, but, as shown in [16, 17], these rules can be expressed in terms of the basic construction rules by adding an extra point and two constraints derived from the initial set of constraints. We give a description of the set of construction rules according to the type of the constraints sets they merge.

3.3.1 Rules involving three CD sets

These rules merge three **CD** sets that pairwise share a point. The general rule is

$$\{\mathrm{cd}_1, \mathrm{cd}_2, \mathrm{cd}_3\} \longrightarrow \{\mathrm{cd}'\}$$

and there is only one of such rules.

Rule DDD

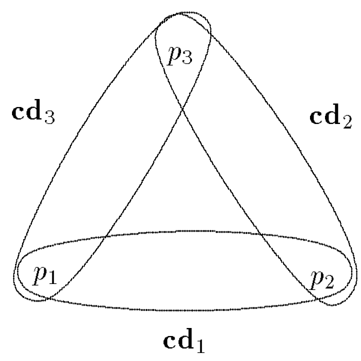
Three **CD** sets pairwise share a point.

If

$$\begin{aligned} \mathrm{cd}_1 &= \{p_1, p_2, \cdots\} \\ \mathrm{cd}_2 &= \{p_2, p_3, \cdots\} \\ \mathrm{cd}_3 &= \{p_3, p_1, \cdots\} \end{aligned}$$

then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \mathrm{cd}_2 \cup \mathrm{cd}_3$$



3.3.2 Rules involving two CD sets and one CA set

These rules merge two **CD** sets that share a point and that are constrained by a **CA** set. The general rule is

$$\{\mathrm{cd}_1, \mathrm{cd}_2, \mathrm{ca}_1\} \longrightarrow \{\mathrm{cd}'\}$$

Since a **CA** set can constrain two **CD** sets in three ways there are three different rules.

Rule DDA1

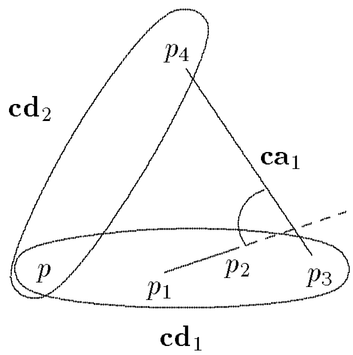
The points of one segment in the **CA** set belong to one **CD** set and the other segment of the **CA** set shares each of its points with a different **CD** set.

If

$$\begin{aligned} \mathrm{cd}_1 &= \{p, p_1, p_2, p_3, \cdots\} \\ \mathrm{cd}_2 &= \{p, p_4, \cdots\} \\ \mathrm{ca}_1 &= \{(p_1, p_2), (p_3, p_4)\} \end{aligned}$$

then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \mathrm{cd}_2$$



Rule DDA2

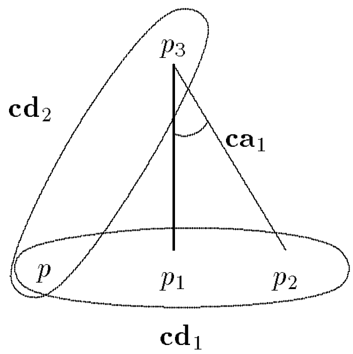
The two segments in the **CA** set have a common point. In addition, this point is shared by one **CD** set. The other two points of the segments in the **CA** set belong to the other **CD** set.

If

$$\begin{aligned} \mathrm{cd}_1 &= \{p, p_1, p_2, \cdots\} \\ \mathrm{cd}_2 &= \{p, p_3, \cdots\} \\ \mathrm{ca}_1 &= \{(p_1, p_3), (p_2, p_3)\} \end{aligned}$$

then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \mathrm{cd}_2$$

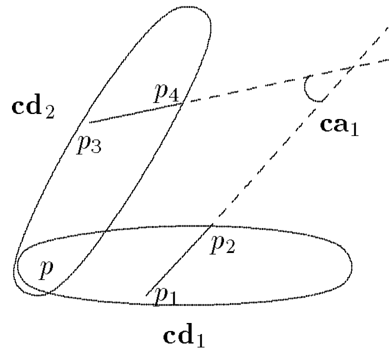


Rule DDA3

The points of one segment in the **CA** set belong to one **CD** set and the points of the other segment belong to the other **CD** set.

If

$$\begin{aligned} \mathrm{cd}_1 &= \{p, p_1, p_2, \cdots\} \\ \mathrm{cd}_2 &= \{p, p_3, p_4, \cdots\} \\ \mathrm{ca}_1 &= \{(p_1, p_2), (p_3, p_4)\} \end{aligned}$$



then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \mathrm{cd}_2$$

3.3.3 Rules involving two CD sets and one CH set

These rules merge two **CD** sets that share a point and are constrained by a **CH** set. The general rule is

$$\{\mathrm{cd}_1, \mathrm{cd}_2, \mathrm{ch}_1\} \longrightarrow \{\mathrm{cd}'\}$$

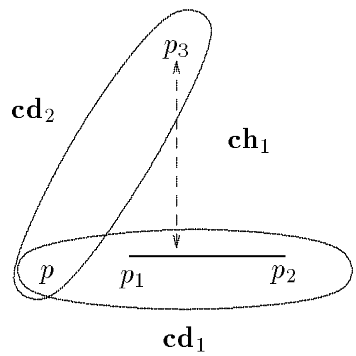
A **CH** set can constrain two **CD** sets in two ways, hence there are two different cases.

Rule DDH1

The two points that define the segment in the **CH** set belong to one **CD** set while the point in the **CH** set is shared by the **CH** and the other **CD** set.

If

$$\begin{aligned} \mathrm{cd}_1 &= \{p, p_1, p_2, \cdots\} \\ \mathrm{cd}_2 &= \{p, p_3, \cdots\} \\ \mathrm{ch}_1 &= \{p_3, (p_1, p_2)\} \end{aligned}$$



then

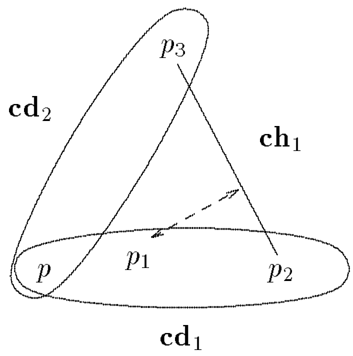
$$\mathrm{cd}' = \mathrm{cd}_1 \cup \mathrm{cd}_2$$

Rule DDH2

Each point of the segment in the **CH** set is shared by the **CH** set and one of the **CD** sets. The point in the **CH** set is shared with one **CD** set.

If

$$
\begin{aligned}
\mathrm{cd}_1 &= \{p, p_1, p_2, \cdots\} \\
\mathrm{cd}_2 &= \{p, p_3, \cdots\} \\
\mathrm{ch}_1 &= \{p_1, (p_2, p_3)\}
\end{aligned}
$$



then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \mathrm{cd}_2$$

3.3.4 Rules involving one CD set and two CA sets

A point shared by two **CA** sets is adjoined to the **CD** set by the rule

$$\{\mathrm{cd}_1, \mathrm{ca}_1, \mathrm{ca}_2\} \longrightarrow \{\mathrm{cd}'\}$$

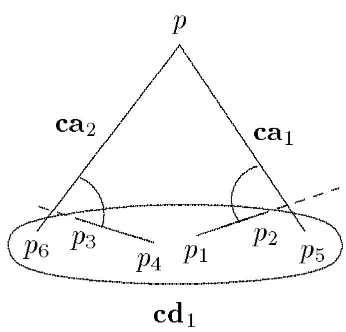
There are three different cases.

Rule DAA1

Three points from each **CA** set belong to the **CD** set. The **CA** sets share their fourth point.

If

$$
\begin{aligned}
\mathrm{cd}_1 &= \{p_1, p_2, p_3, p_4, p_5, p_6, \cdots\} \\
\mathrm{ca}_1 &= \{(p, p_5), (p_1, p_2)\} \\
\mathrm{ca}_2 &= \{(p, p_6), (p_3, p_4)\}
\end{aligned}
$$



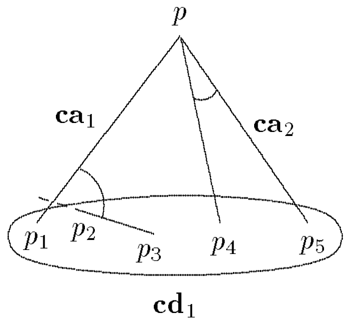
then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \{p\}$$

Rule DAA2

One **CA** set is such that two of its segments share a point and the other two different points belong to the **CD** set. The other **CA** set shares three points with the **CD** set and shares its fourth point with the common point in the first **CA** set.

If



$$\begin{aligned} \mathrm{cd}_1 &= \{p_1, p_2, p_3, p_4, p_5, \cdots\} \\ \mathrm{ca}_1 &= \{(p, p_1), (p_2, p_3)\} \\ \mathrm{ca}_2 &= \{(p, p_4), (p, p_5)\} \end{aligned}$$

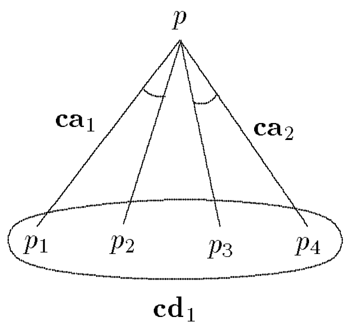
then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \{p\}$$

Rule DAA3

All the segments in the two **CA** sets share a point which does not belong to the **CD** set. The other four points from the **CA** sets belong to the **CD** set.

If



$$\begin{aligned} \mathrm{cd}_1 &= \{p_1, p_2, p_3, p_4, \cdots\} \\ \mathrm{ca}_1 &= \{(p, p_1), (p, p_2)\} \\ \mathrm{ca}_2 &= \{(p, p_3), (p, p_4)\} \end{aligned}$$

then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \{p\}$$

3.3.5 Rules involving one CD set, one CA set and one CH set

Given a **CA** set and a **CH** set with a common point, the rule

$$\{\mathrm{cd}_1, \mathrm{ca}_1, \mathrm{ch}_1\} \longrightarrow \{\mathrm{cd}'\}$$

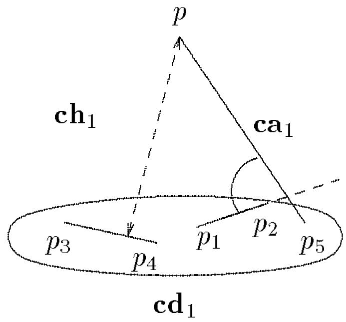
adjoins this common point to the **CD** set. There are four different cases.

Rule DAH1

Three points from the **CA** set and the segment from the **CH** set belong to the **CD** set. The **CA** set and the **CD** set have in common the remaining point.

If

$$
\begin{aligned}
\mathrm{cd}_1 &= \{p_1, p_2, p_3, p_4, p_5, \cdots\} \\
\mathrm{ca}_1 &= \{(p, p_5), (p_1, p_2)\} \\
\mathrm{ch}_1 &= \{p, (p_3, p_4)\}
\end{aligned}
$$



then

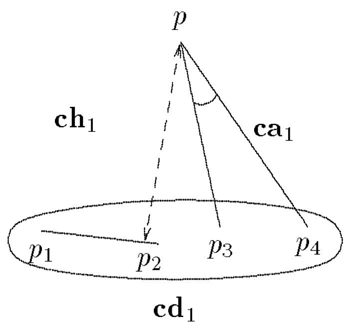
$$\mathrm{cd}' = \mathrm{cd}_1 \cup \{p\}$$

Rule DAH2

Two segments in the **CA** set end in the **CH** set point. The **CD** set has a segment in common with the **CH** set and two different points with the **CA** set.

If

$$
\begin{aligned}
\mathrm{cd}_1 &= \{p_1, p_2, p_3, p_4, \cdots\} \\
\mathrm{ca}_1 &= \{(p, p_3), (p, p_4)\} \\
\mathrm{ch}_1 &= \{p, (p_1, p_2)\}
\end{aligned}
$$



then

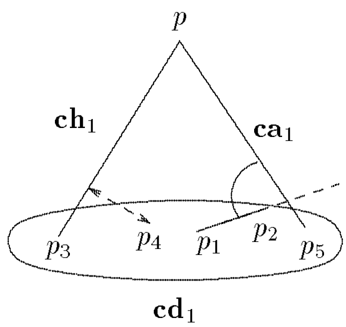
$$\mathrm{cd}' = \mathrm{cd}_1 \cup \{p\}$$

Rule DAH3

The segment in the **CH** set and one segment in the **CA** set have a point in common that is not shared by the **CD** set. The remaining points in both the **CH** set and **CA** set belong to the **CD** set.

If

$$
\begin{aligned}
\mathrm{cd}_1 &= \{p_1, p_2, p_3, p_4, p_5, \cdots\} \\
\mathrm{ca}_1 &= \{(p, p_5), (p_1, p_2)\} \\
\mathrm{ch}_1 &= \{p_4, (p_3, p)\}
\end{aligned}
$$



then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \{p\}$$

Rule DAH4

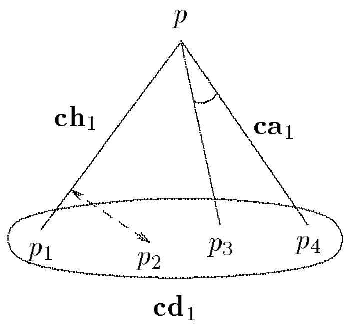
The segment in the **CH** set and the two segments in the **CA** set have a point in common that is not shared by the **CD** set. The remaining points in both the **CH** set and **CA** set belong to the **CD** set.

If

$$\begin{aligned} \mathrm{cd}_1 &= \{p_1, p_2, p_3, p_4, \cdots\} \\ \mathrm{ca}_1 &= \{(p, p_3), (p, p_4)\} \\ \mathrm{ch}_1 &= \{p_2, (p_1, p)\} \end{aligned}$$

then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \{p\}$$



3.3.6 Rules involving one CD set and two CH sets

A point that two **CH** sets have in common is adjoined to the **CD** set. The general rule is

$$\{\mathrm{cd}_1, \mathrm{ch}_1, \mathrm{ch}_2\} \longrightarrow \{\mathrm{cd}'\}$$

and there are three possible cases.

Rule DHH1

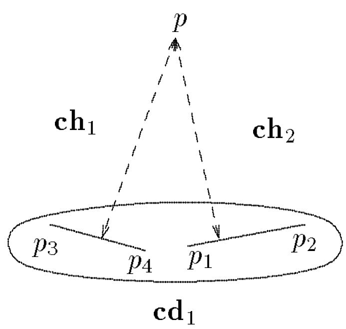
The point in each **CH** set is the common point. The segments in each **CH** set belong to the **CD** set.

If

$$\begin{aligned} \mathrm{cd}_1 &= \{p_1, p_2, p_3, p_4, \cdots\} \\ \mathrm{ch}_1 &= \{p, (p_3, p_4)\} \\ \mathrm{ch}_2 &= \{p, (p_1, p_2)\} \end{aligned}$$

then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \{p\}$$

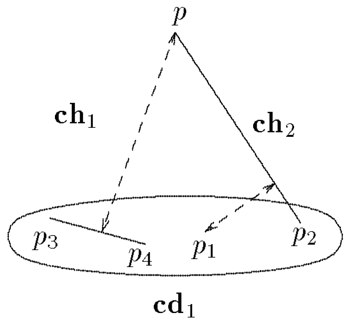


Rule DHH2

The segment from one **CH** set ends in the point from the other **CH** set. The remaining points in both **CH** sets belong to the **CD** set.

If

$$\begin{aligned} \mathrm{cd}_1 &= \{p_1, p_2, p_3, p_4, \cdots\} \\ \mathrm{ch}_1 &= \{p, (p_3, p_4)\} \\ \mathrm{ch}_2 &= \{p_1, (p_2, p)\} \end{aligned}$$



then

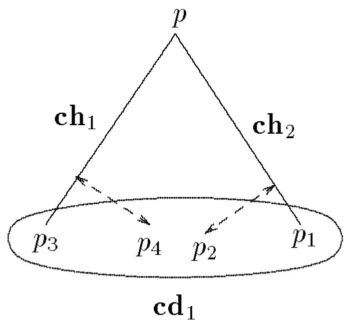
$$\mathrm{cd}' = \mathrm{cd}_1 \cup \{p\}$$

Rule DHH3

The segments in the **CH** sets have a point in common which does not belong to the **CD** set. The remaining points in the **CH** sets belong to the **CD** set.

If

$$\begin{aligned} \mathrm{cd}_1 &= \{p_1, p_2, p_3, p_4, \cdots\} \\ \mathrm{ch}_1 &= \{p_4, (p_3, p)\} \\ \mathrm{ch}_2 &= \{p_2, (p_1, p)\} \end{aligned}$$



then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \{p\}$$

3.3.7 Rules involving two CA sets and one CH set

These rules create a new **CD** set with three points that are constrained by two **CA** sets and one **CH** set. The general rule is

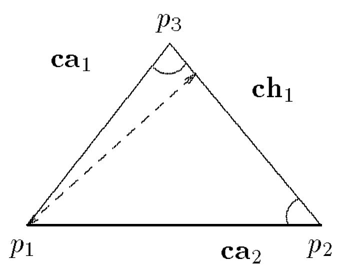
$$\{\mathrm{ca}_1, \mathrm{ca}_2, \mathrm{ch}_1\} \longrightarrow \{\mathrm{cd}'\}$$

and there is only one case.

Rule AAH

If

$$\begin{aligned} \mathrm{ca}_1 &= \{(p_1, p_2), (p_2, p_3)\} \\ \mathrm{ca}_2 &= \{(p_2, p_3), (p_1, p_3)\} \\ \mathrm{ch}_1 &= \{p_1, (p_2, p_3)\} \end{aligned}$$



then

$$\mathrm{cd}' = \{p_1, p_2, p_3\}$$

3.3.8 Rules involving one CA set and two CH sets

These rules create a new **CD** set with three points that are constrained by one **CA** set and two **CH** sets. The general rule is

$$\{\mathrm{ca}_1, \mathrm{ch}_1, \mathrm{ch}_2\} \longrightarrow \{\mathrm{cd}'\}$$

and there are two different cases.

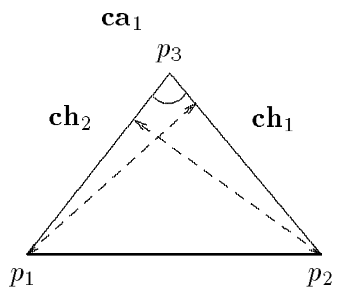
Rule AHH1

If

$$\begin{aligned} \mathrm{ca}_1 &= \{(p_2, p_3), (p_1, p_3)\} \\ \mathrm{ch}_1 &= \{p_1, (p_2, p_3)\} \\ \mathrm{ch}_2 &= \{p_2, (p_1, p_3)\} \end{aligned}$$

then

$$\mathrm{cd}' = \{p_1, p_2, p_3\}$$



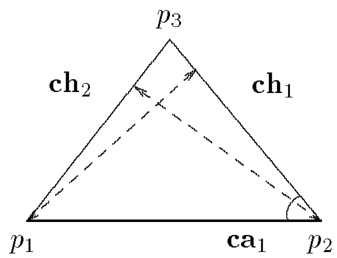
Rule AHH2

If

$$\begin{aligned} \mathrm{ca}_1 &= \{(p_1, p_2), (p_2, p_3)\} \\ \mathrm{ch}_1 &= \{p_1, (p_2, p_3)\} \\ \mathrm{ch}_2 &= \{p_2, (p_1, p_3)\} \end{aligned}$$

then

$$\mathrm{cd}' = \{p_1, p_2, p_3\}$$



3.3.9 Rules involving three CH sets

A new **CD** set is created when three points are constrained by three **CH** sets. The rule is

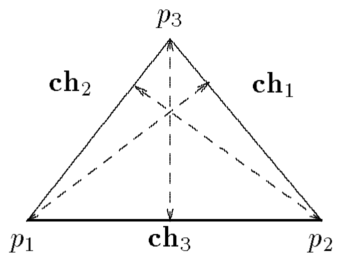
$$\{\mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3\} \longrightarrow \{\mathrm{cd}'\}$$

and there is just one case.

Rule HHH

If

$$\begin{aligned} \mathrm{ch}_1 &= \{p_1, (p_2, p_3)\} \\ \mathrm{ch}_2 &= \{p_2, (p_1, p_3)\} \\ \mathrm{ch}_3 &= \{p_3, (p_1, p_2)\} \end{aligned}$$



then

$$\mathrm{cd}' = \{p_1, p_2, p_3\}$$

3.3.10 Quadrilateral Rules

Three quadrilateral rules were described in [16] and [17]. For each quadrilateral problem first we will give the general rule. Then we will show that the quadrilateral problem can be transformed into an appropiate sequence of triangular problems by adding an extra point and some constraints derived from those constraints already defined in the dimensioning scheme.

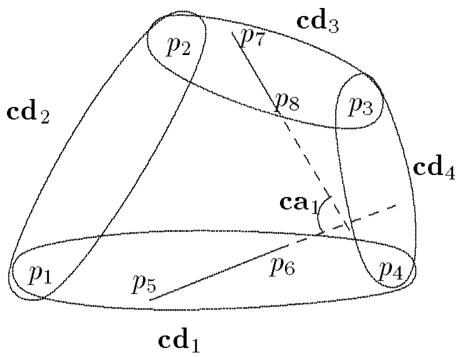
Rule Q1

Four **CD** sets with a pairwise common point and with two non-adjacent **CD** sets constrained by a **CA** set can be merged into one **CD** set. The general rule is

$$\{\mathrm{cd}_1, \mathrm{cd}_2, \mathrm{cd}_3, \mathrm{cd}_4, \mathrm{ca}_1\} \longrightarrow \{\mathrm{cd}'\}$$

If

$$\begin{aligned} \mathrm{cd}_1 &= \{p_1, p_4, p_5, p_6, \cdots\} \\ \mathrm{cd}_2 &= \{p_1, p_2, \cdots\} \\ \mathrm{cd}_3 &= \{p_2, p_3, p_7, p_8, \cdots\} \\ \mathrm{cd}_4 &= \{p_3, p_4, \cdots\} \\ \mathrm{ca}_1 &= \{(p_5, p_6), (p_7, p_8)\} \end{aligned}$$

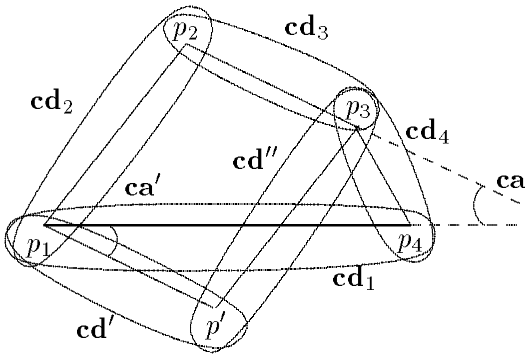


then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \mathrm{cd}_2 \cup \mathrm{cd}_3 \cup \mathrm{cd}_4$$

Let us show how this quadrilateral problem is transformed into a sequence of triangular problems. Assume that the **CD** sets defined by $cd_1 = \{p_1, p_4\}$ and $cd_3 = \{p_2, p_3\}$ are constrained in angle by the **CA** set $ca_1 = \{(p_3, p_2), (p_4, p_1)\}$ as shown in the Figure below. Draw a segment (p_1, p') that is parallel to and has the same orientation and length as segment (p_2, p_3). Note that the segments (p_1, p'), (p_1, p_4) make the same angle as the segments (p_3, p_2), (p_4, p_1). Furthermore, segment (p_1, p_2) and segment (p', p_3) are parallel and congruent. Hence first we can built three new constraint sets $cd' = \{p_1, p'\}, ca' = \{(p_1, p'), (p_1, p_4)\}$ and $cd''\{p', p_3\}$. Now, the quadrilateral problem is solved by the following sequence of construction rules:

$$\begin{array}{rcl} \{\mathrm{cd}', \mathrm{cd}_1, \mathrm{ca}'\} & \xrightarrow{DDA3} & \{\mathrm{cd}^1\} \\ \{\mathrm{cd}^1, \mathrm{cd}_4, \mathrm{cd}''\} & \xrightarrow{DDD} & \{\mathrm{cd}^2\} \\ \{\mathrm{cd}^2, \mathrm{cd}_2, \mathrm{cd}_3\} & \xrightarrow{DDA3} & \{\mathrm{cd}^3\} \end{array}$$



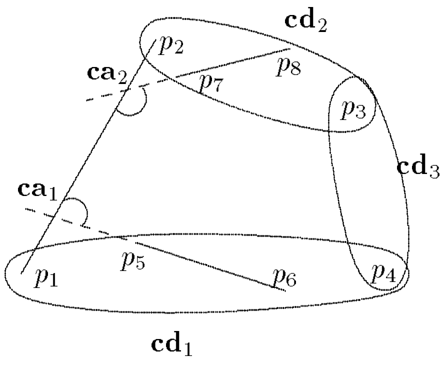
Rule Q2

Assume that three **CD** sets, cd_1, cd_2, cd_3, and two **CA** sets ca_1, ca_2, are given. These constraints sets are such that cd_1 and cd_2 share a point and so do cd_2 and cd_3; ca_1 and ca_2 share a segment; ca_1 shares with cd_1 its segment different from the one it has in common with ca_2; and ca_2 shares with cd_2 its segment different from that it has in common with ca_1. See Figure below. Then, the three **CD** sets can be merged into one **CD** set. The general rule is

$$\{\mathrm{cd}_1, \mathrm{cd}_2, \mathrm{cd}_3, \mathrm{ca}_1, \mathrm{ca}_2\} \longrightarrow \{\mathrm{cd}'\}$$

If

$$\begin{array}{rcl} \mathrm{cd}_1 & = & \{p_1, p_4, p_5, p_6, \cdots\} \\ \mathrm{cd}_2 & = & \{p_2, p_3, p_7, p_8, \cdots\} \\ \mathrm{cd}_3 & = & \{p_3, p_4, \cdots\} \\ \mathrm{ca}_1 & = & \{(p_1, p_2), (p_5, p_7)\} \\ \mathrm{ca}_2 & = & \{(p_1, p_2), (p_7, p_8)\} \end{array}$$



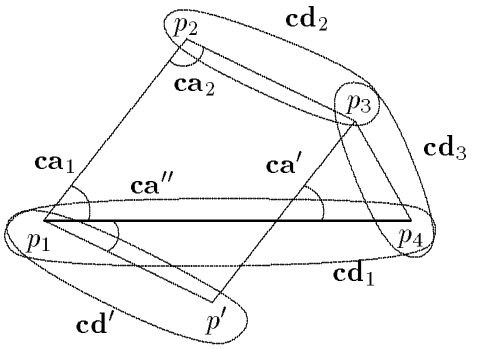
then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \mathrm{cd}_2 \cup \mathrm{cd}_3$$

Assume that the **CD** sets are given by $cd_1 = \{p_1, p_4\}, cd_2 = \{p_2, p_3\}, cd_3 = \{p_4, p_3\}$, and that the angle constraints are $ca_1 = \{(p_1, p_4), (p_1, p_2)\}$ and

$ca_2 = \{(p_2, p_1), (p_2, p_3)\}$, as shown in the Figure below. Draw a segment (p_1, p') that is parallel to and has the same orientation and length as segment (p_2, p_3). First note that the segments (p', p_3), (p_1, p_4) make the same angle as the segments (p_1, p_4), (p_1, p_2). Also note that the angle between segments (p_1, p'), (p_1, p_4) is $180 - (\mathrm{ca}_1 + \mathrm{ca}_2)$. Hence we can built three new constraint sets: $cd' = \{p_1, p'\}, ca' = \{(p_1, p_4), (p', p_3)\}$ and $ca'' = \{(p_1, p'), (p_1, p_4)\}$. Now, the quadrilateral problem is solved by the following sequence of construction rules:

$$
\begin{array}{lcl}
\{\mathrm{cd}', \mathrm{cd}_1, \mathrm{ca}'\} & \stackrel{DDA3}{\longrightarrow} & \{\mathrm{cd}^1\} \\
\{\mathrm{cd}^1, \mathrm{cd}_3, \mathrm{ca}''\} & \stackrel{DDA1}{\longrightarrow} & \{\mathrm{cd}^2\} \\
\{\mathrm{cd}^2, \mathrm{cd}_3, \mathrm{ca}_1\} & \stackrel{DDA1}{\longrightarrow} & \{\mathrm{cd}^3\}
\end{array}
$$



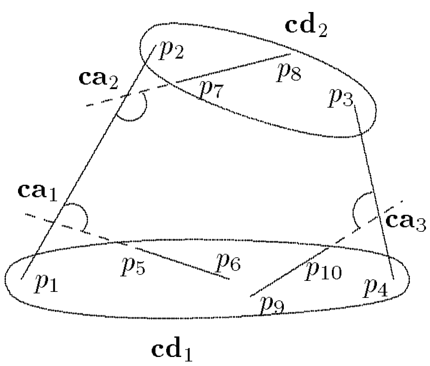
Rule Q3

This rule merges into one **CD** set the points in two **CD** sets constrained by three **CA** sets. Then, the general rule is

$$\{\mathrm{cd}_1, \mathrm{cd}_2, \mathrm{ca}_1, \mathrm{ca}_2, \mathrm{ca}_3\} \longrightarrow \{\mathrm{cd}'\}$$

If

$$
\begin{aligned}
\mathrm{cd}_1 &= \{p_1, p_4, p_5, p_6, p_9, p_{10}, \cdots\} \\
\mathrm{cd}_2 &= \{p_2, p_3, p_7, p_8, \cdots\} \\
\mathrm{ca}_1 &= \{(p_1, p_2), (p_5, p_6)\} \\
\mathrm{ca}_2 &= \{(p_1, p_2), (p_7, p_8)\} \\
\mathrm{ca}_3 &= \{(p_3, p_4), (p_9, p_{10})\}
\end{aligned}
$$

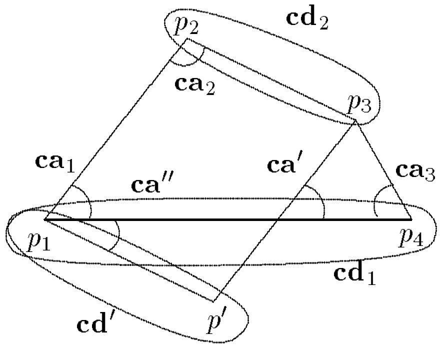


then

$$\mathrm{cd}' = \mathrm{cd}_1 \cup \mathrm{cd}_2$$

The quadrilateral problem is transformed into a sequence of triangular problems in the same way we did it for the **Q2** rule. The sequence of construction rules is now:

$$\{\mathrm{cd}', \mathrm{cd}_1, \mathrm{ca}'\} \overset{DDA3}{\longrightarrow} \{\mathrm{cd}^1\}$$
$$\{\mathrm{cd}^1, \mathrm{ca}_3, \mathrm{ca}''\} \overset{DDA1}{\longrightarrow} \{\mathrm{cd}^2\}$$
$$\{\mathrm{cd}^2, \mathrm{cd}_2, \mathrm{ca}_1\} \overset{DDA1}{\longrightarrow} \{\mathrm{cd}^3\}$$



3.4 General Form of the Constructive Rules

Let $\rightarrow_R$ be the set of the rules given above. Then by transforming the quadrilateral rules we have proved the theorem

Theorem 3.1. Each problem solved by a quadrilateral rule in $\rightarrow_R$ is equivalent to a problem derived from the original one by adding an extra point and two more constraints taken from the original dimensioning scheme. Furthermore, each quadrilateral rule in $\rightarrow_R$ can be defined as a sequence of rules where the quadrilateral rules themselves and the **CA** merging rule are not involved.

and the corollary

Corollary 3.1. The general form of the constructive rules in $\rightarrow_R$ is

$$\{\mathbf{CX}_1, \mathbf{CX}_2, \mathbf{CX}_3\} \longrightarrow \{\mathbf{CD}\}$$

where $\mathbf{CX}_i$ is either a **CD** set or a **CA** set or a **CH** set for $1 \leq i \leq 3$.

4 Conclusion

We have defined the set of constructive rules used by a two-dimensional, variational geometric constraint solver based on a constructive approach. Each rule performs a simple geometric construction. The rules build any triangle defined by side lengths, angles and altitudes, and some quadrilaterals.

Experiments carried out with the solver built on these rules show that the solver domain; e.i, the set of geometric constructions solvable by the set of rules used, is rather large. Work towards a fully characterization of the domain is currently under development.

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