Aggregation Operators and Ruled Surfaces

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Abstract. Aggregation operators in two variables that are ruled quadric surfaces
are studied.

The interest lays in the fact that the most popular aggregation operators are inde-
deed ruled surfaces, either planes or quadric surfaces.

Keywords. Aggregation operator, Idempotent, Symmetric, Ruled quadric surface

Introduction

The most popular aggregation operators (in two variables), namely arithmetic, geo-
metric, quadratic and harmonic means, OWA operators, the Minimum t-norm and Maximum
t-conorm, are ruled surfaces. Geometric, quadratic and harmonic means are quadric sur-
faces while the graphics of the other ones consist of piecewise plane surfaces.

This paper studies other ruled quadric surfaces that correspond to aggregation oper-
ators. In this way new families of aggregation operators, some of them combinations of
the previous ones, are obtained.

Let us recall the definition of aggregation operator (in two variables).

\textbf{Definition 0.1.} [1] An aggregation operator (in two variables) is a map \( h : [0, 1]^2 \to [0, 1] \) satisfying for all \( x, y, x_1, x_2, y_1, y_2 \in [0, 1] \)

1. \( h(0, 0) = 0 \) and \( h(1, 1) = 1 \)
2. \( h(x_1, y_1) \leq h(x_2, y_2) \) if \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \) (monotonicity).

\( h \) is idempotent if and only if \( h(x, x) = x \). \( h \) is symmetric if and only if \( h(x, y) = h(y, x) \).

1. Ruled Quadric Surfaces

\textbf{Definition 1.1.} A quadric surface is a surface defined in implicit form by a second degree
polynomial

\[ ax^2 + by^2 + cz^2 + dxy + exz + fy + gx + hy + iz + j = 0. \] (1)

In order to find the ruled quadric surfaces which are aggregation operators, we will
consider separately the cases \( c \neq 0 \) and \( c = 0 \).
1.1. Case $c = 0$

If $c = 0$, then isolating $z$ from Eq. (1) we obtain

$$z = \frac{ax^2 + by^2 + dx + gy + j}{ex + fy + i}. \quad (2)$$

Replacing $e$ by $-e$, $f$ by $-f$ and $i$ by $-i$, Eq. (2) is

$$z = \frac{ax^2 + by^2 + dx + gy + j}{ex + fy + i}.$$

If we want the last map to be symmetric, we must have $b = a$, $g = h$ and $e = f$, obtaining

$$z = \frac{ax^2 + ay^2 + dx + gy + j}{ex + ey + i}.$$

$z(0, 0)$ must be 0. From this we have $j = 0$.

If $z$ is idempotent, writing explicitly $z(x, x) = x$ we obtain

$$z(x, x) = \frac{2ax^2 + dx + 2gx}{2ex + i} = x$$

or

$$(2a - 2e + d)x^2 = (i - 2g)x.$$  

This equation is satisfied for all $x \in [0, 1]$ if and only if

$$d = 2e - 2a.$$

and

$$i = 2g.$$

The formula of the quadric surface becomes then

$$z = \frac{ax^2 + ay^2 + (2e - 2a)xy + gx + gy}{ex + ey + 2g}. \quad (3)$$

Now we can consider two cases: $e \neq 0$ and $e = 0$. 

1.1.1. Case \( c = 0 \) and \( e \neq 0 \)

In this case we can divide the numerator and the denominator of Eq. (3) by \( e \). Renaming \( a \) by \( a \), and \( g \) by \( g \), we get

\[
z = \frac{a(x - y)^2 + 2xy + gx + gy}{x + y + 2g}.
\]

The denominator must be different from 0 for all \( x, y \in (0, 1) \). This means

\[g \geq 0 \text{ or } g \leq -1.\]

For \( x = 0 \) and \( y = 1 \), we obtain

\[
z(0,1) = \frac{a + g}{1 + 2g}
\]

This value must be between 0 and 1. Imposing that it must be greater or equal than 0, we obtain the following conditions for \( a \) and \( g \).

\[g \geq -\frac{1}{2} \text{ and } a \geq -g\]

or

\[g \leq -\frac{1}{2} \text{ and } a \leq -g\]

Imposing that it must be smaller or equal than 1, we obtain the following conditions for \( a \) and \( g \).

\[g \geq -\frac{1}{2} \text{ and } a \leq g + 1\]

or

\[g \leq -\frac{1}{2} \text{ and } a \geq g + 1\]

The partial derivatives \( \frac{\partial z}{\partial x} \) at \((1,0)\) and \( \frac{\partial z}{\partial x} \) at \((0,1)\) must be greater or equal than 0.

\[
\frac{\partial z}{\partial x}(1,0) = \frac{(2a + g)(1 + 2g) - a - g}{(1 + 2g)^2} \geq 0
\]

is satisfied if and only if

\[g \geq -\frac{1}{4} \text{ and } a \geq -\frac{2g^2}{1 + 4g}\]

or
\[ g \leq -\frac{1}{4} \quad \text{and} \quad a \leq \frac{-2g^2}{1 + 4g} \]

\[
\frac{\partial z}{\partial x}(0, 1) = \frac{(-2a + g + 2)(1 + 2g) - a - g}{(1 + 2g)^2} \geq 0
\]

is satisfied if and only if

\[ g \geq -\frac{3}{4} \quad \text{and} \quad a \leq \frac{2 + 2g^2 + 4g}{3 + 4g} \]

or

\[ g \leq -\frac{3}{4} \quad \text{and} \quad a \geq \frac{2 + 2g^2 + 4g}{3 + 4g}. \]

Sumarizing, the conditions on \( g \) and \( a \) are

\[ g \geq 0 \quad \text{and} \quad \frac{-2g^2}{1 + 4g} \leq a \leq \frac{2 + 2g^2 + 4g}{3 + 4g} \]

or

\[ g \leq -1 \quad \text{and} \quad \frac{2 + 2g^2 + 4g}{3 + 4g} \leq a \leq \frac{-2g^2}{1 + 4g}. \]

1.1.2. Case \( c = 0 \) and \( e = 0 \)

In this case, putting \( \frac{a}{2g} = b \),

\[ z = b(x - y)^2 + \frac{x + y}{2}. \]

\( z(1, 0) \) is then

\[ b + \frac{1}{2}. \]

Imposing again that this value must be between 0 and 1, we get that

\[ -\frac{1}{2} \leq b \leq \frac{1}{2}. \]

Imposing that the partial derivative \( \frac{\partial z}{\partial x}(1, 0) \) must be greater or equal than 0, we get

\[ b \geq -\frac{1}{4}. \]

Imposing that the partial derivative \( \frac{\partial z}{\partial x}(0, 1) \) must be greater or equal than 0, we get
\[ b \leq \frac{1}{4}. \]

Summarizing,

\[ -\frac{1}{4} \leq b \leq \frac{1}{4}. \]

or

\[ g \leq 0 \text{ and } \frac{g}{2} \leq a \leq -\frac{g}{2}. \]

1.2. Case \( c \neq 0 \)

If \( c \neq 0 \), we can divide Eq. (1) by \( c \). Renaming \( \frac{a}{c} \) by \(-a\), \( \frac{b}{c} \) by \(-b\), etc, we obtain

\[ z = \frac{1}{2} \left( ex + fy + i \pm \sqrt{(ex + fy + i)^2 - 4ax^2 - 4by^2 - 4gx - 4dx} - 4hy - 4j \right). \]

If we impose symmetry we get

\[ z = \frac{1}{2} \left( ex + ey + i \pm \sqrt{(e(x + y) + i)^2 - 4ax^2 - 4ay^2 - 4gx - 4dx} - 4gy - 4j \right). \] (4)

We can distinguish the cases where the square root is added or subtracted.

1.2.1. Adding the square root

In this case, Eq. (4) becomes

\[ z = \frac{1}{2} \left( ex + ey + i + \sqrt{(e(x + y) + i)^2 - 4ax^2 - 4ay^2 - 4gx - 4dx} - 4gy - 4j \right). \]

Imposing \( z(0, 0) = 0 \), we get

\[ i + \sqrt{i^2 - 4j} = 0 \]

and therefore \( i \leq 0 \) and \( j = 0 \).

From \( z(1, 1) = 1 \), we get

\[ 2 = 2e + i + \sqrt{(2e + i)^2 - 8a - 8g - 4d} \]

and form this, \( 1 - 2e - i + 2a + 2g + d = 0 \).

From \( z(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} \), we get

\[ 1 = e + i + \sqrt{(e + i)^2 - 2a - 4g - 2d} \]

and form this, \( 1 - 2e - 2i + 2a + 4g + d = 0 \).

So \( i = 2g \) (and \( g \leq 0 \)) and \( d = -1 + 2e - 2a \).

Now imposing \( z(k, k) = k \) we get
\[ k = \frac{1}{2} \left( 2ek + i + \sqrt{(2ke + i)^2 - 8ak^2 - 8gk - 4dk^2} \right). \]

which is equivalent to

\[ k = \frac{1}{2} \left( 2ek + i + \sqrt{(2ke + i)^2 - 8ek^2 - 4ik + 4k^2} \right) = \frac{1}{2} \left( 2ek + i + \sqrt{(2ke + i - 2k)^2} \right). \]

Then

\[ 2ke + i \leq 2k \text{ for all } k \in [0, 1]. \]

This is satisfied for all \( k \in [0, 1] \) if and only if

\[ 2e + i \leq 2. \quad (5) \]

Putting \( b = \frac{e}{2} \), the equation of the quadric surface is then

\[ z = b(x + y) + g + \sqrt{(b(x + y) + g)^2 - a(x - y)^2 - g(x + y) + (1 - 4b)xy}. \]

and Eq. 5 becomes

\[ 2b + g \leq 1. \]

\[ z(1, 0) = b + g + \sqrt{b + g)^2 - a - g} \]

which implies

\[ g + a \leq (b + g)^2. \]

\[ 0 \leq z(1, 0) \leq 1 \text{ gives} \]

\[ b + g \leq 1, g + 2b - a \leq 1 \]

and if \( b + g \leq 0 \), then

\[ a + g \leq 0. \]

Now imposing that \( \frac{\partial z}{\partial x} (1, 0) \geq 0 \) we obtain

\[ b + \frac{1}{2} \frac{2b^2 + 2bg - 2a - g}{\sqrt{(b + g)^2 - a - g}} \geq 0 \]

and \( \frac{\partial z}{\partial x} (0, 1) \geq 0 \) gives

\[ b + \frac{1}{2} \frac{2b^2 + 2bg + 2a - g + 1 - 4b}{\sqrt{(b + g)^2 - a - g}} \geq 0. \]
1.2.2. Subtracting the square root

\[ z(0, 0) = 0 \text{ implies } \]
\[ j = 0 \text{ and } i \geq 0. \]

\[ z(1, 1) = 1 \text{ implies } \]
\[ 1 - 2e - i + 2a + 2g + d = 0 \quad (6) \]

and \[ z\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} \]

\[ 1 - 2e - 2i + 2a + 4g + d = 0. \quad (7) \]

From Eqs. (6) and (7), we get

\[ i - 2g = 0 \text{ (and } g \geq 0) \]

and

\[ d = -1 + 2e - 2a. \]

Now imposing \( z(k, k) = k \) we get

\[ k = \frac{1}{2} \left( 2ek + i - \sqrt{(2ke + i)^2 - 8ak^2 - 8gk - 4dk^2} \right). \]

which is equivalent to

\[ k = \frac{1}{2} \left( 2ek + i - \sqrt{(2ke + i)^2 - 8ek^2 - 4jk^2 + 4k^2} \right) \]

\[ = \frac{1}{2} \left( 2ek + i - \sqrt{(2ke + i - 2k)^2} \right). \]

Then

\[ 2ke + i \geq 2k \text{ for all } k \in [0, 1]. \]

This inequality is satisfied for all \( k \in [0, 1] \) if and only if

\[ 2e + i \geq 2. \]

Putting \( b = \frac{e}{2} \), the equation of the quadric surface is then

\[ z = b(x + y) + g - \sqrt{(b(x + y) + g)^2 - a(x - y)^2 - g(x + y) + (1 - 4b)xy}. \]

and Eq. 1.2.2 becomes

\[ 2b + g \geq 1. \]
\[ z(1, 0) = b + g - \sqrt{b + g}^2 - a - g. \]

From this,
\[ a + g \leq (b + g)^2. \]

If \( 0 \leq z(1, 0) \leq 1 \) gives
\[ g + a \geq 0 \]
and if \( b + g \geq 1 \), then
\[ g + 2b - a \geq 1. \]

Now imposing that \( \frac{\partial z}{\partial x}(1, 0) \geq 0 \) we obtain
\[ b - \frac{1}{2} \frac{2b^2 + 2bg - 2a - g}{\sqrt{(b + g)^2 - a - g}} \geq 0. \]

From \( \frac{\partial z}{\partial x}(0, 1) \geq 0 \) we obtain
\[ b - \frac{1}{2} \frac{2b^2 + 2bg + 2a - g + 1 - 4b}{\sqrt{(b + g)^2 - a - g}} \geq 0. \]

2. Concluding remarks

The quadric ruled surfaces that can be considered as idempotent and symmetric aggregation operators have been studied.

Table 1 summarizes the results obtained in this work.

In forthcoming works, other aggregation operators such as t-norms, t-conorms or uninorms whose graphics are quadric ruled surfaces will be studied (see [2]).

- If in the first equation \( a = g = 0 \), we recover the harmonic mean.
- If in the second equation \( b = 0 \), we recover the arithmetic mean.
- If in the third equation \( b = g = a = 0 \) we recover the geometric mean.
- If in the third equation \( a = -\frac{1}{2} \) and \( b = g = 0 \), we recover the quadratic mean.
- If in the third equation \( a = g = 0 \) and \( b = \frac{1}{2} \), we recover the Maximum aggregation operator.
- If in the fourth equation \( a = g = 0 \) and \( b = \frac{1}{2} \), we recover the Minimum aggregation operator.
- If \( \frac{1}{2} \leq p \leq 1 \) and in the third equation \( a = g = 0 \) and \( b = \frac{p}{2} \), we recover the OWA operator with weights \( p \) and \( 1 - p \).
- If \( 0 \leq p \leq \frac{1}{2} \) and in the fourth equation \( a = g = 0 \) and \( b = \frac{p}{2} \), we recover the OWA operator with weights \( p \) and \( 1 - p \).
Table 1. Ruled quadric surfaces that are idempotent and symmetric aggregation operators

\begin{align*}
1 & \quad z = \frac{a(x-y)^2 + 2xy + gx + gy}{x+y+2g} \\
& \text{or} \\
& \quad g \leq -1 \text{ and } rac{2+2g^2+4g}{3+4g} \leq a \leq -\frac{2g^2}{1+4g} \\
2 & \quad z = b(x-y)^2 + \frac{x+y}{2} \\
& \quad -\frac{1}{4} \leq b \leq \frac{1}{4} \\
3 & \quad z = b(x+y) + g + \sqrt{(b(x+y)+g)^2} - a(x-y)^2 - g(x+y) + (1-4b)xy \\
& \quad g \leq \min \left( 0, 1-2b, 1-b \right) \\
& \quad -1 + g + 2b \leq a \leq (b+g)^2 - g \\
& \quad b + \frac{1}{2} \frac{2b^2 + 2bg - 2a-g}{\sqrt{(b+g)^2-a-g}} \geq 0 \\
& \quad b + \frac{1}{2} \frac{2b^2 + 2bg + 2a-g+1-4b}{\sqrt{(b+g)^2-a-g}} \geq 0 \\
& \quad \text{If } b+g \leq 0, \text{ then } a + g \leq 0. \\
4 & \quad z = b(x+y) + g - \sqrt{(b(x+y)+g)^2} - a(x-y)^2 - g(x+y) + (1-4b)xy \\
& \quad g \geq \max \left( 0, 1-2b, -b \right) \\
& \quad a \leq \min \left( (b+g)^2 - g, 1+g + 2b \right) \\
& \quad b - \frac{1}{2} \frac{2b^2 + 2bg - 2a-g}{\sqrt{(b+g)^2-a-g}} \geq 0 \\
& \quad b - \frac{1}{2} \frac{2b^2 + 2bg + 2a-g+1-4b}{\sqrt{(b+g)^2-a-g}} \geq 0 \\
& \quad \text{If } b+g \geq 1, \text{ then } g + 2b - a \geq 1. \\
\end{align*}

Acknowledgements

Research partially supported by project number TIN2006-14311.

References