Limit cycles bifurcating from a degenerate center

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Abstract

We study the maximum number of limit cycles that can bifurcate from a degenerate center of a cubic homogeneous polynomial differential system. Using the averaging method of second order and perturbing inside the class of all cubic polynomial differential systems we prove that at most three limit cycles can bifurcate from the degenerate center. As far as we know this is the first time that a complete study up to second order in the small parameter of the perturbation is done for studying the limit cycles which bifurcate from the periodic orbits surrounding a degenerate center (a center whose linear part is identically zero) having neither a Hamiltonian first integral nor a rational one. This study needs many computations, which have been verified with the help of the algebraic manipulator Maple.

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1. Introduction

Hilbert in (16) asked for the maximum number of limit cycles which real polynomial differential systems in the plane of a given degree can have. This is actually the well known *16th Hilbert Problem*, see for example the surveys (17; 18) and references therein. Recall that a *limit cycle* of a planar polynomial differential system is a periodic orbit of the system isolated in the set of all periodic orbits of the system.

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Poincaré in (22) was the first to introduce the notion of a center for a vector field defined on the real plane. So according to Poincaré a *center* is a singular point surrounded by a neighborhood filled of periodic orbits with the unique exception of the singular point.

Consider the polynomial differential system

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

and as usually we denote by $\dot{} = d/dt$. Assume that system (1) has a center located at the origin. Then after a linear change of variables and a possible scaling of time system (1) can be written in one of the following forms

$$(A) \quad \begin{array}{l} \dot{x} = -y + F_1(x,y), \\ \dot{y} = x + F_2(x,y), \end{array} \quad (B) \quad \begin{array}{l} \dot{x} = y + F_1(x,y), \\ \dot{y} = F_2(x,y), \end{array} \quad (C) \quad \begin{array}{l} \dot{x} = F_1(x,y), \\ \dot{y} = F_2(x,y), \end{array}$$

with F_1 and F_2 polynomials without constant and linear terms. When system (1) can be written into the form (A) we say that the center is of *linear type*. When system (1) can take the form (B) the center is *nilpotent*, and when system (1) can be transformed into the form (C) the center is *degenerate*.

Due to the difficulty of this problem mathematicians have consider simpler versions. Thus Arnold (1) considered the *weakened 16th Hilbert Problem*, which consists in determining an upper bound for the number of limit cycles which can bifurcate from the periodic orbits of a polynomial Hamiltonian center when it is perturbed inside a class of polynomial differential systems, see for instance (9) and the hundred of references quoted therein. It is known that in a neighborhood of a center always there is a first integral, see (21). When this first integral is not polynomial the computations become more difficult. Moreover, if the center is degenerate the computations become even harder.

In the literature we can basically find the following methods for studying the limit cycles that bifurcate from a center:

- The method that uses the Poincaré return map, like the articles (4; 8).
- The one that uses the Abelian integrals or Melnikov integrals (note that for systems in the plane the two notions are equivalent), see for example section 5 of Chapter 6 of (2) and section 6 of Chapter 4 of (15).

- The one that uses the inverse integrating factor, see (11; 12; 13; 25).
- The averaging theory (6; 14; 19; 23; 24).

The first two methods provide information about the number of limit cycles whereas the last two methods additionally give the shape of the bifurcated limit cycle up to any order in the perturbation parameter.

Almost all the papers studying how many limit cycles can bifurcate from the periodic orbits of a center, work with centers of linear type. There are very few papers studying this problem for nilpotent or degenerate centers. In fact, for degenerate centers as far as we known the bifurcation of limit cycles from the periodic orbits of a degenerate center only have been studying completely using formulas of first order in the small parameter of the perturbation. Here we will provide a complete study of this problem using formulas of second order, and as it occurs with the formulas of second order applied to linear centers that they provide in general more limit cycles than the formulas of first order, the same occurs for the formulas of second order applied to degenerate centers. Of course, the computations from first order to second order increases almost exponentially.

This paper deals with the weakened 16th Hilbert's problem but perturbing non–Hamiltonian degenerate centers using the technique of the averaging method of second order, see (14), and Section 2 for a summary of the results that we need here.

Since we want to study the perturbation of a degenerate center with averaging of second order, from the homogeneous centers the first ones that are degenerate, are the cubic homogeneous centers, see for instance (7). In this class in (20) the authors studied the perturbation of the following cubic homogeneous center

$$\dot{x} = -y(3x^2 + y^2), \qquad \dot{y} = x(x^2 - y^2), \tag{2}$$

inside the class of all cubic polynomial differential systems, using averaging theory of first order. Here we study this problem but using averaging theory of second order.

System (2) has a global center at the origin (i.e. all the orbits contained in $\mathbb{R}^2 \setminus \{(0,0)\}$ are periodic), and it admits the non–rational first integral

$$H(x,y) = (x^2 + y^2)\exp\left(-\frac{2x^2}{x^2 + y^2}\right).$$

The limit cycles bifurcating from the periodic orbits of the global center (2) have already been studied in the following two results, see (20) and (5), respectively.

Theorem 1. *We deal with differential system* (2). *Then the polynomial differential system*

$$\begin{aligned}\dot{x} &= -y(3x^2+y^2)+\varepsilon\left(\sum_{0\le i+j\le 3} a_{ij}x^iy^j\right),\\ \dot{y} &= x(x^2-y^2)+\varepsilon\left(\sum_{0\le i+j\le 3} b_{ij}x^iy^j\right),\end{aligned}$$

has at most one limit cycle bifurcating from the periodic orbits of the center of system (2) *using averaging theory of first order. Moreover, there are examples with 1 and 0 limit cycles.*

Proposition 2. *We consider the homogeneous polynomial differential system* (2). *Let $P_i(x,y)$ and $Q_i(x,y)$ for $i=1,2$ be polynomials of degree at most 3. Then for convenient polynomials P_i and Q_i, the polynomial differential system*

$$\begin{aligned}\dot{x} &= -y(3x^2+y^2)+\varepsilon P_1(x,y)+\varepsilon^2P_2(x,y),\\ \dot{y} &= x(x^2-y^2)+\varepsilon Q_1(x,y)+\varepsilon^2Q_2(x,y),\end{aligned}$$

has at first order averaging one limit cycle, and at second order averaging two limit cycles bifurcating from the periodic solutions of the global center (2).

Our main result is the following one and it do by first time the complete study of the averaging method of second order for a degenerate center having neither a Hamiltonian first integral nor a rational one.

Theorem 3. *We consider the cubic homogeneous differential system* (2). *Then the perturbation of system* (2) *inside the class of all cubic polynomial systems*

$$\begin{aligned}\dot{x} &= -y(3x^2+y^2)+\varepsilon\left(\sum_{0\le i+j\le 3} a_{ij}x^iy^j\right)+\varepsilon^2\left(\sum_{0\le i+j\le 3} b_{ij}x^iy^j\right),\\ \dot{y} &= x(x^2-y^2)+\varepsilon\left(\sum_{0\le i+j\le 3} c_{ij}x^iy^j\right)+\varepsilon^2\left(\sum_{0\le i+j\le 3} d_{ij}x^iy^j\right),\end{aligned}\tag{3}$$

has at most three limit cycles bifurcating from the periodic orbits of the center of system (2) *using averaging theory of second order. Moreover, there are examples with 3, 2, 1 and 0 limit cycles.*

The paper is organized as follows: In section 2 we present a summary of the averaging method of second order following (14). Next in section 3 we provide the proof of our main Theorem 3. In section 4 we provide three examples, of systems (3) with 0, 1, 2 and 3 limit cycles bifurcating from the degenerate center. At the end we present the Appendices A, B and C.

2. The averaging method of second order

In this section we present the averaging method of second order following (14). In that paper the averaging theory for differential equations of one variable is done up to any order in the small parameter of the perturbation. We consider the analytic differential equation

$$\frac{dr}{d\theta} = G_0(\theta, r) + \sum_{k\geq 1} \varepsilon^k G_k(\theta, r), \tag{4}$$

with $r \in \mathbb{R}$, $\theta \in \mathbb{S}^1$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with ε_0 a small positive real value, and the functions $G_k(\theta, r)$ are 2π–periodic in the variable θ. Note that for $\varepsilon = 0$ system (4) is unperturbed. Let $r_s(\theta, r_0)$ be the solution of system (4) with $\varepsilon = 0$ satisfying $r_s(0, r_0) = r_0$ and $r_s(\theta, r_0)$ is 2π periodic for $r_0 \in \mathcal{I}$ with $\mathcal{I}$ a real open interval. We are interested in the limit cycles of equation (4) which bifurcate from the periodic orbits of the unperturbed system with initial condition $r_0 \in \mathcal{I}$. So, we define by $r_\varepsilon(\theta, r_0)$ the solution of equation (4) satisfying $r_\varepsilon(0, r_0) = r_0$.

In what follows we denote by $u = u(\theta, r_0)$ the solution of the variational equation

$$\frac{\partial u}{\partial \theta} = \frac{\partial G_0}{\partial r}(\theta, r_s(\theta, r_0))u,$$

satisfying $u(0, r_0) = 1$.

We define

$$\begin{aligned} u_1(\theta, r_0) &= \int_0^{\theta} \frac{G_1(\phi,\ r_s(\phi, r_0))}{u(\phi, r_0)} d\phi = \int_0^{\theta} \frac{G_1(w,\ r_s(w, r_0))}{u(w, r_0)} dw, \\ G_{10}(r_0) &= \int_0^{2\pi} \frac{G_1(\theta,\ r_s(\theta, r_0))}{u(\theta, r_0)} d\theta, \\ G_{20}(r_0) &= \int_0^{2\pi} \left(\frac{G_2(\theta,\ r_s(\theta, r_0))}{u(\theta, r_0)} + \frac{\partial G_1}{\partial r}(\theta,\ r_s(\theta, r_0))\ u_1(\theta, r_0) \right. \\ &\qquad \left. + \frac{1}{2} \frac{\partial^2 G_0}{\partial r^2}(\theta,\ r_s(\theta, r_0))\ u_1(\theta, r_0)^2 \right) d\theta. \end{aligned} \tag{5}$$

In statement (b) of Corollary 5 of (14) it is proved the following result.

Theorem 4. *Assume that the solution $r_s(\theta, r_0)$ of the unperturbed equation (4) such that $r_s(0, r_0) = r_0$ is 2π–periodic for $r_0 \in \mathcal{I}$ with $\mathcal{I}$ a real open interval. If $G_{10}(r_0)$ is identically zero in $\mathcal{I}$ and $G_{20}(r_0)$ is not identically zero in $\mathcal{I}$, then for each simple zero $r^* \in \mathcal{I}$ of $G_{20}(r_0) = 0$ there exists a periodic solution $r_\varepsilon(\theta, r_0)$ of (4) such that $r_\varepsilon(0, r_0) \to r^*$ when $\varepsilon \to 0$.*

3. Proof of Theorem 3

System (2) in polar coordinates becomes

$$\dot{r} = -2r^3 \cos\theta \sin\theta, \qquad \dot{\theta} = r^2, \tag{6}$$

or equivalently,

$$\frac{dr}{d\theta} = -2r \cos\theta \sin\theta,$$

and it has the solution $r_s(\theta, r_0) = r_0 \exp(-\sin^2\theta)$ satisfying that $r_s(0, r_0) = r_0$.

Now we perturb system (2) inside the class of all cubic polynomial differential systems as in (3). System (3) in polar coordinates give rise to the differential equation

$$\frac{d\,r}{d\,\theta} = G_0(\theta, r) + \varepsilon G_1(\theta, r) + \varepsilon^2 G_2(\theta, r) + O(\varepsilon^3),$$

with

$$
\begin{aligned}
G_0(\theta, r) = &\quad -2\, r \cos\theta \sin\theta, \\
G_1(\theta, r) = &\quad g_{1,1}(\theta)\, \frac{1}{r^2} + g_{1,2}(\theta)\, \frac{1}{r} + g_{1,3}(\theta)\, + g_{1,4}(\theta)\, r, \\
G_2(\theta, r) = &\quad g_{2,1}(\theta)\, \frac{1}{r^5} + g_{2,2}(\theta)\, \frac{1}{r^4} + g_{2,3}(\theta)\, \frac{1}{r^3} + g_{2,4}(\theta)\, \frac{1}{r^2} \\
&\quad + g_{2,5}(\theta)\, \frac{1}{r} + g_{2,6}(\theta)\, + g_{2,7}(\theta)\, r,
\end{aligned}
$$

where the expressions of the coefficients $g_{1,i}(\theta)$ for $i = 1, 2, 3, 4$ and $g_{2,j}(\theta)$ for $j = 1, 2, \cdots, 7$ are given in the Appendix A.

Additionally, we consider the variational equation

$$
\frac{\partial u}{\partial \theta} = \frac{\partial G_0}{\partial r}(\theta, r_s(\theta, r_0)),
$$

and its solution $u(\theta, r_0)$ satisfying $u(0, r_0) = 1$, namely $u_s(\theta) = \exp(-\sin^2\theta)$.

We define

$$
\begin{aligned}
I_1 &= \int_0^{2\pi} \exp(2\sin^2\theta)\; \cos^4\theta\; d\theta = 3.572403292..., \\
I_2 &= \int_0^{2\pi} \exp(2\sin^2\theta)\; \cos^2\theta\; d\theta = 5.985557563..., \\
I_3 &= \int_0^{2\pi} \exp(2\sin^2\theta)\; d\theta = 21.62373221...
\end{aligned} \tag{7}
$$

Lemma 5. *Consider I_1, I_2, I_3 defined in (7). Then for $a_{10} = -(I_3 - 2I_1 + I_2)/(2I_1 - I_2)c_{01}$ and $a_{30} = -2c_{03} - c_{21}$ we have that the function $G_{10}(r_0)$ defined in (5) is identically zero.*

Proof. We have

$$
\frac{G_1(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} = A(\theta)\, \frac{1}{r_0^2} + B(\theta)\, \frac{1}{r_0} + C(\theta)\, + D(\theta)\, r_0,
$$

with

$$
\begin{aligned}
A(\theta) \; &= [\sin\theta\,(2\,\cos^2\theta+1)\,c_{00}+\cos\theta\,(-1+2\,\cos^2\theta)\,a_{00}]\,\mathrm{e}^{3\,\sin^2\theta},\\[1ex]
B(\theta) \; &= [\cos\theta\sin\theta\,(-1+2\,\cos^2\theta)\,a_{01}+\cos^2\theta\,(-1+2\,\cos^2\theta)\,a_{10}\\
&-(-\cos^2\theta+2\,\cos^4\theta-1)\,c_{01}+\cos\theta\sin\theta\,(2\,\cos^2\theta+1)\,c_{10}]\,\mathrm{e}^{2\,\sin^2\theta},\\[1ex]
C(\theta) \; &= [-\cos\theta\,(-3\,\cos^2\theta+1+2\,\cos^4\theta)\,a_{02}+\cos^3\theta\,(-1+2\,\cos^2\theta)\,a_{20}\\
&+\cos^2\theta\sin\theta\,(-1+2\,\cos^2\theta)\,a_{11}+\sin^3\theta\,(2\,\cos^2\theta+1)\,c_{02}\\
&+\cos^2\theta\sin\theta\,(2\,\cos^2\theta+1)\,c_{20}-\cos\theta\,(-\cos^2\theta+2\,\cos^4\theta-1)\,c_{11}]\,\mathrm{e}^{\sin^2\theta},\\[1ex]
D(\theta) \; &= \cos\theta\sin^3\theta\,(-1+2\,\cos^2\theta)\,a_{03}+\cos^4\theta\,(-1+2\,\cos^2\theta)\,a_{30}\\
&+\cos^3\theta\sin\theta\,(-1+2\,\cos^2\theta)\,a_{21}-\cos^2\theta\,(-3\,\cos^2\theta+1+2\,\cos^4\theta)\,a_{12}\\
&+(1-3\,\cos^4\theta+2\,\cos^6\theta)\,c_{03}+\cos^3\theta\sin\theta\,(2\,\cos^2\theta+1)\,c_{30}\\
&-\cos^2\theta\,(-\cos^2\theta+2\,\cos^4\theta-1)\,c_{21}+\cos\theta\sin^3\theta\,(2\,\cos^2\theta+1)\,c_{12}.
\end{aligned}
$$

Now

$$
G_{10}(r_0)=\int_0^{2\pi}\frac{G_1(\theta,r_s(\theta,r_0))}{u(\theta,r_0)}d\theta=\int_0^{2\pi}\left(A(\theta)\,\frac{1}{r_0^2}+B(\theta)\,\frac{1}{r_0}+C(\theta)\;+D(\theta)\,r_0\right)d\theta,
$$

and considering the change of coordinates $\theta=\phi+\pi$ in the interval $[0,2\pi]$ and the symmetries

$$
\sin(\theta+\pi)=-\sin\theta,\qquad \cos(\theta+\pi)=-\cos\theta, \tag{8}
$$

we have that

$$
\int_0^{2\pi}A(\theta)d\theta=\int_{-\pi}^{\pi}A(\phi)d\phi=0,\qquad \int_0^{2\pi}C(\theta)d\theta=\int_{-\pi}^{\pi}C(\phi)d\phi=0.
$$

So we have

$$
\begin{aligned}
G_{10}(r_0) \;&=\; \int_0^{2\pi}\frac{B(\theta)}{r_0}d\theta+\int_0^{2\pi}D(\theta)\,r_0 d\theta\\
&=\; [(2a_{10}-2c_{01})I_1+(c_{01}-a_{10})I_2+c_{01}I_3]\,\frac{1}{r_0}+\frac{\pi}{2}\,(2c_{03}+a_{30}+c_{21})\,r_0,
\end{aligned}
$$

and therefore $G_{10} \equiv 0$ if

$$a_{10} = -\frac{I_3 - 2I_1 + I_2}{2I_1 - I_2} c_{01} = -17.65322447..c_{01}, \qquad a_{30} = -2c_{03} - c_{21}.$$

This completes the proof of the lemma. □

Now we have

$$\frac{G_2(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} = A_5(\theta)\,\frac{1}{r_0^5} + A_4(\theta)\,\frac{1}{r_0^4} + A_3(\theta)\,\frac{1}{r_0^3} + A_2(\theta)\,\frac{1}{r_0^2} + A_1(\theta)\,\frac{1}{r_0} + A_0(\theta) + \tilde{A}_1(\theta)\, r_0,$$

with $A_5(\theta)\,, A_4(\theta)\,, A_3(\theta)\,, A_2(\theta)\,, A_1(\theta)\,, A_0(\theta)\,, \tilde{A}_1(\theta)$ are given in the Appendix B.

We note that

$$\int_0^{2\pi} A_4(\theta)\, d\theta = 0, \qquad \int_0^{2\pi} A_2(\theta)\, d\theta = 0, \qquad \int_0^{2\pi} A_0(\theta)\, d\theta = 0,$$

because of the symmetries (8). So we have

$$\int_0^{2\pi} \frac{G_2(\theta, r_s(\theta, r_0))}{u(\theta, r_0)} d\theta = \frac{1}{r_0^5} \int_0^{2\pi} A_5(\theta)\, d\theta + \frac{1}{r_0^3} \int_0^{2\pi} A_3(\theta)\, d\theta + \frac{1}{r_0} \int_0^{2\pi} A_1(\theta)\, d\theta + \left(\int_0^{2\pi} \tilde{A}_1(\theta)\, d\theta \right) r_0, \tag{9}$$

and we recall that the expressions of $A_5(\theta)\,, A_3(\theta)\,, A_1(\theta)\,, \tilde{A}_1(\theta)$ are given in the Appendix B. We have

$$\begin{aligned}
\int_0^{2\pi} A_5(\theta)\, d\theta &= \left(-4 \int_0^{2\pi} \mathrm{e}^{6\,\sin^2\theta} \cos^4\theta d\theta + 2 \int_0^{2\pi} \mathrm{e}^{6\,\sin^2\theta} \cos^2\theta d\theta + \int_0^{2\pi} \mathrm{e}^{6\,\sin^2\theta} d\theta \right) a_{00} c_{00} \\
&= 665.2264930..a_{00}c_{00}, \\
\int_0^{2\pi} A_3(\theta)\, d\theta &= 239.0000390..a_{01}c_{01} + 97.83745135..a_{00}c_{02} + 97.83745135..a_{02}c_{00} \\
&\quad -257.2692783..c_{01}c_{10} + 12.93483815..a_{00}c_{20} - 7.99641945..a_{00}a_{11} \\
&\quad +12.93483815..a_{20}c_{00} - 28.92767705..c_{00}c_{11},
\end{aligned}$$

$$
\begin{aligned}
\int_0^{2\pi} A_1(\theta)\, d\theta \quad &= 1.159249021..b_{10} + 20.46448319..d_{01} + 2.318498043..a_{11}c_{11} \\
&-4.73165232..c_{02}c_{11} + 2.318498043..a_{20}c_{0,2} + 2.318498045..a_{02}c_{20} \\
&-3.761715750..c_{11}c_{20} + 14.47892563..a_{02}c_{02} - 1.253905250..a_{02}a_{11} \\
&+0.189312456..a_{20}c_{20} + 0.094656226..a_{11}a_{20} + 0.647510463..a_{21}c_{01} \\
&-5.110277230..c_{03}c_{10} + 2.318498043..a_{12}c_{10} + 2.22384182..a_{01}c_{21} \\
&+36.6143963..a_{03}c_{01} - 3.95102821..c_{10}c_{21} - 1.25390525..a_{01}a_{12} \\
&-45.6606188..c_{01}c_{12} - 7.1036910..c_{01}c_{30} + 14.28961317..a_{01}c_{03}, \\
\int_0^{2\pi} \tilde{A}_1(\theta)\, d\theta \quad &= -0.3926990817..c_{30}c_{21} + 0.1963495408..c_{30}a_{12} + 2.159844949..a_{03}c_{03} \\
&-0.1963495408..a_{03}a_{12} - 0.7853981634..c_{12}c_{21} + 0.3926990817..a_{03}c_{21} \\
&-1.178097245..c_{03}c_{12} + 0.3926990817..c_{12}a_{12} + 0.9817477042..c_{03}c_{30} \\
&+1.570796327..d_{21} + 3.141592654..\, d_{03} + 1.570796327..b_{30}.
\end{aligned}
$$

Remark 6. *(a) Looking at the expressions of $A_3, A_1, \tilde{A}_1$ in the Appendix B we can have the exact definition for the numerical coefficients which appear in the previous integrals. Thus for instance*

$$
239.0000390\cdots = -\int_0^{2\pi} \frac{e^{4\,\sin^2\theta}\,(\cos^2\theta - 1)\,(2\,I_1 - 4I_3\,\cos^4\theta + 2I_3\,\cos^2\theta - I_2)}{2\,I_1 - I_2} d\theta,
$$

and I_1, I_2, I_3 satisfying relations (7)*.*

(b) All the computations of this paper have been verified with the algebraic manipulator Maple.

We additionally have

$$
\frac{\partial G_1}{\partial r}(\theta,\ r_s(\theta, r_0)) = B_0(\theta)\ + \frac{B_1(\theta)}{r_0^2} + \frac{B_2(\theta)}{r_0^3},
$$

with

$$
\begin{aligned}
B_0(\theta)\ =\ \ & \cos^2\theta\,(1 + 2\,\cos^2\theta - 4\,\cos^4\theta)\ c_{21} \\
& + (-2\,\cos^6\theta + 1 - \cos^4\theta)\,c_{03} \\
& + \cos\theta\,(\,\sin\theta\cos^2\theta - 2\,\cos^4\theta\sin\theta + \,\sin\theta)\,c_{12} \\
& + \cos^3\theta\,(\,\sin\theta + 2\,\cos^2\theta\sin\theta)\,c_{30} \\
& + \sin\theta\,\cos\theta\,(3\,\cos^2\theta - 1 - 2\,\cos^4\theta)\,a_{03} \\
& + \cos^2\theta\,(-1\, + 3\,\cos^2\theta - 2\,\cos^4\theta)\,a_{12} \\
& + \,\sin\theta\cos^3\theta\,(-1 + 2\,\cos^2\theta)\,a_{21},
\end{aligned}
$$

$$B_1(\theta) = (-1 - 18.65322447...\cos^2\theta + 37.30644894...\cos^4\theta)\, e^{2\sin^2\theta} c_{01} - \sin\theta\cos\theta\,(2\cos^2\theta + 1)\, e^{2\sin^2\theta} c_{10} + \sin\theta\,\cos\theta\,(-2\cos^2\theta + 1)\, e^{2\sin^2\theta} a_{01},$$

$$B_2(\theta) = -2\sin\theta\,(2\cos^2\theta + 1)\, c_{00} + 2\left(-2 + e^{3\sin^2\theta}\cos\theta\right) a_{00}.$$

Now we have

$$\frac{G_1(w,\ r_s(w, r_0))}{u(w, r_0)} = \frac{C_2(w)}{r_0^2} + \frac{C_1(w)}{r_0} + C_0(w) + \tilde{C}_1(w)\, r_0,$$

with

$$C_2(w) = e^{3\sin^2 w}\left[\cos w\,(2\cos^2 w - 1)\, a_{00} + \sin w\,(1 + 2\cos^2 w)\, c_{00}\right],$$

$$C_1(w) = \Big[\cos w \sin w\,(-1 + 2\cos^2 w)\, a_{01} + \left(1 + \frac{I_3}{2I_1 - I_2}(\cos^2 w - 2\cos^4 w)\right) c_{01} + \cos w \sin w\ (1 + 2\cos^2 w)\, c_{10}\Big]\, e^{2\sin^2 w},$$

$$C_0(w) = [\cos^3 w\,(2\cos^2 w - 1)\, a_{20} - \cos w\,(-3\cos^2 w + 2\cos^4 w + 1)\, a_{02} + \sin w\cos^2 w\,(2\cos^2 w - 1)\, a_{11} + \sin w\cos^2 w\,(1 + 2\cos^2 w)\, c_{20} + c_{02}\sin^3 w\,(1 + 2\cos^2 w) - \cos w\,(2\cos^4 w - 1 - \cos^2 w)\, c_{11}]\, e^{\sin^2 w},$$

$$\tilde{C}_1(w) = -(\cos^4 w + 2\cos^6 w - 1)\, c_{03} + \sin^3 w\cos w\,(1 + 2\cos^2 w)\, c_{12} - \cos^2 w\,(-1 - 2\cos^2 w + 4\cos^4 w)\, c_{21} + \cos^3 w\sin w\,(1 + 2\cos^2 w)\, c_{30} + \sin^3 w\cos w\,(2\cos^2 w - 1)\, a_{03} + \sin^2 w\cos^2 w\,(2\cos^2 w - 1)\, a_{12} + \cos^3 w\sin w\,(2\cos^2 w - 1)\, a_{21}.$$

Additionally, from (5) we obtain

$$u_1(\theta, r_0) = \frac{1}{r_0^2}\int_0^\theta C_2(w)\, dw + \frac{1}{r_0}\int_0^\theta C_1(w)\, dw + \int_0^\theta C_0(w)\, dw + r_0\int_0^\theta \tilde{C}_1(w)\, dw,$$

and so

$$\frac{\partial G_1}{\partial r}(\theta,\ r_s(\theta, r_0))\, u_1(\theta, r_0) = s_5(\theta)\,\frac{1}{r_0^5} + s_4(\theta)\,\frac{1}{r_0^4} + s_3(\theta)\,\frac{1}{r_0^3} + s_2(\theta)\,\frac{1}{r_0^2} + s_1(\theta)\,\frac{1}{r_0} + s_0(\theta) + \tilde{s}_1(\theta)\, r_0, \tag{10}$$

and the explicit expressions of $s_i(\theta)$ for $i = 0, 1, \cdots, 5$ and $\tilde{s}_1(\theta)$ are given in the Appendix C.

Since $\dfrac{\partial^2 G_0}{\partial r^2} = 0$ from (5) we have that

$$G_{20}(r_0) = \int_0^{2\pi} \left(\frac{G_2(\theta,\ r_s(\theta, r_0))}{u(\theta, r_0)} + \frac{\partial G_1}{\partial r}(\theta,\ r_s(\theta, r_0))\ u_1(\theta, r_0) \right) d\theta,$$

and we obtain

$$r_0^5 G_{20}(r_0) = v_6 r_0^6 + v_4 r_0^4 + v_2 r_0^2 + v_0,$$

with

$$\begin{aligned}
v_6 = \ & -0.3926990800\cdots\ c_{21}c_{30} + 0.1963495397\cdots\ a_{12}c_{30} + 2.159844949\cdots\ a_{03}c_{03} \\
& -0.1963495365\cdots\ a_{03}a_{12} - 0.7853981634\cdots\ c_{12}c_{21} + 0.3926990817\cdots\ a_{03}c_{21} \\
& -1.178097245\cdots\ c_{03}c_{12} + 0.3926990817\cdots\ a_{12}c_{12} + 0.9817477042\cdots\ c_{03}c_{30} \\
& +1.570796327\cdots\ d_{21} + 3.141592654\cdots\ d_{03} + 1.570796327\cdots\ b_{30},
\end{aligned}$$

$$\begin{aligned}
v_4 = \ & -3.155691751\cdots\ a_{21}c_{01} + 6.612510180\cdots\ c_{03}c_{10} + 1.786201647\cdots\ a_{12}c_{10} \\
& +2.413154277\cdots\ a_{01}c_{21} + 38.88726613\cdots\ a_{03}c_{01} - 1.253905255\cdots\ c_{10}c_{21} \\
& -1.206577137\cdots\ a_{01}a_{12} - 68.30745733\cdots\ c_{01}c_{12} - 38.88726635\cdots\ c_{01}c_{30} \\
& +13.27234849\cdots\ a_{01}c_{03} + 2.318498043\cdots\ a_{11}c_{11} - 4.73165232\cdots\ c_{02}c_{11} \\
& +2.318498043\cdots\ a_{20}c_{02} + 2.318498045\cdots\ a_{02}c_{20} - 3.761715750\cdots\ c_{11}c_{20} \\
& +14.47892563\cdots\ a_{02}c_{02} - 1.253905250\cdots\ a_{02}a_{11} + 0.189312456\cdots\ a_{20}c_{20} \\
& +0.094656226\cdots\ a_{11}a_{20} + 1.159249021\cdots\ b_{10} + 20.46448319\cdots\ d_{01},
\end{aligned}$$

$$\begin{aligned}
v_2 = \ & 95.95703341\cdots\ a_{00}c_{02} + 105.7377762\cdots\ a_{02}c_{00} + 239.0000390\cdots\ a_{01}c_{01} \\
& -7.649140220\cdots\ a_{00}a_{11} + 16.68739736\cdots\ a_{00}c_{20} - 110.9164314\cdots\ c_{00}c_{11} \\
& -257.2692783\cdots\ c_{01}c_{10} - 0.000001\cdots\ {c_{01}}^2 - 31.79348852\cdots\ a_{20}c_{00},
\end{aligned}$$

$$v_0 = \ 665.2264933\cdots\ a_{00}c_{00}.$$

We have that the coefficients v_6, v_4, v_2, v_0 are independent because d_{03} only appears in v_6, b_{10} only appears in v_4, $a_{00}c_{02}$ only appears in v_2, and $a_{00}c_{00}$ only appears in v_0.

Now we are going to use Descartes Theorem:

Theorem 7 (Descartes Theorem)**.** *Consider the real polynomial* $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} \cdots + a_{i_r}x^{i_r}$ *with* $0 \leq i_1 < i_2 < \cdots < i_r$ *and* $a_{i_j} \neq 0$ *real* *constants for $j \in \{1, 2, \cdots, r\}$. When $a_{i_j} a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m, then $p(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $r-1$ positive real roots.*

For a proof of Descartes Theorem see pages 82–83 of (3).

So from Descartes Theorem we can choose v_6, v_4, v_2, v_0 in order that the G_{20} has 3,2,1 or 0 real positive roots. This completes the proof of the first part of Theorem 3.

Remark 8. *Again the exact definition for the numerical coefficients which appear in v_6, v_4, v_2 and v_0 are given in Appendices B and C. For instance*

$$v_0 = \int_0^{2\pi} A_5 d\theta + \int_0^{2\pi} s_{5,3} d\theta = \int_0^{2\pi} A_5 d\theta = 665.2264933...$$

For completing the proof of Theorem 3 we shall provide examples of system (3) with 3, 2, 1 and 0 limit cycles. In fact, strictly speaking it is not necessary to provide examples with 3, 2, 1 and 0 limit cycles but we want to provide such examples.

4. Examples

Example with 3 limit cycles
In Figure 1 we see that for $\varepsilon = 0.001$ the system

$$\begin{aligned} \dot{x} = & -y(3x^2 + y^2) + \varepsilon + \varepsilon^2(3570.576292x - 752.8823806x^3) \\ & = y(3x^2 + y^2) + 0.001 + 0.003570576292x - 0.0007528823806x^3, \\ \\ \dot{y} = & \; x(x^2 - y^2) + \varepsilon(1 - 37.74385845y^2) \\ & = x(x^2 - y^2) + 0.001 - 0.03774385845y^2, \end{aligned} \tag{11}$$

has three limit cycles, since for system (11) we have

$$G_{20}(r_0) = -1182.624878\, r_0 + 4139.187071 \frac{1}{r_0} - 3621.788686 \frac{1}{r_0^3} + 665.2264933 \frac{1}{r_0^5},$$

and from $G_{20}(r_0) = 0$ we obtain the three positive roots near to $r_0 = 0.5, 1, 1.5$.

$\varepsilon = 0$ $\varepsilon = 0.001$

Figure 1: For $\varepsilon = 0$ we have the degenerate center of system (2), and for $\varepsilon = 0.001$ the perturbed system (11) has three limit cycles.

We have used the program P4 described in Chapters 9 and 10 of (10) for doing the phase portraits in the Poincaré disc which appear in this paper.

Example with 2 limit cycles

For $\varepsilon = 0.001$ the system

$$
\begin{aligned}
\dot{x} = & \ -y(3x^2+y^2)+\varepsilon(1+y+x^2y)+\varepsilon^2(-856.6373973x+y^3) \\
& = -y(3x^2+y^2)+0.001+0.001y+0.001x^2y-0.0008566373973x+0.000001y^3, \\
\dot{y} = & \ x(x^2-y^2)+\varepsilon(1+y^2)+\varepsilon^2(x+73.80732101y^3) \\
& = x(x^2-y^2)+0.001+0.001y^2+0.000001x+0.00007380732101y^3,
\end{aligned}
\tag{12}
$$

gives

$$G_{20}(r_0) = 231.8725375\, r_0 - 993.0560642\, \frac{1}{r_0} + 95.95703341\, \frac{1}{r_0^3} + 665.2264933\, \frac{1}{r_0^5},$$

and $G_{20}(r_0) = 0$ has the two positive zeros $r_0 = 1$ and $r_0 = 2$. In Figure 2 we see the two limit cycles bifurcated from the degenerate center of the unperturbed system (12).

$\varepsilon = 0.001$

Figure 2: Two limit cycles bifurcate from the degenerate center of the unperturbed system (12).

Example with 1 limit cycle

For $\varepsilon = 0.001$ the system

$$
\begin{aligned}
\dot{x} & = -y(3x^2+y^2)+\varepsilon(1-176.5322447x)+\varepsilon^2(x+x^3) \\
& = -y\,(3\,x^2+y^2)+0.001-0.1765312447\,x+0.000001\,x^3, \\
\dot{y} & = x(x^2-y^2)+\varepsilon(10+10y+5xy)+\varepsilon^2(y-y^3) \\
& = x\,(x^2-y^2)+0.010+0.010001\,y+0.005\,xy-0.000001\,y^3,
\end{aligned}
\tag{13}
$$

has

$$G_{20}(r_0) = -1.570796327 r_0 + 21.62373221 \frac{1}{r_0} - 5545.821670 \frac{1}{r_0^3} + 6652.264933 \frac{1}{r_0^5}.$$

From relation $G_{20}(r_0) = 0$ we obtain -1.097575824, 1.097575824, $-5.725902515-5.148324797i$, $5.725902515+5.148324797i$,$-5.725902515+5.148324797i$,$5.725902515-5.148324797i$. So only one limit cycle can bifurcate from a periodic orbit of the center of the unperturbed system (13), as we can see in Figure 3.

$$\varepsilon = 0.001$$

Figure 3: The limit cycle bifurcated from the degenerate center of the unperturbed system (13).

Example with zero limit cycles

Now for $\varepsilon = 0.001$ we consider system

$$\begin{aligned} \dot{x} &= -y(3x^2 + y^2) + \varepsilon + \varepsilon^2 x \\ &= -y\,(3\,x^2 + y^2) + 0.001 + 0.000001\,x, \\ \dot{y} &= x(x^2 - y^2) + \varepsilon(1 + y^2) + \varepsilon^2 y^3 \\ &= x\,(x^2 - y^2) + 0.001 + 0.001\,y^2 + 0.000001\,y^3, \end{aligned} \tag{14}$$

with

$$G_{20}(r_0) = 3.141592654 r_0 + 1.159249021 \frac{1}{r_0} + 95.95703341 \frac{1}{r_0^3} + 665.2264933 \frac{1}{r_0^5}.$$

We have that $G_{20}(r0) = 0$ has solutions $-2.116012294-1.570359831i$, $2.116012294+1.570359831i$, $-2.095699520i$, $2.095699520i$, $-2.116012294+1.570359831i$, $2.116012294-1.570359831i$. So no limit cycles can bifurcate from the degenerate center, see also Figure 4.

$$\varepsilon = 0.001$$

Figure 4: No limit cycle bifurcates from the degenerate center of the unperturbed system (14).

5. Appendix A

$$
\begin{aligned}
g_{1,1}(\theta) = \;& \cos\theta\left(2\cos^5\theta - 1\right) a_{00} + \sin\theta\left(1 + 2\cos^2\theta\right)^2 c_{00},
\end{aligned}
$$

$$
\begin{aligned}
g_{1,2}(\theta) = \;& \sin^2\theta\left(1 + 2\cos^2\theta\right) c_{01} + \sin\theta\cos\theta\left(1 + 2\cos^2\theta\right) c_{10} \\
& + \sin\theta\cos\theta\left(2\cos^2\theta - 1\right) a_{01} + \cos^2\theta\left(2\cos^2\theta - 1\right) a_{10},
\end{aligned}
$$

$$
\begin{aligned}
g_{1,3}(\theta) = \;& \sin^3\theta\left(1 + 2\cos^2\theta\right) c_{02} + \left(-2\cos^5\theta + \cos^3\theta + \cos\theta\right) c_{11} \\
& + \cos^2\theta\sin\theta\left(1 + 2\cos^2\theta\right) c_{20} + \left(-2\cos^5\theta + 3\cos^3\theta - \cos\theta\right) a_{02} + \\
& \cos^2\theta\sin\theta\left(2\cos^2\theta - 1\right) a_{11} + \cos^3\theta\left(2\cos^2\theta - 1\right) a_{20},
\end{aligned}
$$

$$
\begin{aligned}
g_{1,4}(\theta) = \;& \sin^4\theta\left(1 + 2\cos^2\theta\right) c_{03} + \sin^3\theta\cos\theta\left(1 + 2\cos^2\theta\right) c_{12} \\
& + \sin^2\theta\cos^2\theta\left(1 + 2\cos^2\theta\right) c_{21} + \sin\theta\cos^3\theta\left(1 + 2\cos^2\theta\right) c_{30} \\
& + \sin^3\theta\cos\theta\left(2\cos^2\theta - 1\right) a_{03} + \sin^2\theta\cos^2\theta\left(2\cos^2\theta - 1\right) a_{12} \\
& + \sin\theta\cos^3\theta\left(2\cos^2\theta - 1\right) a_{21} + \cos^4\theta\left(2\cos^2\theta - 1\right) a_{30},
\end{aligned}
$$

$$
\begin{aligned}
g_{2,1}(\theta) = \;& \left(2\cos^2\theta + 1 - 4\cos^4\theta\right) a_{00}c_{00} - {c_{00}}^2\sin\theta\cos\theta\left(2\cos^2\theta + 1\right) {c_{00}}^2 \\
& + \sin\theta\cos\theta\left(2\cos^2\theta - 1\right) {a_{00}}^2,
\end{aligned}
$$

$$
\begin{aligned}
g_{2,2}(\theta) = \;& -\cos\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) c_{10}a_{00} - 2\cos^2\theta\sin\theta\left(2\cos^2\theta + 1\right) c_{10}c_{00} \\
& + 2\cos^2\theta\sin\theta\left(2\cos^2\theta - 1\right) a_{10}a_{00} - \sin\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) c_{01}a_{00} \\
& - \sin\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) c_{00}a_{01} - \cos\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) c_{00}a_{10} \\
& + \left(4\cos^5\theta - 2\cos\theta - 2\cos^3\theta\right) c_{01}c_{00} + \left(6\cos^3\theta - 4\cos^5\theta - 2\cos\theta\right) a_{01}a_{00},
\end{aligned}
$$

$$
\begin{aligned}
g_{2,3}(\theta) = \;& \left(\cos^2\theta + 4\cos^6\theta - 6\cos^4\theta + 1\right) a_{01}c_{01} + \left(\cos^2\theta + 4\cos^6\theta - 6\cos^4\theta + 1\right) a_{00}c_{02} \\
& + \left(\cos^2\theta + 4\cos^6\theta - 6\cos^4\theta + 1\right) a_{02}c_{00} - c_{11}a_{00}\sin\theta\cos\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) \\
& - c_{00}a_{11}\sin\theta\cos\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) - 2\,c_{20}c_{00}\sin\theta\cos^3\theta\left(2\cos^2\theta + 1\right) \\
& + 2\,a_{20}a_{00}\sin\theta\cos^3\theta\left(2\cos^2\theta - 1\right) + 2\,a_{02}a_{00}\sin^3\theta\cos\theta\left(2\cos^2\theta - 1\right) \\
& - 2\,c_{02}c_{00}\sin^3\theta\cos\theta\left(2\cos^2\theta + 1\right) - c_{10}a_{01}\sin\theta\cos\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) \\
& - c_{01}a_{10}\sin\theta\cos\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) - c_{20}a_{00}\cos^2\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) \\
& - c_{00}a_{20}\cos^2\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) + \left(-4\cos^6\theta + 6\cos^4\theta - 2\cos^2\theta\right) a_{11}a_{00} \\
& + \left(4\cos^6\theta - 2\cos^2\theta - 2\cos^4\theta\right) c_{11}c_{00} - c_{10}a_{10}\cos^2\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) \\
& + \left(-4\cos^6\theta + 6\cos^4\theta - 2\cos^2\theta\right) a_{10}a_{01} + \left(4\cos^6\theta - 2\cos^2\theta - 2\cos^4\theta\right) c_{10}c_{01} \\
& + {a_{10}}^2\sin\theta\cos^3\theta\left(2\cos^2\theta - 1\right) - {c_{01}}^2\sin^3\theta\cos\theta\left(2\cos^2\theta + 1\right) \\
& + {a_{01}}^2\sin^3\theta\cos\theta\left(2\cos^2\theta - 1\right) - {c_{10}}^2\sin\theta\cos^3\theta\left(2\cos^2\theta + 1\right),
\end{aligned}
$$

$$
\begin{aligned}
g_{2,4}(\theta) = \;& -c_{03}a_{00}\sin^3\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) - c_{00}a_{03}\sin^3\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) \\
& - c_{30}a_{00}\cos^3\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) - c_{00}a_{30}\cos^3\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) \\
& + d_{00}\sin\theta\left(2\cos^2\theta + 1\right) + b_{00}\cos\theta\left(2\cos^2\theta - 1\right) - c_{20}a_{10}\cos^3\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) \\
& - c_{10}a_{20}\cos^3\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) - c_{02}a_{01}\sin^3\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) \\
& - c_{01}a_{02}\sin^3\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) - c_{21}a_{00}\cos^2\theta\sin\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) \\
& - c_{00}a_{21}\cos^2\theta\sin\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right) - 2\,c_{12}c_{00}\sin^3\theta\cos^2\theta\left(2\cos^2\theta + 1\right) \\
& - 2\,c_{30}c_{00}\cos^4\theta\sin\theta\left(2\cos^2\theta + 1\right) + 2\,a_{12}a_{00}\sin^3\theta\cos^2\theta\left(2\cos^2\theta - 1\right) \\
& + 2\,a_{03}a_{00}\sin^4\theta\cos\theta\left(2\cos^2\theta - 1\right) + 2\,a_{30}a_{00}\cos^4\theta\sin\theta\left(2\cos^2\theta - 1\right) \\
& + 2\,a_{02}a_{01}\sin^4\theta\cos\theta\left(2\cos^2\theta - 1\right) \\
& - 2\,c_{10}c_{02}\sin^3\theta\cos^2\theta\left(2\cos^2\theta + 1\right) - 2\,c_{20}c_{10}\cos^4\theta\sin\theta\left(2\cos^2\theta + 1\right) \\
& + 2\,a_{11}a_{01}\sin^3\theta\cos^2\theta\left(2\cos^2\theta - 1\right) - c_{20}a_{01}\cos^2\theta\sin\theta\left(-2\cos^2\theta - 1 + 4\cos^4\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-4\,\cos^7\theta-2\,\cos^3\theta+6\,\cos^5\theta\right)a_{21}a_{00}+\left(-2\,\cos^5\theta-2\,\cos^3\theta+4\,\cos^7\theta\right)c_{21}c_{00}\\
&+\left(-2\,\cos\theta+6\,\cos^5\theta-4\,\cos^7\theta\right)c_{03}c_{00}+\left(4\,\cos^7\theta-6\,\cos^5\theta+\cos^3\theta+\cos\theta\right)c_{11}a_{01}\\
&+\left(4\,\cos^7\theta-6\,\cos^5\theta+\cos^3\theta+\cos\theta\right)c_{10}a_{02}+\left(4\,\cos^7\theta-6\,\cos^5\theta+\cos^3\theta+\cos\theta\right)c_{02}a_{10}\\
&+\left(4\,\cos^7\theta-6\,\cos^5\theta+\cos^3\theta+\cos\theta\right)c_{01}a_{11}+\left(-2\,\cos\theta+6\,\cos^5\theta-4\,\cos^7\theta\right)c_{02}c_{01}\\
&+\left(-4\,\cos^7\theta-2\,\cos^3\theta+6\,\cos^5\theta\right)a_{20}a_{01}+\left(-4\,\cos^7\theta-2\,\cos^3\theta+6\,\cos^5\theta\right)a_{11}a_{10}\\
&+\left(-2\,\cos^5\theta-2\,\cos^3\theta+4\,\cos^7\theta\right)c_{20}c_{01}+\left(-2\,\cos^5\theta-2\,\cos^3\theta+4\,\cos^7\theta\right)c_{11}c_{10},
\end{aligned}
$$

$$
\begin{aligned}
g_{2,5}(\theta)=\;&-c_{10}a_{30}\cos^4\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)-c_{30}a_{10}\cos^4\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right),\\
&-{c_{02}}^2\sin^5\theta\cos\theta\left(2\,\cos^2\theta+1\right)-{c_{11}}^2\sin^3\theta\cos^3\theta\left(2\,\cos^2\theta+1\right)\\
&-{c_{20}}^2\cos^5\theta\sin\theta\left(2\,\cos^2\theta+1\right)+{a_{02}}^2\sin^5\theta\cos\theta\left(2\,\cos^2\theta-1\right)\\
&+\left(-7\,\cos^4\theta+1-4\,\cos^8\theta+10\,\cos^6\theta\right)a_{01}c_{03}+\left(-7\,\cos^4\theta+1-4\,\cos^8\theta+10\,\cos^6\theta\right)a_{03}c_{01}\\
&+\left(-7\,\cos^4\theta+1-4\,\cos^8\theta+10\,\cos^6\theta\right)a_{02}c_{02}+d_{10}\sin\theta\cos\theta\left(2\,\cos^2\theta+1\right)\\
&+b_{10}\cos^2\theta\left(2\,\cos^2\theta-1\right)+d_{01}\sin^2\theta^2\left(2\,\cos^2\theta+1\right)+{a_{11}}^2\sin^3\theta\cos^3\theta\left(2\,\cos^2\theta-1\right)\\
&-c_{01}a_{12}\sin^3\theta\cos\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)-c_{30}a_{01}\sin\theta\cos^3\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)\\
&-c_{21}a_{10}\sin\theta\cos^3\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)-c_{20}a_{11}\sin\theta\cos^3\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)\\
&-c_{11}a_{20}\sin\theta\cos^3\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)+2\,a_{10}a_{03}\sin^4\theta\cos^2\theta\left(2\,\cos^2\theta-1\right)\\
&-c_{10}a_{21}\sin\theta\cos^3\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)-c_{01}a_{30}\sin\theta\cos^3\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)\\
&-c_{12}a_{01}\sin^3\theta\cos\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)-c_{11}a_{02}\sin^3\theta\cos\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)\\
&-c_{10}a_{03}\sin^3\theta\cos\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)-c_{03}a_{10}\sin^3\theta\cos\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)\\
&-c_{02}a_{11}\sin^3\theta\cos\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)+2\,a_{12}a_{01}\sin^4\theta\cos^2\theta\left(2\,\cos^2\theta-1\right)\\
&+2\,a_{11}a_{02}\sin^4\theta\cos^2\theta\left(2\,\cos^2\theta-1\right)+{a_{20}}^2\cos^5\theta\sin\theta\left(2\,\cos^2\theta-1\right)\\
&+2\,a_{21}a_{01}\sin^3\theta\cos^3\theta\left(2\,\cos^2\theta-1\right)+2\,a_{20}a_{02}\sin^3\theta\cos^3\theta\left(2\,\cos^2\theta-1\right)\\
&+2\,a_{30}a_{10}\cos^5\theta\sin\theta\left(2\,\cos^2\theta-1\right)-2\,c_{30}c_{10}\cos^5\theta\sin\theta\left(2\,\cos^2\theta+1\right)\\
&-2\,c_{03}c_{01}\sin^5\theta\cos\theta\left(2\,\cos^2\theta+1\right)-c_{20}a_{20}\cos^4\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)\\
&-2\,c_{21}c_{01}\sin^3\theta\cos^3\theta\left(2\,\cos^2\theta+1\right)-2\,c_{20}c_{02}\sin^3\theta\cos^3\theta\left(2\,\cos^2\theta+1\right)\\
&-2\,c_{12}c_{10}\sin^3\theta\cos^3\theta\left(2\,\cos^2\theta+1\right)+2\,a_{12}a_{10}\sin^3\theta\cos^3\theta\left(2\,\cos^2\theta-1\right)\\
&+2\,a_{03}a_{01}\sin^5\theta\cos\theta\left(2\,\cos^2\theta-1\right)+\left(6\,\cos^6\theta-4\,\cos^8\theta-2\,\cos^2\theta\right)c_{10}c_{03}\\
&+\left(-4\,\cos^8\theta-2\,\cos^4\theta+6\,\cos^6\theta\right)a_{30}a_{01}+\left(-4\,\cos^8\theta-2\,\cos^4\theta+6\,\cos^6\theta\right)a_{21}a_{10}\\
&+\left(-4\,\cos^8\theta-2\,\cos^4\theta+6\,\cos^6\theta\right)a_{20}a_{11}+\left(\cos^2\theta-6\,\cos^6\theta+\cos^4\theta+4\,\cos^8\theta\right)c_{11}a_{11}\\
&+\left(\cos^2\theta-6\,\cos^6\theta+\cos^4\theta+4\,\cos^8\theta\right)c_{01}a_{21}+\left(-2\,\cos^6\theta-2\,\cos^4\theta+4\,\cos^8\theta\right)c_{30}c_{01}\\
&+\left(-2\,\cos^6\theta-2\,\cos^4\theta+4\,\cos^8\theta\right)c_{21}c_{10}+\left(-2\,\cos^6\theta-2\,\cos^4\theta+4\,\cos^8\theta\right)c_{20}c_{11}\\
&+\left(\cos^2\theta-6\,\cos^6\theta+\cos^4\theta+4\,\cos^8\theta\right)c_{20}a_{02}+\left(\cos^2\theta-6\,\cos^6\theta+\cos^4\theta+4\,\cos^8\theta\right)c_{12}a_{10}\\
&+\left(\cos^2\theta-6\,\cos^6\theta+\cos^4\theta+4\,\cos^8\theta\right)c_{10}a_{12}+\left(\cos^2\theta-6\,\cos^6\theta+\cos^4\theta+4\,\cos^8\theta\right)c_{02}a_{20}\\
&+\left(\cos^2\theta-6\,\cos^6\theta+\cos^4\theta+4\,\cos^8\theta\right)c_{21}a_{01}+\left(6\,\cos^6\theta-4\,\cos^8\theta-2\,\cos^2\theta\right)c_{12}c_{01}\\
&+\left(6\,\cos^6\theta-4\,\cos^8\theta-2\,\cos^2\theta\right)c_{11}c_{02}+b_{01}\sin\theta\cos\theta\left(2\,\cos^2\theta-1\right),
\end{aligned}
$$

$$
\begin{aligned}
g_{2,6}(\theta)=\;&-2\,c_{21}c_{11}\sin^3\theta\cos^4\theta\left(2\,\cos^2\theta+1\right)-2\,c_{20}c_{12}\sin^3\theta\cos^4\theta\left(2\,\cos^2\theta+1\right)\\
&-2\,c_{30}c_{20}\cos^6\theta\sin\theta\left(2\,\cos^2\theta+1\right)-c_{20}a_{03}\sin^3\theta\cos^2\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)\\
&+2\,a_{21}a_{11}\sin^3\theta\cos^4\theta\left(2\,\cos^2\theta-1\right)\\
&+2\,a_{20}a_{12}\sin^3\theta\cos^4\theta\left(2\,\cos^2\theta-1\right)-c_{30}a_{20}\cos^5\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)\\
&-c_{20}a_{30}\cos^5\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)+\left(\cos^3\theta+\cos\theta-2\,\cos^5\theta\right)d_{11}\\
&+\left(-\cos\theta+3\,\cos^3\theta-2\,\cos^5\theta\right)b_{02}+d_{02}\sin^3\theta\left(2\,\cos^2\theta+1\right)\\
&-c_{20}a_{21}\cos^4\theta\sin\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)+2\,a_{21}a_{02}\sin^4\theta\cos^3\theta\left(2\,\cos^2\theta-1\right)\\
&+2\,a_{20}a_{03}\sin^4\theta\cos^3\theta\left(2\,\cos^2\theta-1\right)+2\,a_{12}a_{11}\sin^4\theta\cos^3\theta\left(2\,\cos^2\theta-1\right)
\end{aligned}
$$

$$
\begin{aligned}
&+2\,a_{03}a_{02}\sin^6\theta\cos\theta\left(2\,\cos^2\theta-1\right)-2\,c_{03}c_{02}\sin^6\theta\cos\theta\left(2\,\cos^2\theta+1\right)\\
&+2\,a_{11}a_{03}\sin^5\theta\cos^2\theta\left(2\,\cos^2\theta-1\right)-2\,c_{11}c_{03}\sin^5\theta\cos^2\theta\left(2\,\cos^2\theta+1\right)\\
&-2\,c_{30}c_{02}\sin^3\theta\cos^4\theta\left(2\,\cos^2\theta+1\right)+2\,a_{12}a_{02}\sin^5\theta\cos^2\theta\left(2\,\cos^2\theta-1\right)\\
&-c_{11}a_{30}\cos^4\theta\sin\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)\\
&-c_{11}a_{12}\sin^3\theta\cos^2\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)-c_{21}a_{02}\sin^3\theta\cos^2\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)\\
&+b_{20}\cos^3\theta\left(2\,\cos^2\theta-1\right)-2\,c_{12}c_{02}\sin^5\theta\cos^2\theta\left(2\,\cos^2\theta+1\right)\\
&-c_{12}a_{11}\sin^3\theta\cos^2\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)-c_{30}a_{11}\cos^4\theta\sin\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)\\
&-c_{21}a_{20}\cos^4\theta\sin\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)+2\,a_{30}a_{20}\cos^6\theta\sin\theta\left(2\,\cos^2\theta-1\right)\\
&-c_{03}a_{20}\sin^3\theta\cos^2\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)-c_{02}a_{21}\sin^3\theta\cos^2\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)\\
&+2\,a_{30}a_{02}\sin^3\theta\cos^4\theta\left(2\,\cos^2\theta-1\right)-c_{03}a_{02}\sin^5\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)\\
&+d_{20}\cos^2\theta\sin\theta\left(2\,\cos^2\theta+1\right)+b_{11}\cos^2\theta\sin\theta\left(2\,\cos^2\theta-1\right)\\
&-c_{02}a_{03}\sin^5\theta\left(-2\,\cos^2\theta-1+4\,\cos^4\theta\right)+\left(\cos\theta+10\,\cos^7\theta-7\,\cos^5\theta-4\,\cos^9\theta\right)c_{11}a_{03}\\
&+\left(\cos\theta+10\,\cos^7\theta-7\,\cos^5\theta-4\,\cos^9\theta\right)c_{03}a_{11}+\left(\cos^5\theta-6\,\cos^7\theta+4\,\cos^9\theta+\cos^3\theta\right)c_{21}a_{11}\\
&+\left(\cos^5\theta-6\,\cos^7\theta+4\,\cos^9\theta+\cos^3\theta\right)c_{11}a_{21}+\left(\cos^5\theta-6\,\cos^7\theta+4\,\cos^9\theta+\cos^3\theta\right)c_{20}a_{12}\\
&+\left(\cos^5\theta-6\,\cos^7\theta+4\,\cos^9\theta+\cos^3\theta\right)c_{12}a_{20}+\left(\cos^5\theta-6\,\cos^7\theta+4\,\cos^9\theta+\cos^3\theta\right)c_{02}a_{30}\\
&+\left(\cos\theta+10\,\cos^7\theta-7\,\cos^5\theta-4\,\cos^9\theta\right)c_{12}a_{02}+\left(\cos\theta+10\,\cos^7\theta-7\,\cos^5\theta-4\,\cos^9\theta\right)c_{02}a_{12}\\
&+\left(\cos^5\theta-6\,\cos^7\theta+4\,\cos^9\theta+\cos^3\theta\right)c_{30}a_{02}+\left(-2\,\cos^3\theta+6\,\cos^7\theta-4\,\cos^9\theta\right)c_{21}c_{02}\\
&+\left(-2\,\cos^3\theta+6\,\cos^7\theta-4\,\cos^9\theta\right)c_{20}c_{03}+\left(-2\,\cos^3\theta+6\,\cos^7\theta-4\,\cos^9\theta\right)c_{12}c_{11}\\
&+\left(-4\,\cos^9\theta+6\,\cos^7\theta-2\,\cos^5\theta\right)a_{30}a_{11}+\left(-4\,\cos^9\theta+6\,\cos^7\theta-2\,\cos^5\theta\right)a_{21}a_{20}\\
&+\left(4\,\cos^9\theta-2\,\cos^5\theta-2\,\cos^7\theta\right)c_{30}c_{11}+\left(4\,\cos^9\theta-2\,\cos^5\theta-2\,\cos^7\theta\right)c_{21}c_{20},
\end{aligned}
$$

$$
\begin{aligned}
g_{2,7}(\theta)=\;&-{c_{21}}^2\sin^3\theta\cos^5\theta\left(2\,\cos^2\theta+1\right)+{a_{12}}^2\sin^5\theta\cos^3\theta\left(-1+2\,\cos^2\theta\right)\\
&-{c_{03}}^2\sin^7\theta\cos\theta\left(2\,\cos^2\theta+1\right)+{a_{03}}^2\sin^7\theta\cos\theta\left(-1+2\,\cos^2\theta\right)\\
&+{a_{21}}^2\sin^3\theta\cos^5\theta\left(-1+2\,\cos^2\theta\right)-{c_{12}}^2\sin^5\theta\cos^3\theta\left(2\,\cos^2\theta+1\right)-{c_{30}}^2\\
&\cos^7\theta\sin\theta\left(2\,\cos^2\theta+1\right)+2\,a_{30}a_{12}\sin^3\theta\cos^5\theta\left(-1+2\,\cos^2\theta\right)\\
&-c_{03}a_{12}\sin^5\theta\cos\theta\left(4\,\cos^4\theta-2\,\cos^2\theta-1\right)-c_{30}a_{21}\cos^5\theta\sin\theta\left(4\,\cos^4\theta-2\,\cos^2\theta-1\right)\\
&-c_{12}a_{21}\sin^3\theta\cos^3\theta\left(4\,\cos^4\theta-2\,\cos^2\theta-1\right)-c_{30}a_{03}\sin^3\theta\cos^3\theta\left(4\,\cos^4\theta-2\,\cos^2\theta-1\right)\\
&-2\,c_{21}c_{03}\sin^5\theta\cos^3\theta\left(2\,\cos^2\theta+1\right)-2\,c_{12}c_{03}\sin^6\theta\cos^2\theta\left(2\,\cos^2\theta+1\right)\\
&+2\,a_{30}a_{03}\sin^4\theta\cos^4\theta\left(-1+2\,\cos^2\theta\right)+2\,a_{12}a_{03}\left(\sin\theta\right)^6\cos^2\theta\left(-1+2\,\cos^2\theta\right)\\
&+2\,a_{21}a_{12}\sin^4\theta\cos^4\theta\left(-1+2\,\cos^2\theta\right)-c_{21}a_{30}\cos^5\theta\sin\theta\left(4\,\cos^4\theta-2\,\cos^2\theta-1\right)\\
&-c_{03}a_{30}\sin^3\theta\cos^3\theta\left(4\,\cos^4\theta-2\,\cos^2\theta-1\right)-c_{21}a_{12}\sin^3\theta\cos^3\theta\left(4\,\cos^4\theta-2\,\cos^2\theta-1\right)\\
&+b_{03}\sin^3\theta\cos\theta\left(-1+2\,\cos^2\theta\right)+b_{12}\sin^2\theta\cos^2\theta\left(-1+2\,\cos^2\theta\right)\\
&-c_{30}a_{30}\cos^6\theta\left(4\,\cos^4\theta-2\,\cos^2\theta-1\right)-a_{03}c_{03}\sin^6\theta\left(4\,\cos^4\theta-2\,\cos^2\theta-1\right)\\
&+d_{12}\sin^3\theta\cos\theta\left(2\,\cos^2\theta+1\right)+d_{21}\sin^2\theta\cos^2\theta\left(2\,\cos^2\theta+1\right)\\
&+d_{30}\sin\theta\cos^3\theta\left(2\,\cos^2\theta+1\right)-2\,c_{30}c_{12}\sin^3\theta\cos^5\theta\left(2\,\cos^2\theta+1\right)\\
&+2\,a_{21}a_{03}\sin^5\theta\cos^3\theta\left(-1+2\,\cos^2\theta\right)-c_{12}a_{03}\sin^5\theta\cos\theta\left(4\,\cos^4\theta-2\,\cos^2\theta-1\right)\\
&+d_{03}\sin^4\theta\left(2\,\cos^2\theta+1\right)+b_{30}\cos^4\theta\left(-1+2\,\cos^2\theta\right)+{a_{30}}^2\cos^7\theta\sin\theta\left(-1+2\,\cos^2\theta\right)\\
&+\left(6\,\cos^8\theta-4\,\cos^{10}\theta-2\,\cos^4\theta\right)c_{30}c_{03}+\left(6\,\cos^8\theta-4\,\cos^{10}\theta-2\,\cos^4\theta\right)c_{21}c_{12}\\
&+\left(4\,\cos^{10}\theta-2\,\cos^8\theta-2\,\cos^6\theta\right)c_{30}c_{21}+b_{21}\sin\theta\cos^3\theta\left(-1+2\,\cos^2\theta\right)\\
&+\left(-6\,\cos^8\theta+\cos^4\theta+\cos^6\theta+4\,\cos^{10}\theta\right)c_{21}a_{21}+\left(-4\,\cos^{10}\theta-2\,\cos^6\theta+6\,\cos^8\theta\right)a_{30}a_{21}\\
&+\left(\cos^2\theta-7\,\cos^6\theta+10\,\cos^8\theta-4\,\cos^{10}\theta\right)c_{03}a_{21}+\left(-6\,\cos^8\theta+\cos^4\theta+\cos^6\theta+4\,\cos^{10}\theta\right)c_{12}a_{30}\\
&+\left(\cos^2\theta-7\,\cos^6\theta+10\,\cos^8\theta-4\,\cos^{10}\theta\right)c_{12}a_{12}+\left(-6\,\cos^8\theta+\cos^4\theta+\cos^6\theta+4\,\cos^{10}\theta\right)c_{30}a_{12}\\
&+\left(\cos^2\theta-7\,\cos^6\theta+10\,\cos^8\theta-4\,\cos^{10}\theta\right)c_{21}a_{03}.
\end{aligned}
$$

6. Appendix B

We present the expressions of $A_4, A_2, A_0, A_5, A_3, A_1, \tilde{A}_1$ that appear in equation (9).

$$A_4(\theta) = e^{5\sin^2\theta}\Bigg[\Bigg[-\frac{(4\,a_{01}c_{00}I_2 - 8\,I_1\,a_{01}c_{00} - 8\,I_1\,c_{00}c_{10} - 4\,a_{00}c_{01}I_3 + 4\,c_{00}c_{10}I_2)\cos^4\theta}{I_2 - 2\,I_1}$$
$$-\frac{(-4\,I_1\,c_{00}c_{10} + 2\,a_{00}c_{01}I_3 - 2\,a_{01}c_{00}I_2 + 2\,c_{00}c_{10}I_2 + 4\,I_1\,a_{01}c_{00})\cos^2\theta}{I_2 - 2\,I_1}$$
$$-\frac{(-a_{01}c_{00}I_2 + 2\,I_1\,a_{01}c_{00} + 2\,I_1\,a_{00}c_{01} - a_{00}c_{01}I_2)}{I_2 - 2\,I_1}\Bigg]\sin\theta$$
$$-\frac{(4\,c_{00}c_{01}I_3 + 4\,a_{00}c_{10}I_2 - 8\,I_1\,a_{00}a_{01} - 8\,I_1\,a_{00}c_{10} + 4\,a_{00}a_{01}I_2)(\cos\theta)^5}{I_2 - 2\,I_1}$$
$$-\frac{(12\,I_1\,a_{00}a_{01} + 4\,I_1\,a_{00}c_{10} - 2\,c_{00}c_{01}I_3 - 2\,a_{00}c_{10}I_2 - 6\,a_{00}a_{01}I_2)(\cos\theta)^3}{I_2 - 2\,I_1}$$
$$-\frac{(-2\,c_{00}c_{01}I_1 + c_{00}c_{01}I_2 - 4\,I_1\,a_{00}a_{01} + 2\,a_{00}a_{01}I_2 + 2\,I_1\,a_{00}c_{10} - c_{00}c_{01}I_3 - a_{00}c_{10}I_2)\cos\theta}{I_2 - 2\,I_1}\Bigg],$$

$$A_2(\theta) = \big[A_{2,7}\cos^7\theta + A_{2,6}\cos^6\theta\sin\theta + A_{2,5}\cos^5\theta + A_{2,4}\cos^4\theta\sin\theta + A_{2,3}\cos^3\theta + A_{2,2}\cos^2\theta\sin\theta$$
$$+ A_{2,1}\cos\theta + A_{2,0}\theta\sin\theta\big]\exp(3\sin^2\theta),$$

$$A_{2,7} = 4\,a_{03}a_{00} - 4\,a_{21}a_{00} - 4\,c_{30}a_{00} + 4\,c_{12}a_{00} - 4\,a_{20}a_{01} + 4\,a_{02}a_{01} + 4\,c_{11}a_{01} - 4\,c_{10}a_{20}$$
$$+4\,c_{10}a_{02} + 4\,\frac{a_{11}c_{01}I_3}{2\,I_1 - I_2} + 4\,c_{00}a_{12} + 4\,c_{03}c_{00} + 8\,c_{21}c_{00} + 4\,\frac{c_{01}c_{20}I_3}{2\,I_1 - I_2}$$
$$-4\,\frac{c_{01}c_{02}I_3}{2\,I_1 - I_2} + 4\,c_{11}c_{10},$$

$$A_{2,6} = -4\,a_{12}a_{00} - 4\,c_{03}a_{00} - 8\,c_{21}a_{00} - 4\,a_{11}a_{01} - 4\,c_{20}a_{01} + 4\,c_{02}a_{01} - 4\,\frac{a_{20}c_{01}I_3}{2\,I_1 - I_2}$$
$$+4\,\frac{a_{02}c_{01}I_3}{2\,I_1 - I_2} - 4\,a_{11}c_{10} + 4\,a_{03}c_{00} - 4\,a_{21}c_{00} - 4\,c_{30}c_{00} + 4\,c_{12}c_{00}$$
$$+4\,\frac{c_{01}c_{11}I_3}{2\,I_1 - I_2} - 4\,c_{20}c_{10} + 4\,c_{10}c_{02},$$

$$A_{2,5} = -10\,a_{03}a_{00} + 6\,a_{21}a_{00} + 2\,c_{30}a_{00} - 6\,c_{12}a_{00} + 6\,a_{20}a_{01} - 10\,a_{02}a_{01} - 6\,c_{11}a_{01} + 2\,c_{10}a_{20}$$
$$-6\,c_{10}a_{02} - 6\,\frac{a_{11}c_{01}I_3}{2\,I_1 - I_2} - 6\,c_{00}a_{12} + 2\,c_{03}c_{00} - 4\,c_{21}c_{00} - 2\,\frac{c_{01}c_{20}I_3}{2\,I_1 - I_2}$$
$$+6\,\frac{c_{01}c_{02}I_3}{2\,I_1 - I_2} - 2\,c_{11}c_{10},$$

$$A_{2,4} = 6\,a_{00}a_{12} - 2\,a_{00}c_{03} + 4\,a_{00}c_{21} + 6\,a_{01}a_{11} + 2\,a_{01}c_{20} - 6\,a_{01}c_{02} + 2\,\frac{a_{20}c_{01}I_3}{2\,I_1 - I_2}$$
$$-6\,\frac{a_{02}c_{01}I_3}{2\,I_1 - I_2} + 2\,a_{11}c_{10} - 6\,a_{03}c_{00} + 2\,a_{21}c_{00} - 2\,c_{00}c_{30} - 2\,c_{00}c_{12} - 2\,\frac{c_{01}c_{11}I_3}{2\,I_1 - I_2}$$
$$-2\,c_{10}c_{20} - 2\,c_{02}c_{10},$$

$$A_{2,3} = 2\,b_{00} + 8\,a_{00}a_{03} - 2\,a_{00}a_{21} + a_{00}c_{30} + a_{00}c_{12} - 2\,a_{01}a_{20} + 8\,a_{01}a_{02} + a_{01}c_{11} + a_{20}c_{10}$$
$$+a_{02}c_{10} - \frac{(-2\,I_3 - I_2 + 2\,I_1)\,c_{01}a_{11}}{2\,I_1 - I_2} + a_{12}c_{00} - 2\,c_{00}c_{03} - 3\,c_{00}c_{21}$$
$$-\frac{(-I_2 + 2\,I_1 + I_3)\,c_{01}c_{20}}{2\,I_1 - I_2} + \frac{(2\,I_1 - I_3 - I_2)\,c_{02}c_{01}}{-I_2 + 2\,I_1} - 2\,c_{10}c_{11},$$

$$
\begin{aligned}
A_{2,2} = \;& 2\,d_{00} - 2\,a_{00}a_{12} + a_{00}c_{03} + a_{00}c_{21} - 2\,a_{01}a_{11} + a_{01}c_{20} + a_{01}c_{02} + a_{20}c_{01} \\
& - \frac{(-2\,I_3 - I_2 + 2\,I_1)\,c_{01}a_{02}}{2\,I_1 - I_2} + a_{11}c_{10} + a_{03}c_{00} + a_{21}c_{00} - 2\,c_{00}c_{12} \\
& - \frac{(-I_2 + 2\,I_1 + I_3)\,c_{11}c_{01}}{2\,I_1 - I_2} - 2\,c_{02}c_{10}, \\
A_{2,1} = \;& -b_{00} - 2\,a_{00}a_{03} + a_{00}c_{12} - 2\,a_{01}a_{02} + a_{01}c_{11} + a_{02}c_{10} + a_{11}c_{01} + a_{12}c_{00} - 2\,c_{00}c_{03} \\
& - \frac{(-I_2 + 2\,I_1 + I_3)\,c_{01}c_{02}}{2\,I_1 - I_2}, \\
A_{2,0} = \;& a_{03}c_{00} + d_{00} + a_{00}c_{03} + a_{01}c_{02} + a_{02}c_{01}.
\end{aligned}
$$

Now we have

$$
\begin{aligned}
& A_0(\theta) = A_{0,9}\cos^9\theta + A_{0,8}\cos^8\theta + A_{0,7}\cos^7\theta + A_{0,6}\cos^6\theta + A_{0,5}\cos^5\theta + A_{0,4}\cos^4\theta \\
& + A_{0,3}\cos^3\theta + A_{0,2}\cos^2\theta + A_{0,1}\cos\theta + A_{0,0},
\end{aligned}
$$

with

$$
\begin{aligned}
A_{0,9}(\theta) = \;& -(-4\,a_{21}c_{11} + 4\,c_{02}c_{03} + 8\,c_{02}c_{21} - 4\,c_{03}c_{20} + 4\,c_{11}c_{12} - 4\,c_{11}c_{30} - 8\,c_{20}c_{21} \\
& -8\,a_{11}c_{21} + 4\,a_{12}c_{02} - 4\,a_{12}c_{20} + 4\,a_{20}a_{21} - 4\,a_{20}c_{12} + 4\,a_{20}c_{30} + 4\,a_{02}a_{03} - 4\,a_{02}a_{21} \\
& +4\,a_{02}c_{12} - 4\,a_{02}c_{30} - 4\,a_{03}a_{20} + 4\,a_{03}c_{11} - 4\,a_{11}a_{12} - 4\,a_{11}c_{03})\,\mathrm{e}^{\sin^2\theta}, \\
A_{0,8}(\theta) = \;& -(-4\,a_{03}c_{20} + 4\,a_{11}a_{21} - 4\,a_{11}c_{12} + 4\,a_{12}a_{20} - 4\,a_{02}c_{03} - 8\,a_{02}c_{21} - 4\,a_{03}a_{11} \\
& +4\,a_{03}c_{02} - 8\,c_{11}c_{21} - 4\,c_{12}c_{20} + 4\,c_{20}c_{30} + 4\,a_{11}c_{30} - 4\,a_{02}a_{12} - 4\,a_{12}c_{11} \\
& +4\,a_{20}c_{03} + 8\,a_{20}c_{21} - 4\,a_{21}c_{02} + 4\,a_{21}c_{20} + 4\,c_{02}c_{12} - 4\,c_{02}c_{30} - 4\,c_{03}c_{11})\,\mathrm{e}^{\sin^2\theta}\sin\theta, \\
A_{0,7}(\theta) = \;& -(2\,c_{11}c_{30} + 4\,c_{20}c_{21} - 14\,a_{02}a_{03} + 10\,a_{02}a_{21} - 10\,a_{02}c_{12} + 6\,a_{02}c_{30} + 10\,a_{03}a_{20} \\
& -10\,a_{03}c_{11} + 10\,a_{11}a_{12} + 2\,a_{11}c_{03} + 12\,a_{11}c_{21} - 10\,a_{12}c_{02} + 6\,a_{12}c_{20} - 6\,a_{20}a_{21} + 6\,a_{20}c_{12} \\
& -2\,a_{20}c_{30} + 6\,a_{21}c_{11} - 2\,c_{02}c_{03} - 12\,c_{02}c_{21} - 2\,c_{03}c_{20} - 6\,c_{11}c_{12})\,\mathrm{e}^{\sin^2\theta}, \\
A_{0,6}(\theta) = \;& -(2\,c_{20}c_{30} + 2\,a_{02}c_{03} + 6\,a_{12}c_{11} + 2\,a_{20}c_{03} - 4\,a_{20}c_{21} + 6\,a_{21}c_{02} - 2\,a_{21}c_{20} - 6\,c_{02}c_{12} \\
& +2\,c_{02}c_{30} - 2\,c_{03}c_{11} + 4\,c_{11}c_{21} + 2\,c_{12}c_{20} - 6\,a_{11}a_{21} + 6\,a_{11}c_{12} - 2\,a_{11}c_{30} - 6\,a_{12}a_{20} \\
& +6\,a_{03}c_{20} + 12\,a_{02}c_{21} + 10\,a_{03}a_{11} + 10\,a_{02}a_{12} - 10\,a_{03}c_{02})\,\mathrm{e}^{\sin^2\theta}\sin\theta,
\end{aligned}
$$

$$
\begin{aligned}
A_{0,5}(\theta) = \;& -[-2\,b_{20} + 18\,a_{02}a_{03} - 8\,a_{11}a_{12} + 2\,b_{02} - a_{20}c_{12} - a_{20}c_{30} - a_{21}c_{11} - 4\,c_{02}c_{03} + c_{02}c_{21} \\
& +2\,c_{03}c_{20} + 2\,c_{1,1}c_{30} + 3\,c_{20}c_{21} + 3\,a_{11}c_{03} - 3\,a_{11}c_{21} + 7\,a_{12}c_{02} - a_{12}c_{20} + 2\,a_{20}a_{21} \\
& +7\,a_{02}c_{12} - a_{02}c_{30} + 2\,d_{11} - 8\,a_{02}a_{21} - 8\,a_{03}a_{20} + 7\,a_{03}c_{11}]\,\mathrm{e}^{\sin^2\theta}, \\
A_{0,4}(\theta) = \;& -[7\,a_{03}c_{02} - a_{03}c_{20} - 8\,a_{03}a_{11} - a_{11}c_{30} + 2\,a_{12}a_{20} - a_{12}c_{11} - a_{11}c_{12} - a_{20}c_{21} \\
& -8\,a_{02}a_{12} + 3\,a_{02}c_{0,3} - 3\,a_{02}c_{21} + 2\,c_{03}c_{11} + 3\,c_{11}c_{21} + 2\,c_{12}c_{20} + 2\,a_{11}a_{21} - a_{20}c_{03} \\
& -2\,b_{11} + 2\,d_{02} - 2\,d_{2,0} + 2\,c_{02}c_{30} - a_{21}c_{02} - a_{21}c_{20})\,\mathrm{e}^{\sin^2\theta}\sin\theta,
\end{aligned}
$$

$$
\begin{aligned}
A_{0,3}(\theta) = \;& -[-a_{11}c_{21} - d_{11} - a_{12}c_{20} - 3\,b_{02} + b_{20} + 2\,a_{11}a_{12} + 3\,c_{02}c_{21} + 2\,c_{03}c_{20} - a_{20}c_{12} \\
& -a_{21}c_{11} - 10\,a_{02}a_{03} + 2\,a_{02}a_{21} + 2\,c_{11}c_{12} + 2\,a_{03}a_{20} - a_{02}c_{30}]\,\mathrm{e}^{\sin^2\theta},
\end{aligned}
$$

$$
\begin{aligned}
A_{0,2}(\theta) = \;& -[2\,a_{03}a_{11} - a_{03}c_{20} - a_{02}c_{21} - a_{20}c_{03} - a_{21}c_{02} - a_{11}c_{12} - a_{12}c_{11} - d_{02} - d_{20} \\
& +2\,c_{02}c_{12} + 2\,c_{03}c_{11} + b_{11} + 2\,a_{02}a_{12}]\,\mathrm{e}^{\sin^2\theta}\sin\theta,
\end{aligned}
$$

$$
A_{0,1}(\theta) = -(2\,c_{02}c_{03} - a_{02}c_{12} + b_{02} - d_{11} + 2\,a_{02}a_{03} - a_{03}c_{11} - a_{11}c_{03} - a_{12}c_{02})\,\mathrm{e}^{\sin^2\theta},
$$

$$
A_{0,0}(\theta) = -[-d_{02} - a_{02}c_{03} - a_{03}c_{02}]\,\mathrm{e}^{\sin^2\theta}\sin\theta.
$$

Additionally, we have

$$
\begin{aligned}
A_5(\theta) = \;& \Big[\sin\theta\cos\theta\left(-1+2\,\cos^2\theta\right)a_{00}{}^2+\left(-4\,\cos^4\theta+2\,\cos^2\theta+1\right)c_{00}a_{00} \\
& -\sin\theta\cos\theta\left(2\,\cos^2\theta+1\right)c_{00}{}^2\Big]\,\mathrm{e}^{6\,\sin^2\theta}, \\
A_3(\theta) = \;& 2\,\cos^3\theta\sin\theta\left(-1+2\,\cos^2\theta\right)a_{00}a_{20}-2\,\cos\theta\sin\theta\left(\cos^2\theta-1\right)\left(-1+2\,\cos^2\theta\right)a_{00}a_{02} \\
& -2\,\cos^2\theta\left(\cos^2\theta-1\right)\left(-1+2\,\cos^2\theta\right)a_{00}a_{11}-\cos^2\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)a_{00}c_{20} \\
& +\left(\cos^2\theta-1\right)\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)a_{00}c_{02}-\cos\theta\sin\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)c_{11}a_{00} \\
& -\cos\theta\sin\theta\left(\cos^2\theta-1\right)\left(-1+2\,\cos^2\theta\right)a_{01}{}^2-\left(\cos^2\theta-1\right)\frac{2\,I_1-4\,\cos^4\theta I_3+2\,\cos^2\theta I_3-I_2}{2\,I_1-I_2}a_{01}c_{01} \\
& -\cos\theta\sin\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)c_{10}a_{01}-\cos^2\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)c_{00}a_{20} \\
& +\left(\cos^2\theta-1\right)\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)c_{00}a_{02}-\cos\theta\sin\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)c_{00}a_{11} \\
& -2\,\cos^3\theta\sin\theta\left(2\,\cos^2\theta+1\right)c_{00}c_{20}+2\,\cos\theta\sin\theta\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta+1\right)c_{00}c_{02} \\
& +2\,\cos^2\theta\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta+1\right)c_{11}c_{00}-\cos\theta\sin\theta\, I_3\,\frac{2\,I_1-I_2+\cos^2\theta I_3-2\,\cos^4\theta I_3}{2\,I_1-I_2}c_{01}{}^2 \\
& -\cos^2\theta\,\frac{2\,I_1+I_3+2\,\cos^2\theta I_3-I_2-4\,\cos^4\theta I_3}{2\,I_1-I_2}c_{10}c_{01}-\cos^3\theta\sin\theta\left(2\,\cos^2\theta+1\right)c_{10}{}^2\Big]\exp(4\sin^2\theta), \\
A_1(\theta) = \;& \Big[-\cos^4\theta\,\frac{2\,I_1+I_3+2\,\cos^2\theta I_3-I_2-4\,\cos^4\theta I_3}{2I_1-I_2}c_{01}c_{30}+\cos^4\theta\left(8\,\cos^4\theta-3-4\,\cos^2\theta\right)c_{10}c_{21} \\
& +2\,\cos^2\theta\left(2\,\cos^2\theta-1\right)\left(\cos^2\theta-1\right)^2a_{02}a_{11}+\cos^2\theta\left(\cos^2\theta-1\right)\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)a_{20}c_{02} \\
& +\sin\theta\cos\theta\left(2\,\cos^2\theta-1\right)\left(\cos^2\theta-1\right)^2a_{02}{}^2-\sin\theta\cos^3\theta\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta-1\right)a_{11}{}^2 \\
& -2\,\cos^4\theta\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta-1\right)a_{11}a_{20} \\
& +\mathrm{e}^{2\,\sin^2\theta}\cos^2\theta\left(\cos^2\theta-1\right)\left(8\,\cos^4\theta-4\,\cos^2\theta-1\right)a_{01}c_{21} \\
& +2\,\cos^2\theta\left(2\,\cos^2\theta-1\right)\left(\cos^2\theta-1\right)^2a_{01}a_{12}+\cos^2\theta\left(2\,\cos^2\theta-1\right)b_{10} \\
& +\cos^2\theta\left(\cos^2\theta-1\right)\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)c_{10}a_{12}+2\,\cos^4\theta\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta+1\right)c_{11}c_{20} \\
& -\sin\theta\cos\theta\left(2\,\cos^2\theta+1\right)\left(\cos^2\theta-1\right)^2c_{02}{}^2-2\,\cos^2\theta\left(2\,\cos^2\theta+1\right)\left(\cos^2\theta-1\right)^2c_{11}c_{02} \\
& +\sin\theta\cos^3\theta\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta+1\right)c_{11}{}^2+\cos^2\theta\left(\cos^2\theta-1\right)\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)c_{11}a_{11} \\
& +\cos^2\theta\left(\cos^2\theta-1\right)\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)a_{02}c_{20} \\
& -\sin\theta\cos\theta\left(\cos^2\theta-1\right)\frac{2\,I_1-4\,\cos^4\theta I_3+2\,\cos^2\theta I_3-I_2}{2I_1-I_2}a_{12}c_{01} \\
& -\sin\theta\cos\theta\,\frac{-4\,\cos^6\theta I_3-I_2+2\,I_1+2\,\cos^2\theta I_1+\cos^2\theta I_3+I_3-2\,\cos^4\theta I_3-\cos^2\theta I_2}{2I_1-I_2}c_{01}c_{03} \\
& -\sin\theta\cos^5\theta\left(2\,\cos^2\theta+1\right)c_{20}{}^2-\sin\theta\cos^3\theta\,\frac{4\,\cos^2\theta I_3+4\,I_1-8\,\cos^4\theta I_3+I_3-2\,I_2}{2I_1-I_2}c_{01}c_{21} \\
& +2\,\cos^2\theta\left(\cos^4\theta-\cos^2\theta-1+2\,\cos^6\theta\right)c_{03}c_{10}+\sin\theta\cos^5\theta\left(-1+2\,\cos^2\theta\right)a_{20}{}^2 \\
& -\cos^4\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)a_{20}c_{20}+\left(\cos^2\theta-1\right)^2\frac{2\,I_1-4\,\cos^4\theta I_3+2\,\cos^2\theta I_3-I_2}{2I_1-I_2}a_{03}c_{01} \\
& -\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta+1\right)d_{01}+\sin\theta\cos\theta\left(2\,\cos^2\theta+1\right)d_{10}+\sin\theta\cos\theta\left(-1+2\,\cos^2\theta\right)b_{01} \\
& -\sin\theta\cos^3\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)a_{11}c_{20}-\sin\theta\cos^3\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)a_{01}c_{30} \\
& +\left(\cos^2\theta-1\right)\left(4\,\cos^6\theta+2\,\cos^4\theta-(\cos\theta)^2-1\right)a_{01}c_{03} \\
& -\cos^2\theta\left(\cos^2\theta-1\right)\frac{2\,I_1-4\,\cos^4\theta I_3+2\,\cos^2\theta I_3-I_2}{2I_1-I_2}a_{21}c_{01} \\
& -\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)\left(\cos^2\theta-1\right)^2a_{02}c_{02}-\sin\theta\cos^3\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)c_{11}a_{20} \\
& +\cos^2\theta\left(\cos^2\theta-1\right)\frac{2\,I_1+I_3+2\,\cos^2\theta I_3-I_2-4\,\cos^4\theta I_3}{2I_1-I_2}c_{12}c_{01} \\
& -\sin\theta\cos^3\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)c_{10}a_{21}-2\,\sin\theta\cos^5\theta\left(2\,\cos^2\theta+1\right)c_{10}c_{30} \\
& +\sin\theta\cos\theta\left(\cos^2\theta-1\right)\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)c_{12}a_{01} \\
& -2\,\sin\theta\cos^3\theta\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta-1\right)a_{01}a_{21} \\
& +2\,\sin\theta\cos\theta\left(-1+2\,\cos^2\theta\right)\left(\cos^2\theta-1\right)^2a_{01}a_{03}-2\,\sin\theta\cos^3\theta\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta-1\right)a_{02}a_{20} \\
& +\sin\theta\cos\theta\left(\cos^2\theta-1\right)\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)c_{11}a_{02} \\
& +\sin\theta\cos\theta\left(\cos^2\theta-1\right)\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)a_{11}c_{02} \\
& +\sin\theta\cos\theta\left(\cos^2\theta-1\right)\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)c_{10}a_{03} \\
& +2\,\sin\theta\cos^3\theta\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta+1\right)c_{12}c_{10} \\
& +2\,\sin\theta\cos^3\theta\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta+1\right)c_{02}c_{20}\Big]\,\mathrm{e}^{2\,\sin^2\theta},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{A}_1(\theta) = \quad & \cos\theta\sin\theta\left(\cos^4\theta-\cos^2\theta+2\,\cos^8\theta+3\,\cos^6\theta-1\right){c_{03}}^2\\
& -\left(4\,\cos^6\theta+2\,\cos^4\theta-\cos^2\theta-1\right)\left(\cos^2\theta-1\right)^2 a_{03}c_{03}\\
& -\cos^5\theta\sin\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)a_{21}c_{30}+\cos^6\theta\left(8\,\cos^4\theta-3-4\,\cos^2\theta\right)c_{21}c_{30}\\
& +2\,\cos^4\theta\left(\cos^4\theta-\cos^2\theta-1+2\,\cos^6\theta\right)c_{03}c_{30}\\
& +\cos^3\theta\sin\theta\left(2\,\cos^2\theta+1\right)d_{30}+\cos^3\theta\sin\theta\left(-1+2\,\cos^2\theta\right)b_{21}\\
& +2\,{c_{21}}^2\cos^5\theta\sin\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)+{a_{12}}^2\sin^5\theta\cos^3\theta\left(-1+2\,\cos^2\theta\right)\\
& -\cos^7\theta\sin\theta\left(2\,\cos^2\theta+1\right){c_{30}}^2+\left(2\,\cos^2\theta+1\right)\left(\cos^2\theta-1\right)^2 d_{03}\\
& +\cos^3\theta\sin\theta\left(4\,\cos^4\theta+8\,\cos^6\theta-3-3\,\cos^2\theta\right)c_{03}c_{21}-\cos^2\theta\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta+1\right)d_{21}\\
& -\cos^2\theta\left(\cos^2\theta-1\right)\left(-1+2\,\cos^2\theta\right)b_{12}+\cos^4\theta\left(-1+2\,\cos^2\theta\right)b_{30}\\
& -\cos\theta\sin\theta\left(-1+2\,\cos^2\theta\right)\left(\cos^2\theta-1\right)^3{a_{03}}^2+\cos^4\theta\left(\cos^2\theta-1\right)\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)a_{12}c_{30}\\
& -2\,\cos^2\theta\left(\cos^2\theta-1\right)\left(\cos^4\theta-\cos^2\theta-1+2\,\cos^6\theta\right)c_{03}c_{12}\\
& -\cos\theta\sin\theta\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta+1\right)d_{12}\\
& -\cos\theta\sin\theta\left(\cos^2\theta-1\right)\left(-1+2\,\cos^2\theta\right)b_{03}+2\,\cos^4\theta\left(-1+2\,\cos^2\theta\right)\left(\cos^2\theta-1\right)^2 a_{12}a_{21}\\
& +\cos^4\theta\left(\cos^2\theta-1\right)\left(8\,\cos^4\theta-4\,\cos^2\theta-1\right)a_{21}c_{21}\\
& -\cos^4\theta\left(\cos^2\theta-1\right)\left(8\,\cos^4\theta-3-4\,\cos^2\theta\right)c_{12}c_{21}\\
& -2\,\cos^2\theta\left(-1+2\,\cos^2\theta\right)\left(\cos^2\theta-1\right)^3 a_{03}a_{12}-\cos^2\theta\left(8\,\cos^4\theta-4\,\cos^2\theta-1\right)\left(\cos^2\theta-1\right)^2 a_{03}c_{21}\\
& -\cos^2\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)\left(\cos^2\theta-1\right)^2 a_{12}c_{12}\\
& +\cos^2\theta\left(\cos^2\theta-1\right)\left(4\,\cos^6\theta+2\,\cos^4\theta-\cos^2\theta-1\right)a_{21}c_{03}\\
& -\cos^5\theta\sin\theta\left(\cos^2\theta-1\right)\left(-1+2\,\cos^2\theta\right){a_{21}}^2\\
& -\cos^3\theta\sin\theta\left(2\,\cos^2\theta+1\right)\left(\cos^2\theta-1\right)^2{c_{12}}^2+2\,\cos^5\theta\sin\theta\left(\cos^2\theta-1\right)\left(2\,\cos^2\theta+1\right)c_{12}c_{30}\\
& +2\,\cos^3\theta\sin\theta\left(-1+2\,\cos^2\theta\right)\left(\cos^2\theta-1\right)^2 a_{03}a_{21}\\
& +\cos^3\theta\sin\theta\left(\cos^2\theta-1\right)\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)a_{03}c_{30}\\
& +\cos^3\theta\sin\theta\left(\cos^2\theta-1\right)\left(8\,\cos^4\theta-4\,\cos^2\theta-1\right)a_{12}c_{21}\\
& +\cos^3\theta\sin\theta\left(\cos^2\theta-1\right)\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)a_{21}c_{12}\\
& -\cos\theta\sin\theta\left(4\,\cos^4\theta-1-2\,\cos^2\theta\right)\left(\cos^2\theta-1\right)^2 a_{03}c_{12}\\
& +\cos\theta\sin\theta\left(\cos^2\theta-1\right)\left(4\,\cos^6\theta+2\,\cos^4\theta-\cos^2\theta-1\right)a_{12}c_{03}.
\end{aligned}
$$

7. Appendix C

Here we present the explicit expressions of $s_5(\theta), s_4(\theta), s_3(\theta), s_2(\theta), s_1(\theta), \tilde{s}_1(\theta)$ that appear in relation (10). Thus $s_5(\theta) = s_{5,1}(\theta)c_{00}^2 + s_{5,2}(\theta)a_{00}^2 + s_{5,3}(\theta)a_{00}c_{00}$, with

$$
\begin{aligned}
s_{5,1}(\theta) = \quad & -2\,\mathrm{e}^{3\,\sin^2\theta}\left(\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw+2\,\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw\right)\\
& \left(\sin\theta+2\,\sin\theta\cos^2\theta\right),\\
s_{5,2}(\theta) = \quad & -2\,\mathrm{e}^{3\,\sin^2\theta}\left(-\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos w dw+2\,\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos^3 w dw\right)\\
& \left(-\cos\theta+2\,\cos^3\theta\right),
\end{aligned}
$$

$$
\begin{aligned}
s_{5,3}(\theta) = \quad & -2\,\mathrm{e}^{3\,\sin^2\theta}\left(\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\sin w dw + 2\,\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw\right) \\
& \left(-\cos\theta + 2\,\cos^3\theta\right) \\
& -2\,\mathrm{e}^{3\,\sin^2\theta}\left(-\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\cos w dw + 2\,\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\cos^3 w dw\right) \\
& \left(\sin\theta + 2\,\sin\theta\cos^2\theta\right).
\end{aligned}
$$

Additionally, we have

$$
s_4(\theta) = s_{4,1}(\theta)a_{00}c_{10} - s_{4,2}(\theta)a_{00}a_{01} - s_{4,3}(\theta)c_{00}c_{01} - s_{4,4}(\theta)c_{00}c_{10} - s_{4,5}(\theta)a_{00}c_{01} - s_{4,6}(\theta)a_{01}c_{00},
$$

and $s_{4,i}(\theta)$ for $i = 1\cdots, 6$ are the following:

$$
\begin{aligned}
s_{4,1}(\theta) = \quad & -\Bigg(-\sin\theta\mathrm{e}^{-\sin^2\theta}\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\cos w dw + 4\,\cos^2\theta\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw \\
& +2\,\sin\theta\mathrm{e}^{-\sin^2\theta}\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\cos^3 w dw + 4\,\cos^2\theta\sin\theta\mathrm{e}^{-\sin^2\theta}\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\cos^3 w dw \\
& +8\,\cos^2\theta\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw - 2\,\cos^2\theta\sin\theta\mathrm{e}^{-\sin^2\theta}\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\cos w dw \\
& -4\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw - 2\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\Bigg)\mathrm{e}^{3\,\sin^2\theta}\cos\theta, \\
s_{4,2}(\theta) = \quad & -\Bigg(\sin\theta\mathrm{e}^{-\sin^2\theta}\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\cos w dw - 2\,\sin\theta\mathrm{e}^{-\sin^2\theta}\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\cos^3 w dw \\
& -4\,\cos^2\theta\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw - 4\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw \\
& -2\,\cos^2\theta\sin\theta\mathrm{e}^{-\sin^2\theta}\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\cos w dw \\
& +4\,\cos^2\theta\sin\theta\mathrm{e}^{-\sin^2\theta}\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\cos^3 w dw + 2\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw \\
& +8\,\cos^2\theta\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\Bigg)\mathrm{e}^{3\,\sin^2\theta}\cos\theta, \\
s_{4,3}(\theta) = \quad & -\mathrm{e}^{2\,\sin^2\theta}\Bigg(8\,\mathrm{e}^{\sin^2\theta}I_1\,\cos^2\theta\sin\theta\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}dw \\
& -2\,I_2\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw + 2\,I_1\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\sin w dw \\
& -2\,\mathrm{e}^{\sin^2\theta}I_2\,\sin\theta\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}dw - 2\,\cos^4\theta I_3\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\sin w dw \\
& -4\,\mathrm{e}^{\sin^2\theta}I_2\,\cos^2\theta\sin\theta\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}dw + 2\,\mathrm{e}^{\sin^2\theta}I_3\,\sin\theta\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}\cos^2 w dw \\
& +4\,I_1\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw - 4\,\mathrm{e}^{\sin^2\theta}I_3\,\sin\theta\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}\cos^4 w dw \\
& +2\,\cos^2\theta I_3\int_0^{\theta}\mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw + 4\,\mathrm{e}^{\sin^2\theta}I_3\,\cos^2\theta\sin\theta\int_0^{\theta}\mathrm{e}^{2\,\sin^2 w}\cos^2 w dw
\end{aligned}
$$

$$
\begin{aligned}
&+4\,\mathrm{e}^{\sin^2\theta} I_1\ \sin\theta\int_0^\theta \mathrm{e}^{2\ \sin^2 w}dw+\cos^2\theta I_3\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\sin w dw\\
&-I_2\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\sin w dw-4\ \cos^4\theta I_3\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\sin w\cos^2 w dw\\
&-8\,\mathrm{e}^{\sin^2\theta} I_3\ \cos^2\theta\sin\theta\textstyle\int_0^\theta \mathrm{e}^{2\ \sin^2 w}\cos^4 w dw\Big)\dfrac{1}{2\,I_1-I_2},
\end{aligned}
$$

$$
\begin{aligned}
s_{4,4}(\theta)=\ &-\Bigg(8\,\mathrm{e}^{\sin^2\theta}\cos^2\theta\int_0^\theta \mathrm{e}^{2\ \sin^2 w}\cos^3 w\sin w dw\\
&+4\,\mathrm{e}^{\sin^2\theta}\cos^2\theta\int_0^\theta \mathrm{e}^{2\ \sin^2 w}\sin w\cos w dw+2\left(\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\sin w\cos^2 w dw\right)\cos\theta\\
&+\left(\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\sin w dw\right)\cos\theta+2\,\mathrm{e}^{\sin^2\theta}\int_0^\theta \mathrm{e}^{2\ \sin^2 w}\sin w\cos w dw\\
&+4\left(\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\sin w\cos^2 w dw\right)\cos^3\theta+4\,\mathrm{e}^{\sin^2\theta}\int_0^\theta \mathrm{e}^{2\ \sin^2 w}\cos^3 w\sin w dw\\
&+2\left(\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\sin w dw\right)\cos^3 w\Bigg)\mathrm{e}^{2\ \sin^2\theta}\sin\theta,
\end{aligned}
$$

$$
\begin{aligned}
s_{4,5}(\theta)=\ &-\mathrm{e}^{2\ \sin^2\theta}\Bigg(-2\,I_1\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\cos w dw+2\ \cos^4\theta I_3\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\cos w dw\\
&-2\,I_2\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\cos^3 w dw+4\,\mathrm{e}^{\sin^2\theta} I_3\ \cos\theta\int_0^\theta \mathrm{e}^{2\ \sin^2 w}\cos^4 w dw\\
&-2\,\mathrm{e}^{\sin^2\theta} I_3\ \cos\theta\int_0^\theta \mathrm{e}^{2\ \sin^2 w}\cos^2 w dw-4\ \cos^4\theta I_3\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\cos^3 w dw\\
&+2\,\mathrm{e}^{\sin^2\theta} I_2\ \cos\theta\int_0^\theta \mathrm{e}^{2\ \sin^2 w}dw+I_2\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\cos w dw\\
&-4\,\mathrm{e}^{\sin^2\theta} I_2\ \cos^3 w\int_0^\theta \mathrm{e}^{2\ \sin^2 w}dw-8\,\mathrm{e}^{\sin^2\theta} I_3\ \cos^3 w\int_0^\theta \mathrm{e}^{2\ \sin^2 w}\cos^4 w dw\\
&+4\,\mathrm{e}^{\sin^2\theta} I_3\ \cos^3 w\int_0^\theta \mathrm{e}^{2\ \sin^2 w}\cos^2 w dw+4\,I_1\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\cos^3 w dw\\
&+8\,\mathrm{e}^{\sin^2\theta} I_1\ \cos^3 w\int_0^\theta \mathrm{e}^{2\ \sin^2 w}dw+2\ \cos^2\theta I_3\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\cos^3 w dw\\
&-\cos^2\theta I_3\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\cos w dw-4\,\mathrm{e}^{\sin^2\theta} I_1\ \cos\theta\int_0^\theta \mathrm{e}^{2\ \sin^2 w}dw\Bigg)\frac{1}{2\,I_1-I_2},
\end{aligned}
$$

$$
\begin{aligned}
s_{4,6}(\theta)=\ &-\Bigg(4\int_0^\theta \mathrm{e}^{2\ \sin^2 w}\cos^3 w\sin w dw+4\ \cos^3 w\mathrm{e}^{-\sin^2\theta}\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\sin w\cos^2 w dw\\
&+8\ \cos^2\theta\int_0^\theta \mathrm{e}^{2\ \sin^2 w}\cos^3 w\sin w dw-4\ \cos^2\theta\int_0^\theta \mathrm{e}^{2\ \sin^2 w}\sin w\cos w dw\\
&+2\ \cos^3 w\mathrm{e}^{-\sin^2\theta}\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\sin w dw-\cos\theta\mathrm{e}^{-\sin^2\theta}\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\sin w dw\\
&-2\ \cos\theta\mathrm{e}^{-\sin^2\theta}\int_0^\theta \mathrm{e}^{3\ \sin^2 w}\sin w\cos^2 w dw-2\int_0^\theta \mathrm{e}^{2\ \sin^2 w}\sin w\cos w dw\Bigg)\mathrm{e}^{3\ \sin^2\theta}\sin\theta.
\end{aligned}
$$

Now we have

$$s_3(\theta) = s_{3,1}(\theta)a_{00}c_{02} + s_{3,2}(\theta)a_{02}c_{00} + s_{3,3}(\theta)a_{01}c_{01} + s_{3,4}(\theta)a_{00}a_{11} + s_{3,5}(\theta)a_{00}c_{20}$$
$$+s_{3,6}(\theta)c_{00}c_{11} + s_{3,7}(\theta)c_{01}c_{10} + s_{3,8}(\theta)a_{00}a_{02} + s_{3,9}(\theta)a_{00}a_{20}$$
$$+s_{3,10}(\theta)a_{00}c_{11} + s_{3,11}(\theta)a_{01}c_{10} + s_{3,12}(\theta)a_{11}c_{00} + s_{3,13}(\theta)c_{00}c_{02} + s_{3,14}(\theta)c_{00}c_{20}$$
$$+s_{3,15}(\theta)a_{01}^2 + s_{3,16}(\theta)c_{10}^2 + s_{3,17}(\theta)c_{01}^2 + s_{3,18}(\theta)a_{20}c_{00},$$

and $s_{3,i}(\theta)$ for $i = 1\cdots,18$ satisfying the following expressions:

$$\begin{aligned}
s_{3,1}(\theta) = \quad & -2\,\mathrm{e}^{3\,\sin^2\theta}\cos\theta\left[\left(4\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w\cos^2 w dw\right)\cos^2\theta\right.\\
& \left.-2\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w\cos^2 w dw + 2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw\right)\cos^2\theta - \int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw\right],\\
s_{3,2}(\theta) = \quad & 2\,\mathrm{e}^{3\,\sin^2\theta}\sin\theta\left[2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw\right)\cos^2\theta\right.\\
& +2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw + 4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta + \int_0^\theta \mathrm{e}^{\sin^2 w^2}\cos w dw\\
& \left.-\,6\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\right)\cos^2\theta - 3\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\right),\\
s_{3,3}(\theta) = \quad & -\mathrm{e}^{2\,\sin^2\theta}\left[-2\,I_1\,\cos\theta\sin\theta\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw + 4\,I_1\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right.\\
& -2\,I_1\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw + 4\,I_1\,\sin\theta\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\right)\cos^3\theta\\
& +2\,I_3\,\cos\theta\sin\theta\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw\\
& +2\,I_3\,\sin\theta\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw\right)\cos^3\theta + 2\,I_3\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^2\theta\\
& -2\,I_2\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw + I_2\,\cos\theta\sin\theta\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\\
& -4\,I_3\,\sin\theta\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw\right)\cos^3\theta - I_3\,\cos\theta\sin\theta\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw\\
& +2\,I_3\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^4\theta - 2\,I_2\,\sin\theta\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\right)\cos^3\theta\\
& -4\,I_3\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^3 w\cos^4\theta - I_3\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^2\theta\\
& \left.+I_2\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right]\frac{1}{2\,I_1 - I_2},
\end{aligned}$$

$$
\begin{aligned}
s_{3,4}(\theta) = \;& 2\,\mathrm{e}^{3\,\sin^2\theta}\cos\theta\left[\left(2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw\right)\cos^2\theta\right.\\
&-\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw+2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\\
&\left.-4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\right)\cos^2\theta\right],\\
s_{3,5}(\theta) = \;& -2\,\mathrm{e}^{3\,\sin^2\theta}\cos\theta\left[2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw\right)\cos^2\theta\right.\\
&+4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\right)\cos^2\theta\\
&\left.-\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw-2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\right],
\end{aligned}
$$

$$
\begin{aligned}
s_{3,6}(\theta) = \;& -2\,\mathrm{e}^{3\,\sin^2\theta}\sin\theta\left[\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw-2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\right.\\
&+\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw+2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw\right)\cos^2\theta\\
&\left.+\,2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\right)\cos^2\theta-4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta\right],
\end{aligned}
$$

$$
\begin{aligned}
s_{3,7}(\theta) = \;& -\mathrm{e}^{2\,\sin^2\theta}\left[2\,I_1\,\cos\theta\sin\theta\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw+4\,I_1\,\sin\theta\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\right)\cos^3\theta\right.\\
&+2\,I_1\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw+4\,I_1\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\\
&+I_3\,\cos\theta\sin\theta\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw-I_2\,\cos\theta\sin\theta\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\\
&-2\,I_3\,\cos\theta\sin\theta\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw-I_2\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\\
&+I_3\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^2\theta-2\,I_2\,\sin\theta\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\right)\cos^3\theta\\
&-2\,I_2\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw-4\,I_3\,\sin\theta\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw\right)\cos^3\theta\\
&+2\,I_3\,\sin\theta\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw\right)\cos^3\theta+2\,I_3\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^2\theta\\
&\left.-4\,I_3\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^4\theta-2\,I_3\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^4\theta\right]\frac{1}{2\,I_1-I_2},
\end{aligned}
$$

$$s_{3,8}(\theta) = -2\,\mathrm{e}^{3\,\sin^2\theta}\cos\theta\left[-3\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw - 2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw\right)\cos^2\theta\right.$$
$$+6\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w dw\right)\cos^2\theta + 2\int_0^\theta \mathrm{e}^{\sin^2 w^2}\cos^5 w dw$$
$$\left.+\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw - 4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta\right],$$
$$s_{3,9}(\theta) = 2\,\mathrm{e}^{3\,\sin^2\theta}\cos\theta\left[2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw - 4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta\right.$$
$$\left.-\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw + 2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\right)\cos^2\theta\right],$$

$$s_{3,10}(\theta) = -2\,\mathrm{e}^{3\,\sin^2\theta}\cos\theta\left[2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\right)\cos^2\theta\right.$$
$$+2\left(\int_0^\theta \mathrm{e}^{\sin w^2}\cos w dw\right)\cos^2\theta - \int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw$$
$$\left.-4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta - \int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw + 2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\right],$$
$$s_{3,11}(\theta) = 2\,\mathrm{e}^{2\,\sin^2\theta}\sin\theta\cos\theta\left[-4\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^2\theta\right.$$
$$\left.+\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right],$$

$$s_{3,12}(\theta) = 2\,\mathrm{e}^{3\,\sin^2\theta}\sin\theta\left[2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw\right)\cos^2\theta\right.$$
$$+\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw - 2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw$$
$$\left.-4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\right)\cos^2\theta\right],$$

$$s_{3,13}(\theta) = -2\,\mathrm{e}^{3\,\sin^2\theta}\sin\theta\left[2\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w\cos^2 w dw\right.$$
$$+\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw + 4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w\cos^2 w dw\right)\cos^2\theta$$
$$\left.+2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw\right)\cos^2\theta\right],$$

$$
\begin{aligned}
s_{3,14}(\theta) = \quad & -2\,\mathrm{e}^{3\,\sin^2\theta}\sin\theta\left[2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\right.\\
&+4\,\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\right)\cos^2\theta+\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw\\
&\left.+\,2\,\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw\right)\cos^2\theta\right],\\
s_{3,15}(\theta) = \quad & \mathrm{e}^{2\,\sin^2\theta}\sin\theta\cos\theta\left[\left(-4\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^2\theta\right.\\
&-\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw+2\,\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^2\theta\\
&\left.+\,2\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right),
\end{aligned}
$$

$$
\begin{aligned}
s_{3,16}(\theta) = \quad & -\mathrm{e}^{2\,\sin^2\theta}\sin\theta\cos\theta\left[4\,\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^2\theta\right.\\
&+\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw+2\,\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^2\theta\\
&\left.+\,2\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right],
\end{aligned}
$$

$$
\begin{aligned}
s_{3,17}(\theta) = \quad & -\mathrm{e}^{2\,\sin^2\theta}\left[-4\,I_1\,I_2\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw-2\,{I_3}^2\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw\right)\cos^2\theta\right.\\
&-I_2\,I_3\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw+{I_2}^2\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\\
&+2\,I_1\,I_3\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw+2\,I_2\,I_3\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw\\
&+2\,I_1\,I_3\,\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\right)\cos^2\theta+{I_3}^2\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw\right)\cos^2\theta\\
&-I_2\,I_3\,\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\right)\cos^2\theta+4\,{I_1}^2\textstyle\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw-4\,I_1\,I_3\displaystyle\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw\\
&-2\,{I_3}^2\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw\right)\cos^4\theta+2\,I_2\,I_3\,\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\right)\cos^4\theta\\
&\left.-4\,I_1\,I_3\,\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\right)\cos^4\theta+4\,{I_3}^2\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw\right)\cos^4\theta\right]\frac{1}{\left(-I_2+2\,I_1\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
s_{3,18}(\theta) = \quad & 2\,\mathrm{e}^{3\,\sin^2\theta}\sin\theta\left[-2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw+\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\right.\\
&\left.+2\,\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\right)\cos^2\theta-4\,\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta\right].
\end{aligned}
$$

Now we have

$$s_2(\theta) = s_{2,1}(\theta)\, a_{00}c_{30} + s_{2,2}(\theta)\, a_{20}c_{10} + s_{2,3}(\theta)\, a_{00}c_{12} + s_{2,4}(\theta)\, a_{02}c_{10} + s_{2,5}(\theta)\, a_{12}c_{0,0} + s_{2,6}(\theta)\, a_{00}a_{21}$$
$$+s_{2,7}(\theta)\, a_{01}a_{20} + s_{2,8}(\theta)\, a_{01}c_{11} + s_{2,9}(\theta)\, a_{11}c_{01} + s_{2,10}(\theta)\, c_{00}c_{21} + s_{2,11}(\theta)\, c_{01}c_{20} + s_{2,12}(\theta)\, c_{10}c_{11}$$
$$+s_{2,13}(\theta)\, a_{00}a_{03} + s_{2,14}(\theta)\, a_{01}a_{02} + s_{2,15}(\theta)\, c_{00}c_{03} + s_{2,16}(\theta)\, c_{01}c_{02} + s_{2,17}(\theta)\, a_{00}a_{12} + s_{2,18}(\theta)\, a_{03}c_{00}$$
$$+s_{2,19}(\theta)\, a_{11}c_{10} + s_{2,20}(\theta)\, a_{01}c_{02} + s_{2,21}(\theta)\, a_{01}c_{20} + s_{2,22}(\theta)\, a_{01}a_{11} + s_{2,23}(\theta)\, a_{00}c_{03}$$
$$+s_{2,24}(\theta)\, a_{00}c_{21} + s_{2,25}(\theta)\, a_{02}c_{01} + s_{2,26}(\theta)\, a_{20}c_{01} + s_{2,27}(\theta)\, a_{21}c_{00}$$
$$+s_{2,28}(\theta)\, c_{00}c_{12} + s_{2,29}(\theta)\, c_{00}c_{30} + s_{2,30}(\theta)\, c_{01}c_{11} + s_{2,31}(\theta)\, c_{02}c_{10} + s_{2,32}(\theta)\, c_{10}c_{20},$$

and $s_{2,i}(\theta)$ for $i = 1, \cdots, 32$ satisfying the following expressions:

$$s_{2,1}(\theta) = \frac{1}{6}\Bigg(12\cos^2\theta\sin\theta e^{-3\sin^2\theta}\int_0^\theta e^{3\sin^2 w}\cos^3 w dw - 12\cos^4\theta\sin\theta\left(\int_0^\theta e^{3\sin^2 w}\cos w dw\right)e^{-3\sin^2\theta}$$
$$+2\cos^6\theta - 14\cos^2\theta - 6\cos^2\theta\sin\theta\left(\int_0^\theta e^{3\sin^2 w}\cos w dw\right)e^{-3\sin^2\theta}$$
$$+24\cos^4\theta\sin\theta e^{-3\sin^2\theta}\int_0^\theta e^{3\sin^2 w}\cos^3 w dw - 3\cos^4\theta + 7 + 8\cos^8\theta\Bigg)e^{3\sin^2\theta}\cos\theta,$$

$$s_{2,2}(\theta) = \Bigg(-2\int_0^\theta e^{\sin^2 w}\cos^5 w dw + \int_0^\theta e^{\sin^2 w}\cos^3 w dw$$
$$+2\left(\int_0^\theta e^{\sin^2 w}\cos^3 w dw\right)\cos^2\theta - 4\left(\int_0^\theta e^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta\Bigg)e^{2\sin^2\theta}\sin\theta\cos\theta,$$

$$s_{2,3}(\theta) = -\frac{1}{6}\Bigg(-12\sin\theta e^{-3\sin^2\theta}\int_0^\theta e^{3\sin^2 w}\cos^3 w dw - 9\cos^4\theta$$
$$-12\cos^4\theta\sin\theta\left(\int_0^\theta e^{3\sin^2 w}\cos w dw\right)e^{-3\sin^2\theta} + 6\cos^2\theta\sin\theta\left(\int_0^\theta e^{3\sin^2 w}\cos w dw\right)e^{-3\sin^2\theta}$$
$$-10\cos^6\theta - 12\cos^2\theta\sin\theta e^{-3\sin^2\theta}\int_0^\theta e^{3\sin^2 w}\cos^3 w dw + 6\sin\theta\left(\int_0^\theta e^{3\sin^2 w}\cos w dw\right)e^{-3\sin^2\theta}$$
$$+8\cos^8\theta + 16\cos^2\theta + 24\cos^4\theta\sin\theta e^{-3\sin^2\theta}\int_0^\theta e^{3\sin^2 w}\cos^3 w dw - 5\Bigg)e^{3\sin^2\theta}\cos\theta,$$

$$s_{2,4}(\theta) = -\Bigg(-4\left(\int_0^\theta e^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta + 3\int_0^\theta e^{\sin^2 w}\cos^3 w dw + 6\left(\int_0^\theta e^{\sin^2 w}\cos^3 w dw\right)\cos^2\theta$$
$$-\int_0^\theta e^{\sin^2 w}\cos w dw - 2\left(\int_0^\theta e^{\sin^2 w}\cos w dw\right)\cos^2\theta - 2\int_0^\theta e^{\sin^2 w}\cos^5 w dw\Bigg)e^{2\sin^2\theta}\sin\theta\cos\theta,$$

$$s_{2,5}(\theta) = -\frac{1}{3}\Bigg(6\cos^4\theta\int_0^\theta e^{3\sin^2 w}\sin w dw + 4e^{3\sin^2\theta}\cos^7\theta - 18\left(\int_0^\theta e^{3\sin^2 w}\sin w\cos^2 w dw\right)\cos^2\theta$$
$$-6e^{3\sin^2\theta}\cos^5\theta - 9\left(\int_0^\theta e^{3\sin^2 w}\sin w dw\right)\cos^2\theta + 2e^{3\sin^2\theta}\cos\theta + 3\int_0^\theta e^{3\sin^2 w}\sin w dw$$
$$+12\cos^4\theta\int_0^\theta e^{3\sin^2 w}\sin w\cos^2 w dw + 6\int_0^\theta e^{3\sin^2 w}\sin w\cos^2 w dw\Bigg)\cos^2\theta,$$

$$
\begin{aligned}
s_{2,6}(\theta) = \quad & \frac{1}{6}\Bigg(-12\,\cos^2\theta\sin\theta \mathrm{e}^{-3\,\sin^2\theta}\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos^3 w dw + 1 - 2\,\cos^2\theta \\
& -10\,\cos^6\theta - 12\,\cos^4\theta\sin\theta\left(\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos w dw\right)\mathrm{e}^{-3\,\sin^2\theta} + 3\,\cos^4\theta \\
& +24\,\cos^4\theta\sin\theta \mathrm{e}^{-3\,\sin^2\theta}\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos^3 w dw \\
& +6\,\cos^2\theta\sin\theta\left(\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos w dw\right)\mathrm{e}^{-3\,\sin^2\theta} + 8\,\cos^8\theta\Bigg)\mathrm{e}^{3\,\sin^2\theta}\cos\theta, \\
s_{2,7}(\theta) = \quad & \Bigg(2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw - 4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta \\
& -\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw + 2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\right)\cos^2\theta\Bigg)\mathrm{e}^{2\,\sin^2\theta}\sin\theta\cos\theta, \\
s_{2,8}(\theta) = \quad & -\Bigg(2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\right)\cos^2\theta + 2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw\right)\cos^2\theta - \int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw \\
& -4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta - \int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw + 2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\Bigg)\mathrm{e}^{2\,\sin^2\theta}\sin\theta\cos\theta, \\
s_{2,9}(\theta) = \quad & \mathrm{e}^{2\,\sin^2\theta}\Bigg(-I_2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw + 2\,I_1\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw \\
& -4\,I_1\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw - 2\,I_3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw \\
& +I_3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw - 2\,I_3\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw \\
& +4\,I_3\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw + 2\,I_2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\Bigg)\frac{1}{2\,I_1 - I_2}, \\
s_{2,10}(\theta) = \quad & -\frac{1}{3}\,\cos^2\theta\Bigg(-6\left(\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw\right)\cos^2\theta - 2\,\mathrm{e}^{3\,\sin^2\theta}\cos\theta - 3\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw \\
& -6\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw - 12\left(\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw\right)\cos^2\theta - 6\,\mathrm{e}^{3\,\sin^2\theta}\cos^3\theta \\
& +8\,\mathrm{e}^{3\,\sin^2\theta}\cos^7\theta + 12\,\cos^4\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw + 24\,\cos^4\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw\Bigg), \\
s_{2,11}(\theta) = \quad & -\mathrm{e}^{2\,\sin^2\theta}\Bigg(2\,I_1\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw + 2\,I_3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw \\
& -I_2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw + I_3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw \\
& -2\,I_3\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw - 4\,I_3\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw \\
& -2\,I_2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw + 4\,I_1\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\Bigg)\frac{1}{2\,I_1 - I_2},
\end{aligned}
$$

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\begin{aligned}
s_{2,12}(\theta) = \quad & -\Bigg(\int_0^\theta e^{\sin^2 w}\cos w dw - 2\int_0^\theta e^{\sin^2 w}\cos^5 w dw + \int_0^\theta e^{\sin^2 w}\cos^3 w dw \\
& +2\left(\int_0^\theta e^{\sin^2 w}\cos w dw\right)\cos^2\theta + 2\left(\int_0^\theta e^{\sin^2 w}\cos^3 w dw\right)\cos^2\theta \\
& -4\left(\int_0^\theta e^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta\Bigg) e^{2\sin^2\theta}\cos\theta\sin\theta, \\
s_{2,13}(\theta) = \quad & -\frac{1}{6}\Bigg(-12\cos^4\theta\sin\theta\int_0^\theta e^{3\sin^2 w}\cos w dw - 6\sin\theta\int_0^\theta e^{3\sin^2 w}\cos w dw \\
& +24\cos^4\theta\sin\theta\int_0^\theta e^{3\sin^2 w}\cos^3 w dw + 12\sin\theta\int_0^\theta e^{3\sin^2 w}\cos^3 w dw \\
& -36\cos^2\theta\sin\theta\int_0^\theta e^{3\sin^2 w}\cos^3 w dw - 22 e^{3\sin^2\theta}\cos^6\theta + e^{3\sin^2\theta} \\
& -8 e^{3\sin^2\theta}\cos^2\theta + 8 e^{3\sin^2\theta}\cos^8\theta + 18\sin\theta\cos^2\theta\int_0^\theta e^{3\sin^2 w}\cos w dw \\
& +21 e^{3\sin^2\theta}\cos^4\theta\Bigg)\cos\theta, \\
s_{2,14}(\theta) = \quad & -\Bigg(-3\int_0^\theta e^{\sin^2 w}\cos^3 w dw - 2\left(\int_0^\theta e^{\sin^2 w}\cos w dw\right)\cos^2\theta + 6\left(\int_0^\theta e^{\sin^2 w}\cos^3 w dw\right)\cos^2\theta \\
& +2\int_0^\theta e^{\sin^2 w}\cos^5 w dw + \int_0^\theta e^{\sin^2 w}\cos w dw - 4\left(\int_0^\theta e^{\sin^2 w}\cos^5 w dw\right)\cos^2\theta\Bigg) e^{2\sin^2\theta}\cos\theta\sin\theta, \\
s_{2,15}(\theta) = \quad & -2 e^{3\sin^2\theta}\cos^7\theta - \cos^4\theta\int_0^\theta e^{3\sin^2 w}\sin w dw - 2\cos^6\theta\int_0^\theta e^{3\sin^2 w}\sin w dw \\
& -2 e^{3\sin^2\theta}\cos^5\theta - 2\cos^4\theta\int_0^\theta e^{3\sin^2 w}\sin w\cos^2 w dw \\
& +\frac{10}{3} e^{3\sin^2\theta}\cos^3\theta - 4\cos^6\theta\int_0^\theta e^{3\sin^2 w}\sin w\cos^2 w dw - \frac{4}{3}\cos^9\theta e^{3\sin^2\theta} \\
& +2 e^{3\sin^2\theta}\cos\theta + \int_0^\theta e^{3\sin^2 w}\sin w dw + 2\int_0^\theta e^{3\sin^2 w}\sin w\cos^2 w dw, \\
s_{2,16}(\theta) = \quad & -e^{2\sin^2\theta}\Bigg(-2 I_2\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw + 2 I_3\cos^2\theta\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw \\
& +2 I_1\int_0^\theta e^{\sin^2 w}\sin^3 w dw + 4 I_1\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw - 4 I_3\cos^4\theta\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw \\
& +I_3\cos^2\theta\int_0^\theta e^{\sin^2 w}\sin^3 w dw - I_2\int_0^\theta e^{\sin^2 w}\sin^3 w dw - 2 I_3\cos^4\theta\int_0^\theta e^{\sin^2 w}\sin^3 w dw\Bigg)\frac{1}{2 I_1 - I_2}, \\
s_{2,17}(\theta) = \quad & \frac{1}{3}\Bigg(-6\int_0^\theta e^{3\sin^2 w}\cos^3 w dw + 3\int_0^\theta e^{3\sin^2 w}\cos w dw + 2 e^{3\sin^2\theta}\cos^2\theta\sin\theta \\
& +6\cos^4\theta\int_0^\theta e^{3\sin^2 w}\cos w dw + 4 e^{3\sin^2\theta}\cos^6\theta\sin\theta - 6 e^{3\sin^2\theta}\cos^4\theta\sin\theta \\
& -9\cos^2\theta\int_0^\theta e^{3\sin^2 w}\cos w dw + (18\cos^2\theta - 12\cos^4\theta)\int_0^\theta e^{3\sin^2 w}\cos^3 w dw\Bigg)\cos^2\theta,
\end{aligned}
$$

$$
\begin{aligned}
s_{2,18}(\theta) = & -\frac{1}{6}\Big(8\,\mathrm{e}^{3\,\sin^2\theta}\cos^8\theta - 14\,\mathrm{e}^{3\,\sin^2\theta}\cos^6\theta - \mathrm{e}^{3\,\sin^2\theta} + 12\,\cos^5\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw \\
& +24\,\cos^5\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw + 4\,\mathrm{e}^{3\,\sin^2\theta}\cos^2\theta - 36\,\cos^3\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw \\
& -18\,\cos^3\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw + 6\,\cos\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw \\
& + 12\,\cos\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw + 3\,\mathrm{e}^{3\,\sin^2\theta}\cos^4\theta\Big)\sin\theta, \\
s_{2,19}(\theta) = & \left(2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw\right)\cos^2\theta + \int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw\right. \\
& \left. - 2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw - 4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\right)\cos^2\theta\right)\mathrm{e}^{2\,\sin^2\theta}\cos\theta\sin\theta, \\
s_{2,20}(\theta) = & -\left(4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w\cos^2 w dw\right)\cos^2\theta - 2\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w\cos^2 w dw\right. \\
& \left. -\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw + 2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw\right)\cos^2\theta\right)\mathrm{e}^{2\,\sin^2\theta}\cos\theta\sin\theta, \\
s_{2,21}(\theta) = & -\left(2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw\right)\cos^2\theta + 4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\right)\cos^2\theta\right. \\
& \left. - \int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw - 2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\right)\mathrm{e}^{2\,\sin^2\theta}\cos\theta\sin\theta \\
s_{2,22}(\theta) = & \left(2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw\right)\cos^2\theta - \int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw\right. \\
& \left. +2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw - 4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\right)\cos^2\theta\right)\mathrm{e}^{2\,\sin^2\theta}\cos\theta\sin\theta, \\
s_{2,23}(\theta) = & 2\,\mathrm{e}^{3\,\sin^2\theta}\cos^6\theta\sin\theta + \frac{4}{3}\,\mathrm{e}^{3\,\sin^2\theta}\cos^8\theta\sin\theta + 2\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos^3 w dw + \frac{8}{3}\,\mathrm{e}^{3\,\sin^2\theta}\cos^4\theta\sin\theta \\
& -2\,\cos^4\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos^3 w dw - 2\,\mathrm{e}^{3\,\sin^2\theta}\cos^2\theta\sin\theta + 2\,\cos^6\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos w dw \\
& -4\,\cos^6\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos^3 w dw - \int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos w dw + \cos^4\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos w dw, \\
s_{2,24}(\theta) = & \frac{1}{3}\left(6\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos^3 w dw + 12\,\cos^4\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos w dw + 12\,\cos^2\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos^3 w dw\right. \\
& -3\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos w dw + 8\,\mathrm{e}^{3\,\sin^2\theta}\cos^6\theta\sin\theta - 24\,\cos^4\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos^3 w dw \\
& \left. -6\,\cos^2\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\cos w dw - 2\,\mathrm{e}^{3\,\sin^2\theta}\cos^2\theta\sin\theta\right)\cos^2\theta,
\end{aligned}
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\begin{aligned}
s_{2,25}(\theta) = \;& -\mathrm{e}^{2\,\sin^2\theta}\Bigg(3\,I_3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw + 2\,I_3\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw\\
& +6\,I_1\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw - 4\,I_1\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw + 4\,I_3\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\\
& +2\,I_2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw + I_2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw - 6\,I_3\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\\
& -2\,I_3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw - 2\,I_1\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw\\
& -3\,I_2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw - I_3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw\Bigg)\frac{1}{2\,I_1 - I_2},\\
s_{2,26}(\theta) = \;& \mathrm{e}^{2\,\sin^2\theta}\Bigg(-2\,I_3\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw + 4\,I_3\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\\
& -I_2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw - 2\,I_3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw + 2\,I_2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\\
& +2\,I_1\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw - 4\,I_1\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw + I_3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\Bigg)\frac{1}{2\,I_1 - I_2},\\
s_{2,27}(\theta) = \;& \frac{1}{6}\Bigg(-6\,\cos^3\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw + 12\,\cos^5\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw - 3\,\mathrm{e}^{3\,\sin^2\theta}\cos^4\theta\\
& -2\,\mathrm{e}^{3\,\sin^2\theta}\cos^2\theta - 2\,\mathrm{e}^{3\,\sin^2\theta}\cos^6\theta - \mathrm{e}^{3\,\sin^2\theta} + 24\,\cos^5\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw\\
& -12\,\cos^3\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw + 8\,\mathrm{e}^{3\,\sin^2\theta}\cos^8\theta\Bigg)\sin\theta,\\
s_{2,28}(\theta) = \;& -\frac{1}{6}\Bigg(5\,\mathrm{e}^{3\,\sin^2\theta} + 24\,\cos^5\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw\\
& -12\,\cos^3\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw - 12\,\cos\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw\\
& -6\,\cos^3\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw - 15\,\mathrm{e}^{3\,\sin^2\theta}\cos^4\theta - 6\,\cos\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw\\
& +4\,\mathrm{e}^{3\,\sin^2\theta}\cos^2\theta + 8\,\mathrm{e}^{3\,\sin^2\theta}\cos^8\theta + 12\,\cos^5\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw\\
& -2\,\mathrm{e}^{3\,\sin^2\theta}\cos^6\theta\Bigg)\sin\theta,\\
s_{2,29}(\theta) = \;& \frac{1}{6}\Bigg(-14\,\mathrm{e}^{3\,\sin^2\theta}\cos^2\theta + 10\,\mathrm{e}^{3\,\sin^2\theta}\cos^6\theta + 12\,\cos^5\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw\\
& +6\,\cos^3\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w dw + 12\,\cos^3\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw + 8\,\mathrm{e}^{3\,\sin^2\theta}\cos^8\theta\\
& +3\,\mathrm{e}^{3\,\sin^2\theta}\cos^4\theta - 7\,\mathrm{e}^{3\,\sin^2\theta} + 24\,\cos^5\theta\int_0^\theta \mathrm{e}^{3\,\sin^2 w}\sin w\cos^2 w dw\Bigg)\sin\theta,
\end{aligned}
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\begin{aligned}
s_{2,30}(\theta) = \ & -\mathrm{e}^{2\sin^2\theta}\Bigg(-4\,I_1\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw - 2\,I_3\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw \\
& -2\,I_3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw + 2\,I_2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw - 2\,I_3\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw \\
& +2\,I_1\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw - I_2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw + I_3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw \\
& +2\,I_1\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw + I_3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw \\
& - I_2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw + 4\,I_3\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\Bigg)\frac{1}{2\,I_1 - I_2}, \\
s_{2,31}(\theta) = \ & -\Bigg(2\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w\cos^2 w dw + \int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw + 4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w\cos^2 w dw\right)\cos^2\theta \\
& +2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw\right)\cos^2\theta\Bigg)\mathrm{e}^{2\sin^2\theta}\cos\theta\sin\theta, \\
s_{2,32}(\theta) = \ & -\Bigg(2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw + 4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\right)\cos^2\theta \\
& + \int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw + 2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw\right)\cos^2\theta\Bigg)\mathrm{e}^{2\sin^2\theta}\cos\theta\sin\theta.
\end{aligned}
$$

We also have

$$
\begin{aligned}
s_1(\theta) = \ & s_{1,1}(\theta)a_{03}c_{10} + s_{1,2}(\theta)a_{01}c_{03} + s_{1,3}(\theta)a_{03}c_{01} + s_{1,4}(\theta)a_{01}c_{30} + s_{1,5}(\theta)c_{10}c_{21} + s_{1,6}(\theta)a_{01}a_{12} \\
& + s_{1,7}(\theta)c_{01}c_{12} + s_{1,8}(\theta)c_{01}a_{11} + s_{1,9}(\theta)c_{03}c_{10} + s_{1,10}(\theta)a_{12}c_{10} + s_{1,11}(\theta)c_{01}c_{21} \\
& + s_{1,12}(\theta)c_{10}c_{12} + s_{1,13}(\theta)a_{21}c_{10} + s_{1,14}(\theta)a_{01}a_{21} + s_{1,15}(\theta)c_{10}c_{30} + s_{1,16}(\theta)a_{01}c_{21} \\
& + s_{1,17}(\theta)a_{01}a_{03} + s_{1,18}(\theta)a_{01}c_{12} + s_{1,19}(\theta)a_{12}c_{01} + s_{1,20}(\theta)c_{01}c_{03} + s_{1,21}(\theta)a_{21}c_{01},
\end{aligned}
$$

and $s_{1,i}(\theta)$ for $i = 1\cdots, 21$ are given in the following expressions:

$$
\begin{aligned}
s_{1,1}(\theta) = \ & -\frac{1}{12}\Bigg(24\left(\int_0^\theta \mathrm{e}^{2\sin^2 w}\sin w\cos w dw\right)\cos^4\theta - \mathrm{e}^{2\sin^2\theta} + 12\int_0^\theta \mathrm{e}^{2\sin^2 w}\sin w\cos w dw \\
& +4\,\mathrm{e}^{2\sin^2\theta}\cos^2\theta - 36\left(\int_0^\theta \mathrm{e}^{2\sin^2 w}\sin w\cos w dw\right)\cos^2\theta - 72\left(\textstyle\int_0^\theta \mathrm{e}^{2\sin^2 w}\cos^3 w\sin w dw\right)\cos^2\theta \\
& +24\int_0^\theta \mathrm{e}^{2\sin^2 w}\cos^3 w\sin w dw + 8\,\mathrm{e}^{2\sin^2\theta}\cos^8\theta + 48\left(\int_0^\theta \mathrm{e}^{2\sin^2 w}\cos^3 w\sin w dw\right)\cos^4\theta \\
& +3\,\mathrm{e}^{2\sin^2\theta}\cos^4\theta - 14\,\mathrm{e}^{2\sin^2\theta}\cos^6\theta\Bigg)\cos\theta\sin\theta,
\end{aligned}
$$

$$
\begin{aligned}
s_{1,2}(\theta) = \quad & -\frac{1}{3}\,\mathrm{e}^{2\,\sin^2\theta}\cos^6\theta + 2\,\cos^6\theta\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw - \frac{2}{3}\,\mathrm{e}^{2\,\sin^2\theta}\cos^{10}\theta \\
& -2\,\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^4\theta - \int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw \\
& + \left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^4\theta + \frac{7}{3}\,\mathrm{e}^{2\,\sin^2\theta}\cos^4\theta - 4\,\cos^6\theta\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw \\
& -\mathrm{e}^{2\,\sin^2\theta}\cos^2\theta - \frac{1}{3}\,\mathrm{e}^{2\,\sin^2\theta}\cos^8\theta + 2\,\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw, \\
s_{1,3}(\theta) = \quad & \mathrm{e}^{2\,\sin^2\theta}\,(\cos\theta - 1)\,(\cos\theta + 1)\,\big(-8\,\cos^8\theta I_3 + 14\,\cos^6\theta I_3 - 7\,\cos^4\theta I_3 + 8\,\cos^4\theta I_1 \\
& -4\,\cos^4\theta I_2 + 24\,I_3\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw \\
& +48\,I_1\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w} dw - 48\,I_3\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw \\
& -24\,I_2\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w} dw + \cos^2\theta I_3 - 10\,\cos^2\theta I_1 + 5\,\cos^2\theta I_2 \\
& +12\,I_2\,\cos\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w} dw - 12\,I_3\,\cos\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw \\
& -24\,I_1\,\cos\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w} dw \\
& \left. + \,24\,I_3\,\cos\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw - I_2 + 2\,I_1\right)\,\tfrac{1}{12(I_2 - 2\,I_1)}, \\
s_{1,4}(\theta) = \quad & -\frac{1}{12}\,\left(24\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw - 7 + 3\,\cos^4\theta\right. \\
& -8\,\cos^8\theta + 14\,\cos^2\theta - 24\,\cos^2\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw \\
& -2\,\cos^6\theta + 12\,\cos^2\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw \\
& \left. -48\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\,\mathrm{e}^{2\,\sin^2\theta}\sin\theta\cos\theta, \\
s_{1,5}(\theta) = \quad & -\frac{1}{3}\,\left[-\mathrm{e}^{2\,\sin^2\theta}\cos^2\theta + 12\,\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^4\theta\right. \\
& +24\,\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^4\theta - 3\,\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw \\
& -6\,\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^2\theta - 6\,\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw \\
& -12\,\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^2\theta + 4\,\mathrm{e}^{2\,\sin^2\theta}\cos^8\theta \\
& \left. -\,3\,\mathrm{e}^{2\,\sin^2\theta}\cos^4\theta\right]\cos^2\theta,
\end{aligned}
$$

$$
\begin{aligned}
s_{1,6}(\theta) = \ & -\frac{1}{3}\left(12\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\cos^4\theta + 9\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^2\theta\right. \\
& +6\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw - 3\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw + 4\,\mathrm{e}^{2\,\sin^2\theta}\cos^4\theta \\
& -6\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^4\theta - 5\,\mathrm{e}^{2\,\sin^2\theta}\cos^6\theta + 2\,\mathrm{e}^{2\,\sin^2\theta}\cos^8\theta \\
& \left.-18\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^2\theta - \mathrm{e}^{2\,\sin^2\theta}\cos^2\theta\right)\cos^2\theta, \\
s_{1,7}(\theta) = \ & -\frac{1}{12}\,\mathrm{e}^{2\,\sin^2\theta}\left(12\,I_2\,\cos\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\right. \\
& -12\,I_3\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw - 24\,I_1\,\cos\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw \\
& +12\,I_2\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw + 24\,I_3\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw \\
& +24\,I_3\,\cos\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw - 12\,I_3\,\cos\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw \\
& -24\,I_1\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw + 24\,I_3\,\cos^5\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\left(\cos^4 w\right)^2 dw \\
& +48\,I_1\,\cos^5\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw - 24\,I_2\,\cos^5\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw \\
& -48\,I_3\,\cos^5\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw - 16\,\cos^4\theta I_3 + 9\,\cos^6\theta I_3 + 5\,\cos^2\theta I_3 \\
& -12\,\cos^2\theta I_1 - 8\,I_3\,\cos^{10}\theta + 8\,\cos^6\theta I_1 - 4\,\cos\theta^6 I_2 - 5\,I_2 + 10\,I_1 + 6\,\cos^2\theta I_2 \\
& \left.-6\,\cos^4\theta I_1 + 3\,\cos^4\theta I_2 + 10\,\cos^8\theta I_3\right)\frac{1}{2\,I_1 - I_2}, \\
s_{1,8}(\theta) = \ & \frac{1}{12}\,\mathrm{e}^{2\,\sin\theta^2}\left(-4\,\cos^6\theta I_2 - 48\,I_3\,\cos^5\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw\right. \\
& -3\,\cos^4\theta I_2 + 14\,\cos^4\theta I_3 - 24\,I_3\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw \\
& +3\,\cos^6\theta I_3 - 2\,\cos^8\theta I_3 + 24\,I_3\,\cos^5\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw \\
& +6\,\cos^4\theta I_1 + 8\,\cos^6\theta I_1 + 7\,I_2 - 24\,I_2\,\cos^5\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw \\
& -12\,I_2\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw + 48\,I_1\,\cos^5\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw - 8\,I_3\,\cos^{10}\theta \\
& -7\,\cos^2\theta I_3 + 12\,I_3\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw \\
& \left.+\,24\,I_1\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw - 14\,I_1\right)\frac{1}{-I_2 + 2\,I_1},
\end{aligned}
$$

$$
\begin{aligned}
s_{1,9}(\theta) = \quad & -\left(\int_0^\theta e^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^4\theta + \int_0^\theta e^{2\,\sin^2 w}\sin w\cos w dw \\
& -e^{2\,\sin^2\theta}\cos^8\theta - 2\,\cos^6\theta\int_0^\theta e^{2\,\sin^2 w}\sin w\cos w dw - e^{2\,\sin^2\theta}\cos^6\theta \\
& +5/3\,e^{2\,\sin^2\theta}\cos^4\theta + 2\int_0^\theta e^{2\,\sin^2 w}\cos^3 w\sin w dw \\
& -4\,\cos^6\theta\int_0^\theta e^{2\,\sin^2 w}\cos^3 w\sin w dw - \frac{2}{3}\,e^{2\,\sin^2\theta}\cos^{10}\theta \\
& +e^{2\,\sin^2\theta}\cos^2\theta - 2\left(\int_0^\theta e^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^4\theta, \\
s_{1,10}(\theta) = \quad & -\frac{1}{3}\left(e^{2\,\sin\theta^2}\cos^2\theta + 12\left(\int_0^\theta e^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^4\theta - 3\,e^{2\,\sin^2\theta}\cos^6\theta\right. \\
& -9\left(\int_0^\theta e^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^2\theta + 6\left(\int_0^\theta e^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^4\theta \\
& +6\int_0^\theta e^{2\,\sin^2 w}\cos^3 w\sin w dw - 18\left(\int_0^\theta e^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^2\theta \\
& \left.+3\int_0^\theta e^{2\,\sin^2 w}\sin w\cos w dw + 2\,e^{2\,\sin^2\theta}\cos^8\theta\right)\cos^2\theta, \\
s_{1,11}(\theta) = \quad & -\frac{1}{3}\,e^{2\,\sin\theta^2}\cos^2\theta\left(-6\,I_1\,e^{-2\,\sin^2\theta}\int_0^\theta e^{2\,\sin^2 w}dw + 4\,\cos^7\theta\sin\theta I_3\right. \\
& -4\,\cos^3\theta\sin\theta I_1 + 6\,I_2\,\cos^2\theta e^{-2\,\sin^2\theta}\int_0^\theta e^{2\,\sin^2 w}dw \\
& -12\,I_1\,\cos^2\theta e^{-2\,\sin^2\theta}\int_0^\theta e^{2\,\sin^2 w}dw - 3\,I_3\,e^{-2\,\sin^2\theta}\int_0^\theta e^{2\,\sin^2 w}\cos^2 w dw \\
& +2\,\cos^3\theta\sin\theta I_2 + 24\,I_1\,\cos^4\theta e^{-2\,\sin^2\theta}\int_0^\theta e^{2\,\sin^2 w}dw + 12\,I_3\,\cos^2\theta e^{-2\,\sin^2\theta}\int_0^\theta e^{2\,\sin^2 w}\cos^4 w dw \\
& +12\,I_3\,\cos^4\theta e^{-2\,\sin^2\theta}\int_0^\theta e^{2\,\sin^2 w}\cos^2 w dw - 2\,\cos\theta\sin\theta I_1 \\
& -12\,I_2\,\cos^4\theta e^{-2\,\sin\theta^2}\int_0^\theta e^{2\,\sin^2 w}dw + \cos\theta\sin\theta I_2 \\
& +6\,I_3\,e^{-2\,\sin^2\theta}\int_0^\theta e^{2\,\sin^2 w}\cos^4 w dw \\
& -24\,I_3\,\cos^4\theta e^{-2\,\sin\theta^2}\int_0^\theta e^{2\,\sin^2 w}\cos^4 w dw + 3\,I_2\,e^{-2\,\sin^2\theta}\int_0^\theta e^{2\,\sin^2 w}dw \\
& \left.-\cos^3\theta\sin\theta I_3 - 6\,I_3\,\cos^2\theta e^{-2\,\sin^2\theta}\int_0^\theta e^{2\,\sin^2 w}\cos^2 w dw\right)\frac{1}{2\,I_1 - I_2},
\end{aligned}
$$

$$
\begin{aligned}
s_{1,12}(\theta) = \quad & -\frac{1}{12}\Bigg(-12\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw - 15\,\cos^4\theta - 2\,\cos^6\theta \\
& +24\,\cos^4\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw - 24\,\cos^2\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw \\
& -24\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw + 8\,\cos^8\theta + 4\,\cos^2\theta \\
& +48\,\cos^4\theta \mathrm{e}^{-2\,\sin\theta^2}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw \\
& -12\,\cos^2\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw + 5\Bigg)\,\mathrm{e}^{2\,\sin^2\theta}\cos\theta\sin\theta, \\
s_{1,13}(\theta) = \quad & \frac{1}{12}\Bigg(-12\,\cos^2\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw \\
& +24\,\cos^4\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw + 8\,\cos^8\theta - 3\,\cos^4\theta \\
& -24\,\cos^2\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw + 48\,\cos^4\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw \\
& -1 - 2\,\cos^2\theta\ -2\,\cos^6\theta\big)\,\mathrm{e}^{2\,\sin^2\theta}\cos\theta\sin\theta, \\
s_{1,14}(\theta) = \quad & \frac{1}{12}\Bigg(1 + 48\,\cos^4\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw + 8\,\cos^8\theta \\
& -24\,\cos^2\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw - 10\,\cos^6\theta \\
& -24\,\cos\theta^4 \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw - 2\,\cos^2\theta \\
& +12\,\cos^2\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw + 3\,\cos^4\theta\Bigg)\,\mathrm{e}^{2\,\sin^2\theta}\cos\theta\sin\theta, \\
s_{1,15}(\theta) = \quad & \frac{1}{12}\Bigg(24\,\cos^2\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw + 10\,\cos^6\theta \\
& +24\,\cos^4\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw \\
& +48\,\cos^4\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw \\
& +3\,\cos^4\theta + 12\,\cos^2\theta \mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw + 8\,\cos^8\theta \\
& -14\,\cos^2\theta - 7\big)\,\mathrm{e}^{2\,\sin^2\theta}\cos\theta\sin\theta,
\end{aligned}
$$

$$
\begin{aligned}
s_{1,16}(\theta) = \; & -\frac{1}{3}\biggl(-6\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw - 12\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^4\theta \\
& + \mathrm{e}^{2\,\sin^2\theta}\cos^2\theta + 4\,\mathrm{e}^{2\,\sin^2\theta}\cos^8\theta - 12\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^2\theta \\
& + 24\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\right)\cos^4\theta - 4\,\mathrm{e}^{2\,\sin^2\theta}\cos^6\theta \\
& + 6\left(\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\right)\cos^2\theta - \mathrm{e}^{2\,\sin^2\theta}\cos^4\theta \\
& + 3\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw\biggr)\cos^2\theta,
\end{aligned}
$$

$$
\begin{aligned}
s_{1,17}(\theta) = \; & -\frac{1}{12}\biggl(48\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw \\
& - 12\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw - 24\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw \\
& + 36\,\cos^2\theta\mathrm{e}^{-2\,\sin\theta^2}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw + 8\,\cos^8\theta + 1 + 21\,\cos^4\theta \\
& + 24\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw - 72\,\cos^2\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw - 8\,\cos^2\theta \\
& - 22\,\cos^6\theta\biggr)\,\mathrm{e}^{2\,\sin\theta^2}\cos\theta\sin\theta,
\end{aligned}
$$

$$
\begin{aligned}
s_{1,18}(\theta) = \; & -\frac{1}{12}\biggl(-24\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw \\
& - 24\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw + 16\,\cos^2\theta - 5 \\
& + 12\,\cos^2\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw \\
& - 9\,\cos^4\theta - 24\,\cos^2\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw - 10\,\cos^6\theta \\
& + 12\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\sin w\cos w dw + 8\,\cos^8\theta \\
& + 48\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^3 w\sin w dw\biggr)\,\mathrm{e}^{2\,\sin^2\theta}\cos\theta\sin\theta,
\end{aligned}
$$

$$
\begin{aligned}
s_{1,19}(\theta) = \quad & -\frac{1}{3}\,\mathrm{e}^{2\,\sin\theta^2}\cos^2\theta\Bigg(2\,\cos\theta\sin\theta I_1 - 18\,I_1\,\cos^2\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw \\
& -\cos\theta\sin\theta I_2 - 3\,\cos^5\theta\sin\theta I_3 + \cos^3\theta\sin\theta I_2 \\
& +9\,I_2\,\cos^2\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw + 18\,I_3\,\cos^2\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw \\
& +\cos^3\theta\sin\theta I_3 - 6\,I_2\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw \\
& +6\,I_3\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw + 6\,I_1\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw \\
& +2\,\cos\theta^7\sin\theta I_3 - 3\,I_2\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw - 9\,I_3\,\cos^2\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw \\
& -12\,I_3\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw + 12\,I_1\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw \\
& -2\,\cos^3\theta\sin\theta I_1 - 6\,I_3\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw + 3\,I_3\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw\Bigg)\frac{1}{2\,I_1 - I_2},
\end{aligned}
$$

$$
\begin{aligned}
s_{1,20}(\theta) = \quad & -\frac{1}{3}\,\mathrm{e}^{2\,\sin^2\theta}\Bigg(-6\,I_3\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw \\
& +2\,\cos^3\theta\sin\theta I_2 + 12\,I_1\,\cos^6\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw \\
& -6\,I_2\,\cos^6\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw - 3\,I_2\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw \\
& -3\,\cos^3\theta\sin\theta I_3 - 12\,I_3\,\cos^6\theta\mathrm{e}^{-2\,\sin\theta^2}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw \\
& +6\,I_1\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw - 6\,I_1\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw + \cos^5\theta\sin\theta I_2 \\
& +4\,\cos^5\theta\sin\theta I_3 + 3\,\cos\theta\sin\theta I_2 + 3\,\cos^7\theta\sin\theta I_3 \\
& -3\,I_3\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw + 3\,I_3\,\cos^4\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw \\
& +3\,I_2\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw - 6\,\cos\theta\sin\theta I_1 - 2\,\cos^5\theta\sin\theta I_1 \\
& +6\,I_3\,\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw - 4\,\cos^3\theta\sin\theta I_1 + 2\,I_3\,\cos^9\theta\sin\theta \\
& +6\,I_3\,\cos^6\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw\Bigg)\frac{1}{2\,I_1 - I_2},
\end{aligned}
$$

$$
\begin{aligned}
s_{1,21}(\theta) = \quad & \frac{1}{12}\,\mathrm{e}^{2\,\sin\theta^2}\Bigg(2\,\cos^4\theta I_3 - 48\,I_3\,\cos^5\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw\\
& -3\,\cos^6\theta I_3 - 4\,\cos^6\theta I_2 + I_2 - 12\,I_3\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin\theta^2}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw\\
& -6\,\cos^4\theta I_1 - 8\,I_3\,\cos^{10}\theta - 2\,I_1 + 24\,I_3\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^4 w dw\\
& -24\,I_2\,\cos^5\theta\sin\theta\mathrm{e}^{-2\,\sin\theta^2}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw + 12\,I_2\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\\
& -24\,I_1\,\cos^3\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw + 10\,\cos^8\theta I_3\\
& +8\,\cos^6\theta I_1 - \cos^2\theta I_3 + 24\,I_3\,\cos^5\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}\cos^2 w dw\\
& +3\,\cos^4\theta I_2 + 48\,I_1\,\cos^5\theta\sin\theta\mathrm{e}^{-2\,\sin^2\theta}\int_0^\theta \mathrm{e}^{2\,\sin^2 w}dw\Bigg)\,\frac{1}{2\,I_1 - I_2}.
\end{aligned}
$$

Now we present the expression of $s_0(\theta)$.

$$
\begin{aligned}
s_0(\theta) = \quad & s_{0,1}(\theta)a_{12}c_{11} + s_{0,2}(\theta)a_{20}c_{03} + s_{0,3}(\theta)a_{20}c_{21} + s_{0,4}(\theta)a_{21}c_{11} + s_{0,5}(\theta)a_{21}c_{02} + s_{0,6}(\theta)a_{21}c_{20}\\
& +s_{0,7}(\theta)c_{02}c_{30} + s_{0,8}(\theta)a_{02}a_{12} + s_{0,9}(\theta)a_{02}c_{03} + s_{0,10}(\theta)a_{02}c_{21} + s_{0,11}(\theta)a_{03}a_{11} + s_{0,12}(\theta)a_{03}c_{02}\\
& +s_{0,13}(\theta)a_{03}c_{20} + s_{0,14}(\theta)a_{11}a_{21} + s_{0,15}(\theta)a_{11}c_{12} + s_{0,16}(\theta)a_{11}c_{30} + s_{0,17}(\theta)a_{12}a_{20}\\
& +s_{0,18}(\theta)a_{11}a_{12} + s_{0,19}(\theta)a_{03}a_{20} + s_{0,20}(\theta)a_{02}a_{21} + s_{0,21}(\theta)a_{11}c_{21} + s_{0,22}(\theta)c_{20}c_{30}\\
& +s_{0,23}(\theta)c_{11}c_{03} + s_{0,24}(\theta)c_{02}c_{03} + s_{0,25}(\theta)c_{11}c_{21} + s_{0,26}(\theta)c_{12}c_{20} + s_{0,27}(\theta)c_{02}c_{12} + s_{0,28}(\theta)a_{02}c_{12}\\
& +s_{0,29}a_{02}c_{30} + s_{0,30}(\theta)a_{02}a_{03} + s_{0,31}(\theta)a_{20}a_{21} + s_{0,32}(\theta)c_{11}c_{30} + s_{0,33}(\theta)c_{20}c_{21} + s_{0,34}(\theta)c_{02}c_{21}\\
& +s_{0,35}(\theta)c_{03}c_{20} + s_{0,36}(\theta)c_{11}c_{12} + s_{0,37}(\theta)a_{20}c_{30} + s_{0,38}(\theta)a_{12}c_{20} + s_{0,39}(\theta)a_{20}c_{12}\\
& +s_{0,40}(\theta)a_{11}c_{03} + s_{0,41}(\theta)a_{12}c_{02} + s_{0,42}(\theta)a_{03}c_{11}.
\end{aligned}
$$

and $s_{0,i}(\theta)$ for $i = 1\cdots, 42$ are the following:

$$
\begin{aligned}
s_{0,1}(\theta) = \quad & -\cos^2\theta\Bigg(2\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw + \int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw - 3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw\\
& -4\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw - 2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw + 6\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\\
& +2\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw + \int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw - 3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\Bigg),\\
s_{0,2}(\theta) = \quad & -4\,\cos^6\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw + 2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw - 2\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\\
& +2\,\cos^6\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw - \int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw + \cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw,\\
s_{0,3}(\theta) = \quad & \cos^2\theta\Bigg(2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw + 4\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw - 8\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\\
& -\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw - 2\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw + 4\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
s_{0,4}(\theta) = \quad & \cos^3\theta\sin\theta\left(2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos w dw - \int_0^\theta e^{\sin^2 w}\cos w dw\ - 4\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw\right. \\
& \left. +2\int_0^\theta e^{\sin^2 w}\cos^5 w dw\ +2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw - \int_0^\theta e^{\sin^2 w}\cos^3 w dw\right), \\
s_{0,5}(\theta) = \quad & \cos^3\theta\sin\theta\left(2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\sin^3 w dw - \int_0^\theta e^{\sin^2 w}\sin^3 w dw\right. \\
& \left. +4\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw - 2\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw\right), \\
s_{0,6}(\theta) = \quad & \cos^3\theta\sin\theta\left(2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw\ - \int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw\right. \\
& \left. +4\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\ -2\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\right), \\
s_{0,7}(\theta) = \quad & \cos^3\theta\sin\theta\left(\int_0^\theta e^{\sin^2 w}\sin^3 w dw + 2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\sin^3 w dw\right. \\
& \left. +2\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw + 4\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw\right), \\
s_{0,8}(\theta) = \quad & -\cos^2\theta\left(6\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw\ + 3\int_0^\theta e^{\sin^2 w}\cos^3 w dw - 9\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw\right. \\
& -4\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw - 2\int_0^\theta e^{\sin^2 w}\cos^5 w dw + 6\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw \\
& \left. -2\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos w dw\ - \int_0^\theta e^{\sin^2 w}\cos w dw + 3\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos w dw\right), \\
s_{0,9}(\theta) = \quad & -6\,\cos^6\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw + 3\int_0^\theta e^{\sin^2 w}\cos^3 w dw - 3\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw \\
& +4\,\cos^6\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw - 2\int_0^\theta e^{\sin^2 w}\cos^5 w dw + 2\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw \\
& +2\,\cos^6\theta\int_0^\theta e^{\sin^2 w}\cos w dw\ - \int_0^\theta e^{\sin^2 w}\cos w dw + \cos^4\theta\int_0^\theta e^{\sin^2 w}\cos w dw, \\
s_{0,10}(\theta) = \quad & -\cos^2\theta\left(-3\int_0^\theta e^{\sin^2 w}\cos^3 w dw - 6\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw\ + 12\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw\right. \\
& +2\int_0^\theta e^{\sin^2 w}\cos^5 w dw + 4\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw - 8\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw \\
& \left. +\int_0^\theta e^{\sin^2 w}\cos w dw + 2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos w dw - 4\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos w dw\right), \\
s_{0,11}(\theta) = \quad & \cos\theta\sin\theta\left(2\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw\ + \int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw\right. \\
& -3\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw - 4\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw \\
& \left. -2\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\ +6\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\right),
\end{aligned}
$$

$$
\begin{aligned}
s_{0,12}(\theta) = \quad & -\cos\theta\sin\theta\left(2\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\sin^3 w dw + \int_0^\theta e^{\sin^2 w}\sin^3 w dw\right.\\
& -3\,\cos^2\theta\int_0^\theta e^{\sin w^2}\sin^3 w dw + 4\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw\\
& \left. +2\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw - 6\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw\right),\\
s_{0,13}(\theta) = \quad & -\cos\theta\sin\theta\left(2\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw + \int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw\right.\\
& -3\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw + 4\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\\
& \left. +2\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw -6\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\right),\\
s_{0,14}(\theta) = \quad & -\cos^3\theta\sin\theta\left(2\,\cos^2\theta\int_0^\theta e^{\sin w^2}\cos^2 w\sin w dw - \int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw\right.\\
& \left. -4\,\cos^2\theta\textstyle\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw +2\displaystyle\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\right),\\
s_{0,15}(\theta) = \quad & \cos\theta\sin\theta\left(-\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw - \int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw\right.\\
& +2\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw + 2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\\
& \left. +2\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw -4\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\right),\\
s_{0,16}(\theta) = \quad & -\cos^3\theta\sin\theta\left(\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw + 2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw\right.\\
& \left. -2\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw -4\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\right),\\
s_{0,17}(\theta) = \quad & \cos^2\theta\left(-4\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw - 2\int_0^\theta e^{\sin^2 w}\cos^5 w dw\right.\\
& +6\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw + 2\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw + \int_0^\theta e^{\sin^2 w}\cos^3 w dw\\
& \left. -3\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw\right),\\
s_{0,18}(\theta) = \quad & \cos^2\theta\left(2\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw\,,\right.\\
& +\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw - 3\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw\\
& -4\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw - 2\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\\
& \left. +6\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\right),
\end{aligned}
$$

$$
\begin{aligned}
s_{0,19}(\theta) = \; & \cos\theta\sin\theta\Bigg(-4\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw\,-2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw \\
& +6\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin w^2}\cos^5 w dw+2\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw+\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw \\
& -3\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\Bigg), \\
s_{0,20}(\theta) = \; & \cos^3\theta\sin\theta\Bigg(6\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin w^2}\cos^3 w dw \\
& -3\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw-4\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw \\
& +2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw-2\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw+\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw\Bigg), \\
s_{0,21}(\theta) = \; & \cos^2\theta\Bigg(-\textstyle\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw-2\,\cos^2\theta\displaystyle\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw \\
& +4\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw+2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw \\
& +4\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw-8\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\Bigg), \\
s_{0,22}(\theta) = \; & \cos^3\theta\sin\theta\Bigg(\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw\,+2\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw \\
& +2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\,+4\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\Bigg), \\
s_{0,23}(\theta) = \; & -2\,\cos^6\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw+\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw-\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw \\
& +4\,\cos^6\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw-2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw+2\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw \\
& -2\,\cos^6\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw+\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw-\cos\theta^4\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw, \\
s_{0,24(\theta)} = \; & -2\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw\right)\cos^6\theta+\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw-\cos\theta^4\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw \\
& -4\left(\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w\cos^2 w dw\right)\cos^6\theta+2\int_0^\theta \mathrm{e}^{\sin w^2}\sin^3 w\cos^2 w dw \\
& -2\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w\cos^2 w dw, \\
s_{0,25}(\theta) = \; & -\cos^2\theta\Bigg(-\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw-2\,\cos\theta^2\int_0^\theta \mathrm{e}^{\sin w^2}\cos w dw\,+4\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw \\
& +2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw+4\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw-8\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw \\
& -\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw-2\,\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\,+4\,\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
s_{0,26}(\theta) = \quad & -\cos\theta\sin\theta\Bigg(-\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw - \int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw \\
& +2\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw - 2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw - 2\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw \\
& +4\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
s_{0,27}(\theta) = \quad & -\cos\theta\sin\theta\Bigg(-\cos^2\theta\int_0^\theta e^{\sin^2 w}\sin^3 w dw\Bigg) - \int_0^\theta e^{\sin^2 w}\sin^3 w dw \\
& +2\,\cos^4\theta\int_0^\theta e^{\sin w^2}\sin^3 w dw - 2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw - 2\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw \\
& +4\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
s_{0,28}(\theta) = \quad & -\cos\theta\sin\theta\Bigg(-3\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw - 3\int_0^\theta e^{\sin^2 w}\cos^3 w dw \\
& +6\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw + 2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw \\
& +2\int_0^\theta e^{\sin^2 w}\cos^5 w dw - 4\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw + \cos^2\theta\int_0^\theta e^{\sin^2 w}\cos w dw \\
& +\int_0^\theta e^{\sin^2 w}\cos w dw - 2\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos w dw\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
s_{0,29}(\theta) = \quad & \cos^3\theta\sin\theta\Bigg(3\int_0^\theta e^{\sin^2 w}\cos^3 w dw + 6\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw - 2\int_0^\theta e^{\sin^2 w}\cos^5 w dw \\
& -4\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw - \int_0^\theta e^{\sin w^2}\cos w dw - 2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos w dw\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
s_{0,30}(\theta) = \quad & -\cos\theta\sin\theta\Bigg(6\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw + 3\int_0^\theta e^{\sin^2 w}\cos^3 w dw \\
& -9\,\cos^2\theta\int_0^\theta e^{\sin w^2}\cos^3 w dw - 4\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw - 2\int_0^\theta e^{\sin^2 w}\cos^5 w dw \\
& +6\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw - 2\,\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos w dw - \int_0^\theta e^{\sin^2 w}\cos w dw \\
& +3\,\cos^2\theta\textstyle\int_0^\theta e^{\sin^2 w}\cos w dw\Big),
\end{aligned}
$$

$$
\begin{aligned}
s_{0,31}(\theta) = \quad & -\cos^3\theta\sin\theta\Bigg(-4\,\cos^2\theta\int_0^\theta e^{\sin w^2}\cos^5 w dw + 2\int_0^\theta e^{\sin^2 w}\cos^5 w dw \\
& +2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw - \int_0^\theta e^{\sin^2 w}\cos^3 w dw\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
s_{0,32}(\theta) = \quad & \cos^3\theta\sin\theta\Bigg(\int_0^\theta e^{\sin^2 w}\cos w dw + 2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos w dw - 2\int_0^\theta e^{\sin w^2}\cos^5 w dw \\
& -4\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw + \int_0^\theta e^{\sin^2 w}\cos^3 w dw + 2\,\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
s_{0,33}(\theta) = \;& -\cos^2\theta\Big(-\textstyle\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw - 2\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw \\
& +4\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw - 2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw \\
& -4\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw + 8\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\Big),\\
s_{0,34}(\theta) = \;& -\cos^2\theta\Big(-\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw - 2\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw \\
& +4\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w dw - 2\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w\cos^2 w dw \\
& -4\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w\cos^2 w dw + 8\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\sin^3 w\cos^2 w dw\Big),\\
s_{0,35}(\theta) = \;& -2\cos^6\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw + \int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw \\
& -\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw - 4\cos^6\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw \\
& +2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw - 2\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw,\\
s_{0,36}(\theta) = \;& -\cos\theta\sin\theta\Big(-\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw - \int_0^\theta \mathrm{e}^{\sin w^2}\cos w dw \\
& +2\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos w dw + 2\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw + 2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw \\
& -4\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw - \cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw - \int_0^\theta \mathrm{e}^{\sin w^2}\cos^3 w dw \\
& +2\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\Big),\\
s_{0,37}(\theta) = \;& -\cos^3\theta\sin\theta\Big(-2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw - 4\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw \\
& +\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw + 2\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\Big),\\
s_{0,38}(\theta) = \;& -\cos^2\theta\Big(2\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw + \int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw \\
& -3\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^2 w\sin w dw + 4\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw \\
& +2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw - 6\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^4 w\sin w dw\Big),
\end{aligned}
$$

$$
\begin{aligned}
s_{0,39}(\theta) = \;& \cos\theta\sin\theta\Big(2\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw + 2\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^5 w dw - 4\cos^4\theta\int_0^\theta \mathrm{e}^{\sin w^2}\cos^5 w dw \\
& -\cos^2\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw - \int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw + 2\cos^4\theta\int_0^\theta \mathrm{e}^{\sin^2 w}\cos^3 w dw\Big),
\end{aligned}
$$

$$s_{0,40}(\theta) = 2\cos^6\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw - \int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw + \cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^2 w\sin w dw$$
$$-4\cos^6\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw + 2\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw - 2\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^4 w\sin w dw,$$
$$s_{0,41}(\theta) = -\cos^2\theta\Bigg(2\cos^4\theta\int_0^\theta e^{\sin^2 w}\sin^3 w dw + \int_0^\theta e^{\sin^2 w}\sin^3 w dw - 3\cos^2\theta\int_0^\theta e^{\sin^2 w}\sin^3 w dw$$
$$+(4\cos^4\theta + 2 - 6\cos^2\theta)\int_0^\theta e^{\sin^2 w}\sin^3 w\cos^2 w dw,$$
$$s_{0,42}(\theta) = -\cos\theta\sin\theta\Bigg(2\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos w dw + \int_0^\theta e^{\sin w^2}\cos w dw - 3\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos w dw$$
$$-4\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw - 2\int_0^\theta e^{\sin^2 w}\cos^5 w dw + 6\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^5 w dw$$
$$+2\cos^4\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw + \int_0^\theta e^{\sin^2 w}\cos^3 w dw - 3\cos^2\theta\int_0^\theta e^{\sin^2 w}\cos^3 w dw\Bigg).$$

Now we have

$$\begin{aligned}\tilde{s}_1(\theta) = {} & \tilde{s}_{1,1}(\theta)c_{03}c_{12} + \tilde{s}_{1,2}(\theta)c_{03}c_{30} + \tilde{s}_{1,3}(\theta)a_{03}c_{03} + \tilde{s}_{1,4}(\theta)a_{21}c_{03} + \tilde{s}_{1,5}(\theta)c_{30}c_{12} + \tilde{s}_{1,6}(\theta)c_{12}a_{21}\\ & +\tilde{s}_{1,7}(\theta)c_{30}a_{21} + \tilde{s}_{1,8}(\theta)a_{03}a_{21} + \tilde{s}_{1,9}(\theta)a_{03}c_{30} + \tilde{s}_{1,10}(\theta)a_{03}c_{12} + \tilde{s}_{1,11}(\theta)a_{12}c_{03} + \tilde{s}_{1,12}(\theta)a_{12}c_{21}\\ & +\tilde{s}_{1,13}(\theta)c_{03}c_{21} + \tilde{s}_{1,14}(\theta)a_{21}c_{21} + \tilde{s}_{1,15}(\theta)a_{21}a_{12} + \tilde{s}_{1,16}(\theta)c_{12}a_{12} + \tilde{s}_{1,17}(\theta)a_{03}c_{21}\\ & +\tilde{s}_{1,18}(\theta)a_{03}a_{12} + \tilde{s}_{1,19}(\theta)c_{30}c_{21} + \tilde{s}_{1,20}(\theta)c_{30}a_{12} + \tilde{s}_{1,21}(\theta)c_{12}c_{21} + \tilde{s}_{1,22}(\theta)a_{12}^2\\ & +\tilde{s}_{1,23}(\theta)c_{21}^2 + \tilde{s}_{1,24}(\theta)c_{30}^2 + \tilde{s}_{1,25}(\theta)a_{03}^2 + \tilde{s}_{1,26}(\theta)c_{03}^2 + \tilde{s}_{1,27}(\theta)a_{21}^2 + \tilde{s}_{1,28}(\theta)c_{12}^2,\end{aligned}$$

and $\tilde{s}_{1,i}(\theta)$ for $i = 1\cdots, 28$ satisfying the following expressions:

$$\tilde{s}_{1,1}(\theta) = -\frac{1}{12}\left(16\cos^8\theta + 34\cos^6\theta + 37\cos^4\theta + 8\cos^2\theta - 5\right)\left(\cos^2\theta - 1\right)^2,$$
$$\tilde{s}_{1,2}(\theta) = \frac{1}{12}\left(\cos^2\theta - 1\right)\left(16\cos^{10}\theta + 38\cos^8\theta + 53\cos^6\theta + 15\cos^4\theta - 7\cos^2\theta - 7\right),$$
$$\tilde{s}_{1,3}(\theta) = -\frac{1}{12}\left(16\cos^8\theta + 14\cos^6\theta + 15\cos^4\theta - 16\cos^2\theta + 1\right)\left(\cos^2\theta - 1\right)^2,$$
$$\tilde{s}_{1,4}(\theta) = \frac{1}{12}\left(\cos^2\theta - 1\right)\left(16\cos^{10}\theta + 18\cos^8\theta + 19\cos^6\theta - 15\cos^4\theta - \cos^2\theta - 1\right),$$
$$\tilde{s}_{1,5}(\theta) = \frac{1}{12}\cos\theta\sin^5\theta\left(2\cos^2\theta + 1\right)\left(8\cos^4\theta + 12\cos^2\theta + 7\right),$$
$$\tilde{s}_{1,6}(\theta) = \frac{1}{12}\cos\theta\sin^5\theta\left(16\cos^6\theta + 12\cos^4\theta - 2\cos^2\theta + 1\right),$$
$$\tilde{s}_{1,7}(\theta) = \frac{1}{6}\cos^3\theta\sin^3\theta\left(8\cos^6\theta + 8\cos^4\theta + 5\cos^2\theta - 3\right),$$
$$\tilde{s}_{1,8}(\theta) = \frac{1}{12}\cos\theta\sin^5\theta\left(-1 + 2\cos^2\theta\right)\left(8\cos^4\theta + 1\right),$$
$$\tilde{s}_{1,9}(\theta) = \frac{1}{12}\cos\theta\sin^5\theta\left(16\cos^6\theta + 12\cos^4\theta + 6\cos^2\theta - 7\right),$$
$$\tilde{s}_{1,10}(\theta) = \frac{1}{6}\cos\theta\sin^7\theta\left(8\cos^4\theta + 4\cos^2\theta - 3\right),$$
$$\tilde{s}_{1,11}(\theta) = -\frac{4}{3}\cos^3\theta\sin^3\theta\left(\cos^6\theta + \cos^4\theta + \cos^2\theta - 1\right),$$
$$\tilde{s}_{1,12}(\theta) = -\frac{2}{3}\cos^5\theta\sin^3\theta\left(4\cos^4\theta - \cos^2\theta - 1\right),$$
$$\tilde{s}_{1,13}(\theta) = \frac{2}{3}\cos^3\theta\sin\theta\left(-5\cos^2\theta - 2 + 5\cos^6\theta + 4\cos^4\theta + 4\cos^8\theta\right),$$

$$\tilde{s}_{1,14}(\theta) = -\frac{1}{12}\sin^2\theta\cos^2\theta\left(32\cos^8\theta - 4\cos^6\theta - 6\cos^4\theta - 3\cos^2\theta - 1\right),$$
$$\tilde{s}_{1,15}(\theta) = \frac{1}{12}\sin^4\theta\cos^2\theta\left(-1 + 2\cos^2\theta\right)\left(8\cos^4\theta + \cos^2\theta + 1\right),$$
$$\tilde{s}_{1,16}(\theta) = \frac{1}{12}\sin^6\theta\cos^2\theta\left(16\cos^4\theta + 10\cos^2\theta - 5\right),$$
$$\tilde{s}_{1,17}(\theta) = -\frac{1}{12}\sin^4\theta\cos^2\theta\left(32\cos^6\theta - 12\cos^4\theta - 6\cos^2\theta + 1\right),$$
$$\tilde{s}_{1,18}(\theta) = \frac{1}{12}\sin^6\theta\cos^2\theta\left(8\cos^2\theta - 1\right)\left(-1 + 2\cos^2\theta\right),$$
$$\tilde{s}_{1,19}(\theta) = -\frac{1}{12}\sin^2\theta\cos^2\theta\left(32\cos^8\theta + 36\cos^6\theta + 14\cos^4\theta - 21\cos^2\theta - 7\right),$$
$$\tilde{s}_{1,20}(\theta) = \frac{1}{12}\sin^4\theta\cos^2\theta\left(16\cos^6\theta + 14\cos^4\theta + 7\cos^2\theta - 7\right),$$
$$\tilde{s}_{1,21}(\theta) = -\frac{1}{12}\sin^4\theta\cos^2\theta\left(32\cos^6\theta + 28\cos^4\theta - 10\cos^2\theta - 5\right),$$
$$\tilde{s}_{1,22}(\theta) = \frac{1}{3}\cos^5\theta\sin^5\theta\left(-1 + 2\cos^2\theta\right),$$
$$\tilde{s}_{1,23}(\theta) = \frac{1}{3}\cos^5\theta\sin\theta\left(2\cos^2\theta + 1\right)\left(4\cos^4\theta - 1 - 2\cos^2\theta\right),$$
$$\tilde{s}_{1,24}(\theta) = \frac{1}{12}\cos^3\theta\sin^3\theta\left(2\cos^2\theta + 1\right)\left(4\cos^4\theta + 7\cos^2\theta + 7\right),$$
$$\tilde{s}_{1,25}(\theta) = \frac{1}{12}\cos\theta\sin^7\theta\,(2\cos\theta + 1)\,(2\cos\theta - 1)\left(-1 + 2\cos^2\theta\right),$$
$$\tilde{s}_{1,26}(\theta) = \frac{1}{3}\cos\theta\sin\theta\left(\cos^4\theta + 2\cos^2\theta + 3\right)\left(2\cos^6\theta - 1 + \cos^4\theta\right),$$
$$\tilde{s}_{1,27}(\theta) = \frac{1}{12}\cos^3\theta\sin^3\theta\left(-1 + 2\cos^2\theta\right)\left(4\cos^4\theta + \cos^2\theta + 1\right),$$
$$\tilde{s}_{1,28}(\theta) = \frac{1}{12}\cos\theta\sin^7\theta\left(4\cos^2\theta + 5\right)\left(2\cos^2\theta + 1\right).$$

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