Excitation of Multiple Spatial Modes of a MEMS Cantilever in the Pulsed Digital Oscillator

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Abstract—The aim of this paper is to apply an approach that will allow us to consider different mechanical modes of a MEMS cantilever in the form of separate mass-spring-damper equations with the appropriate form of an external driving. In the paper, we focus on a specific MEMS cantilever and use a Pulsed Digital Oscillator (PDO) to keep self-sustained oscillations of the mechanical structure. By applying the order-reduction procedure to a partial differential equation that describes the transversal deflections, we obtain a system of coupled ordinary differential equations that describes the excitation of multiple spatial modes. On the basis of these ordinary differential equations, we formulate a set of iterative maps to describe the evolution of the modes between two sampling events. The numerical simulations we present are focused on the most common case when the first two mechanical modes are taken into consideration.

I. INTRODUCTION

Resonant sensors exploit mechanical structures that are excited in one of their mechanical modes. These mechanical structures can be beams [1]–[3] or plates (membranes) [4] with additional supporting elements (springs, arms, etc.). Typically, a plate that is clamped across one of its edges (usually at the short end(s)) displays predominantly one-dimensional oscillations and can be modelled by using equations for elastic beams.

The dynamics of continuous structures in MEMS are described by partial differential equations (PDEs) with appropriate boundary conditions. The most typical way to deal with the problem is to reduce distributed systems in the form of PDEs to lumped systems in the form of ordinary differential equations (ODEs). These ODEs usually take the form of the mass-spring-damper equation. Linear or nonlinear mass-spring-damper equations are a typical and yet effective way to model the dynamics of a variety of MEMS systems [4]–[7].

There are a number of procedures that allow us to develop the reduced-order models from the original PDEs (see [8], [9] and the references cited there). A wide class of methods is based on eliminating the spatial dependencies in the PDEs by means of a set of spatial eigenmode functions \( \psi_i(x, y, z) \), which in a general case is infinite. As a result, we will obtain a set of ordinary differential equations that represent the dynamics of each spatial mode as the mass-spring-damper equation with appropriate parameters.

In this paper, we focus on a specific type of resonant MEMS device embedded inside a pulsed digital oscillator (PDO). The PDO is a large signal oscillator that utilises a micromechanical resonator in a feedback loop of a type well known from the domain of sigma-delta modulation. The general single-feedback topology of this oscillator is shown in Fig. 1. The position of the resonator is evaluated at each sampling time, and short force pulses (which can take two possible values depending on the position of the resonator) are applied to actuate the resonator. The output of the system is the bitstream provided by the 1-bit quantizer.

We start with the partial differential equation that describes the transversal deflections of a beam [10]. This PDE provides a more general description than a model based only on mass-spring-damper equations. We apply the procedure based on presenting a solution using linear mode shapes. As a result, we obtain a system of ordinary differential equations that are coupled via the external driving. We formulate a set of iterative equations (maps) to describe the evolution of the spatial modes between two sampling events. In this paper, we focus mostly on the simplest case when only the first two mechanical modes are taken into consideration. We present results of numerical simulations and show that depending on values of the sampling frequency, the system displays oscillations that can be associated with various spatial mode.

II. STATEMENT OF THE PROBLEM

In this section we focus on deriving the reduced-order model from the partial differential equation. Let us assume that the MEMS we study utilises a clamped-free cantilever of the type shown in Fig. 2. We consider that the beam is excited by the external force \( F(\xi, \tau) \). The equation and the boundary conditions that describe the transverse vibrations of the beam...
in the dimensionless form are as follows [10]:

\[
\frac{\partial^4 u}{\partial \xi^4} + \frac{\partial^2 u}{\partial \tau^2} + \gamma \frac{\partial u}{\partial \tau} = F(\xi, \tau),
\]

\[u(0, \tau) = \frac{\partial u(0, \tau)}{\partial \xi} = 0, \quad \frac{\partial^2 u(1, \tau)}{\partial \xi^2} = \frac{\partial^3 u(1, \tau)}{\partial \xi^4} = 0,
\]

with the dimensionless variables and parameters

\[\xi = x/l, \quad \tau = \frac{t}{l^2} \sqrt{\frac{EI}{\rho A}}, \quad \gamma = \frac{\eta l^2}{\sqrt{\rho AE}}.
\]

In (1), (2) and (3) \(u(x, t)\) is the transverse displacement at the position \(\xi\) and time \(\tau\), \(E\) is the Young’s modulus, \(I\) is the area of the cross-section, \(A\) is the moment of inertia of the cross-section (for beams with a rectangular cross-section \(I = bh^3/12\)), \(\rho\) is the density, \(A\) is the area of the cross-section, \(c\) is the dissipation coefficient, \(\gamma\) is the dimensionless dissipation parameter, \(l\) is the length of the beam, \(b\) is its width and \(h\) is its thickness.

Next, we discretise the continuous equation (1) by using a truncated set of linear mode shapes

\[u(\xi, \tau) = \sum_{i=1}^{M} \xi_i(\tau) \psi_i(\xi).
\]

For a clamped-free beam, these functions are [10]

\[\psi_i(\xi) = \cos(\Omega_i \xi) - \cos(\Omega_i \xi) + \nu_i \sin(\Omega_i \xi) - \sin(\Omega_i \xi),
\]

where \(\nu_i = (\cos(\Omega_i \xi) + \sin(\Omega_i \xi))/\sin(\Omega_i \xi) + \sin(\Omega_i \xi)\), and \(\Omega_i\) are the roots of the equation \(\cos \Omega \cosh \Omega = -1\). The first three spatial functions \(\psi_{1,2,3}(\xi)\) as functions of the axial coordinate are shown in Fig. 3. Later in the paper, we will refer to vibrations that correspond to these functions as spatial modes with the spatial frequencies \(\Omega_i\) and the temporal frequencies \(\Omega_i^2\).

Substituting (4) into (1), multiplying the latter equation by \(\psi_j\), integrating over the beam domain and taking into account the orthogonality of the functions \(\psi_j\), we obtain the system of ordinary differential equations

\[\ddot{x}_i + \gamma \dot{x}_i + \Omega_i^4 x_i = F_i(\tau), \quad i = 1, M.
\]

Note that the system (6) presents each spatial mode as the mass-spring-damper system with appropriate parameters such as the dissipation parameter \(\gamma\) and the temporal frequency \(\omega_0 = \Omega_i\).

The functions \(F_i\) in the right hand of the equations are the ‘projections’ of the external force to each eigenmode. They define what quantity of the force is transmitted to excite the oscillations of a mode:

\[F_i(\tau) = \int_0^1 \psi_i(\xi) F(\xi, \tau) d\xi.
\]

Now let us return to the PDO and rewrite the system of equations (6) for our specific case. The external driving has the following form:

\[F(\xi, \tau) = -\sigma F_0 \sum_{n} \text{sign}(u(\xi, \tau_n-L-1)) \delta(\xi - \xi_a) \delta(\tau - \tau_n),
\]

where \(L + 1\) is the number of delays in the feedback (see Fig. 1), \(\tau_n = n T_s\) is the instants of time at which we apply the force pulse, \(T_s = 1/\beta\) is the time between two impulses, \(\delta(x)\) is the Dirac delta function and \(\text{sign}(x)\) is the signum function.

In eq. (8), \(\xi_a\) is the point of the microcantilever at which the sensing system measures the resonator position and \(\xi_a\) is the point at which the actuating system applies the external driving in order to maintain the oscillations. If we restrict ourselves to the case of the first two spatial modes, the set of equations becomes

\[\ddot{x}_1 + \gamma \dot{x}_1 + \Omega_1^4 x_1 = -\sigma F_0 \sum_{n} b_{n-L-1} \delta(\tau - \tau_n),
\]

\[\ddot{x}_2 + \gamma \dot{x}_2 + \Omega_2^4 x_2 = -\sigma F_0 \sum_{n} b_{n-L-1} \delta(\tau - \tau_n),
\]

where \(b_n\) is the sequence of signs

\[b_n = \text{sign}(x_1(\tau_n) + \beta x_2(\tau_n)),
\]

the coefficients \(\beta\) and \(\mu\) describe the sensing and actuating systems

\[\beta = \psi_2(\xi_a)/\psi_1(\xi_a), \quad \mu = \psi_2(\xi_a)/\psi_1(\xi_a),
\]

and \(F_0\) are the amplitudes of the external driving for the \(i\)th spatial mode in the corresponding equation

\[F_{0i} = F_0 \psi_i(\xi_a).
\]

In this paper we assume that \(\xi_a = \xi_0 = 1\), i.e. \(\beta = \mu = -1\), and the sign of the feedback loop \(\sigma = -1\).
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of the external driving, the coefficients (11) and the temporal frequency $\Omega_2^2$. Thus, in the given range of the sampling ratio and at the chosen values of the coefficients $\beta$, $\mu$ and $\sigma$, the first spatial mode receives a larger ‘portion’ of the external driving and should be the predominant one.

Now let us give examples of the system dynamics in terms of the first and the second spatial modes. The results presented below were obtained by iterating the system of equation (13). Figure 4(a) shows typical limit cycles displayed by the variables $x_{1,n}, v_{1,n}$ (the limit cycle denoted as “1”) and $x_{2,n}, v_{2,n}$ (the limit cycle denoted as “2”). The profile of the beam is given in Fig. 4(b). The dashed lines show the corresponding profiles of the first and the second eigenmode and the solid line illustrates the actual profile of the beam that is formed as the superposition of the modes. The amplitude of the second mode is smaller than the amplitude of the first one, however it affects the overall profile of the cantilever. The values of parameters for the example are given in the figure caption.

However, at high values of the sampling frequency, one can find dominance of the second mode, as shown in Fig. 5. As can be seen, in this case the amplitude of the first spatial mode is smaller than that of the second one (see the limit cycles in Fig. 5(a)) Note that normally the amplitude of suppressed mode is significantly smaller than the amplitude of the dominating one, as is demonstrated in Fig. 5(a). The actual profile of the cantilever corresponds to the second mode and the distortion of the profile caused by the first one is negligible (see Fig. 5(b)).

V. CONCLUSIONS

In this paper we have developed a theory that can demonstrate the extraction of various spatial modes in the Pulsed Digital Oscillator for MEMS application. In order to obtain a system of mass-spring-damper equations that defines the dynamics of the spatial modes, we have applied the order-reduction procedure to a partial differential equation that describes the transversal deflections of a beam. Due to the pulsed nature of the external excitation, we have formulated a set of iterative maps to describe the evolution of the modes between two sampling events. We have presented the results of numerical simulations in terms of the spacial modes of the system. Depending on values of the sampling frequency, the system displays oscillations that can be associated with various spatial modes.

REFERENCES