An example of a non acyclic Koszul complex of a module

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In his paper [3], F. Sancho de Salas defines the universal Koszul complex of a module $M$ over a sheaf of rings $\mathcal{O}$ as $\text{Kos}(M) = \Lambda(M) \otimes_{\mathcal{O}} S(M)$, where $\Lambda(M)$ and $S(M)$ stand for the exterior and symmetric algebras of $M$, endowed with the usual differential, and he conjectures (Conjecture 2.3.) that $\text{Kos}(M)$ is always acyclic. It is well known that for $M$ flat or $\mathcal{O}$ an algebra over a field of characteristic zero, this is true (see [1] and [2] for definitions and proofs). We give now an example that it fails in characteristic 2. Recall that $\text{Kos}^2(M)$, the homogeneous component of degree 2 of the Koszul complex $\text{Kos}(M)$, is

$$0 \to \Lambda^2(M) \xrightarrow{\partial_2,0} M \otimes M \xrightarrow{\partial_1,1} S^2(M) \to 0,$$

where $\partial_{2,0}(u \wedge v) = v \otimes u - u \otimes v$ and $\partial_{1,1}(u \otimes v) = uv$. Let $A$ be a local ring containing a field of characteristic 2 and let $x, y, z$ be a system of parameters. Let $I = (x, y, z)$ be the ideal they generate and take $u = x(y \wedge z)$ in $\Lambda^2(I)$. To see $u \neq 0$, consider the bilinear surjective map $f : I \times I \to I^2/I^{[2]}$ defined by $f(a,b) = ab + I^{[2]}$, where $I^{[2]}$ is the ideal generated by the 2th powers of all elements of $I$. Since $f$ vanishes over the elements $(a,a)$, it extends to an epimorphism $f : \Lambda^2(I) \to I^2/I^{[2]}$. Remark that if 2 were invertible, $I^2 = I^{[2]}$ and $f = 0$. Since the characteristic is 2, $I^{[2]} = (x^2, y^2, z^2)$ and $f(u) = xyz + I^{[2]} \neq 0$ (by The Monomial Conjecture, see for instance, Theorem 9.2.1 in [2], or simply take $A$ a regular ring and $x, y, z$ a regular sequence). Hence $u \neq 0$. On the other hand,

$$x(y \otimes z) = (xy) \otimes z = y(x \otimes z) = x \otimes (yz) = z(x \otimes y) = (xz) \otimes y = x(z \otimes y).$$

Therefore, $\partial_{2,0}(u) = x(z \otimes y) - x(y \otimes z) = 0$ and $H_2(\text{Kos}^2(M)) = \text{Ker}(\partial_{2,0}) \neq 0$. Remark that from general properties of the symmetric functor, it follows that $H_1(\text{Kos}(M)) = 0$. Thus, $\text{Kos}(M)$ is not a rigid complex.

References


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