VARIENTIONAL PRINCIPLES FOR MULTISYMPLECTIC
SECOND-ORDER CLASSICAL FIELD THEORIES

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Abstract

We state a unified geometrical version of the variational principles for second-order classical field theories. The standard Lagrangian and Hamiltonian variational principles and the corresponding field equations are recovered from this unified framework.

Key words: Second-order classical field theories; Variational principles; Unified, Lagrangian and Hamiltonian formalisms

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1 Introduction

As stated in [9], the field equations of a classical field theory arising from a partial differential Hamiltonian system (in the sense of [9]) are locally variational, that is, they can be derived using a variational principle. In this work we use the geometric Lagrangian-Hamiltonian formulation for second-order classical field theories given in [4] to state the variational principles for this kind of theories from a geometric point of view, thus giving a different point of view and completing previous works on higher-order classical field theories [1,8].

(All the manifolds are real, second countable and $C^\infty$. The maps and the structures are assumed to be $C^\infty$. Usual multi-index notation introduced in [6] is used).

2 Higher-order jet bundles

(See [6] for details). Let $M$ be an orientable $m$-dimensional smooth manifold, and let $\eta \in \Omega^m(M)$ be a volume form for $M$. Let $E \xrightarrow{\pi} M$ be a bundle with $\dim E = m + n$. If $k \in \mathbb{N}$, the $k$th-order jet bundle of the projection $\pi$, $J^k \pi$, is the manifold of the $k$-jets of local sections $\phi \in \Gamma(\pi)$; that is, equivalence classes of local sections of $\pi$ by the relation of equality on every partial derivative up to order $k$. A point in $J^k \pi$ is denoted by $j^k \phi$, where $x \in M$ and $\phi \in \Gamma(\pi)$ is a representative of the equivalence class. We have the following natural projections: if $r \leq k$,

$$
\begin{align*}
\pi^k_r: J^k \pi &\rightarrow J^r \pi \\
j^k_r \phi &\rightarrow j^r \phi
\end{align*}
$$

$$
\begin{align*}
\pi^k: J^k \pi &\rightarrow E \\
j^k \phi &\rightarrow x
\end{align*}
$$

Observe that $\pi^k_r \circ \pi^k_s = \pi^k_r$, $\pi^k_0 = \pi^k$, $\pi^k_k = \text{Id}_{j^k \pi}$, and $\pi^k = \pi \circ j^k$.

If local coordinates in $E$ adapted to the bundle structure are $(x^i, u^\alpha)$, $1 \leq i \leq m$, $1 \leq \alpha \leq n$, then local coordinates in $J^k \pi$ are denoted $(x^i, u^\alpha_I)$, with $0 \leq |I| \leq k$.

If $\psi \in \Gamma(\pi)$, we denote the $k$th prolongation of $\phi$ to $J^k \pi$ by $j^k \psi \in \Gamma(\pi)$.
Definition 1 A section $\psi \in \Gamma(\tilde{\pi}^k)$ is holonomic if $j^k(\pi^k \circ \psi) = \psi$; that is, $\psi$ is the $k$th prolongation of a section $\phi = \pi^k \circ \psi \in \Gamma(\pi)$.

In the following we restrict ourselves to the case $k = 2$. According to [1], consider the subbundle of fiber-affine maps $J^1\pi^1 \to \mathbb{R}$ which are constant on the fibers of the affine subbundle $(\tilde{\pi}^1)^*(\Lambda^2T^*M) \otimes (\pi^1)^*(V)\pi^1$ of $J^1\pi^1$ over $J^1\pi$. This subbundle is canonically diffeomorphic to the $\pi_{J^1\pi}$-transverse submanifold $J^2\pi^2$ of $\Lambda_2^m(J^1\pi)$ defined locally by the constraints $p_i^{jij} = p_i^{jij}$, which fibers over $J^1\pi$ and $M$ with projections $\pi_{J^1\pi}^1 : J^2\pi^2 \to J^1\pi$ and $\pi_{J^1\pi}^2 : J^2\pi^2 \to M$, respectively. The submanifold $j_s : J^2\pi^2 \hookrightarrow \Lambda_2^m(J^1\pi)$ is the extended 2-symmetric multimomentum bundle.

All the canonical geometric structures in $\Lambda_2^m(J^1\pi)$ restrict to $J^2\pi^2$. Denote $\Theta^1 = j_s^*\Theta_1 \in \Omega^m(J^2\pi^2)$ and $\Theta^2 = j_s^*\Omega_1 \in \Omega^{m+1}(J^2\pi^2)$ the pull-back of the Liouville forms in $\Lambda_2^m(J^1\pi)$, which we call the symmetrized Liouville forms.

Finally, let us consider the quotient bundle $J^2\pi^2 = J^2\pi^2 / \Lambda_2^m(J^1\pi)$, which is called the restricted 2-symmetric multimomentum bundle. This bundle is endowed with a natural quotient map, $\mu : J^2\pi^2 \to J^2\pi^2$, and the natural projections $\pi_{J^2\pi^2}^1 : J^2\pi^2 \to J^1\pi$ and $\pi_{J^2\pi^2}^2 : J^2\pi^2 \to M$. Observe that $\dim J^2\pi^2 = \dim J^2\pi^2 - 1$.

3 Lagrangian-Hamiltonian unified formalism

(See [1] for details). Let $\pi : E \to M$ be the configuration bundle of a second-order field theory, where $M$ is an orientable $m$-dimensional manifold with volume form $\eta \in \Omega^m(M)$, and $\dim E = m + n$. Let $\mathcal{L} \in \Omega^m(J^2\pi)$ be a second-order Lagrangian density for this field theory. The 2-symmetric jet-multimomentum bundles are

$$\mathcal{W} = J^3\pi \times_{J^1\pi} J^2\pi^2 \ ; \ \mathcal{W}_r = J^3\pi \times_{J^1\pi} J^2\pi^2.$$

These bundles are endowed with the canonical projections $\rho^1_1 : \mathcal{W}_r \to J^3\pi$, $\rho^2 : \mathcal{W} \to J^2\pi^2$, $\rho_2^2 : \mathcal{W}_r \to J^2\pi^2$, and $\rho^3_M : \mathcal{W}_r \to M$. In addition, the natural quotient map $\mu : J^2\pi^2 \to J^2\pi^2$ induces a natural submersions $\mu_\mathcal{W} : \mathcal{W} \to \mathcal{W}_r$.

Using the canonical structures in $\mathcal{W}$ and $\mathcal{W}_r$, we define a Hamiltonian section $\dot{h} \in \Gamma(\mu_\mathcal{W})$, which is specified by giving a local Hamiltonian function $\dot{H} \in C^\infty(\mathcal{W}_r)$. Then we define the forms $\Theta_r = (\rho_2 \circ \dot{h})^*\Theta \in \Omega^m(\mathcal{W}_r)$ and $\Omega_r = -d\Theta_r \in \Omega^{m+1}(\mathcal{W}_r)$. Finally, $\psi \in \Gamma(\rho^3_M)$ is holonomic in $\mathcal{W}_r$ if $\rho^3_M \circ \psi \in \Gamma(\bar{\pi}^3)$ is holonomic in $J^3\pi$.

The Lagrangian-Hamiltonian problem for sections associated with the system $(\mathcal{W}_r, \Omega_r)$ consists in finding holonomic sections $\psi \in \Gamma(\rho^3_M)$ satisfying

$$\psi^* i(X)\Omega_r = 0 \ , \ \text{for every } X \in \mathfrak{X}(\mathcal{W}_r). \quad (1)$$

Proposition 1 A section $\psi \in \Gamma(\rho^3_M)$ solution to the equation [1] takes values in a $n(m + m(m + 1)/2)$-codimensional submanifold $j_L : \mathcal{W}_r \to \mathcal{W}_r$ which is identified with the graph of a bundle map $\mathcal{F}\mathcal{L} : J^3\pi \to J^2\pi^2$ over $J^1\pi$ defined locally by

$$\mathcal{F}\mathcal{L}^*p_{\alpha}^i = \frac{\partial L}{\partial u_{\alpha}^i} - \sum_{j=1}^m \frac{1}{n(ij)} \frac{d}{dx^j} \left( \frac{\partial L}{\partial u_{\alpha+1}^{i+1,j}} \right) \ ; \ \mathcal{F}\mathcal{L}^*p_{\alpha}^i = \frac{\partial L}{\partial u^i}. $$

The map $\mathcal{F}\mathcal{L}$ is the restricted Legendre map associated with $\mathcal{L}$, and it can be extended to a map $\tilde{\mathcal{F}}\mathcal{L} : J^3\pi \to J^2\pi^2$, which is called the extended Legendre map.
4 Variational Principle for the unified formalism

If $\Gamma(\rho^r_M)$ is the set of sections of $\rho^r_M$, we consider the following functional (where the convergence of the integral is assumed)

$$\text{LH}: \Gamma(\rho^r_M) \rightarrow \mathbb{R}, \quad \psi \mapsto \int_M \psi^* \Theta_r$$

**Definition 2 (Generalized Variational Principle)** The Lagrangian-Hamiltonian variational problem for the field theory $(\mathcal{W}_r, \Omega_r)$ is the search for the critical holonomic sections of the functional $\text{LH}$ with respect to the variations of $\psi$ given by $\psi_t = \sigma_t \circ \psi$, where $\{\sigma_t\}$ is a local one-parameter group of any compact-supported $\rho^r_M$-vertical vector field $Z$ in $\mathcal{W}_r$, that is,

$$\left. \frac{d}{dt} \right|_{t=0} \int_M \psi_t^* \Theta_r = 0.$$

**Theorem 1** A holonomic section $\psi \in \Gamma(\rho^r_M)$ is a solution to the Lagrangian-Hamiltonian variational problem if, and only if, it is a solution to equation (1).

**(Proof)** This proof follows the patterns in [2] (see also [3]). Let $Z \in \mathfrak{X}^{V(\rho^r_M)}(\mathcal{W}_r)$ be a compact-supported vector field, and $V \subset M$ an open set such that $\partial V$ is a $(m-1)$-dimensional manifold and $\rho^r_M(\text{supp}(Z)) \subset V$. Then,

$$\left. \frac{d}{dt} \right|_{t=0} \int_M \psi_t^* \Theta_r = \left. \frac{d}{dt} \right|_{t=0} \int_V \psi_t^* \Theta_r = \int_V \psi^* \left( \lim_{t \rightarrow 0} \frac{\sigma_t^r \Theta_r - \Theta_r}{t} \right) = \int_V \psi^* L(Z) \Theta_r = \int_V \psi^* (i(Z) d \Theta_r + d i(Z) \Theta_r) = \int_V \psi^* (-i(Z) \Omega_r + d i(Z) \Theta_r)$$

$$= -\int_V \psi^* i(Z) \Omega_r + \int_V d (\psi^* i(Z) \Theta_r) = -\int_V \psi^* i(Z) \Omega_r + \int_{\partial V} \psi^* i(Z) \Theta_r$$

$$= -\int_V \psi^* i(Z) \Omega_r,$$

as a consequence of Stoke’s theorem and the assumptions made on the supports of the vertical vector fields. Thus, by the fundamental theorem of the variational calculus, we conclude

$$\left. \frac{d}{dt} \right|_{t=0} \int_M \psi_t^* \Theta_r = 0 \iff \psi^* i(Z) \Omega_r = 0,$$

for every compact-supported $Z \in \mathfrak{X}^{V(\rho^r_M)}(\mathcal{W}_r)$. However, since the compact-supported vector fields generate locally the $C^\infty(\mathcal{W}_r)$-module of vector fields in $\mathcal{W}_r$, it follows that the last equality holds for every $\rho^r_M$-vertical vector field $Z$ in $\mathcal{W}_r$. Now, for every $w \in \text{Im} \psi$, we have a canonical splitting of the tangent space of $\mathcal{W}_r$ at $w$ in a $\rho^r_M$-vertical subspace and a $\rho^r_M$-horizontal subspace,

$$T_w \mathcal{W}_r = V_w(\rho^r_M) \oplus T_w(\text{Im} \psi).$$

Thus, if $Y \in \mathfrak{X}(\mathcal{W}_r)$, then

$$Y_w = (Y_w - T_w(\psi \circ \rho^r_M)(Y_w)) + T_w(\psi \circ \rho^r_M)(Y_w) \equiv Y^V_w + Y^\psi_w,$$

with $Y^V_w \in V_w(\rho^r_M)$ and $Y^\psi_w \in T_w(\text{Im} \psi)$. Therefore

$$\psi^* i(Y) \Omega_r = \psi^* i(Y^V) \Omega_r + \psi^* i(Y^\psi) \Omega_r = \psi^* i(Y^\psi) \Omega_r,$$
since $\psi^* i(Y^*) \Omega_r = 0$, by the conclusion in the above paragraph. Now, as $Y^w_\psi \in T_w(\text{Im} \psi)$ for every $w \in \text{Im} \psi$, then the vector field $Y^\psi$ is tangent to $\text{Im} \psi$, and hence there exists a vector field $X \in \mathfrak{X}(M)$ such that $X$ is $\psi$-related with $Y^\psi$; that is, $\psi_* X = Y^\psi|_{\text{Im} \psi}$. Then $\psi^* i(Y^\psi) \Omega_r = i(X) \psi^* \Omega_r$. However, as $\dim \text{Im} \psi = \dim M = m$ and $\Omega_r$ is a $(m+1)$-form, we obtain that $\psi^* i(Y^\psi) \Omega_r = 0$. Hence, we conclude that $\psi^* i(Y) \Omega_r = 0$ for every $Y \in \mathfrak{X}(W_r)$.

Taking into account the reasoning of the first paragraph, the converse is obvious since the condition $\psi^* i(Y) \Omega_r = 0$, for every $Y \in \mathfrak{X}(W_r)$, holds, in particular, for every $Z \in \mathfrak{X}^V(\rho^*_M)(W_r)$.

\section{Lagrangian variational problem}

Consider the submanifold $j_L: W_L \hookrightarrow W_r$. Since $W_L$ is the graph of the restricted Legendre map, the map $\rho^L_1 = \rho^L_1 \circ j_L: W_L \rightarrow J^3 \pi$ is a diffeomorphism. Then we can define the Poincaré-Cartan $m$-form as $\Theta_L = (j_L \circ (\rho^L_1)^{-1})^* \Theta_r \in \Omega^m(J^3 \pi)$. This form coincides with the usual Poincaré-Cartan $m$-form derived in [5, 7].

Given the Lagrangian field theory $(J^3 \pi, \Omega_L)$, consider the following functional

$$
\mathbf{L}: \Gamma(\pi) \rightarrow \mathbb{R}
\phi \mapsto \int_M (j^3 \phi)^* \Theta_L
$$

**Definition 3 (Generalized Hamilton Variational Principle)** The Lagrangian variational problem (or Hamilton variational problem) for the second-order Lagrangian field theory $(J^3 \pi, \Omega_L)$ is the search for the critical sections of the functional $\mathbf{L}$ with respect to the variations of $\phi$ given by $\phi_t = \sigma_t \circ \phi$, where $\{\sigma_t\}$ is a local one-parameter group of any compact-supported $\phi \in \mathfrak{X}^V(\pi)(E)$; that is,

$$
\frac{d}{dt} \bigg|_{t=0} \int_M (j^3 \phi_t)^* \Theta_L = 0.
$$

**Theorem 2** Let $\psi \in \Gamma(\rho^*_M)$ be a holonomic section which is critical for the functional $\mathbf{LH}$. Then, $\phi = \pi^3 \circ \rho^L_1 \circ \psi \in \Gamma(\pi)$ is critical for the functional $\mathbf{L}$.

Conversely, if $\phi \in \Gamma(\pi)$ is a critical section for the functional $\mathbf{L}$, then the section $\psi = j_L \circ (\rho^L_1)^{-1} \circ j^3 \phi \in \Gamma(\rho^*_M)$ is holonomic and it is critical for the functional $\mathbf{LH}$.

(Proof) The proof follows the same patterns as in Theorem 1. The same reasoning also proves the converse.

\section{Hamiltonian variational problem}

Let $\mathcal{P} = \text{Im}(\mathcal{F} \mathcal{L}) \hookrightarrow J^2 \pi^\dagger$ and $\mathcal{P} = \text{Im}(\mathcal{F} \mathcal{L}) \hookrightarrow J^2 \pi^\dagger$ the image of the extended and restricted Legendre maps, respectively; $\pi_P: \mathcal{P} \rightarrow M$ the natural projection, and $\mathcal{F} \mathcal{L}_o: J^3 \pi \rightarrow \mathcal{P}$ the map defined by $\mathcal{F} \mathcal{L} = j \circ \mathcal{F} \mathcal{L}_o$.

A Lagrangian density $L \in \Omega^m(J^2 \pi)$ is almost-regular if (i) $\mathcal{P}$ is a closed submanifold of $J^2 \pi^\dagger$, (ii) $\mathcal{F} \mathcal{L}$ is a submersion onto its image, and (iii) for every $j^3_2 \phi \in J^3 \pi$, the fibers $\mathcal{F} \mathcal{L}^{-1}(\mathcal{F} \mathcal{L}(j^3_2 \phi))$ are connected submanifolds of $J^3 \pi$. 

The Hamiltonian section \( \hat{h} \in \Gamma(\mu_W) \) induces a Hamiltonian section \( h \in \Gamma(\mu) \) defined by \( \rho_\alpha \circ \hat{h} = h \circ \rho_\alpha^2 \). Then, we define the Hamilton-Cartan m-form in \( P \) as \( \Theta_h = (h \circ j)^* \Theta^1 \in \Omega^m(P) \). Observe that \( F\mathcal{L}_o \Theta_h = \Theta_L \).

In what follows, we consider that the Lagrangian density \( L \in \Omega^m(J^2\pi) \) is, at least, almost-regular. Given the Hamiltonian field theory \( (P, \Omega_h) \), let \( \Gamma(\bar{\pi}_P) \) be the set of sections of \( \bar{\pi}_P \). Consider the following functional

\[
H : \Gamma(\bar{\pi}_P) \rightarrow \mathbb{R} \\
\psi_h \mapsto \int_M \psi_h^* \Theta_P
\]

**Definition 4 (Generalized Hamilton-Jacobi Variational Principle)** The Hamiltonian variational problem (or Hamilton-Jacobi variational problem) for the second-order Hamiltonian field theory \( (P, \Omega_h) \) is the search for the critical sections of the functional \( H \) with respect to the variations of \( \psi_h \) given by \( (\psi_h)_t = \sigma_t \circ \psi_h \), where \( \{\sigma_t\} \) is a local one-parameter group of any compact-supported \( Z \in X^{V(\bar{\pi}_P)}(P) \),

\[
\left. \frac{d}{dt} \right|_{t=0} \int_M (\psi_h)_t^* \Theta_h = 0.
\]

**Theorem 3** Let \( \psi \in \Gamma(\rho^r_M) \) be a critical section of the functional \( L\mathcal{H} \). Then, the section \( \psi_h = F\mathcal{L}_o \circ \rho^r_1 \circ \psi \in \Gamma(\bar{\pi}_P) \) is a critical section of the functional \( H \).

Conversely, if \( \psi_h \in \Gamma(\bar{\pi}_P) \) is a critical section of the functional \( H \), then the section \( \psi = j_L \circ (\rho^r_1)^{-1} \circ \gamma \circ \psi_h \in \Gamma(\rho^r_M) \) is a critical section of the functional \( L\mathcal{H} \), where \( \gamma \in \Gamma_P(F\mathcal{L}_o) \) is a local section of \( F\mathcal{L}_o \).

(Proof) The proof follows the same patterns as in Theorem 1. The same reasoning also proves the converse, bearing in mind that \( \gamma \in \Gamma_P(F\mathcal{L}_o) \) is a local section.

\[\blacksquare\]

7 The higher-order case

As stated in [4], this formulation fails when we try to generalize it to a classical field theory of order greater or equal than 3. The main obstruction to do so is the relation among the multimomentum coordinates used to define the submanifold \( J^2\pi^1, p_{ij} = p_{ji}^{\alpha} \) for every \( 1 \leq i, j \leq m \) and every \( 1 \leq \alpha \leq n \). Although this “symmetry” relation on the multimomentum coordinates can indeed be generalized to higher-order field theories, it only holds for the highest-order multimomenta. That is, this relation on the multimomenta is not invariant under change of coordinates for lower orders, and hence we do not obtain a submanifold of \( \Lambda_2^m(J^{k-1}\pi) \).

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